

Appendix to Juhani Yli-Vakkuri's “Epistemicism and Modality”

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Relative to an assignment function, the two-dimensional models of section 4 evaluate formulas as true or false at pairs of elements of a set W , and so formulas can be understood as interpreted using sets of pairs of elements of W . Several specific ways of interpreting Δ based on a binary relation R on W are explored and rejected. The aim of this appendix is to provide the formal basis for generalizations of these arguments. To make these generalizations as strong as possible, Δ will be allowed to be interpreted in the most general way which respects the following compositionality constraint: under an assignment function, the interpretation of a formula of the form $\Delta\phi$ is determined by the interpretation of ϕ . Thus each pair $\langle w, v \rangle$ will be associated with a set of sets of pairs – the set of interpretations of formulas to which Δ is to apply in $\langle w, v \rangle$. Such functions are commonly studied under the labels of “neighborhood” and “Scott-Montague” semantics. So define a *generalized 2D model* to be a structure $\langle W, D, \llbracket \cdot \rrbracket \rangle$ such that W and $\llbracket \cdot \rrbracket$ are as in standard 2D models, and $D : W^2 \rightarrow \mathcal{P}(\mathcal{P}(W^2))$.

It will be useful to be able to refer easily to the set of pairs at which a given formula is true under a given assignment, so let $\llbracket \phi \rrbracket_g^{\mathfrak{A}} = \{\langle w, v \rangle \in W^2 : |\phi|_{w,v}^{\mathfrak{A}} = 1\}$. With this, extend the interpretation of formulas described in section 4 by the following clause for Δ :

$$|\Delta\phi|_{w,v}^{\mathfrak{A}} = 1 \text{ iff } \llbracket \phi \rrbracket_g^{\mathfrak{A}} \in D(\langle w, v \rangle)$$

The first result to be established concerns formulas of the language L_{VMQ} including propositional variables and quantifiers. For a class \mathcal{C} of generalized 2D models, let the *logic of* \mathcal{C} be the set of L_{VMQ} -formulas true at each proper point under each variable assignment in each model in \mathcal{C} . In the following, p will be used to indicate a propositional variable; note that assignment functions map such variables to “barcodes”, whereas atomic sentences which are not variables may be interpreted as arbitrary sets of pairs.

Proposition 1. *If \mathcal{C} is a class of generalized 2D models whose logic includes all instances of the schemas*

$$(1) \Delta(\phi \supset \psi) \supset (\Delta\phi \supset \Delta\psi) \text{ and}$$

$$(2) p \supset \Delta p,$$

and is closed under the schematic rule

$$(3) \phi / \Delta\phi,$$

then the logic of \mathbf{C} includes all instances of the schema

$$(4) \phi \supset \Delta\phi.$$

Proof. Consider any formula ϕ , model $\mathfrak{A} = \langle W, D, [\![\cdot]\!] \rangle$ in \mathbf{C} , $w \in W$ and assignment function g such that $|\phi|_{w,w}^{\mathfrak{A}^g} = 1$; we show that $|\Delta\phi|_{w,w}^{\mathfrak{A}^g} = 1$. For any formula ψ , we write $\vdash \psi$ for ψ being in the logic of \mathbf{C} .

That $|\Delta\phi|_{w,w}^{\mathfrak{A}^g} = 1$ will be shown as follows: We produce an assignment function f which agrees with g on the free variables in ϕ and a formula ψ such that $|\Delta\psi|_{w,w}^{\mathfrak{A}^f} = 1$ and $|\Delta(\psi \supset \phi)|_{w,w}^{\mathfrak{A}^f} = 1$. It follows by (1) that $|\Delta\phi|_{w,w}^{\mathfrak{A}^f} = 1$, and so with the fact that f agrees with g on the free variables in ϕ that $|\Delta\phi|_{w,w}^{\mathfrak{A}^g} = 1$. The idea behind the construction of ψ and f is to let ψ be a conjunction $p \wedge \chi$, where p is a propositional variable not free in ϕ which f maps to the barcode true at $\langle w, w \rangle$ and no other proper point, and χ is a formula interpreted as the set of proper points. We show that Δ applies to p and χ at $\langle w, w \rangle$, and conclude that it also applies to their conjunction ψ , which is true only at $\langle w, w \rangle$. Since ϕ is true there, $\psi \supset \phi$ is interpreted as the set of all points, to which Δ applies, which completes the the proof.

In more detail, let p be a propositional variable not free in ϕ , $b = \{\langle v, w \rangle : v \in W\}$, and f the assignment function which differs from g only in that $f(p) = b$. Since $|p|_{w,w}^{\mathfrak{A}^f} = 1$, it follows with (2) that $|\Delta p|_{w,w}^{\mathfrak{A}^f} = 1$.

To construct χ , let $\delta = \forall p(Ap \supset p)$, and $\chi = A\delta \supset \delta$. By the semantics of A , $\vdash \chi$, and so by (3), $\vdash \Delta\chi$; in particular $|\Delta\chi|_{w,w}^{\mathfrak{A}^f} = 1$. Furthermore, letting $d = \{\langle v, v \rangle : v \in W\}$, $[\![\delta]\!]_f^{\mathfrak{A}} = d$, and so also $[\![\chi]\!]_f^{\mathfrak{A}} = d$.

Since the logic of any class of generalized 2D models includes all substitution instances of tautologies and is closed under modus ponens, it is routine to derive the schema $\vdash (\Delta\theta \wedge \Delta\eta) \supset \Delta(\theta \wedge \eta)$ using (1) and (3). Letting $\psi = p \wedge \chi$, it follows that $|\Delta\psi|_{w,w}^{\mathfrak{A}^f} = 1$.

$[\![\psi]\!]_f^{\mathfrak{A}} = b \cap d = \{\langle w, w \rangle\}$. By assumption $\langle w, w \rangle \in [\![\phi]\!]_g^{\mathfrak{A}}$, which is identical to $[\![\phi]\!]_f^{\mathfrak{A}}$ since f and g agree on the free variables in ϕ . So $[\![\psi \supset \phi]\!]_f^{\mathfrak{A}} = W^2$. By (3), $\vdash \Delta\top$, so $W^2 \in D(\langle w, w \rangle)$. Hence $|\Delta(\psi \supset \phi)|_{w,w}^{\mathfrak{A}^f} = 1$, and so with (1), $|\Delta\phi|_{w,w}^{\mathfrak{A}^f} = 1$. Since f and g agree on the free variables in ϕ , $|\Delta\phi|_{w,w}^{\mathfrak{A}^g} = 1$. \square

Although propositional quantifiers are not explicitly mentioned in the statement of Proposition 1, the proof relies on an application of rule (3) to the quantified formula $\chi = A\forall p(Ap \supset p) \supset \forall p(Ap \supset p)$. However, a variant of the result can be established which does not rely on the presence of quantifiers, by considering classes of frames rather than classes of models. Let a *generalized 2D frame* be a structure $\langle W, D \rangle$, with W and D as above; let the *quantifier-free logic of* a class \mathbf{C} of generalized 2D frames be the set of formulas in the language L_{VM} (which omits propositional variables and quantifiers) which are in the logic of the class of generalized 2D models whose underlying frames are in \mathbf{C} .

Proposition 2. *For every class \mathbf{C} of generalized 2D frames $\langle W, D \rangle$ which satisfy*

$$(2^*) \langle w, w \rangle \in b \text{ only if } b \in D(\langle w, w \rangle), \text{ for all barcodes } b \text{ and } w \in W,$$

if the quantifier-free logic of \mathbf{C} includes all instances of schema (1) and is closed under schematic rule (3), then it includes all instances of schema (4).

Proof. Like the proof of Proposition 1, but using atomic formulas α and β not occurring in ϕ instead of p and δ , respectively, considering a variant interpretation function mapping α to b and β to d instead of f , and appealing to (2*) instead of (2). \square

Further variants of these two results can immediately be obtained by replacing the requirement of the relevant logic being closed under rule (3) with the requirement of containing all instances of the schemas

$$(3a) \quad \Delta(A\phi \supset \phi),$$

$$(3b) \quad (\Delta\phi \wedge \Delta\psi) \supset \Delta(\phi \wedge \psi), \text{ and}$$

$$(3c) \quad \Delta\top.$$