Journal of Philosophical Logic manuscript No.
(will be inserted by the editor)

Paths to Triviality

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March 22, 2015

Abstract This paper presents a range of new triviality proofs pertaining to naïve truth theory formulated in paraconsistent relevant logics. It is shown that excluded middle together with various permutation principles such as $A \to (B \to C) \vdash B \to (A \to C)$ trivialize naïve truth theory. The paper also provides some new triviality proofs which utilize the axioms $((A \to B) \wedge (B \to C)) \to (A \to C)$ and $(A \to \neg A) \to \neg A$, the fusion connective and the Ackermann constant. An overview over various ways to formulate Leibniz's law in non-classical logics and two new triviality proofs for naïve set theory are also provided.

Keywords Contraction · Curry's paradox · identity · Leibniz's law · naïve truth theory · naïve set theory · paraconsistent logic · relevant logic

1 Introduction

Recent years have seen an increasing interest in non-classical logics in which it is possible to uphold the naïve theories of truth, properties and sets—the theories consisting of, respectively, every instance of the T-schema $T\langle A\rangle \leftrightarrow A$, the unrestricted abstraction schema $a \in \{x|A\} \leftrightarrow A(x/a)$, and the unrestricted abstraction schema together with the claim that coextensional sets are identical. Prominent amongst such logics are the contraction-free and paraconsistent relevant logics.

Ross Brady showed in the late seventies that there are logics in which naïve set theory is non-trivial ([9], [10]).[1] John Slaney showed around the same time that there is a definite limit on how strong such a paraconsistent logic can be ([51]); although the contraction axiom $(A \to (A \to B)) \to (A \to B)$ is not derivable in

[1] There were earlier attempts at showing that the naïve theories can be non-trivial, notably [8], [24], [31], [47], [48], [49] and [50]. However, these results either restrict abstraction or lack a decent conditional, one satisfying at least identity and modus ponens—$A \to A$ and $A, A \to B \vdash B$—and so at best show that A & $T\langle A\rangle$ and $A(a)$ & $a \in \{x|A\}$ are intersubstitutable without delivering the biconditionals $A \leftrightarrow T\langle A\rangle$ and $a \in \{x|A\} \leftrightarrow A(x/a)$.

the logic **RWX**, the instance $(A \to (A \to \bot)) \to (A \to \bot)$ is, and so Curry's paradox trivialize any naïve theory based upon it.[2]

Contraction wreaks havoc on any naïve theory. The reason for this is that for each sentence α, a naïve theory suffices for the existence of a sentence C such that both $C \to (C \to \alpha)$ and $(C \to \alpha) \to C$ hold. The following proof, a slight variant of Haskell Curry's proof in [15], shows how the contraction rule $A \to (A \to B) \vdash A \to B$ together with modus ponens suffice for deriving α:

(1)	$C \to (C \to \alpha)$	naïve theory
(2)	$C \to \alpha$	1, contraction
(3)	$(C \to \alpha) \to C$	naïve theory
(4)	C	2, 3, modus ponens
(5)	α	2, 4, modus ponens

Since $\bot$ is a sentence the defining axiom of which is $\bot \to A$, it follows that if $\bot$ is derivable, then so is every sentence. By replacing α with $\bot$ in the above proof and using the instance $A \to (A \to \bot) \vdash A \to \bot$ of the contraction rule one therefore gets a triviality proof as good as any other. Thus Slaney's proof that $(A \to (A \to \bot)) \to (A \to \bot)$ is derivable in **RWX** shows that **RWX** validates too much contraction for any naïve theory.

Whereas Brady's constructions fail to validate any form of permutation—all the principles

$$(A \to (B \to C)) \to (B \to (A \to C)) \qquad A \to (B \to C), B \vdash B \to (A \to C)$$
$$A \to (B \to C) \vdash B \to (A \to C) \qquad A \to (B \to C), B \vdash A \to C$$

fail in them—the axiom form of permutation holds in **RWX**. This makes for a considerable gap between what has been shown to work and what has been shown not to work. After having defined the logics that will be under scrutiny throughout this paper and given a brief review of the available positive results concerning naïve theories, section 4 closes the aforementioned gap by showing that virtually any form of permutation has to go if excluded middle is to be part of the logic.

Reductio, the axiom $(A \to \neg A) \to \neg A$, is a strong form of excluded middle. It is as of yet unknown if any logic can validate this axiom while retaining naïve theories as non-trivial. Section 5 of this paper shows, among other things, that the logic **TL**, a logic which includes reductio, is too strong for naïve theories in the sense that a propositional schema which is not a a theorem of classical logic becomes provable.

Brady's constructions do not cater for the connective $\circ$ called *fusion*. The defining rule for $\circ$ is the two-way residuation rule

$$(A \circ B) \to C \dashv\vdash A \to (B \to C).$$

Brady's construction in [10] does however validate the rule defining the *Ackermann constant* $\mathbf{t}$, namely

$$A \dashv\vdash \mathbf{t} \to A.$$

[2] Both Brady's [10] and Slaney's [51] were published in [37] which came out in 1989. The results in these articles were however discovered about a decade earlier. Slaney tells me that he discovered his proof around the end/beginning of 1978/1979. The earliest reference to this result that I have been able to find is in Graham Priest's 1983 paper [34, fn. 6]. Brady on the other hand has informed me that he completed [10] in late 1979 and that the results in [9] were proved in 1980. Brady gave two seminars on the results in the latter paper that year.

Section 6 presents some triviality proofs which utilize the fusion connective and the Ackermann constant.

The proofs in sections 4–6 are proofs relating to naïve truth theory, $\mathcal{T}$. This paper also offers two new triviality proofs for naïve set theory, $\mathcal{S}$. The first one shows that the mere presence of the fusion connective together with the Ackermann constant is sufficient for trivializing naïve set theory, and the latter shows that any logic with the weak permutation rule $A \to (B \to C), B \vdash A \to C$ trivializes $\mathcal{S}$. These proofs are to be found in section 9. The two preceding sections, section 7 and section 8, provide, respectively, an overview over various versions of Leibniz's law and a quick discussion of restricted quantification and coextensionality. Section 10 gives some perspectives on the prospects of naïve set theory. Lastly, appendix A shows that giving up structural contraction is not sufficient for saving naïve set theory from triviality, while appendix B casts doubt on the possibility of deriving unrestricted abstraction for pairs in any logic which does not trivialize naïve set theory.

Triviality proofs, like limitative results more generally, are interesting in their own right. This is not to say that some kind of explanation is uncalled for. It would however take this paper too far afield to engage in the philosophical debate concerning naïve theories and non-classical logics. The purpose of this paper is therefore neither more nor less than to present the new triviality proofs in the hope that both friend and foe of naïve theories may find them both interesting and useful.

2 Logics and naïve theories

The logics of interest in this paper are all extensions of the relevant logic **BB**. Despite being rather weak, **BB** is still a decent logic in the sense that it is possible to prove, where Ψ_B is obtained from Ψ_A by replacing any number of instances of A by B, that the intersubstitutability rule $A \leftrightarrow B \vdash \Psi_A \leftrightarrow \Psi_B$ holds. This section shows how **BB** and its extensions are pieced together, and provides a handful of definitions which will be used throughout this paper.

Definition 1 A proof of a formula A from a set of formulas Γ in the logic $\mathbf{L}$ is defined to be a finite list $A_1, \ldots, A_n$ such that $A_n = A$ and every $A_{i \leq n}$ is either a member of Γ, a logical axiom of $\mathbf{L}$, or there is a set $\Delta \subseteq \{A_j | j < i\}$ such that $\Delta \vdash A_i$ is an instance of a rule of $\mathbf{L}$. The existential claim that there is such a proof is is written $\Gamma \vdash_{\mathbf{L}} A$.

To improve readability, I will use the convention that $\neg$ binds most strongly of every connective and the connectives $\wedge, \vee$ and $\circ$ bind equally strong, but more strongly than $\to$. Thus $\neg A \circ B \to C \vee D$ is parsed as $((\neg A) \circ B) \to (C \vee D)$. $\leftrightarrow$ will throughout the paper be a defined connective:

Definition 2 $A \leftrightarrow B =_{df} (A \to B) \wedge (B \to A)$

Ax1	$A \to A$	
Ax2	$A \to A \vee B$ and $B \to A \vee B$	
Ax3	$A \wedge B \to A$ and $A \wedge B \to B$	
Ax4	$\neg\neg A \to A$	
Ax5	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	
Ax6	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$	strong lattice $\wedge$
Ax7	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$	strong lattice $\vee$
Ax8	$(A \to \neg B) \to (B \to \neg A)$	contraposition axiom
Ax9	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing axiom
Ax10	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing axiom
Ax11	$(A \to (B \to C)) \to (B \to (A \to C))$	permutation axiom
Ax12	$(A \to B) \wedge (B \to C) \to (A \to C)$	conjunctive syllogism
Ax13	$A \vee \neg A$	excluded middle
Ax14	$(A \to \neg A) \to \neg A$	reductio
Ax15	$(A \to (A \to B)) \to (A \to B)$	contraction axiom
Ax16	$A \to (B \to A)$	weakening
Ax17	$(A \to B) \vee (B \to A)$	Dummett's axiom
Ax18	$((A \to B) \to B) \to A \vee B$	$\mathbf{Ł}_{\aleph}$-axiom
R1	$A, B \vdash A \wedge B$	adjunction
R2	$A, A \to B \vdash B$	modus ponens
R3	$A \to B \vdash (B \to C) \to (A \to C)$	suffixing rule
R4	$A \to B \vdash (C \to A) \to (C \to B)$	prefixing rule
R5	$A \to \neg B \vdash B \to \neg A$	contraposition rule
R6	$A \to B, A \to C \vdash A \to B \wedge C$	lattice $\wedge$
R7	$A \to C, B \to C \vdash A \vee B \to C$	lattice $\vee$
R8	$A \to (B \to C), B \vdash A \to C$	δ
R9	$A \to (B \to C), B \vdash B \to (A \to C)$	
R10	$A \to (B \to C) \vdash B \to (A \to C)$	permutation rule
R11	$A \vdash \neg(A \to \neg A)$	counter-example rule
R12	$A \dashv\vdash \mathbf{t} \to A$	$\mathbf{t}$-rule
R13	$A \circ B \to C \dashv\vdash A \to (B \to C)$	residuation
R14	$A \to (A \to B) \vdash A \to B$	contraction rule
R15	$A, \neg A \vdash B$	explosion
R16	$A, \neg A \vee B \vdash B$	γ, disjunctive syllogism
MR1	$\dfrac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C}$	reasoning by cases

Table 1 shows how the most familiar relevant logics and some of their irrelevant siblings are pieced together. Fig. 1 is a maps of these logics in terms of the sublogic relation. The depicted logics between **BB** and **R** are relevant logics, whereas those between **BBK** and **CL** are not.[3] The four-valued logic **FDE** is the extensional fragment of **BB**. The three-valued logics $\mathbf{K}_3$ and **LP** are got from **FDE** by adding, respectively, the explosion rule $A, \neg A \vdash B$ and excluded middle. A presentation of these logics can be found in [20, pp. 79–82].

[3] Ax5, Ax17 and Ax18 will not concern us in this paper. The only reason for mentioning them is to give the reader a better picture of where in the logical landscape the relevant logics fit in. Notice that the logics **DJX** and **TJX** are more commonly known as *DK* and *TK*.

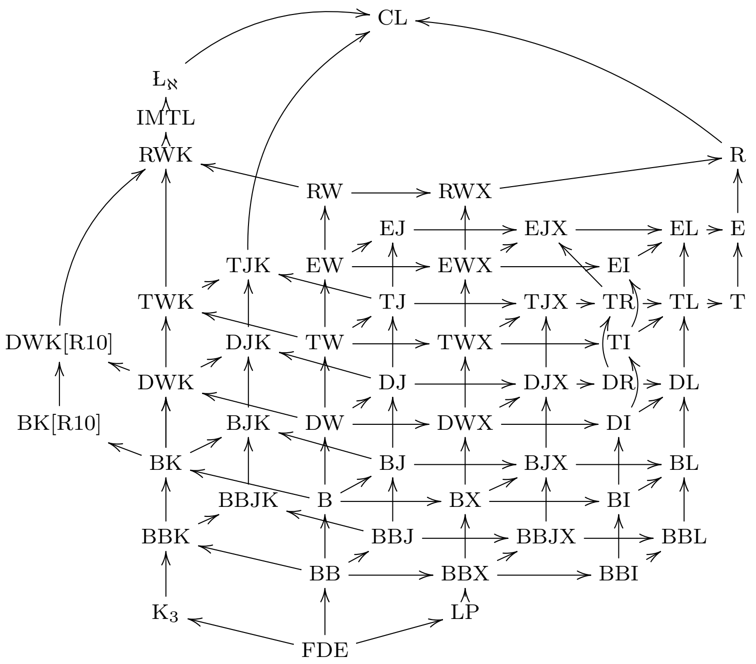

Fig. 1 Map of logics and their sublogic relations

Not every logic of interest in this paper can be found in Fig. 1. For instance the logic **DWX** strengthened by the permutation rule R10 is nowhere to be found. The following definition makes it easier to talk about such logics:

Definition 3 If **L** is a logic, then $\mathbf{L}^d$ is **L** with the meta-rule MR1 added. Furthermore, $\mathbf{L}^\mathbf{t}$ and $\mathbf{L}^\circ$ are the logics **L** with, respectively, the Ackermann constant together with rule R12 and the fusion connective together with rule R13 added. The logic obtained from a logic **L** by adding the rules and axioms $R_{m_1}, \ldots, R_{m_i}$ and $Ax_{n_1}, \ldots, Ax_{n_j}$ will be denoted $\mathbf{L}[R_{m_1}, \ldots, R_{m_i}, Ax_{n_1}, \ldots, Ax_{n_j}]$.

BB	Ax1–Ax5, R1–R7	**T**	**TL** +Ax15
B	**BB** +A6 +A7 −R6 −R7	**E**	**EL** +Ax15
DW	**B** +A8 −R5	**R**	**RW** +Ax12/Ax14/Ax15
TW	**DW** +A10 +A11 −R3 −R4	**DR**	**DJX** +R11
EW	**TW** +R8	**TR**	**TJX** +R11
RW	**TW** +Ax11	**CL**	**BBK** +Ax13
IMTL	**RWK** +Ax17	**Ł**$_{\aleph}$	**RWK** +Ax18
J	+Ax12	**K**	+Ax16
X	+Ax13	**L**	+**J** +**I**
I	+Ax14		

Table 1 Relevant logics and associates

I will occasionally make use of quantification and identity principles, and so a quick presentation on how to extend the various logics from propositional to first-order quantificational logics with identity is in order.

Q1	$\forall xA \to A(^x/_a)$	a free for x
Q2	$\forall x(A \lor B) \to A \lor \forall xB$	$x \notin FV\{A\}$
Q3	$\forall x(A \to B) \to (A \to \forall xB)$	$x \notin FV\{A\}$
Q4	$A(^x/_a) \to \exists xA$	a free for x
Q5	$A \land \exists xB \to \exists x(A \land B)$	$x \notin FV\{A\}$
Q6	$\forall x(B \to A) \to (\exists xB \to A)$	$x \notin FV\{A\}$
Q7	$\forall x(A \to B) \vdash A \to \forall xB$	$x \notin FV\{A\}$
Q8	$\forall x(B \to A) \vdash \exists xB \to A$	$x \notin FV\{A\}$
RQ	$\dfrac{\Gamma \vdash A(^x/_y)}{\Gamma \vdash \forall xA}$	$y \notin FV(\Gamma \cup \{\forall xA\})$
MR2	$\dfrac{A(^x/_y) \vdash B}{\exists xA \vdash B}$	$y \notin FV\{\exists xA, B\}$

Definition 4 If **L** is a logic extending **BB** but not **B**, then $\forall$**L** is **L** augmented with Q1–Q2, Q4–Q5, Q7–Q8 and RQ. If **L** extends **B**, then $\forall$**L** is **L** augmented with Q1–Q6 and RQ. $\forall\mathbf{L}^d$ is got from $\forall$**L** by adding MR1 and MR2.

Definition 5 $\overset{=}{\forall}\mathbf{L}$ is $\forall$**L** augmented with the identity principles[4]

(Ax=)	$\forall x(x = x)$	
$(LL_{2\vdash\vdash})$	$a = b, A(^x/_a) \vdash A(^x/_b)$	$a\&b$ free for x.

Definition 6 A logic in which the explosion rule $A, \neg A \vdash B$ is not derivable is called *paraconsistent*. A paraconsistent logic in which excluded middle is derivable is called *strongly paraconsistent*. Any logic in which excluded middle is not derivable is called *paracomplete*. A paracomplete logic in which $A, \neg A \vdash B$ is derivable is called *strongly paracomplete*.

Writing out a proof in full detail is tedious and makes for quite an onerous read. Some corners will therefore be cut. Double negation introduction and elimination will be such corners throughout this paper. It will also be convenient to make use of derivable rules. The following four rules, one simple transitivity rule, and three "embedding rules" will be used extensively throughout the paper.

Lemma 1

(transitivity)	$A \to B, B \to C \vdash_{\mathbf{BB}} A \to C$
(leftER)	$A \to (B \to C), D \to B \vdash_{\mathbf{BB}} A \to (D \to C)$
(rightER)	$A \to (B \to C), C \to D \vdash_{\mathbf{BB}} A \to (B \to D)$
(left/rightER)	$A \to (B \to C), B_1 \to B, C \to C_1 \vdash_{\mathbf{BB}} A \to (B_1 \to C_1)$

[4] Stronger versions of Leibniz's law than $LL_{2\vdash\vdash}$ will be presented in section 7.

Proof *I'll show that* (*leftER*) *holds. The others are left for the reader.*

(1)	$A \to (B \to C)$	*assumption*
(2)	$D \to B$	*assumption*
(3)	$(B \to C) \to (D \to C)$	*2, R3*
(4)	$A \to (D \to C)$	*1, 3, transitivity*

□

Lemma 2 *The following pairs are interderivable in* **BB**

(1)	$A \vee \neg A$	*excluded middle (Ax13)*
(2)	$A \to \neg A \vdash \neg A$	*reductio rule*
(3)	$A \wedge \neg B \to \neg(A \to B)$	*counter-example axiom*
(4)	$(A \to \neg A) \to \neg A$	*reductio (Ax14)*
(5)	$A, \neg B \vdash \neg(A \to B)$	*counter-example rule v.2*
(6)	$A \vdash \neg(A \to \neg A)$	*counter-example rule (R11).*

Proof *Quite trivially we have that* (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) *and* (5) $\Rightarrow$ (6). *That* (2) $\Rightarrow$ (1), (4) $\Rightarrow$ (3) *and* (6) $\Rightarrow$ (5) *is easily seen by noting that* $(A \to B) \to (A \wedge \neg B \to \neg(A \wedge \neg B))$ *is a logical theorem of* **BB** *(use left/rightER).* □

The Church constants $\top$ and $\bot$ are not definable in relevant logics. I will throughout this paper simply assume that these constants are available, and that they obey their defining axioms $A \to \top$ and $\bot \to A$. The justification given in proofs will be *def. of* $\top$ and *def. of* $\bot$.[5]

Many of the triviality proofs in this paper are like Slaney's **RWX**-proof in that they show that some instance of a version of contraction holds. It will in such cases be possible to read off the proof which instance of which version of contraction is in play. I will make a comment on this after presenting each proof or cluster of such. If a triviality proof needs $\bot$ to be present, the proof will be a proof of $\bot$, and if not it will be a proof of the arbitrary formula α.

Definition 7 (Naïve theories)

- $\mathcal{N}$, our minimal naïve theory, can throughout this paper be taken to be any theory which yields fixed-point sentences; for any formula $B(p)$, where p is a propositional variable, there is a sentence C such that
$$\mathcal{N} \vdash C \leftrightarrow B(C)$$
where $B(C)$ is obtained from $B(p)$ by replacing every occurrence of p by C.
- $\mathcal{T}$, *naïve truth theory*, is the set of T-biconditionals $A \leftrightarrow T\langle A\rangle$. $\langle\cdot\rangle$ is here a naming device and $\mathcal{T}$ is assumed to be such that for any logic **L** which extends **BB** and every open formula $B(x)$, there is a sentence C such that
$$\mathcal{T} \vdash_{\mathbf{L}} C \leftrightarrow B(x/\langle C\rangle).$$
- $\mathcal{P}$, *naïve property theory*, is the universal closure of the schema of unrestricted abstraction
$$\forall x(x \in \{x|A\} \leftrightarrow A).$$

[5] $\top$ and $\bot$ can in naïve truth theory be defined as $\exists x T(x)$ and $\forall x T(x)$.

– $\mathcal{S}$, *naïve set theory*, extends $\mathcal{P}$ with the axiom of extensionality

$$\forall x \forall y(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)).$$

Theorem 1 [6]
$\mathcal{N}$ is a sub-theory of $\mathcal{T}$ which is interpretable in $\mathcal{P}$.

Proof *That $\mathcal{N}$ is a sub-theory of $\mathcal{T}$: Let $B(p)$ be any formula in which p is a propositional variable. Replace every p by $T(x)$ where x is a variable not occurring in B. $\mathcal{T}$ now entails that there is a sentence C such that $\mathcal{T} \vdash C \leftrightarrow B({}^{p}/_{T\langle C\rangle})$. Since $\mathcal{T} \vdash C \leftrightarrow T\langle C\rangle$ and the intersubstitutability rule $A \leftrightarrow D \vdash \Psi_A \leftrightarrow \Psi_D$ is derivable for* **BB**, *it follows that $\mathcal{T} \vdash C \leftrightarrow B(C)$.*

That $\mathcal{T}$ is interpretable in $\mathcal{P}$: Let $\emptyset =_{df} \{z|\forall x\forall y(x \in y)\}$. For every sentence A let $\langle A\rangle =_{df} \{x|A\}$ where x is a fixed variable. By defining $T(x) =_{df} \emptyset \in x$ we get that $\mathcal{P} \vdash A \leftrightarrow T\langle A\rangle$. To show that $\mathcal{P}$ suffices for the diagonalization theorem, let $B(x)$ be any formula in which x is a free variable and let for simplicity w_1 and w_2 be variables not occurring in B.

$$\begin{aligned}
&\sigma_B =_{df} \{w_2|B({}^{x}/_{\{x|\emptyset\in\{w_1|w_2\in w_2\}\}})\}\\
&\tau_B =_{df} \{w_1|\sigma_B \in \sigma_B\}\\
&C =_{df} \emptyset \in \tau_B\\
&\langle C\rangle =_{df} \{x|\emptyset \in \tau_B\}
\end{aligned}$$

It is easily seen that both τ_B and σ_B are closed terms.

(1)	$C \leftrightarrow \emptyset \in \tau_B$	*def. of C*
(2)	$\emptyset \in \tau_B \leftrightarrow (\sigma_B \in \sigma_B)({}^{w_1}/_{\emptyset})$	$\mathcal{P}$
(3)	$(\sigma_B \in \sigma_B)({}^{w_1}/_{\emptyset}) \leftrightarrow \sigma_B \in \sigma_B$	*σ_B is closed*
(4)	$\sigma_B \in \sigma_B \leftrightarrow B({}^{x}/_{\{x\|\emptyset\in\{w_1\|w_2\in w_2\}\}})({}^{w_2}/_{\sigma_B})$	$\mathcal{P}$
(5)	$B({}^{x}/_{\{x\|\emptyset\in\{w_1\|w_2\in w_2\}\}})({}^{w_2}/_{\sigma_B}) \leftrightarrow B({}^{x}/_{\{x\|\emptyset\in\{w_1\|\sigma_B\in\sigma_B\}\}})$	*def. of $({}^{w_2}/_{\sigma_B})$*
(6)	$B({}^{x}/_{\{x\|\emptyset\in\{w_1\|\sigma_B\in\sigma_B\}\}}) \leftrightarrow B({}^{x}/_{\{x\|\emptyset\in\tau_B\}})$	*def. of τ_B*
(7)	$B({}^{x}/_{\{x\|\emptyset\in\tau_B\}}) \leftrightarrow B({}^{x}/_{\langle C\rangle})$	*def. of $\langle C\rangle$*
(8)	$C \leftrightarrow B({}^{x}/_{\langle C\rangle})$	*1–7, trans. of $\leftrightarrow$*

□

The Ackermann constant **t** is, like the Church constants, not definable in relevant logics. It is however a definable truth-constant in $\mathcal{T}$ provided the logic in question validates the rule $A \to (B \to C), B \vdash A \to C$:

Theorem 2 *$\mathcal{T}$ formulated in any logic extending* $\forall$**BB**[*R8*] *has a definable truth-constant* **t** *such that the theory is closed under the two-way rule*

$$A \dashv\vdash \mathbf{t} \to A.$$

[6] Priest showed in [35, p. 363] that $\mathcal{T}$ is interpretable in $\mathcal{P}$ provided it is extended to its absolutely unrestricted form $\forall x(x \in \{x; y|A\} \leftrightarrow A({}^{y}/_{\{x;y|A\}}))$ where $\{x; y|A\}$ is free for y in A. See Appendix B for more on this version of abstraction.

Proof

(1)	$\mathbf{t} \leftrightarrow \forall x(T(x) \wedge \mathbf{t} \rightarrow T(x))$	$\mathcal{N}$
(2)	$\forall x(T(x) \wedge \mathbf{t} \rightarrow T(x))$	*Ax3 + RQ*
(3)	$\mathbf{t}$	*1, 2, R2*
(4)	$\mathbf{t} \rightarrow A$	*assumption*
(5)	A	*3, 4, R2*
(6)	A	*assumption*
(7)	$T\langle A\rangle \wedge \mathbf{t}$	*3, 6,* $\mathcal{T}$ *+ R1*
(8)	$\mathbf{t} \rightarrow (T\langle A\rangle \wedge \mathbf{t} \rightarrow T\langle A\rangle)$	*1, Q1*
(9)	$\mathbf{t} \rightarrow A$	*7, 8, R8 +* $\mathcal{T}$

□

3 State of the art

The purpose of this short section is to give an overview of which logics has been shown to support which naïve theory non-trivially in the sense that a model validating the axioms and rules of both the logic and the naïve theory has been constructed. Table 2 gives such an overview.[7]

Extensional identity $\quad a \overset{e}{=} b =_{df} \forall z(z \in a \leftrightarrow z \in b)$

Intensional identity $\quad a \overset{i}{=} b =_{df} \forall z(a \in z \leftrightarrow b \in z)$

$(Ext_{\overset{i}{=}}) \quad \forall x \forall y(x \overset{e}{=} y \rightarrow x \overset{i}{=} y)$

$(Ext_B) \quad \forall x \forall y \forall w((x \overset{e}{=} y \wedge w \overset{e}{=} w) \rightarrow (x \in w \leftrightarrow y \in w))$

$(Ext_r) \quad a \overset{e}{=} b \vdash a \overset{i}{=} b$

Table 2 Models for naïve theories

Author	Year	Work	Logic	Theory	Ext?
Brady	1983	[9]	$\overset{=}{\forall}\mathbf{TW}^d[R11, R16]$	$\mathcal{S}$	Ext_r
Brady	1989	[10]	$\overset{=}{\forall}\mathbf{DR}^{d\mathbf{t}}$	$\mathcal{S}$	Ext_B
Hájek et al.	2000	[29]	$\overset{=}{\forall}\mathbf{Ł}_{\aleph}$	$\mathcal{T}$	—
Field	2002	[16]	$\overset{=}{\forall}\mathbf{TJ}^d$	$\mathcal{T}$	—
Brady	2006	[11]	$\overset{=}{\forall}\mathbf{TJ}^d[R11, R16]$, $\overset{=}{\forall}\mathbf{TJX}^d$	$\mathcal{S}$	$Ext_{\overset{i}{=}}$
Field	2003/11	[17]+[21]	$\overset{=}{\forall}\mathbf{BBK}^d[R11, Ax8]$	$\mathcal{T}$	—
Field	2004/11	[18]+[21]	$\overset{=}{\forall}\mathbf{BBK}^d[R11, Ax8]$	$\mathcal{P}$	—

[7] I defined naïve set theory above to be $\mathcal{P}$ together with the extensionality axiom $\forall x \forall y(x = y \leftrightarrow x \overset{e}{=} y)$. Given this it is easy to see that $(Ext_{\overset{i}{=}})$ is interderivable with the version $\forall x \forall y(x = y \rightarrow (A(x) \rightarrow A(y)))$ of Leibniz's law, that (Ext_B) is slightly stronger than $\forall x \forall y((x = y \wedge \mathbf{t}) \rightarrow (A(x) \rightarrow A(y)))$, whereas (Ext_r) is interderivable with the rule, $a = b \vdash A(a) \rightarrow A(b)$. This latter version of Leibniz's law is interderivable with the seemingly weaker rule $a = b, A(a) \vdash A(b)$ (see section 7). From this it is easy to see that $\mathcal{S}$ is non-trivial in $\overset{=}{\forall}\mathbf{L}$ if and only if $\mathcal{P} + (Ext_r)$ is non-trivial in $\forall\mathbf{L}$ for any logic $\mathbf{L}$ which extends $\mathbf{BB}$.

Brady's 1983 and 2006 non-triviality proofs do not cater for the Ackermann constant, and so even though both $\forall\mathbf{TJX}^d$ and $\forall\mathbf{TJ}^d[R11]$ have been shown to treat naïve theories non-trivially, nothing is known about $\forall\mathbf{TJ}^{\mathbf{t}}$. Brady's method in [10] does however make room for $\mathbf{t}$.[8]

Richard White's paper [54] purportedly showed that $\mathcal{P}$ is consistent in $\forall\mathbf{Ł}_{\aleph}$. Petr Hájek remarked in [28] that Kazushige Terui ([52]) has found a gap in White's proof, and so the consistency of $\mathcal{P}$ in logics between $\forall\mathbf{EW}$ & $\forall\mathbf{Ł}_{\aleph}$ and between $\forall\mathbf{BK}$ & $\forall\mathbf{Ł}_{\aleph}$ is as of yet unknown.[9]

Field's constructions in [17] and [18] validates the rule $A \vdash B \to A$, but not the stronger $\mathbf{K}$ axiom $A \to (B \to A)$. However, as Field notes in [21], it is easy to simply define a new conditional which will validate the $\mathbf{K}$-axiom; $A \rightharpoonup B =_{df} (A \to B) \vee (\neg A \vee B)$ is such a conditional. The logic with $\to$ replaced by $\rightharpoonup$ validated in both [17] and [18] extends $\forall\mathbf{BBK}^d[R11, Ax8]$.

It should be noted that the constructions in [9], [10], [11] and [18] either do, or can easily be modified so as to validate the absolutely unrestricted form of $\mathcal{P}$, $\forall x(x \in \{x; y|A\} \leftrightarrow A(y/\{x;y|A\}))$, mentioned in fn. 6. Furthermore, there are interesting proof-theoretic results for linear logics. For instance, White's paper [55] shows that $\forall\mathbf{RWK}$ minus the distribution axioms Ax5, Q2 and Q5, but with an added implication connective, does treat the absolutely unrestricted version of $\mathcal{P}$ non-trivially.

4 Proofs involving permutation and excluded middle

Slaney showed in his paper *RWX is not Curry Paraconsistent* that **RWX** suffice, given the availability of $\bot$, for proving the contraction formula $(A \to (A \to \bot)) \to (A \to \bot)$. His proof appeals to both the permutation axiom (Ax11), the strong lattice $\vee$ axiom (Ax7) and the contraposition axiom (Ax8). In chapter 18 of *In Contraction*, after having described a logic extending **DWX**, Graham Priest raised the following problem:

> Various natural arguments require the use of principles that involve nested $\to$s, such as Permutation, $\{\alpha \to (\beta \to \gamma)\} \vdash \beta \to (\alpha \to \gamma)$. The logic just described does not contain this principle. Whether it can be added while maintaining non-triviality is not known. There is certainly triviality in the area. See Slaney (1989). [36, p. 253 fn. 11]

Slaney ended his paper with the sentence "Meanwhile it seems some more investigation would be appropriate, and of course some more theorems would be absolutely splendid." [51, p. 479]. This section answers Slaney's call for more theorems by showing three new proofs to the effect that if excluded middle is to be part of the logic, then (1) the permutation rule (R10) has to go, (2) if the logic

[8] Andrew Bacon's paper [2], Brady's paper [12], and the paper [23] of Field et al. should be mentioned in relation to the logic $\forall\mathbf{TJK}$. Bacon shows that the positive fragment of $\forall\mathbf{TJK}[MR1]$ does treat naïve truth theory non-trivially. Furthermore, it is shown in [23] that the positive fragment of $\forall\mathbf{TJK}^d$ treats $\mathcal{P}{+}Ext_{\underline{i}}$, and thus $\mathcal{S}$, non-trivially. Brady claims in [12, Cor. 4–5] that the construction made use of there validates $\mathcal{P}{+}Ext_{\underline{i}}$ over the logic full logic $\forall\mathbf{TJK}^d$. This is sadly not the case. See [13] and section 5 below for further comments.

[9] $\mathcal{T}$ formulated in $\forall\mathbf{Ł}_{\aleph}$ is, although a non-trivial theory, riddled with ω-troubles. See [1], [27], [29] and [38].

has both the contraposition axiom (Ax8) and the strong lattice axioms (Ax6–7), then R9 has to go, and (3) that if the logic has the meta-rule of reasoning by cases (MR1), then even R8 has to go.

Theorem 3 $\mathcal{N} \vdash_{\mathbf{BBX}[R10]} \bot$

Proof

(1)	$C \leftrightarrow (\neg(C \to \bot) \to \bot)$	$\mathcal{N}$
(2)	$\neg(C \to \bot) \to (C \to \bot)$	*1, R10*
(3)	$C \to \bot$	*2, reductio rule*
(4)	$C \to (\top \to \bot)$	*3 + def. of* $\bot$
(5)	$\top \to (C \to \bot)$	*4, R10*
(6)	$\neg(C \to \bot) \to \bot$	*5, R5*
(7)	C	*1, 6, R2*
(8)	$\bot$	*3, 7, R2*

□

Lemma 3 $\vdash_{\mathbf{BX}[R8]} (A \to B) \wedge (\neg A \to B) \to B$

Proof

(1)	$(A \to C) \wedge (\neg A \to B) \to (A \vee \neg A \to B)$	*Ax7*
(2)	$A \vee \neg A$	*Ax13*
(3)	$(A \to B) \wedge (\neg A \to B) \to B$	*1, 2, Ax8*

□

Theorem 4 $\mathcal{N} \vdash_{\mathbf{DWX}[R9]} \bot$

Proof

(1)	$C \leftrightarrow (\top \to (C \to \bot))$	$\mathcal{N}$
(2)	$\top \to (C \to (C \to \bot))$	*1, R9*
(3)	$\top \to (\neg(C \to \bot) \to \neg C)$	*2, Ax8*
(4)	$(C \to \bot) \to (\top \to \neg C)$	*Ax8*
(5)	$\top \to ((C \to \bot) \to \neg C)$	*4, R9*
(6)	$\top \to ((C \to \bot) \to \neg C) \wedge (\neg(C \to \bot) \to \neg C)$	*3, 5, R6*
(7)	$((C \to \bot) \to \neg C) \wedge (\neg(C \to \bot) \to \neg C) \to \neg C$	*Lem. 3*
(8)	$\top \to \neg C$	*6, 7, transitivity*
(9)	$C \to \bot$	*8, R5*
(10)	$C \to (\top \to \bot)$	*9 + def. of* $\bot$
(11)	$\top \to (C \to \bot)$	*10, R9*
(12)	C	*1, 11, R2*
(13)	$\bot$	*9, 12, R2*

□

The two triviality proofs above both show that some form of contraction is at work. Lines 1–6 of the first proof show that the rule

$$A \to (\neg(A \to \bot) \to \bot) \vdash \neg(A \to \bot) \to \bot$$

holds in **BBX**[*R*10], whereas lines 1–11 of the second proof show that the rule

$$A \to (\top \to (A \to \bot)) \vdash \top \to (A \to \bot)$$

holds in **DWX**[*R*9].

Theorem 5 $\mathcal{N} \vdash_{\mathbf{BBX}^d[R8]} (\neg(A \to A) \to A) \to A$

Proof

(1)	$C \leftrightarrow (\neg(C \to A) \to A)$	$\mathcal{N}$
(2)	$(C \to A) \vee \neg(C \to A)$	*Ax13*
(3)	$\quad \neg(C \to A)$	*assumption for MR1*
(4)	$\quad C \to A$	*1, 3, R8*
(5)	$C \to A$	*2, 3–4, MR1*
(6)	$(A \to A) \to (C \to A)$	*5, R3*
(7)	$\neg(C \to A) \to \neg(A \to A)$	*6, R5*
(8)	$(\neg(A \to A) \to A) \to (\neg(C \to A) \to A)$	*7, R3*
(9)	$(\neg(A \to A) \to A) \to A$	*1, 5, 8, transitivity*

□

Although the above proof is far short of a triviality proof, I take it that deriving a propositional schema which is not a theorem of classical logic is bad enough. Thus I take it that the strong paraconsistentist ought to either drop reasoning by cases, or the weak permutation rule R8.

This section has focused on the **X**-logics—the logics with excluded middle. The next section will shift the focus onto the **J**-logics—the logics with conjunctive syllogism (Ax12).

5 Proofs involving conjunctive syllogism

Prominent paraconsistentists such as Brady, Priest, Richard Routley and Zach Weber have all at one time or another opted for extensions of the logic **DJ**.[10] **DJ** contains the axiom called *conjunctive syllogism*. The proofs in this section show that even if excluded middle is dropped, the permutation rule can't be added if conjunctive syllogism is to be part of the logic and that both **TL** and **EJ** are too strong for $\mathcal{N}$.

Both R8 and R9 are variants of the permutation rule $A \to (B \to C) \vdash B \to (A \to C)$ got by imposing restrictions on B.[11] One may of course consider weakenings of the permutation rule got by imposing conditions of C instead. Brady considered in [12] so-called *M1-logics*—logics contained in $\forall\mathbf{TJK}^d$ strengthened by the permutation axiom $(A \to (B \to (C_0 \to C_1))) \to (B \to (A \to (C_0 \to C_1)))$.[12] Brady furthermore claimed that any such M1-logic can consistently support naïve set theory even as strong as $\mathcal{P}{+}Ext_{\underline{\underline{i}}}$ ([12, Cor. 4–5]). This is not the case as I will now show.[13]

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10 See for instance [11], [36], [44] and [53].

11 This is easily seen in the case of R9. In the case of R8 it should suffice to note that in the presence of $\mathbf{t}$ it is interderivable with $A \to (\mathbf{t} \to C) \vdash \mathbf{t} \to (A \to C)$. Thus whereas R9 licenses permutation under the condition that B is true, R8 does so only if B is the particular true sentence $\mathbf{t}$.

12 He also allows a M1-logic to have the axiom $(A \to B \vee C) \wedge (A \wedge B \to C) \to (A \to C)$. M1-logics are contrasted to *M2-logics* which defines to be any logic between $\forall\mathbf{B}^d[R11]$ and $\forall\mathbf{RWK}^d$.

13 The reader is referred to the correction note [13] for further comments.

Definition 8

$$(R10_n) \quad A \to (B \to (C_0 \to (\ldots \to C_n))) \vdash B \to (A \to (C_0 \to (\ldots \to C_n)$$

$$A \overset{0}{\to} B =_{df} B \qquad A \overset{n+1}{\to} B =_{df} A \to (A \overset{n}{\to} B)$$

Theorem 6 $\vdash_{\mathbf{BBJ}[R10_n]} (A \overset{n+2}{\to} B) \to (A \overset{n+1}{\to} B)$

Proof

(1)	$(A \overset{n+1}{\to} B) \to (A \to (A \overset{n}{\to} B))$	*Ax1*, *def. of* $\overset{n}{\to}$
(2)	$A \to ((A \overset{n+1}{\to} B) \to (A \overset{n}{\to} B))$	*2*, $R10_n$
(3)	$(A \overset{n+2}{\to} B) \to (A \to (A \overset{n+1}{\to} B))$	*Ax1*, *def. of* $\overset{n}{\to}$
(4)	$A \to ((A \overset{n+2}{\to} B) \to (A \overset{n+1}{\to} B))$	*4*, $R10_n$
(5)	$A \to ((A \overset{n+2}{\to} B) \to (A \overset{n+1}{\to} B)) \land ((A \overset{n+1}{\to} B) \to (A \overset{n}{\to} B))$	*2, 4, R6*
(6)	$((A \overset{n+2}{\to} B) \to (A \overset{n+1}{\to} B)) \land ((A \overset{n+1}{\to} B) \to (A \overset{n}{\to} B)) \to$ $((A \overset{n+2}{\to} B) \to (A \overset{n}{\to} B))$	*Ax12*
(7)	$A \to ((A \overset{n+2}{\to} B) \to (A \overset{n}{\to} B))$	*5, 6, transitivity*
(8)	$(A \overset{n+2}{\to} B) \to (A \overset{n+1}{\to} B)$	*7*, $R10_n$, *def. of* $\overset{n}{\to}$

□

For $n = 0$, the rule $A \overset{n+2}{\to} B \vdash A \overset{n+1}{\to} B$ is simply the contraction rule which Curry showed to trivialize naïve theories. It is however easy to see that if the logic validates $A \overset{n+2}{\to} B \vdash A \overset{n+1}{\to} B$ for any n, then by using the sentence $C \leftrightarrow (C \overset{n+1}{\to} \alpha)$ and modifying Curry's proof slightly, one obtains yet another triviality proof.[14] We therefore have the following corollary:

Corollary 1 *For any* n, $\mathcal{N} \vdash_{\mathbf{BBJ}[R10_n]} \alpha$.

Theorem 7 *(***EJ** *trivialize* $\mathcal{N}$*)*
$\mathcal{N} \vdash_{\mathbf{BBJ}[R8,Ax10]} \alpha$

Proof Let $\Box A =_{df} (A \to A) \to A$ *and* $D =_{df} C \to C$.

(1)	$C \leftrightarrow (\Box C \to \alpha)$	$\mathcal{N}$
(2)	$\Box C \to ((C \to C) \to C)$	*Ax1*, *def. of* $\Box C$
(3)	$C \to C$	*Ax1*
(4)	$\Box C \to C$	*2, 3, R8*
(5)	$\Box C \to (\Box C \to \alpha)$	*1, 4, transitivity*
(6)	$\Box C \to ((D \to D) \to \Box C)$	*Ax10*
(7)	$\Box C \to ((D \to D) \to \Box C) \land (\Box C \to \alpha)$	*5, 6, R6*
(8)	$((D \to D) \to \Box C) \land (\Box C \to \alpha) \to ((D \to D) \to \alpha)$	*Ax12*
(9)	$\Box C \to ((D \to D) \to \alpha)$	*7, 8, transitivity*
(10)	$D \to D$	*Ax1*
(11)	$\Box C \to \alpha$	*9, 10, R8*
(12)	C	*1, 11, R2*
(13)	$(C \to C) \to (C \to C)$	*Ax1*
(14)	$\Box C$	*12, 13, R8*
(15)	α	*11, 14, R2*

□

[14] This was first shown by Moh Shaw-Kwei in [46, Thm. 1].

Lines 1–11 show that **EJ** validates the the contraction rule $A \to (\Box A \to B) \vdash \Box A \to B$.[15]

The next theorem shows that $\mathcal{N}$ is badly non-conservative in **TL**; although a triviality proof has yet to be discovered, **TL**, like $\mathbf{BBX}^d[R8]$ in Thm. 5, proves a propositional schema which is not a theorem of classical logic and certainly not a theorem of **TL**.

Lemma 4 $A \leftrightarrow (A \to B) \vdash_{\mathbf{BBJ}[Ax9]} A \to ((B \to B) \to B)$

Proof

(1)	$A \leftrightarrow (A \to B)$	*assumption*
(2)	$(A \to B) \to ((B \to B) \to (A \to B))$	*Ax9*
(3)	$A \to ((B \to B) \to A)$	*1, 2, rightER+trans.*
(4)	$A \to ((B \to B) \to A) \wedge (A \to B)$	*1, 3, R6*
(5)	$((B \to B) \to A) \wedge (A \to B) \to ((B \to B) \to B)$	*Ax12*
(6)	$A \to ((B \to B) \to B)$	*4, 5, transitivity*

□

Theorem 8 *($\mathcal{N}$ is non-classical in* **TL***)*[16]
$\mathcal{N} \vdash_{\mathbf{BBL}[Ax9]} (A \to (\neg A \to \neg A)) \to \neg A$

Proof

(1)	$C \leftrightarrow (C \to \neg A)$	$\mathcal{N}$
(2)	$C \to ((\neg A \to \neg A) \to \neg A)$	*1, Lem. 4*
(3)	$(A \to (\neg A \to \neg A)) \to (((\neg A \to \neg A) \to \neg A) \to (A \to \neg A))$	*Ax9*
(4)	$(A \to (\neg A \to \neg A)) \to (C \to (A \to \neg A))$	*2, 3, leftER*
(5)	$(A \to \neg A) \to \neg A$	*Ax14*
(6)	$(A \to (\neg A \to \neg A)) \to (C \to \neg A)$	*4, 5, rightER*
(7)	$(A \to (\neg A \to \neg A)) \to C$	*1, 6, trans.*
(8)	$(A \to (\neg A \to \neg A)) \to ((\neg A \to \neg A) \to \neg A)$	*2, 7, trans.*
(9)	$(A \to (\neg A \to \neg A)) \to$ $(A \to (\neg A \to \neg A)) \wedge ((\neg A \to \neg A) \to \neg A)$	*8, Ax1, R6*
(10)	$(A \to (\neg A \to \neg A)) \wedge ((\neg A \to \neg A) \to \neg A) \to (A \to \neg A)$	*Ax12*
(11)	$(A \to (\neg A \to \neg A)) \to (A \to \neg A)$	*9, 10, trans.*
(12)	$(A \to (\neg A \to \neg A)) \to \neg A$	*5, 11, trans.*

□

I have so far shown that conjunctive syllogism and permutation makes for a potent mix—too potent for $\mathcal{N}$—and that $\mathcal{N}$ is non-conservative over **TL**. The three logics $\mathbf{DL}^{dt}$, $\mathbf{TR}^{dt}$ and $\mathbf{TI}^{dt}$ seem to be some of the strongest strongly paraconsistent logics which may treat $\mathcal{N}$ non-trivially. The next section shows that these logics are altogether too strong if the the fusion connective is added.

[15] A similar proof shows that the strongly paracomplete logic **BBJK** can't be extended by R11, $A \vdash \neg(A \to \neg A)$: let C be the Curry sentence. Then by the **K**-axiom and R6 one gets $C \to (\top \to C) \wedge (C \to \bot)$. Conjunction syllogism (Ax12) delivers $(\top \to C) \wedge (C \to \bot) \to (\top \to \bot)$ and R11 together with the **K**-rule $(\top \to \bot) \to \bot$. Transitivity then yields $C \to \bot$ from which $\bot$ easily follows.

[16] I owe Weber thanks in regards to this theorem. My original proof was to the effect that $(\top \to (\bot \to \bot)) \to \bot$ is derivable in $\mathbf{BBL}[Ax9]$. It was Weber who noticed that the proof sufficed for $(A \to (\neg A \to \neg A)) \to \neg A$.

6 Proofs involving fusion

In intuitionistic and classical logic conjunction residuates the implication connective; the two-way rule $A \wedge B \to C \dashv\vdash A \to (B \to C)$ holds in these logics. It is easily seen that the rule $A \to (B \to C) \vdash A \wedge B \to C$ is interderivable with the contraction rule, and so $\wedge$ can't residuate $\to$ in logics fit for $\mathcal{N}$. For logics in the vicinity of relevant logics it is however common to enrich the set of logical connectives with an *intensional* conjunction, $\circ$, called *fusion* which by definition (R13) residuates $\to$. This section presents one old and three new triviality proofs which utilize $\circ$.

The first theorem is due to J. Michael Dunn and Slaney and was first published in the first volume of *Relevant Logics and their Rivals* ([45, pp. 366–367]).

Theorem 9 *(Dunn/Slaney)* $\mathcal{N} \vdash_{\mathbf{BBJ}^\circ} \alpha$

Proof

(1)	$C \leftrightarrow (C \circ C \to \alpha)$	$\mathcal{N}$
(2)	$C \circ C \to C \circ C$	*Ax1*
(3)	$C \to (C \to C \circ C)$	*2, R13*
(4)	$C \to (C \to C \circ C) \wedge (C \circ C \to \alpha)$	*1, 3, R6*
(5)	$(C \to C \circ C) \wedge (C \circ C \to \alpha) \to (C \to \alpha)$	*Ax12*
(6)	$C \to (C \to \alpha)$	*4, 5, transitivity*
(7)	$C \circ C \to \alpha$	*6, R13*
(8)	C	*1, 7, R2*
(9)	α	*6, 8, R2*

□

Lines 1–7 show that $\mathbf{BBJ}^\circ$ validates the contraction rule

$$A \to (A \circ A \to B) \vdash A \circ A \to B.$$

Dunn and Slaney's result shows that the **J**-logics are off the table if fusion is present. Since naïve truth theory is non-trivial in $\mathbf{Ł}_{\aleph}$ and fusion is definable in **B**[R10, *Ax8*] ($A \circ B =_{df} \neg(A \to \neg B)$), we're left with investigating whether or not fusion can be added to the **X**- and **I**-logics—the strongly paraconsistent logics.

Lemma 5 *($\top/\bot$-lemma)*

$$\vdash_{\mathbf{BB}^\circ[Ax8]} \top \to (\bot \to \bot) \qquad \vdash_{\mathbf{BBI}^\circ[Ax9]} \top \to (\bot \to \bot)$$

Proof

(1)	$(\top \to \bot) \leftrightarrow \bot$	*Ax13 + def. of* $\bot$
(2)	$(\top \to \top) \to ((\top \to \bot) \to (\top \to \bot))$	*Ax9*
(3)	$(\top \to \top) \to (\bot \to \bot)$	*(1, 2, left/rightER) or Ax8*
(4)	$\top \circ \top \to \top$	*def. of* $\top$
(5)	$\top \to (\top \to \top)$	*4, R13*
(6)	$\top \to (\bot \to \bot)$	*3, 5, transitivity*

□

Theorem 10 $\mathcal{N} \vdash_{\mathbf{BBX}^{d\mathbf{t}\circ}[Ax8]} \bot$

Proof

(1)	$C \leftrightarrow (\neg(C \circ \mathbf{t} \to \bot) \to \bot)$	$\mathcal{N}$
(2)	$(C \circ \mathbf{t} \to \bot) \vee \neg(C \circ \mathbf{t} \to \bot)$	*Ax13*
(3)	$\quad \neg(C \circ \mathbf{t} \to \bot)$	*assumption for MR1*
(4)	$\quad \mathbf{t} \to \neg(C \circ \mathbf{t} \to \bot)$	*3, R12*
(5)	$\quad (\neg(C \circ \mathbf{t} \to \bot) \to \bot) \to (\mathbf{t} \to \bot)$	*4, R3*
(6)	$\quad C \to (\mathbf{t} \to \bot)$	*1, 5, transitivity*
(7)	$\quad C \circ \mathbf{t} \to \bot$	*6, R13*
(8)	$C \circ \mathbf{t} \to \bot$	*2, 3–7, MR1*
(9)	$\top \to (\bot \to \bot)$	*$\top/\bot$-lemma*
(10)	$\top \to (C \circ \mathbf{t} \to \bot)$	*8, 10, leftER*
(11)	$\neg(C \circ \mathbf{t} \to \bot) \to \bot$	*10, R5*
(12)	C	*1, 11, R2*
(13)	$C \to (\mathbf{t} \to \bot)$	*8, R13*
(14)	$\mathbf{t}$	*Ax1 + R12*
(15)	$\bot$	*12, 13, 14, R2*

□

Lines 1–11 show that the contraction rule

$$A \to (\neg(A \circ \mathbf{t} \to \bot) \to \bot) \vdash \neg(A \circ \mathbf{t} \to \bot) \to \bot$$

holds in $\mathbf{BBX}^{d\mathbf{t}\circ}[Ax8]$.

Theorem 11 $\mathcal{N} \vdash_{\mathbf{BBX}^{\circ}[R8,Ax8]} \bot$

Proof

(1)	$C \leftrightarrow (C \circ \top \to \bot)$	$\mathcal{N}$
(2)	$C \to (\top \to \neg(C \circ \top))$	*1, Ax8*
(3)	$C \circ \top \to \neg(C \circ \top)$	*2, R13*
(4)	$\neg(C \circ \top)$	*3, reductio rule*
(5)	$C \circ \top \to C \circ \top$	*Ax1*
(6)	$C \to (\top \to C \circ \top)$	*5, R13*
(7)	$C \to (\neg(C \circ \top) \to \bot)$	*6, Ax8*
(8)	$C \to \bot$	*4, 7, R8*
(9)	$C \to (\top \to \bot)$	*8 + def. of $\bot$*
(10)	$C \circ \top \to \bot$	*9, R13*
(11)	C	*1, 10, R2*
(12)	$\bot$	*8, 11, R2*

□

Theorem 12 $\mathcal{N} \vdash_{\mathbf{BBI}^{\circ}[Ax8]} \bot \qquad \mathcal{N} \vdash_{\mathbf{BBI}^{\circ}[Ax9]} \bot$

Proof

(1)	$C \leftrightarrow (\top \circ C \rightarrow \bot)$	$\mathcal{N}$
(2)	$\bot \rightarrow \neg(\top \circ C)$	*def. of* $\bot$
(3)	$(\top \circ C \rightarrow \bot) \rightarrow (\top \circ C \rightarrow \neg(\top \circ C))$	*2, R4*
(4)	$(\top \circ C \rightarrow \neg(\top \circ C)) \rightarrow \neg(\top \circ C)$	*Ax14*
(5)	$C \rightarrow \neg(\top \circ C)$	*1, 3, 4, transitivity*
(6)	$(\top \circ C) \rightarrow \neg C$	*5, R5*
(7)	$\top \rightarrow (C \rightarrow \neg C)$	*6, R13*
(8)	$(C \rightarrow \neg C) \rightarrow \neg C$	*Ax14*
(9)	$\top \rightarrow \neg C$	*7, 8, transitivity*
(10)	$C \rightarrow \bot$	*9, R5*
(11)	$\top \rightarrow (\bot \rightarrow \bot)$	*$\top/\bot$-lemma*
(12)	$\top \rightarrow (C \rightarrow \bot)$	*10, 11, leftER*
(13)	$(\top \circ C) \rightarrow \bot$	*12, R13*
(14)	C	*1, 12, R2*
(15)	$\bot$	*10, 14, R2*

□

Lines 1–10 in Thm. 11 show that the contraction rule

$$A \rightarrow (A \circ \top \rightarrow \bot) \vdash A \circ \top \rightarrow \bot$$

holds in $\mathbf{BBX}^{\circ}[R8, Ax8]$ and lines 1–13 in Thm. 12 show that

$$A \rightarrow (\top \circ A \rightarrow \bot) \vdash \top \circ A \rightarrow \bot$$

holds in both $\mathbf{BBI}^{\circ}[Ax8]$ and $\mathbf{BBI}^{\circ}[Ax9]$.

I have in this section shown that adding the fusion connective and the Ackermann constant narrows the possible logics a paraconsistentist can adhere to. The proofs leave the logics $\mathbf{BI}^{d\mathbf{t}\circ}$ and $\mathbf{BX}^{d\mathbf{t}\circ}[R11, Ax9, Ax10]$ as two of the strongest candidates among the strongly paraconsistent logics which might treat naïve theories non-trivially. Let me also note that if the paraconsistentist is willing to drop MR1, then both $\mathbf{TWX}^{\mathbf{t}\circ}[R11]$ and $\mathbf{BI}^{\mathbf{t}\circ}[R8]$ are still on the table.

The triviality proofs so far have been proofs that $\mathcal{N}$ is trivial in some logic. I will present two more triviality proof. These are however proofs that naïve set theory, $\mathcal{S}$, is trivial. Since these proofs go beyond $\mathcal{N}$, $\mathcal{T}$ and $\mathcal{P}$ in that they rely on Leibniz's law in some shape or form, it seems fitting to end this section with a clear picture over at least some of the logics which might, but for which nothing is as of yet known, treat $\mathcal{N}$, $\mathcal{T}$ and $\mathcal{P}$ as non-trivial theories.

Strongly paraconsistent:
- $\mathbf{TI}^{d\mathbf{t}}$
- $\mathbf{TR}^{d\mathbf{t}}$
- $\mathbf{TWX}^{\mathbf{t}\circ}[R11]$
- $\mathbf{DL}^{d\mathbf{t}}$
- $\mathbf{BI}^{\mathbf{t}\circ}[R8]$
- $\mathbf{BI}^{d\mathbf{t}\circ}$
- $\mathbf{BX}^{d\mathbf{t}\circ}[R11, Ax9, Ax10]$

Strongly paracomplete:
- $\mathbf{TJK}^{d}$
- $\mathbf{DJ}^{d\mathbf{t}}[R9, R16]$

7 Leibniz's law in non-classical logics

The one principle of naïve set theory which makes it stand out amongst the naïve theories is the principle of extensionality. The added strength of $\mathcal{S}$ over $\mathcal{P}$ does not primarily depend on the formulation of this principle however, but on how it connects up with Leibniz's law. This section shows forth various ways to formulate Leibniz's law in non-classical logics. The next section will then show why the relevantist should formulate the axiom of extensionality in the way that I have done. I then go on to give two triviality proofs for naïve set theory.

In classical first-order logic there is basically just one way to formulate Leibniz's law, the idea behind which is that identical objects co-instantiate every (nameable) property, namely as

$$a = b \to (A(a) \to A(b)).$$

This contrasts to non-classical logics in which there generally are several non-equivalent ways to formulate this idea. Fig. 2 depicts some of these in terms of strength. After some brief comments on the classification I will prove that the arrows in the figure are appropriate; that if a is below b in the map, then b is *at least as strong* as a assuming the logic $\overset{=}{\forall}\mathbf{BB}$. I'll also show that the difference between some of these principles collapse when $\overset{=}{\forall}\mathbf{BB}$ is strengthened in various ways.

Definition 9 $A \mapsto B =_{df} A \wedge \mathbf{t} \to B$

Definition 10 Variations on Leibniz's law ($\rightsquigarrow, \twoheadrightarrow \in \{\to, \mapsto\}$):

$$
\begin{array}{ll}
(\overset{\rightsquigarrow}{L}\overset{\twoheadrightarrow}{L}_1) & A(a) \rightsquigarrow (a = b \twoheadrightarrow A(b)) \\
(\overset{\rightsquigarrow}{L}L_{1\vdash}) & A(a) \vdash a = b \rightsquigarrow A(b) \\
(\overset{\rightsquigarrow}{L}\overset{\twoheadrightarrow}{L}_2) & a = b \rightsquigarrow (A(a) \twoheadrightarrow A(b)) \\
(\overset{\rightsquigarrow}{L}L_{2\vdash}) & a = b \vdash A(a) \rightsquigarrow A(b) \\
(LL_{2\vdash\vdash}) & a = b, A(a) \vdash A(b) \\
(\overset{\rightsquigarrow}{L}L_3) & a = b \wedge A(a) \rightsquigarrow A(b) \\
(LL_{\sqsubset}) & A(a), \neg A(b) \vdash a \neq b
\end{array}
$$

The relevance property can be ascribed to a propositional logic just in case $A \to B$ is a logical theorem only for sentences A and B which share a propositional variable. As such it pertains only to propositional logics, and so to classify identity principles according to relevance seems to be a category mistake. However, there seems to be, at least amongst the relevant logicians, an intuitive notion of relevance which is appealed to when applying *relevant* to other things than propositional logics. For instance, relevant logicians agree on classifying $\vec{L}\vec{L}_2$ as *irrelevant* since it entails instances such as $a = a \to (A \to A)$. In this spirit let me note that $\vec{L}L_3$ also preserves at least some idea of relevance even in the relevant logic $\overset{=}{\forall}\mathbf{R}^{d\mathbf{t}}[R16]$ which $\vec{L}\overset{\mapsto}{L}_2$ does not. As an indication that this classification is a sound one let me note that it is easy to extend the 8-valued model Belnap used in [7] to show that logics such as $\mathbf{E}[R16]$ has the variable sharing property in such a way so that the resulting model validates the axioms and rules of $\overset{=}{\forall}\mathbf{R}^{d\mathbf{t}}[R16, \vec{L}L_3]$, but fails to validate $\vec{L}\overset{\mapsto}{L}_2$.[17] This is the reason why the principles at least as strong

[17] Such a model is shown in [33, Thm. 15].

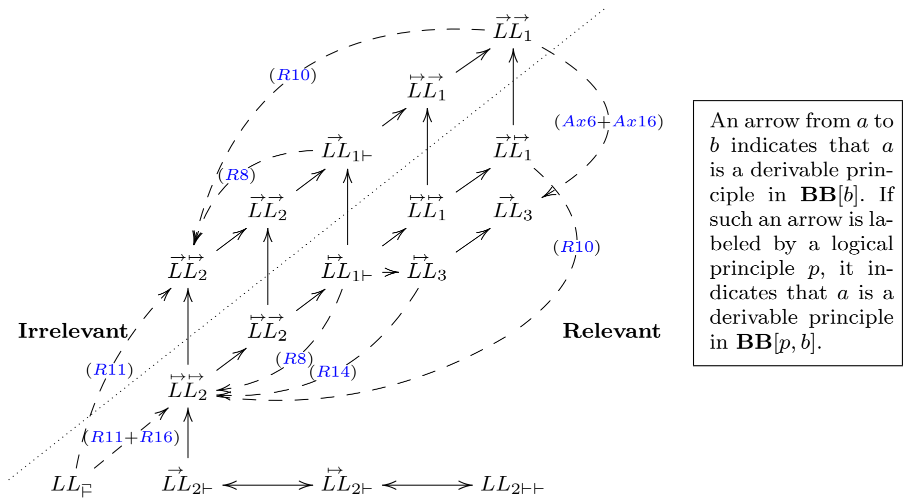

Fig. 2 Map of identity principles

as $\overset{\mapsto}{L}\overset{\mapsto}{L}_2$ have been classified as *irrelevant*, whereas identity principles derivable in $\bar{\bar{\forall}}\mathbf{R}^{\mathbf{dt}}[R16, \vec{L}L_3]$ are on the *relevant* side of the dotted line in Fig. 2.

Lemma 6 *The arrows in Fig. 2 are appropriate.*

Proof
1. $\vec{L}\vec{L}_i \Rightarrow \overset{\mapsto}{L}\vec{L}_i \Rightarrow \overset{\mapsto}{L}\overset{\mapsto}{L}_i$ *and* $\vec{L}\vec{L}_i \Rightarrow \vec{L}\overset{\mapsto}{L}_i \Rightarrow \overset{\mapsto}{L}\overset{\mapsto}{L}_i$*: use Ax3 and leftER.*
2. $\overset{\mapsto}{L}\vec{L}_1 \Rightarrow \vec{L}L_{1\vdash} \Rightarrow \overset{\mapsto}{L}L_{1\vdash}$ *and* $\overset{\mapsto}{L}\overset{\mapsto}{L}_1 \Rightarrow \overset{\mapsto}{L}L_{1\vdash}$*: use Ax3, R1 and R2.*
3. $\overset{\rightsquigarrow}{L}L_{1\vdash} \Rightarrow \overset{\rightsquigarrow}{L}\vec{L}_2$*:* $A(a) \to A(x/a)$ *is a theorem and applying* $\overset{\rightsquigarrow}{L}L_{1\vdash}$ *to it yields* $a = b \rightsquigarrow (A(a) \to A(b))$.
4. $\vec{L}L_{2\vdash} \Leftrightarrow \overset{\mapsto}{L}L_{2\vdash} \Leftrightarrow LL_{2\vdash\vdash}$*: similar to 3.*
5. $\overset{\mapsto}{L}\overset{\mapsto}{L}_2 \Rightarrow \vec{L}L_{2\vdash}$*: obvious given 4.*
6. $\vec{L}L_3 \Rightarrow \overset{\mapsto}{L}L_3$*: obvious.* $\overset{\mapsto}{L}L_3 \Rightarrow \overset{\mapsto}{L}L_{1\vdash}$*:*

(1)	$A(a)$	*assumption*
(2)	$\mathbf{t} \to A(a)$	*1, R12*
(3)	$a = b \wedge \mathbf{t} \to a = b \wedge A(a) \wedge \mathbf{t}$	*2, fiddling*
(4)	$a = b \wedge A(a) \wedge \mathbf{t} \to A(b)$	$\overset{\mapsto}{L}L_3$
(5)	$a = b \mapsto A(b)$	*3, 4, transitivity*

7. *That* $\vec{L}L_3 \Rightarrow \vec{L}\vec{L}_1$ *given for Ax6 & Ax16 (***BK***): this is easily seen by noting that* $A \wedge B \to C \vdash B \to (A \to C)$ *is a derivable rule:*

(1)	$A \wedge B \to C$	*assumption*
(2)	$(A \to B) \wedge (A \to A) \to (A \to B \wedge A)$	*Ax6*
(3)	$(A \to B) \wedge (A \to A) \to (A \to C)$	*1, 2, rightER*
(4)	$B \to (A \to B)$	*Ax16*
(5)	$B \to (A \to A)$	*Ax16 + fiddling*
(6)	$B \to (A \to B) \wedge (A \to A)$	*4, 5, R6*
(7)	$B \to (A \to C)$	*3, 6, transitivity*

8. $\overset{\rightsquigarrow}{L}\overset{\mapsto}{L}_2 \Rightarrow \overset{\rightsquigarrow}{L}L_{1\vdash}$ *given R8:*

(1)	$A(a)$	*assumption*
(2)	$a = b \rightsquigarrow (A(a) \mapsto A(b))$	$\overset{\rightsquigarrow}{L}\overset{\mapsto}{L}_2$
(3)	$A(a) \wedge \mathbf{t}$	*1, R1 +* $\mathbf{t}$*-fiddling*
(4)	$a = b \rightsquigarrow A(b)$	*2, 3, R8*

9. $\overset{\rightsquigarrow}{L}\overset{\mapsto}{L}_2 \Rightarrow \vec{L}\overset{\rightsquigarrow}{L}_1$ *given R10: trivial given 8.*
10. $\overset{\mapsto}{L}\overset{\mapsto}{L}_2 \Rightarrow \overset{\mapsto}{L}L_3$ *given R14:* $A \to (B \to C) \vdash A \wedge B \to (A \wedge B \to C)$ *is a derivable rule in* **BB** *(use Ax3 and leftER). The contraction rule, R14, therefore yields* $A \to (B \to C) \vdash A \wedge B \to C$.[18]
11. $\overset{\rightsquigarrow}{L}\overset{\mapsto}{L}_2 \Rightarrow LL_{\overline{\vdash}}$ *given R11 and, if* $\rightsquigarrow$ *is* $\mapsto$*, also R16:*

(1)	$A(a)$	*assumption*
(2)	$\neg A(b)$	*assumption*
(3)	$\mathbf{t}$	$\mathbf{t}$*-fiddling*
(4)	$A(a) \wedge \mathbf{t}$	*1, 3, R1*
(5)	$\neg(A(a) \mapsto A(b))$	*2, 4, R11 (Lem. 2)*
(6)	$a = b \mapsto (A(a) \mapsto A(b))$	$\overset{\mapsto}{L}\overset{\mapsto}{L}_2$
(7)	$a \neq b \vee \neg\mathbf{t}$	*5, 6, modus tollens*
(8)	$a \neq b$	*3, 7, R16*

□

□

I end this section by noting that, besides preserving an intuitive notion of relevance, one might also want to preserve other features of the underlying propositional logic in question. For instance, $\vec{L}\overset{\mapsto}{L}_1$ entails $A \to (\mathbf{t} \to A)$ which is generally not a theorem of logics without the permutation rule R10. Furthermore, $LL_{\neg\vdash}$ seems to be an unnatural rule for paraconsistent logics; the explosion rule, $A, \neg A \vdash B$, fails to hold in such logics and there seems little reason to accept instances of it such as $A, \neg A \vdash \forall x(x \neq x)$ if explosion does not hold. $LL_{\neg\vdash}$ is however an interesting rule for strongly paracomplete logics.

8 Coextensionality in naïve set theory

The two triviality proofs to be presented in the next section rely on the extensionality rule $a \overset{e}{=} b \vdash a = b$ which intuitively allows one to infer that 'a' denotes the

[18] And similarly that $\vec{L}L_3$ is derivable in $\mathbf{BB}[\vec{L}\vec{L}_2, R14]$.

same object as 'b' provided that a and b are coextensional sets.[19] Since it is far from evident that the coextensionality of a and b is to be expressed as $a \stackrel{e}{=} b$, that is $\forall z(z \in a \leftrightarrow z \in b)$, I will now give a brief discussion of coextensionality and the problem of restricted quantification.

The coextensionality of a and b is simply the claim that every member of a is a member of b and vice versa, and so is a claim utilizing *restricted quantification*. It seems therefore reasonable to claim that if $\Pi x[A(x), B(x)]$ is the way *all A's are B's* is to be formalized, then the coextensionality of a and b should be expressed as $\Pi x[x \in a, x \in b] \wedge \Pi x[x \in b, x \in a]$.

All the logics of interest for naïve theories in this paper have trouble with expressing restricted quantification. For instance, a side effect of the failure of contraction is that if *all A's are B's* is to be expressed as $\forall x(A(x) \rightarrow B(x))$, then the modus ponens like rule that from the sentences *every $\mathcal{D}$ is B if it's A* and *every $\mathcal{D}$ is A* to infer that *every $\mathcal{D}$ is B*,

$$\forall x(\mathcal{D}(x) \rightarrow (A(x) \rightarrow B(x))), \forall x(\mathcal{D}(x) \rightarrow A(x)) \vdash \forall x(\mathcal{D}(x) \rightarrow B(x)),$$

fails unless contraction holds for the predicate $\mathcal{D}$. For both the paraconsistentist and the relevantist this problem is particularly pressing; the failure of the **K**-rule $A \vdash B \rightarrow A$ entails that one is not licensed by logic to infer that *every A is B* follows from *everything is B*.[20] Many authors have thought this counter-intuitive and sought other ways of expressing restricted quantification. I will now show that both the relevantist and the paraconsistentist interested in naïve set theory can't validate this rule—$\forall x B(x) \vdash \Pi x[A(x), B(x)]$—no matter what form $\Pi x[A(x), B(x)]$ takes.

Definition 11 $CoExt(a, b) =_{df} \Pi x[x \in a, x \in b] \wedge \Pi x[x \in b, x \in a]$.

Theorem 13 *$\mathcal{P}$ together with the three rules*

$$\vec{L}L_{2\vdash} \qquad \forall x B(x) \vdash \Pi x[A(x), B(x)] \qquad CoExt(a, b) \vdash a = b$$

*suffice for deriving, where $x \notin FV\{A\}$, the rule $A \vdash B \rightarrow A$ in $\overline{\overline{\forall}}$**BB**.*

Proof

(1)	A	*assumption*
(2)	$\forall x(x \in \{x\|A\})$	*1, $\mathcal{P}$, $x \notin FV\{A\}$, RQ*
(3)	$\Pi x[x \in \{x\|\top\}, x \in \{x\|A\}]$	*2, assumed rule*
(4)	$\top$	*theorem*
(5)	$\Pi x[x \in \{x\|A\}, x \in \{x\|\top\}]$	*4, similar to 3*
(6)	$CoExt(\{x\|\top\}, \{x\|A\})$	*3, 5, R1 & def. of CoExt*
(7)	$\{x\|\top\} = \{x\|A\}$	*6, assumed rule*
(8)	$\{x\|\top\} \in \{x\|\top\} \rightarrow \{x\|A\} \in \{x\|A\}$	*7, $\vec{L}L_{2\vdash}$*
(9)	$\top \rightarrow A$	*8, $\mathcal{P}$; $x \notin FV\{A, \top\}$*
(10)	$B \rightarrow \top$	*def. of $\top$*
(11)	$B \rightarrow A$	*9, 10, transitivity*

□

[19] If one wishes to quantify over other things than sets as well, one could weaken the rule to $Set(a), Set(b), a \stackrel{e}{=} b \vdash a = b$, where $Set(\{x|A\})$ is assumed to hold for every A.

[20] For more discussion on restricted quantification in non-classical logics see [4, pp. 119–126], [5], [6], [14, §13.3] and [22].

The *only* viable option currently on the table for expressing restricted quantification within naïve set theory, and therefore coextensionality, for both the paraconsistentist and the relevantist alike seems to be the standard one; *all A's are B's* is best formalized as $\forall x(A(x) \rightarrow B(x))$ and the coextensionality of a and b therefore as $a \stackrel{e}{=} b$. The next section therefore assume that the naïve set theorist accepts at least the rule $a \stackrel{e}{=} b \vdash a = b$.

9 Two triviality proofs for naïve set theory

If A is a sentence, one may think of $\{x|A\}$ as the proposition expressed by A. It will be convenient to have a simpler notation for such sets:

Definition 12 If A is a sentence, then $\mathfrak{p}_A =_{df} \{x|A\}$.

Roland Hinnion and Thierry Libert introduced in [30] what has become known as the *Hinnion class*:

$$\mathfrak{h} =_{df} \{y|\{x|y \in y\} = \{x|\perp\}\}.$$

If we let $\mathfrak{H} =_{df} \mathfrak{h} \in \mathfrak{h}$ we get by abstraction that $\mathfrak{H} \leftrightarrow \mathfrak{p}_{\mathfrak{H}} = \mathfrak{p}_{\perp}$. Thus the Hinnion sentence $\mathfrak{H}$ says that the proposition expressed by it is identical to the trivial one.

Restall showed in [40] and [41] that the Hinnion class can be used to give a triviality proof.[21] Bacon showed in [3] that the Hinnion sentence can be used to show that virtually any logic with modus ponens will trivialize $\mathcal{N}$ given that there is a propositional identity connective $\equiv$ which satisfies the three rules

$$\begin{array}{ll} (i) & A \vdash B \equiv C \rightarrow A(B/C) \\ (ii) & A \equiv A \\ (iii) & A \leftrightarrow B \vdash A \equiv B, \end{array}$$

where $A(B/C)$ is the result of substituting C everywhere for B. He also remarks that a similar proof can be given using a "normal" identity predicate provided it satisfies the analogues of (i)–(iii). The following proof uses such an identity predicate, but assumes instead of $\vec{L}L_{1\vdash}$ the weaker rule $A(a) \vdash a = b \mapsto (\mathbf{t} \rightarrow A(b))$ which is easily seen to be derivable from $\overleftrightarrow{L}\overleftrightarrow{L}_2$.[22]

Lemma 7 *$\mathcal{P}$ formulated in any logic extending $\forall\mathbf{BB}^{\mathbf{t}\circ}$ is the trivial theory if there is a definable binary relation $\approx$ such that the theory is closed under the rules*

$$\begin{array}{ll} (I) & A \leftrightarrow B \vdash \mathfrak{p}_A \approx \mathfrak{p}_B \\ (II) & A(a) \vdash a \approx b \mapsto (\mathbf{t} \rightarrow A(b)), \end{array}$$

where A & B are sentences in (I).

[21] For a discussion of the proof, see sections 2.3 and 5.2 of Edwin Mares and Francesco Paoli's paper [32].

[22] I should emphasize that the proof of Lem. 7 is at heart quite similar to that given by Bacon in [3, sec. 2.2].

Proof

(1)	$(((\mathfrak{p}_C \approx \mathfrak{p}_\perp \wedge \mathbf{t}) \circ \mathbf{t}) \circ \mathbf{t}) \leftrightarrow C$	$\mathcal{N}$
(2)	$\{x\|((\mathfrak{p}_C \approx \mathfrak{p}_\perp \wedge \mathbf{t}) \circ \mathbf{t}) \circ \mathbf{t}\} \approx \mathfrak{p}_C \wedge \mathbf{t}$	*1, (I),* $\mathbf{t}$*-fiddle*
(3)	$\mathfrak{p}_C \approx \mathfrak{p}_\perp \mapsto (\mathbf{t} \to \{x\|((\mathfrak{p}_\perp \approx \mathfrak{p}_\perp \wedge \mathbf{t}) \circ \mathbf{t}) \circ \mathbf{t}\} \approx \mathfrak{p}_\perp \wedge \mathbf{t})$	*2, (II)*
(4)	$((\mathfrak{p}_\perp \approx \mathfrak{p}_\perp \wedge \mathbf{t}) \circ \mathbf{t}) \circ \mathbf{t}$	*theorem*
(5)	$\mathfrak{p}_\perp \in \{x\|((\mathfrak{p}_\perp \approx \mathfrak{p}_\perp \wedge \mathbf{t}) \circ \mathbf{t}) \circ \mathbf{t}\}$	*4,* $\mathcal{P}$
(6)	$\{x\|(\mathfrak{p}_\perp \approx \mathfrak{p}_\perp \wedge \mathbf{t}) \circ \mathbf{t}) \circ \mathbf{t}\} \approx \mathfrak{p}_\perp \mapsto (\mathbf{t} \to \mathfrak{p}_\perp \in \mathfrak{p}_\perp)$	*5, (II)*
(7)	$\mathfrak{p}_C \approx \mathfrak{p}_\perp \mapsto (\mathbf{t} \to (\mathbf{t} \to \mathfrak{p}_\perp \in \mathfrak{p}_\perp))$	*3, 6, rightER*
(8)	$((\mathfrak{p}_C \approx \mathfrak{p}_\perp \wedge \mathbf{t}) \circ \mathbf{t}) \circ \mathbf{t} \to \mathfrak{p}_\perp \in \mathfrak{p}_\perp$	*7, R13*
(9)	$\mathfrak{p}_\perp \in \mathfrak{p}_\perp \to \perp$	$\mathcal{P}$
(10)	$((\mathfrak{p}_C \approx \mathfrak{p}_\perp \wedge \mathbf{t}) \circ \mathbf{t}) \circ \mathbf{t} \to \perp$	*8, 9, transitivity*
(11)	$C \to \perp$	*1, 10, transitivity*
(12)	$C \leftrightarrow \perp$	*11, fiddling*
(13)	$\mathfrak{p}_C \approx \mathfrak{p}_\perp$	*12, (I)*
(14)	$((\mathfrak{p}_C \approx \mathfrak{p}_\perp \wedge \mathbf{t}) \circ \mathbf{t}) \circ \mathbf{t}$	*13,* $\mathbf{t}$*-fiddling*
(15)	$\perp$	*10, 14, R2*

□

Theorem 14 *$\mathcal{P}$ trivializes if closed under the rule $a \overset{e}{=} b \vdash a \overset{i}{=} b$ (Ext$_r$) in any logic extending $\forall\mathbf{BB}^{\mathbf{t}\circ}$, and so $\mathcal{S} \vdash_{\overset{=}{\forall}\mathbf{BB}^{\mathbf{t}\circ}} \perp$.*

Proof *$\overset{i}{=}$ satisfies conditions (I) and (II) in Lem. 7: since $\mathcal{P}$ is, for sentences A and B, closed under the rule $A \leftrightarrow B \vdash \mathfrak{p}_A \overset{e}{=} \mathfrak{p}_B$ and is furthermore assumed closed under the rule $a \overset{e}{=} b \vdash a \overset{i}{=} b$, we get that (I) in Lem. 7 holds. It is easily seen that $\overset{i}{=}$ is such that $a \overset{i}{=} b \mapsto (A(a) \to A(b))$ is a theorem of $\mathcal{P}$ from which it follows that it is closed under the rule $A(a) \vdash a \overset{i}{=} b \mapsto (\mathbf{t} \to A(b))$. Thus also (II) in Lem. 7 holds.* □

Corollary 2 *$\mathcal{P}$ formulated in any logic extending $\forall\mathbf{BB}$ is the trivial theory if there is a definable binary relation $\approx$ such that the theory is closed under the rules*

$$\begin{array}{ll} (I) & A \leftrightarrow B \vdash \mathfrak{p}_A \approx \mathfrak{p}_B \\ (III) & A(a) \vdash a \approx b \rightsquigarrow A(b), \end{array}$$

where A & B are sentences in (I) and $\rightsquigarrow$ is either $\to$ or $\mapsto$.

Proof *A proof using $A(a) \vdash a \approx b \mapsto A(b)$ instead of $A(a) \vdash a \approx b \mapsto (\mathbf{t} \to A(b))$ is obtained by deleting every $\circ\mathbf{t}$ in the proof of Lem. 7 above. To obtain a proof using $A(a) \vdash a \approx b \to A(b)$, delete additionally every $\wedge\mathbf{t}$.* □

Theorem 15 *$\mathcal{P}$ trivializes if closed under the rules R8 and $a \overset{e}{=} b \vdash a \overset{i}{=} b$ in any logic extending $\forall\mathbf{BB}$, and so $\mathcal{S} \vdash_{\overset{=}{\forall}\mathbf{BB}[R8]} \perp$.*

Proof *$\overset{i}{=}$ satisfies conditions (I) and (III) in Cor. 2: That (I) holds was shown in Thm. 14. Since $a \overset{i}{=} b \to (A(a) \to A(b))$ holds and $\mathcal{P}$ is assumed closed under R8 it follows that it is closed under the rule $A(a) \vdash a \overset{i}{=} b \to A(b)$. Thus also (III) in Cor. 2 holds.* □

In [11, p. 242] Brady stated that it is unknown whether $\forall x\forall y(x\overset{e}{=}y \to x \overset{i}{=} y)$ or $a \overset{e}{=} b \vdash a \overset{i}{=} b$ can consistently be added to $\mathcal{P}$ formulated in the logic $\forall\mathbf{RW}$. The above theorem settles this in the negative.[23]

Furthermore, Weber raised the question whether $\vec{L}L_3$ suffices for a triviality proof ([53, fn. 12]). I showed in Lem. 6 that $\vec{L}L_3$ entails $\vec{L}L_{1\vdash}$ (provided $\mathbf{t}$ is present), and so Cor. 2 settles his question in the negative.[24]

Brady's results in [10] and [11] show that $\mathcal{S}$ is non-trivial in $\overline{\overline{\forall}}\mathbf{DR}^{d\mathbf{t}}[\vec{L}\vec{L}_2]$, $\overline{\overline{\forall}}\mathbf{TJ}[\vec{L}\vec{L}_2, R11, R16]$ and $\overline{\overline{\forall}}\mathbf{TJX}[\vec{L}\vec{L}_2]$. The non-triviality of $\mathcal{S}$ in the following logics is as of yet unknown:[25]

$$\begin{array}{ll} \text{Strongly paraconsistent} & \begin{cases} \overline{\overline{\forall}}\mathbf{TI}^{d\mathbf{t}}[\vec{L}\vec{L}_2] \\ \overline{\overline{\forall}}\mathbf{TR}^{d\mathbf{t}}[\vec{L}\vec{L}_2] \\ \overline{\overline{\forall}}\mathbf{DL}^{d\mathbf{t}} \\ \overline{\overline{\forall}}\mathbf{BX}^{d\mathbf{t}}[\vec{L}\vec{L}_2, R11, Ax9, Ax10] \end{cases} \\ \text{Strongly paracomplete} & \left\{ \overline{\overline{\forall}}\mathbf{TJK}^{d}[\vec{L}\vec{L}_2] \right. \end{array}$$

10 The prospects of naïve set theory

The results in the above two sections show that the cost of upholding naïve set theory is immensely higher than of upholding naïve truth theory; it was shown in [23, sec. 10] that the addition of the rules $A \vdash \neg(A \to \neg A)$ and $A \vdash B \to A$ to $\overline{\overline{\forall}}\mathbf{BB}$ is sufficient for trivializing $\mathcal{S}$.[26]. The above corollaries show that this is so also with R8 and $\circ$ together with $\mathbf{t}$. This in comparison with $\mathcal{T}$ which is non-trivial in $\overline{\overline{\forall}}\mathbf{L}_{\aleph}$ ([29]), a logic in which all these rules and connectives are derivable/definable, and $\mathcal{P}$ which is non-trivial in $\forall\mathbf{BBK}[R11, Ax8]$ ([18]).

Furthermore, Cor. 2 above shows that the committed naïve set theorist needs to take great care when stating Leibniz's law—the relevantist can at best hope for $\vec{L}\vec{L}_2$ and the irrelevantist for $\vec{L}\vec{L}_2$.

Naïve set theorists of the relevant branch have typically wanted to do some mathematics within naïve set theory and in order to do so one typically needs to deal with functions which are normally taken to be sets of ordered pairs. Thus one would expect that it would be possible to find a way to define ordered pairs

[23] In addtion to Bacon's paper [3], Grišin's paper [26] should be mentioned in connection with Thm. 14 and Thm. 15. Grišin shows ([26, §4.5]) that contraction is derivable if $\mathcal{P}$ is extended by the extensionality axiom $\forall u\forall v(\forall x((x \in u \to x \in v) \circ (x \in v \to x \in u)) \to \forall x(u \in x \to v \in x))$ in the linear fragment of $\forall\mathbf{RWK}$ ($\forall\mathbf{RWK}$ minus Ax5, Q2 and Q5) formulated substructurally.

[24] The presence of $\mathbf{t}$ is not required for trivializing $\mathcal{S}$ using $\vec{L}L_3$: use $\mathcal{N}$ to obtain the sentence $C \leftrightarrow \mathfrak{p}_C = \mathfrak{p}_\perp$ and then derive $C \to (\mathfrak{p}_C = \mathfrak{p}_\perp \wedge \mathfrak{p}_C \in \mathfrak{p}_C)$ and, using $\vec{L}L_3$, $(\mathfrak{p}_C = \mathfrak{p}_\perp \wedge \mathfrak{p}_C \in \mathfrak{p}_C) \to \mathfrak{p}_\perp \in \mathfrak{p}_\perp$. From these sentences it is evident that $C \to \perp$ follows. The rest of the proof is then similar to the proof given above.

[25] This is of course not to say that these logics are the only logics of interest for naïve set theory. For instance, the logic $\overline{\overline{\forall}}\mathbf{DL}^{d\mathbf{t}}[\vec{L}\vec{L}_2]$ may also treat $\mathcal{S}$ non-trivially. However, $\vec{L}\vec{L}_2$ entails $(A \leftrightarrow B) \mapsto ((C \leftrightarrow A) \to (C \leftrightarrow B))$ and $(A \leftrightarrow B) \mapsto ((B \leftrightarrow C) \to (A \leftrightarrow C))$ for sentences A, B and C in the case of $\mathcal{S}$, and so $\mathcal{S}$ is propositionally non-conservative in these logics. These formulas are generally not logical theorems of logics without the pre- and suffixing axioms Ax9 and Ax10, and so $\vec{L}\vec{L}_2$ seems in $\mathcal{S}$ only to be appropriate for logics with Ax9 and Ax10.

[26] In fact it can be shown that the addition of the single sentence $(\mathbf{t} \to \perp) \to \perp$ will suffice.

and derive unrestricted abstraction for such objects. It seems however that one needs either permutation principles at least as strong as R8, both $\mathbf{t}$ and $\circ$, or some version of Leibniz's too strong for naïve set theory in order to do this. Appendix B shows one proof of unrestricted pair-abstraction which makes use of both $\mathbf{t}$ and $\circ$. Thus it seems the logics presented so far fall in one of two categories: either it trivializes naïve set theory, or it doesn't, but renders the theory too weak to do basic mathematics. In order to have a useful set theory one seems therefore forced to add to $\mathcal{S}$ the theory of ordered objects and naïve abstraction for them.[27]

In light of this one might be tempted to try to find other logics altogether. The proofs so far have all been assuming that the usual structural rules hold and in particular that the structural rule of *contraction*—that if $A, A \vdash B$, then $A \vdash B$—holds unrestrictedly. Contraction-free substructural logics banish even this form of contraction, and so the question arises whether such logics can do better by $\mathcal{S}$ than structural logics can. Within a substructural framework it is customary to define two notions of consequence—*external* and *internal*. Mares and Paoli have recently argued that the paradoxes of naïve truth and set theory stem from not properly distinguishing between these two notions of consequence. They furthermore claim ([32, §5.2]) that the rules

$$(\mathrm{Ext}_\in)\ \frac{\Gamma, x \in a \vdash x \in b, \Delta \qquad \Gamma, x \in b \vdash x \in a, \Delta}{\Gamma \vdash a = b, \Delta} \qquad \frac{\Gamma, \phi(a) \vdash \Delta}{\Gamma, a = b, \phi(b) \vdash \Delta}\ (=\mathrm{L}_l)$$

are both sound for naïve set theory formulated in a linear logic provided $\vdash$ is in $(\mathrm{Ext}_\in)$ taken to be the internal consequence relation of the logic whereas in $(=\mathrm{L}_l)$ the external consequence relation. These rules however entail that both $a \overset{e}{=} b \vdash a = b$ and $(\vec{L}L_{2\vdash})$ hold for the external consequence relation, which amounts to the multiple conclusion version of the relation $\vdash_{\overline{\overline{\forall}}\mathbf{LRW}^{\mathbf{t}\circ}}$ where $\forall\mathbf{LRW}$ is $\forall\mathbf{RW}$ minus the distribution axioms Ax5, Q2 and Q5. Distribution was not used in Thm. 14 and Thm. 15 which therefore show that $(\mathrm{Ext}_\in)$ and $(=\mathrm{L}_l)$ cannot both be sound in their logic. It is shown in appendix A that any substructural logic which has both $\circ$ and $\mathbf{t}$, and thus are able to interpret the antecedent structure of their sequents, trivialize naïve set theory. The only option seems therefore to be to opt for a substructural logic in which cut, i.e. transitivity of entailment, is restricted. Such an approach to the paradoxes of naïve truth, properties and sets is however beyond the scope of this paper. The interested reader should consult David Ripley's papers [42] and [43].

I end this section by noting that even though $\mathcal{S}$ is trivial in $\overline{\overline{\forall}}\mathbf{BB}^{\mathbf{t}\circ}$ and $\overline{\overline{\forall}}\mathbf{BB}[R8]$, it is still an open question whether $\mathcal{P}$ suffices for triviality or not—the non-triviality of $\mathcal{P}$ in even $\overline{\overline{\forall}}\mathbf{IMTL}[\vec{L}L_3]$ remains to be settled.

11 Summary

This paper has shown a variety of new triviality proofs for naïve theories. Section 4 showed that excluded middle and permutation principles make for a bad mix. Section 5 focused on conjunctive syllogism and section 6 showed that the addition of the fusion connective and the Ackermann constant significantly reduces the

[27] This is the approach taken by Brady in [11] although he gives a different reason for doing so.

options for the strong paraconsistentist. Lastly, section 9 showed that both the fusion connective together with the Ackermann constant and the weak permutation rule $A \to (B \to C), B \vdash A \to C$ (R8) trivialize naïve set theory.

The purpose of this paper has been to narrow the gap between the logics which have been shown to be fit for naïve theories and the logics which has been shown to be unfit for such theories. This paper goes a long way toward narrowing this gap, but a gap still remains. Given this it seems only fitting to end this article with the same sentence Slaney ended his *RWX is not Curry Paraconsistent*, namely:

> Meanwhile it seems some more investigation would be appropriate, and of course some more theorems would be absolutely splendid.

Acknowledgements I am very grateful to Zach Weber for his comments on an early version of this paper and for encouraging conversations with him and Ross Brady. I would also very much like to thank Sam Roberts and the anonymous referees for their comments. The results in sections 4–6 have been aided by William McCune's automated theorem prover Prover9 and model generator Mace4 and by John Slaney's model generator MaGIC.

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Appendices

A Naïve set theory and substructural logics

The goal of this appendix is to show that cutting structural contraction while retaining $\circ$ and $\mathbf{t}$, is not sufficient to avoid trivializing naïve set theory. The structural rule of contraction is the rule that $A \vdash B$ follows from $A, A \vdash B$. Structural contraction holds in Hilbertian proof systems since such systems take the antecentent of $\vdash$ to be *sets*. So in order to restrict this rule, we need a new notion of antecedent structure.

In order to make the transition as smooth as possible I have kept the notion of a proof intact. However, it is now *sequents* and not formulas which are the objects of proofs. $\Vdash$ will in the following be the sequent symbol. The proof system will (by and large) be that of Restall's *An Introduction to Substructural Logics* ([39]).

Definition 13 (Structure)

- 0 is a structure (but not a formula)
- If A is a formula, then A is a structure
- If X and Y are structures, then so is $(X;Y)$
- If X is a structure, and A is a formula, then $X \Vdash A$ is a sequent.

Substructure is defined in the obvious way.

- $X(Y)$ indicates that Y is a substructure of X.
- $X({}^{Y}\!/_{Z})$ is the structure got by replacing every substructure Y in X with Z.

The system $\mathfrak{S}$ consists of the following rules:

Operational rules

$$\frac{}{A \Vdash A}\,(ID)$$

$$\frac{X;A \Vdash B}{X \Vdash A \to B}\,(\to I) \qquad \frac{X \Vdash A \to B \qquad Y \Vdash A}{X;Y \Vdash B}\,(\to E)$$

$$\frac{X \Vdash A \qquad Y \Vdash B}{X;Y \Vdash A \circ B}\,(\circ I) \qquad \frac{X \Vdash A \circ B \qquad Y(A;B) \Vdash C}{Y({}^{A;B}\!/_{X}) \Vdash C}\,(\circ E)$$

$$\frac{}{0 \Vdash \mathbf{t}}\,(\mathbf{t}I) \qquad \frac{X \Vdash \mathbf{t} \qquad Y(0) \Vdash A}{Y({}^{0}\!/_{X}) \Vdash A}\,(\mathbf{t}E)$$

$$\frac{X \Vdash \bot}{X \Vdash A}\,(\bot E)$$

$$\frac{X \Vdash A({}^{x}\!/_{y})}{X \Vdash \forall x A}\,(\forall I) \qquad \frac{X; A({}^{x}\!/_{a}) \Vdash B}{X; \forall x A \Vdash B}\,(\forall E)$$

(y not free in $X \Vdash \forall x A$) (a is any term free for x in A)

Structural rules

$$\frac{X \Vdash A \qquad Y(A) \Vdash B}{Y(A/X) \Vdash B}\ (cut)$$

$$\frac{0; X \Vdash A}{X \Vdash A}\ (Left\,Pop) \qquad \frac{X \Vdash A}{0; X \Vdash A}\ (Left\,Push)$$

Set-theoretic rules

$$\frac{X; A(x/a) \Vdash B}{X; a \in \{x|A\} \Vdash B}\ (\in L) \qquad \frac{X \Vdash A(x/a)}{X \Vdash a \in \{x|A\}}\ (\in R)$$

$$\frac{x \in a \Vdash x \in b \qquad x \in b \Vdash x \in a}{0 \Vdash a = b}\ (=\in)$$

Identity rules

$$\frac{}{0 \Vdash a = a}\ (=ID) \qquad \frac{0 \Vdash a = b \qquad 0 \Vdash A(a)}{0 \Vdash A(b)}\ (subLL)$$

Definition 14 (Proof) A proof of a sequent $X \Vdash A$ from a set of sequents Γ in the system $\mathfrak{S}$ is defined to be a finite list $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that α_n is $X \Vdash A$ and every $\alpha_{m\leq n}$ is either a member of Γ, or there is a set $\Delta \subseteq \{\alpha_i \,|\, i < m\}$ such that α_m follows from Δ by one of the rules of $\mathfrak{S}$. The existential claim that there is such a proof will be written

$$\Gamma \Vvdash_{\mathfrak{S}} (X \Vdash A).$$

I will now show that the sequent $0 \Vdash \bot$ is derivable in the system $\mathfrak{S}$. The proof will mimic the proof of Lemma 7.

Definition 15

$$\begin{aligned} \mathfrak{p}_A &=_{df} \{x|A\} \\ a \rhd b &=_{df} \forall x(a \in x \to b \in x) \\ \mathfrak{c} &=_{df} \{y|(\mathfrak{p}_{y\in y} \rhd \mathfrak{p}_\bot \circ \mathbf{t}) \circ \mathbf{t}\} \\ \mathfrak{C} &=_{df} \mathfrak{c} \in \mathfrak{c} \end{aligned}$$

The following two lemmas are easily proven and are therefore left for the reader.

Lemma 8 *For sentences A and B,* $\{A \Vdash B, B \Vdash A\} \Vvdash_{\mathfrak{S}} (0 \Vdash \mathfrak{p}_A \rhd \mathfrak{p}_B)$

Lemma 9 $\{0 \Vdash A(a)\} \Vvdash_{\mathfrak{S}} (a \rhd b; \mathbf{t} \Vdash A(b))$

Theorem 16 $\emptyset \Vvdash_{\mathfrak{S}} (0 \Vdash \bot)$

Proof

(1)	$\mathfrak{C} \Vdash (\mathfrak{p}_\mathfrak{C} \rhd \mathfrak{p}_\bot \circ \mathbf{t}) \circ \mathbf{t}$	$(\in L)$
(2)	$(\mathfrak{p}_\mathfrak{C} \rhd \mathfrak{p}_\bot \circ \mathbf{t}) \circ \mathbf{t} \Vdash \mathfrak{C}$	$(\in R)$
(3)	$0 \Vdash \{x|(\mathfrak{p}_\mathfrak{C} \rhd \mathfrak{p}_\bot \circ \mathbf{t}) \circ \mathbf{t}\} \rhd \mathfrak{p}_\mathfrak{C}$	*1, 2, Lem. 8*
(4)	$\mathfrak{p}_\mathfrak{C} \rhd \mathfrak{p}_\bot; \mathbf{t} \Vdash \{x|(\mathfrak{p}_\bot \rhd \mathfrak{p}_\bot \circ \mathbf{t}) \circ \mathbf{t}\} \rhd \mathfrak{p}_\bot$	*3, Lem. 9*
(5)	$0 \Vdash \mathfrak{p}_\bot \in \{x|(\mathfrak{p}_\bot \rhd \mathfrak{p}_\bot \circ \mathbf{t}) \circ \mathbf{t}\}$	*fiddling*
(6)	$\{x|(\mathfrak{p}_\bot \rhd \mathfrak{p}_\bot \circ \mathbf{t}) \circ \mathbf{t}\} \rhd \mathfrak{p}_\bot; \mathbf{t} \Vdash \mathfrak{p}_\bot \in \mathfrak{p}_\bot$	*5, Lem. 9*
(7)	$(\mathfrak{p}_\mathfrak{C} \rhd \mathfrak{p}_\bot; \mathbf{t}); \mathbf{t} \Vdash \mathfrak{p}_\bot \in \mathfrak{p}_\bot$	*4, 6, (cut)*
(8)	$(\mathfrak{p}_\mathfrak{C} \rhd \mathfrak{p}_\bot; \mathbf{t}); \mathbf{t} \Vdash \bot$	*7, fiddling*
(9)	$(\mathfrak{p}_\mathfrak{C} \rhd \mathfrak{p}_\bot \circ \mathbf{t}) \circ \mathbf{t} \Vdash \bot$	*8, $(\circ E)$ + fiddling*
(10)	$\mathfrak{C} \Vdash \bot$	*1, 9, (cut)*
(11)	$\bot \Vdash \mathfrak{C}$	*ID + $(\bot E)$*
(12)	$0 \Vdash \mathfrak{p}_\mathfrak{C} \rhd \mathfrak{p}_\bot$	*10, 11, Lem. 8*
(13)	$0 \Vdash (\mathfrak{p}_\mathfrak{C} \rhd \mathfrak{p}_\bot \circ \mathbf{t}) \circ \mathbf{t}$	*12, fiddling*
(14)	$0 \Vdash \mathfrak{C}$	*2, 13, (cut)*
(15)	$0 \Vdash \bot$	*10, 14, (cut)*

□

One could, just as in the structural setting, dispense with $\mathbf{t}$ and $\circ$ provided one adds a substructural counterpart of R8, namely the rule

$$\frac{X; 0 \Vdash A}{X \Vdash A}\ (Right\,Pop).$$

I leave the proof to the reader. However, doing away with **t** and $\circ$ seems quite a drastic measure for the substructuralist. After all, these connectives in a sense *represent* the basic building blocks of a structure, namely 0 and ; respectively. Eliminating them would be comparable to removing $\wedge$ in the structural setting.

I mentioned in section 10 that Mares and Paoli differentiated between the external and internal consequence relation of their logic. The following defines these relations for the system $\mathfrak{S}$.

Definition 16 External vs. internal consequence

- (Internal consequence) $X \vdash^I_{\mathfrak{S}} A$ is defined for any structure X and formula A to hold just in case $\emptyset \Vdash\!\vdash_{\mathfrak{S}} (X \Vdash A)$.
- (External consequence) $\Theta \vdash^E_{\mathfrak{S}} A$ is defined for any *set* of formulas Θ and formula A to hold just in case $\{0 \Vdash \theta | \theta \in \Theta\} \Vdash\!\vdash_{\mathfrak{S}} (0 \Vdash A)$.

Thus A is an *internal* consequence of X just in case the sequent $X \Vdash A$ is derivable without assumptions. That A is a logical truth (according to $\mathfrak{S}$) is recorded as $0 \Vdash A$. Furthermore, A is an *external* consequence of the Θ's if it is provable that A is a logical truth upon assuming the Θ's to be logical truths.

Mares and Paoli wanted the rule

$$(\text{Ext}_\in)\ \frac{\Gamma, x \in a \vdash x \in b, \Delta \qquad \Gamma, x \in b \vdash x \in a, \Delta}{\Gamma \vdash a = b, \Delta}$$

to hold provided $\vdash$ was interpreted as the internal consequence relation of their logic. In their system Γ and Δ are *multiset*. For empty Γ and Δ this then amounts to validating the inference that if the sequents $x \in a \Vdash x \in b$ and $x \in b \Vdash x \in a$ are derivable without using assumptions, then so is $0 \Vdash a = b$ which is precisely what the rule $(=\in)$ above does. To avoid irrelevant formulas such as $a = b \rightarrow (A \rightarrow A)$, Mares and Paoli restricted the use of the rule

$$\frac{\phi(a) \vdash \Delta}{\Gamma, a = b, \phi(b) \vdash \Delta}\ (=\text{L}_l)$$

to context where $\vdash$ is the external consequence relation. It would therefore license the inference of $0 \Vdash B$ from the assumptions $0 \Vdash a = b$ and $0 \Vdash A(b)$, provided one has inferred $0 \Vdash B$ from $0 \Vdash A(a)$. This is however easily seen to be equivalent to what the rule $(subLL)$ licenses.

Substructural approaches to the paradoxes are sometimes deemed more radical than the structural. Restricting structural contraction is arguably a radical approach if the internal consequence relation is used to interpret what *logical entailment* amounts to. With regards to which logic does or does not trivialize the naïve theories of truth, properties and sets, the two approaches are however equivalent—all the logics set forth in section 2 can be formulated as substructural logics in such a way that if $\mathfrak{L}$ is the set of "substructural rules" for **L**, then for any set of formulas Θ,

$$\Theta \vdash_{\mathbf{L}} A \Leftrightarrow \Theta \vdash^E_{\mathfrak{L}} A.^{28}$$

From this it is easy to see that by adding rules for the naïve theory $\mathcal{M}$ to $\mathfrak{L}$ instead of adding its axioms, one obtains that

$$\mathcal{M} \vdash_{\mathbf{L}} \bot \Leftrightarrow \emptyset \Vdash\!\vdash_{\mathfrak{L}} (0 \Vdash \bot).$$

For instance, let $\mathfrak{T}$, in addition to (ID), $(\rightarrow I)$, $(\rightarrow E)$, $(\bot E)$ and (cut), consist of the rules of structural permutation and weak reductio

$$[\text{R10}](\text{Permutation rule})\ \frac{X; Y \Vdash C}{Y; X \Vdash C} \qquad \frac{A \Vdash \neg A}{0 \Vdash \neg A}\ (\text{Weak reductio})[\text{Ax13}]$$

and the rules for a *simple negation* satisfying double negation elimination, that is $\neg$ satisfies the rules

$$[\text{R5}](\neg I/\neg E)\ \frac{A \Vdash \neg B \qquad X \Vdash B}{X \Vdash \neg A} \qquad \frac{X \Vdash \neg\neg A}{X \Vdash A}\ (\neg\neg E)[\text{Ax4}].$$

$\mathfrak{T}$ is the substructural version of the implication-negation fragment of the logic **BBX**[*R*10], and it is easy to show that the sequent $0 \Vdash \bot$ is derivable from the sequents $\neg(C \rightarrow \bot) \rightarrow \bot \Vdash C$ and $C \Vdash \neg(C \rightarrow \bot) \rightarrow \bot$ in it. Thus also substructural **BBX**[*R*10] trivializes any naïve theory (cf. Thm. 3).

[28] See [39, ch. 4] for some ideas on how to prove this.

B Unrestricted pair-abstraction and the fixed-point theorem

Unrestricted abstraction is sometimes generalized so as to allow for impredicative definitions as it were—by generalizing the notion of an abstract to $\{x; y|A\}$, the schema of *generalized unrestricted abstraction* may be stated as the universal closure of

$$\forall x(x \in \{x; y|A\} \leftrightarrow A(y/\{x;y|A\}))$$

where $\{x; y|A\}$ is free for y in A. This schema guarantees the existence of fixed-point terms modulo $\overset{e}{=}$; for every formula A there is a term t_A such that $t_A \overset{e}{=} A(t_A)$. That every formula has such a fixed-point term is however easily derived from $\mathcal{P}$ alone provided the logic is sufficiently strong:

Theorem 17 *(Fixed-point theorem) If $\forall x\forall y(\langle x, y\rangle \in \{\langle x, y\rangle|A\} \leftrightarrow A)$ holds for some definition of $\langle x, y\rangle$ and $\{\langle x, y\rangle|A\}$, then $\mathcal{P}$ suffices for the the existence of fixed-points modulo $\overset{e}{=}$; for every formula A there is a term t_A such that $t_A \overset{e}{=} \{x|A(y/t_A)\}$.*[29]

Proof Let

$$r_A =_{df} \{\langle u, v\rangle|A(x/u, y/\{w|\langle w,v\rangle\in v\})\}$$
$$t_A =_{df} \{w|\langle w, r_A\rangle \in r_A\}.$$

$$\begin{array}{llll}
(1) & x \in t_A & \leftrightarrow \langle x, r_A\rangle \in r_A & \mathcal{P}\ +\ \textit{def. of}\ t_A \\
(2) & & \leftrightarrow A(x/x, y/\{w|\langle w,r_A\rangle\in r_A\}) & \textit{1, assumption} \\
(3) & & \leftrightarrow A(y/t_A) & \textit{2, def. of}\ t_A \\
(4) & & \leftrightarrow x \in \{x|A(y/t_A)\} & \textit{3, }\mathcal{P} \\
(5) & & t_A \overset{e}{=} \{x|A(y/t_A)\} & \textit{1–4, RQ}
\end{array}$$

□

The purpose of this appendix is to show that the logic $\forall\mathbf{B}^{\mathbf{t}\circ}$ is sufficiently strong to provide a definition of both $\langle x, y\rangle$ and $\{\langle x, y\rangle|A\}$ so that unrestricted pair-abstraction,

$$\forall x\forall y(\langle x, y\rangle \in \{\langle x, y\rangle|A\} \leftrightarrow A),$$

is derivable. Since $\forall\mathbf{B}^{\mathbf{t}\circ}$ is a rather weak logic the definitions and proofs will however be quite baroque. After presenting the proofs I will give some quick comments on the prospects of finding other definitions suitable for logics which might treat naïve set theory as a non-trivial theory.

Definition 17

$$\begin{aligned}
a \overset{i}{=} b &=_{df} \forall x(a \in x \leftrightarrow b \in x) \\
\{a\} &=_{df} \{x|x \overset{i}{=} a\} \\
\{a, b\} &=_{df} \{x|x \overset{i}{=} a \vee x \overset{i}{=} b\} \\
\langle a, b\rangle &=_{df} \{\{a\}, \{a, b\}\} \\
\partial(A) &=_{df} \big[(A \circ \mathbf{t}) \circ \mathbf{t}\big] \circ \Bigg(\bigg(\Big(\big(\big[(A \circ \mathbf{t}) \circ \mathbf{t}\big] \circ \big[(A \circ \mathbf{t}) \circ \mathbf{t}\big]\big) \circ \mathbf{t}\Big) \circ \big[(A \circ \mathbf{t}) \circ \mathbf{t}\big]\bigg) \circ \mathbf{t}\Bigg) \\
\{\langle x, y\rangle|A\} &=_{df} \{z|\exists x\exists y(\partial(\langle x, y\rangle \overset{i}{=} z) \circ A)\}
\end{aligned}$$

Lemma 10

$$A \vdash_{\mathbf{BB}^{\mathbf{t}\circ}} \partial(A)$$

Proof *This holds essentially because $A, B \vdash_{\mathbf{BB}^{\mathbf{t}\circ}} A \circ B$ holds. That this is so is easily seen by noting that $A \to (B \to A \circ B)$ is derivable using Ax1 and R13.* □

Lemma 11 *Assuming a and b to be free for x in A, then*

$$(1)\ \mathcal{P} \vdash_{\forall\mathbf{BB}} a \overset{i}{=} b \to (A(x/a) \to A(x/b))$$

$$\textit{(2)}\ \frac{\mathcal{P} \vdash_{\forall\mathbf{BB}^{\mathbf{t}}} A(x/a)}{\mathcal{P} \vdash_{\forall\mathbf{BB}^{\mathbf{t}}} a \overset{i}{=} b \to (\mathbf{t} \to A(x/b))}$$

[29] This result was to my knowledge first proven by Jean-Yves Girard in [25, Prop. 4]. Girard remarks that the result goes back to the fixed-point theorem of λ-calculus.

Proof Obvious. □

Lemma 12

$$\mathcal{P} \vdash_{\forall\mathbf{BB}^{\mathbf{t}\circ}} \big[(\langle a,b\rangle \overset{i}{=} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to a \overset{i}{=} c$$

Proof

(1)	$\{a\} \in \langle a,b\rangle$	*def. of ord. pair +* $\mathcal{P}$
(2)	$\langle a,b\rangle \overset{i}{=} \langle c,d\rangle \to (\mathbf{t} \to \{a\} \in \langle c,d\rangle)$	*1, Lem. 11(2)*
(3)	$\{a\} \in \langle c,d\rangle \to (\{a\} \overset{i}{=} \{c\} \vee \{a\} \overset{i}{=} \{c,d\})$	*def. of ord. pair +* $\mathcal{P}$
(4)	$a \in \{a\}$	*def. of singleton +* $\mathcal{P}$
(5)	$\{a\} \overset{i}{=} \{c\} \to (\mathbf{t} \to a \in \{c\})$	*4, Lem. 11(2)*
(6)	$a \in \{c\} \to a \overset{i}{=} c$	*def. of singleton +* $\mathcal{P}$
(7)	$\{a\} \overset{i}{=} \{c\} \to (\mathbf{t} \to a \overset{i}{=} c)$	*5, 6, rightER*
(8)	$\{a\} \overset{i}{=} \{c,d\} \to (\mathbf{t} \to a \overset{i}{=} c)$	*similar to (7)*
(9)	$(\{a\} \overset{i}{=} \{c\} \vee \{a\} \overset{i}{=} \{c,d\}) \to (\mathbf{t} \to a \overset{i}{=} c)$	*7, 8, R7*
(10)	$\langle a,b\rangle \overset{i}{=} \langle c,d\rangle \to (\mathbf{t} \to (\mathbf{t} \to a \overset{i}{=} c))$	*2, 3, 9, rightER & transitivity*
(11)	$\big[(\langle a,b\rangle \overset{i}{=} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to a \overset{i}{=} c$	*10, R13*

□

Lemma 13

$$(1)\ \mathcal{P} \vdash_{\forall\mathbf{BB}^{\mathbf{t}\circ}} \big[(\langle a,b\rangle \overset{i}{=} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to b \overset{i}{=} c \vee b \overset{i}{=} d$$
$$(2)\ \mathcal{P} \vdash_{\forall\mathbf{BB}^{\mathbf{t}\circ}} \big[(\langle a,b\rangle \overset{i}{=} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to a \overset{i}{=} d \vee b \overset{i}{=} d$$

Proof The proof of (2) is similar to that of (1).

(1)	$\{a,b\} \in \langle a,b\rangle$	*def. of ord. pair +* $\mathcal{P}$
(2)	$\langle a,b\rangle \overset{i}{=} \langle c,d\rangle \to (\mathbf{t} \to \{a,b\} \in \langle c,d\rangle)$	*1, Lem. 11(2)*
(3)	$\{a,b\} \in \langle c,d\rangle \to \{a,b\} \overset{i}{=} \{c\} \vee \{a,b\} \overset{i}{=} \{c,d\}$	*def. of ord. pair +* $\mathcal{P}$
(4)	$b \in \{a,b\}$	*def. of ord. pair +* $\mathcal{P}$
(5)	$\{a,b\} \overset{i}{=} \{c\} \to (\mathbf{t} \to b \in \{c\})$	*4, Lem. 11(2)*
(6)	$b \in \{c\} \to b \overset{i}{=} c \vee b \overset{i}{=} d$	*def. of singleton,* $\mathcal{P}$ *& Ax2*
(7)	$\{a,b\} \overset{i}{=} \{c\} \to (\mathbf{t} \to b \overset{i}{=} c \vee b \overset{i}{=} d)$	*5, 6, rightER*
(8)	$\{a,b\} \overset{i}{=} \{c,d\} \to (\mathbf{t} \to b \overset{i}{=} c \vee b \overset{i}{=} d)$	*similar to (7)*
(9)	$\{a,b\} \overset{i}{=} \{c\} \vee \{a,b\} \overset{i}{=} \{c,d\} \to (\mathbf{t} \to b \overset{i}{=} c \vee b \overset{i}{=} d)$	*4, 5, R7*
(10)	$\langle a,b\rangle \overset{i}{=} \langle c,d\rangle \to (\mathbf{t} \to (\mathbf{t} \to b \overset{i}{=} c \vee b \overset{i}{=} d))$	*2, 3, 6, rightER & transitivity*
(11)	$\big[(\langle a,b\rangle \overset{i}{=} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to b \overset{i}{=} c \vee b \overset{i}{=} d$	*10, R13*

□

Lemma 14

$$\mathcal{P} \vdash_{\forall\mathbf{B}^{\mathbf{t}}} a \overset{i}{=} c \to (a \overset{i}{=} d \vee b \overset{i}{=} d \to (\mathbf{t} \to (b \overset{i}{=} c \vee b \overset{i}{=} d \to (\mathbf{t} \to b \overset{i}{=} d))))$$

Proof

(1)	$b \overset{i}{=} b$	*theorem*
(2)	$b \overset{i}{=} c \to (\mathbf{t} \to b \overset{i}{=} b)$	*1, Lem. 11(2)*
(3)	$b \overset{i}{=} d \to (\mathbf{t} \to (b \overset{i}{=} c \to (\mathbf{t} \to b \overset{i}{=} d)))$	*2, Lem. 11(2)*
(4)	$a \overset{i}{=} c \to (b \overset{i}{=} d \to b \overset{i}{=} d)$	*Lem. 11(1)*
(5)	$a \overset{i}{=} c \to (b \overset{i}{=} d \to (\mathbf{t} \to (b \overset{i}{=} c \to (\mathbf{t} \to b \overset{i}{=} d))))$	*3, 4, rightER*
(6)	$b \overset{i}{=} c \to (\mathbf{t} \to b \overset{i}{=} c)$	*1, Lem. 11(2)*
(7)	$c \overset{i}{=} d \to (\mathbf{t} \to (b \overset{i}{=} c \to (\mathbf{t} \to b \overset{i}{=} d)))$	*6, Lem. 11(2)*
(8)	$a \overset{i}{=} c \to (a \overset{i}{=} d \to c \overset{i}{=} d)$	*Lem. 11(1)*
(9)	$a \overset{i}{=} c \to (a \overset{i}{=} d \to (\mathbf{t} \to (b \overset{i}{=} c \to (\mathbf{t} \to b \overset{i}{=} d))))$	*7, 8, rightER*
(10)	$a \overset{i}{=} c \to (a \overset{i}{=} d \vee b \overset{i}{=} d \to (\mathbf{t} \to (b \overset{i}{=} c \to (\mathbf{t} \to b \overset{i}{=} d))))$	*5, 9, R6 & Ax7*
(11)	$a \overset{i}{=} c \to (a \overset{i}{=} d \vee b \overset{i}{=} d \to (\mathbf{t} \to (b \overset{i}{=} d \to (\mathbf{t} \to b \overset{i}{=} d))))$	*similar to (10)*
(12)	$a \overset{i}{=} c \to (a \overset{i}{=} d \vee b \overset{i}{=} d \to (\mathbf{t} \to (b \overset{i}{=} c \vee b \overset{i}{=} d \to (\mathbf{t} \to b \overset{i}{=} d))))$	*10, 11, Ax6 & Ax7*

□

Lemma 15

$$A \to (B \to (C \to (D \to E))), F \to D \vdash_{\mathbf{BB}} A \to (B \to (C \to (F \to E)))$$

Proof Use *R3*, *R4* and *rightER*. □

Lemma 16

$$\mathcal{P} \vdash_{\forall \mathbf{Bt}\circ} \Bigg(\Bigg(\Big(\Big(\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \circ \big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big]\Big) \circ \mathbf{t}\Big) \circ \big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big]\Bigg) \circ \mathbf{t}\Bigg) \to b \overset{i}{\doteq} d$$

Proof

(1) $a \overset{i}{\doteq} c \to ((a \overset{i}{\doteq} d \vee b \overset{i}{\doteq} d) \to (\mathbf{t} \to (b \overset{i}{\doteq} c \vee b \overset{i}{\doteq} d \to (\mathbf{t} \to b \overset{i}{\doteq} d))))$ — *Lem. 14*

(2) $\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to a \overset{i}{\doteq} d \vee b \overset{i}{\doteq} d$ — *Lem. 13*

(3) $a \overset{i}{\doteq} c \to (\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to (\mathbf{t} \to (b \overset{i}{\doteq} c \vee b \overset{i}{\doteq} d \to (\mathbf{t} \to b \overset{i}{\doteq} d))))$ — *1, 2, leftER*

(4) $\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to b \overset{i}{\doteq} c \vee b \overset{i}{\doteq} d$ — *Lem. 13*

(5) $a \overset{i}{\doteq} c \to (\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to$
$(\mathbf{t} \to (\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to (\mathbf{t} \to b \overset{i}{\doteq} d)))$ — *3, 4, Lem. 15*

(7) $\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to a \overset{i}{\doteq} c$ — *Lem. 12*

(8) $\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to (\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to$
$(\mathbf{t} \to (\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to (\mathbf{t} \to b \overset{i}{\doteq} d)))$ — *6, 7, trans.*

(9) $\Bigg(\Bigg(\Big(\Big(\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \circ \big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big]\Big) \circ \mathbf{t}\Big) \circ \big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big]\Bigg) \circ \mathbf{t}\Bigg) \to b \overset{i}{\doteq} d$ — *8, R13*

□

Lemma 17

$$\mathcal{P} \vdash_{\forall \mathbf{Bt}\circ} \partial\big(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle\big) \circ A(a,b) \to A(c,d), i.e.$$

$$\mathcal{P} \vdash_{\forall \mathbf{Bt}\circ} \Bigg(\Bigg[\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \circ \Bigg(\Bigg(\Big(\Big(\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \circ \big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big]\Big) \circ \mathbf{t}\Big) \circ \big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big]\Bigg) \circ \mathbf{t}\Bigg)\Bigg] \circ A(a,b)\Bigg) \to A(c,d)$$

Proof

(1) $b \overset{i}{\doteq} d \to (A(a,b) \to A(a,d))$ — *Lem. 11(1)*

(2) $a \overset{i}{\doteq} c \to ((A(a,b) \to A(a,d)) \to (A(a,b) \to A(c,d)))$ — *Lem. 11(1)*

(3) $a \overset{i}{\doteq} c \to (b \overset{i}{\doteq} d \to (A(a,b) \to A(c,d)))$ — *1, 2, leftER*

(4) $\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to a \overset{i}{\doteq} c$ — *Lem. 12*

(5) $\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to \big[b \overset{i}{\doteq} d \to (A(a,b) \to A(c,d))\big]$ — *3, 4, trans.*

(6) $\Bigg(\Bigg(\Big(\Big(\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \circ \big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big]\Big) \circ \mathbf{t}\Big) \circ \big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big]\Bigg) \circ \mathbf{t}\Bigg) \to b \overset{i}{\doteq} d$ — *Lem. 16*

(7) $\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \to$
$\Bigg[\Bigg(\Bigg(\Big(\Big(\big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big] \circ \big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big]\Big) \circ \mathbf{t}\Big) \circ \big[(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle \circ \mathbf{t}) \circ \mathbf{t}\big]\Bigg) \circ \mathbf{t}\Bigg) \to (A(a,b) \to A(c,d))\Bigg]$ — *5, 6, leftER*

(8) $\partial\big(\langle a,b\rangle \overset{i}{\doteq} \langle c,d\rangle\rangle\big) \circ A(a,b) \to A(c,d)$ — *7, R13*

□

Theorem 18 $\mathcal{P} \vdash_{\forall \mathbf{B^{t\circ}}} \langle a,b\rangle \in \{\langle x,y\rangle|A\} \leftrightarrow A(a,b)$

Proof

(1)	$\partial\big(\langle x,y\rangle \overset{i}{=} \langle a,b\rangle\big) \circ A(x,y) \to A(a,b)$	*Lem. 17*
(2)	$\exists x\exists y(\partial\big(\langle x,y\rangle \overset{i}{=} \langle a,b\rangle\big) \circ A(x,y)) \to A(a,b)$	*1, RQ & Q8*
(3)	$\langle a,b\rangle \in \{\langle x,y\rangle|A\} \to A(a,b)$	*2, $\mathcal{P}$ + def. of $\{\langle x,y\rangle|A\}$*
(4)	$\partial\big(\langle a,b\rangle \overset{i}{=} \langle a,b\rangle\big) \to (A(a,b) \to \partial\big(\langle a,b\rangle \overset{i}{=} \langle a,b\rangle\big) \circ A(a,b))$	*Ax1 & R13*
(5)	$\partial\big(\langle a,b\rangle \overset{i}{=} \langle a,b\rangle\big)$	*Lem. 10*
(6)	$A(a,b) \to \partial\big(\langle a,b\rangle \overset{i}{=} \langle a,b\rangle\big) \circ A(a,b)$	*4, 5, R2*
(7)	$A(a,b) \to \exists x\exists y(\partial\big(\langle x,y\rangle \overset{i}{=} \langle a,b\rangle\big) \circ A(x,y))$	*6, Q4*
(8)	$A(a,b) \to \langle a,b\rangle \in \{\langle x,y\rangle|A\}$	*7, $\mathcal{P}$ + def. of $\{\langle x,y\rangle|A\}$*
(9)	$\langle a,b\rangle \in \{\langle x,y\rangle|A\} \leftrightarrow A(a,b)$	*3, 8, R1*

□

Corollary 3 $\forall\mathbf{B^{t\circ}}$ *suffices for the fixed-point theorem.*

Proof *This follows from Thm. 17 and Thm. 18.*

$\forall\mathbf{B^{t\circ}}$ trivializes naïve set theory (Thm. 14). The question then is if there are logics weak enough not to trivialize naïve set theory, yet strong enough so as to make unrestricted pair-abstraction derivable. As the definition of $\{\langle x,y\rangle|A\}$ above should make clear, there are countless non-equivalent ways of defining sets of ordered pairs provided the logic is weak enough. What should also be clear is that defining $\{\langle x,y\rangle|A\}$ as $\{z|\forall x\forall y(\partial'\big(\langle x,y\rangle = z\big) \to A)\}$ for some variant ∂' of ∂, would not improve the situation: $\mathcal{P}$ in $\overline{\overline{\forall}}\mathbf{BB}$ suffices for deriving $\langle a,b\rangle \in \{\langle x,y\rangle|A\} \to (\partial'\big(\langle a,b\rangle = \langle a,b\rangle\big) \to A(a,b))$. To get $\langle a,b\rangle \in \{\langle x,y\rangle|A\} \to A(a,b)$ from this one would then most certainly require the weak permutation rule R8 which trivializes naïve set theory (Thm. 15). The only option left as I see it would be to use $\{\langle x,y\rangle|A\} =_{df} \{z|\exists x\exists y(\partial'(\langle x,y\rangle = z) \wedge A)\}$. However, this is not an option for the naïve set theoriest either: in order to have that $A(a,b) \vdash \langle a,b\rangle \in \{\langle x,y\rangle|A\}$, one would need a function ∂' such that $A \vdash \partial'(A)$. Furthermore, in order to prove $\langle a,b\rangle \in \{\langle x,y\rangle|A\} \to A(a,b)$, one would need $\partial'(\langle a,b\rangle = \langle c,d\rangle) \wedge A(a,b) \to A(c,d)$. These two assumptions suffice for a triviality proof:

(1)	$C \leftrightarrow \partial'(\langle \mathfrak{p}_C, \mathfrak{p}_C\rangle = \langle \mathfrak{p}_\perp, \mathfrak{p}_\perp\rangle)$	$\mathcal{N}$
(2)	$C \to \partial'(\langle \mathfrak{p}_C, \mathfrak{p}_C\rangle = \langle \mathfrak{p}_\perp, \mathfrak{p}_\perp\rangle) \wedge \mathfrak{p}_C \in \mathfrak{p}_C$	1, fiddling
(3)	$\partial'(\langle \mathfrak{p}_C, \mathfrak{p}_C\rangle = \langle \mathfrak{p}_\perp, \mathfrak{p}_\perp\rangle) \wedge \mathfrak{p}_C \in \mathfrak{p}_C \to \mathfrak{p}_\perp \in \mathfrak{p}_\perp$	assumed theorem
(4)	$C \to \perp$	2, 3, fiddling
(5)	$\mathfrak{p}_C = \mathfrak{p}_\perp$	4, extensionality + fiddling
(6)	$\langle \mathfrak{p}_C, \mathfrak{p}_C\rangle = \langle \mathfrak{p}_\perp, \mathfrak{p}_\perp\rangle$	5, $LL_{2\vdash\vdash}$
(7)	$\partial'(\langle \mathfrak{p}_C, \mathfrak{p}_C\rangle = \langle \mathfrak{p}_\perp, \mathfrak{p}_\perp\rangle)$	6, assumed rule
(8)	$\perp$	1, 4, 7, R2

I therefore conclude that the prospects of finding a logic and a way of defining $\{\langle x,y\rangle|A\}$ so that the logic does not trivialize naïve set theory, yet is strong enough to make unrestricted pair-abstraction a theorem thereof, are dim, at best.