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Sensitivity of entanglement measures in bipartite pure quantum states

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Entanglement measures quantify the amount of quantum entanglement that is contained in quantum states. Typically, different entanglement measures do not have to be partially ordered. The presence of a definite partial order between two entanglement measures for all quantum states, however, allows for meaningful conceptualization of sensitivity to entanglement, which will be greater for the entanglement measure that produces the larger numerical values. Here, we have investigated the partial order between the normalized versions of four entanglement measures based on Schmidt decomposition of bipartite pure quantum states, namely, concurrence, tangle, entanglement robustness and Schmidt number. We have shown that among those four measures, the concurrence and the Schmidt number have the highest and the lowest sensitivity to quantum entanglement, respectively. Further, we have demonstrated how these measures could be used to track the dynamics of quantum entanglement in a simple quantum toy model composed of two qutrits. Lastly, we have employed state-dependent entanglement statistics to compute measurable correlations between the outcomes of quantum observables in agreement with the uncertainty principle. The presented results could be helpful in quantum applications that require monitoring of the available quantum resources for sharp identification of temporal points of maximal entanglement or system separability.

Keywords: entanglement measure; partial order; Schmidt decomposition.

1. Introduction

Quantum entanglement is an important resource in quantum information technologies.[1–4] Shared quantum entanglement between two distantly located parties allows for the execution of classically impossible tasks, such as quantum teleportation,[5,6] superdense coding,[7] or quantum cryptography.[8–10] Being such a valuable commodity, the amount of quantum entanglement possessed by composite quantum systems has been subject to quantification with a variety of entanglement measures.[11–14] Some of these measures were defined operationally,[11,15–17] whereas others were defined with an explicit formula that is computed from the complex quantum probability amplitudes characterizing the state of the composite system.[18–25] The rapid

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burgeoning of quantum resource theory[26] has generated a zoo of entanglement measures, most of which appeared under different names in the works of different authors. This impedes accessibility of available mathematical results and complicates the conduction of literature searches. Furthermore, the utility and performance of different measures for tracking the entanglement dynamics in composite quantum systems has been rarely compared. To remedy this situation, in this work we analyze a number of entanglement measures based on Schmidt decomposition and systematically explore their ability to resolve maximal entanglement or complete disentanglement of a toy model system consisting of two interacting qutrits. Then we provide a comprehensive introduction to state-dependent entanglement statistics and compute measurable correlations between the outcomes of quantum observables in agreement with the uncertainty principle.

The organization of the presentation is as follows: In Section 2, we briefly summarize how every bipartite state vector can be expressed in the Schmidt basis using singular value decomposition of the complex coefficient matrix given in some explicit basis. Then, we introduce four entanglement measures that can be computed directly from the Schmidt coefficients. The most popular names for these four measures are: concurrence, tangle, entanglement robustness and Schmidt number. In Section 3, we introduce the concept of relative sensitivity to quantum entanglement and prove two main theorems, which establish the existing partial order between different normalized versions of the four entanglement measures. In Section 4, we present a quantum toy model of two interacting qutrits, which ensures the minimal Hilbert space required to avoid reduction of entanglement robustness to concurrence. In Section 5, we report computational results on the performance of each of the four entanglement measures on resolving maximal entanglement or complete disentanglement of the toy quantum system. In Section 6, we introduce the concept of state-dependent entanglement statistics and demonstrate how the Schmidt decomposition features prominently in the computation of measurable correlations between the outcomes of quantum observables. Finally, we conclude with a brief discussion on the computational complexity involved in the evaluation of the presented entanglement measures and their overall utility for tracking the entanglement dynamics in composite quantum systems.

2. Entanglement measures

Quantum entanglement was originally conceptualized by Erwin Schrödinger in the form of probability relations between distant quantum systems.[27] It needs to be emphasized, however, that *quantum probabilities* relate *quantum observables*, which describe potentialities of what could be measured, but not necessarily of what is actually measured.[28] This means that given a quantum state vector $|\Psi\rangle$ of a composite quantum system, one could always compute the *expectation values* of different quantum observables, including incompatible (non-commuting) observables whose simultaneous measurement is physically impossible.[29,30] Furthermore,

even for maximally entangled quantum states there are quantum observables whose measurement outcomes are maximally correlated and quantum observables whose measurement outcomes are not correlated at all. For example, given the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\uparrow_z\rangle + |\downarrow_z\downarrow_z\rangle)$ one could either measure the observable $\hat{\sigma}_z \otimes \hat{\sigma}_z$ obtaining maximally correlated outcomes or measure the observable $\hat{\sigma}_z \otimes \hat{\sigma}_x$ obtaining completely uncorrelated outcomes.[31] This highlights the fact that quantum entanglement is not a genuine property of quantum observables. Instead, the quantum entanglement is a genuine property of the *quantum state vector* $|\Psi\rangle$, which is comprised of *quantum probability amplitudes* rather than *quantum probabilities*, and motivates the following definition valid for non-relativistic quantum mechanics of distinguishable particles.

Definition 2.1. (Entangled state) A bipartite quantum state vector $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is quantum entangled if and only if it cannot be written as a tensor product[25]

$$|\Psi\rangle \neq |\psi\rangle_A \otimes |\psi\rangle_B. \tag{1}$$

Otherwise, the quantum state vector is separable (factorizable).

The Schmidt decomposition provides a straightforward criterion for determining whether a bipartite quantum state vector is entangled or not.

Theorem 2.1. *(Schmidt decomposition) Consider a composite bipartite quantum state vector $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Given any two complete orthonormal bases for the individual Hilbert spaces, respectively $\{|i\rangle_A\}$ for $\mathcal{H}_A$ and $\{|j\rangle_B\}$ for $\mathcal{H}_B$, one can always construct a complete orthonormal tensor product basis $\{|i\rangle_A \otimes |j\rangle_B\}$ for the composite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ in which the bipartite quantum state vector is expressed as[31]*

$$|\Psi\rangle = \sum_i \sum_j c_{ij}|i\rangle_A \otimes |j\rangle_B. \tag{2}$$

Then, singular value decomposition of the complex coefficient matrix $\hat{C} = (c_{ij})$ renders it in the form

$$\hat{C} = \hat{U}\hat{\Lambda}\hat{V}^\dagger, \tag{3}$$

where $\hat{U}$ and $\hat{V}^\dagger$ are unitary matrices, and $\hat{\Lambda}$ is a diagonal matrix with non-negative singular values (Schmidt coefficients) sorted in descending order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s \geq 0$. Finally, using the operations of matrix reshaping and reshuffling in the context of Jamiołkowski isomorphism,[32] the bipartite quantum state vector can always be expressed in the Schmidt basis as

$$|\Psi\rangle = \sum_s \lambda_s \left(\hat{U}|i_s\rangle_A\right) \otimes \left(\hat{V}^\dagger|j_s\rangle_B\right), \tag{4}$$

where the index s runs from 1 to $\min\left[\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)\right]$.

Definition 2.2. (Schmidt rank) The number of non-zero Schmidt coefficients is referred to as the Schmidt rank of a given Schmidt decomposition.

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The Schmidt rank provides a binary, Yes/No, classification of quantum states. The quantum state is entangled if and only if its Schmidt rank is greater than 1. For separable states, the Schmidt rank is exactly 1. Unfortunately, the binary classification of quantum states does not suffice for quantitative evaluation and management of quantum entanglement as a resource. Thus, given an entangled state that has at least two non-zero Schmidt coefficients, it would be useful to have quantitative measures that determine how valuable the state is. Next, we present four such entanglement measures whose numerical values can be computed from explicit formulas involving the Schmidt coefficients, namely, *concurrence*, *tangle*, *entanglement robustness* and *Schmidt number*.

2.1. *Concurrence*

The *concurrence* was first introduced by Hill and Wootters for pure two qubit states using a modified Bell basis $\{|\Phi^+\rangle, \imath|\Phi^-\rangle, \imath|\Psi^+\rangle, |\Psi^-\rangle\}$,[20,21] but was then generalized as *I-concurrence* to include multi-level bipartite quantum systems[23] using the sum of the fourth powers of the Schmidt coefficients

$$\mathcal{C}\left(\Psi\right) = \sqrt{2\left[1 - \sum_{i=1}^{n} \lambda_i^4\right]}, \tag{5}$$

where $n = \min\left[\dim\left(\mathcal{H}_A\right), \dim\left(\mathcal{H}_B\right)\right]$.

In the context of quantum interferometry with entangled particles, the concurrence is manifested as *two-particle visibility*.[33–35] Recently, within the context of a general theory of entanglement, Gudder proposed the *entanglement number*, which is essentially I-concurrence without the scale factor[24,25]

$$e\left(\Psi\right) = \sqrt{1 - \sum_i \lambda_i^4} = \sqrt{1 - \mathrm{Tr}\left(\hat{\rho}_A^2\right)} = \sqrt{1 - \mathrm{Tr}\left(\hat{\rho}_B^2\right)}. \tag{6}$$

We can use the fact that the Schmidt decomposition gives a normalized vector

$$\sum_i \lambda_i^2 = 1 \tag{7}$$

in order to substitute in (6) and obtain

$$e\left(\Psi\right) = \sqrt{\sum_i \lambda_i^2 - \sum_j \lambda_j^4} = \sqrt{\sum_i \left(\lambda_i^2 - \lambda_i^4\right)} = \sqrt{\sum_i \lambda_i^2 \left(1 - \lambda_i^2\right)}. \tag{8}$$

Since (7) implies that

$$1 - \lambda_i^2 = \sum_{j \neq i} \lambda_j^2, \tag{9}$$

one arrives at

$$e\left(\Psi\right) = \sqrt{\sum_{i \neq j} \lambda_i^2 \lambda_j^2} = \sqrt{2 \sum_{i < j} \lambda_i^2 \lambda_j^2}. \tag{10}$$

Alternatively, it is possible to use the identity $1^2 = 1$ to directly obtain

$$e\left(\Psi\right) = \sqrt{1^2 - \sum_j \lambda_j^4} = \sqrt{\left(\sum_i \lambda_i^2\right)^2 - \sum_j \lambda_j^4} = \sqrt{\sum_{i \neq j} \lambda_i^2 \lambda_j^2}. \tag{11}$$

The range of the entanglement number is within the interval $0 \leq e(\Psi) \leq \sqrt{\frac{n-1}{n}}$.

The normalized concurrence is the same as the normalized entanglement number[31]

$$\tilde{\mathcal{C}}\left(\Psi\right) = \tilde{e}\left(\Psi\right) = \sqrt{\frac{n}{n-1}\left(1 - \sum_i \lambda_i^4\right)}. \tag{12}$$

Computationally useful is the fact that the entanglement number and concurrence could be evaluated from squaring the Hermitian matrix $\hat{C}\hat{C}^\dagger$ obtained from the complex coefficient matrix $\hat{C} = (c_{ij})$ without the need of singular value decomposition[24, 25]

$$e\left(\Psi\right) = \sqrt{\left[\mathrm{Tr}\left(\hat{C}\hat{C}^\dagger\right)\right]^2 - \mathrm{Tr}\left[\left(\hat{C}\hat{C}^\dagger\right)^2\right]} = \sqrt{1 - \mathrm{Tr}\left[\left(\hat{C}\hat{C}^\dagger\right)^2\right]}. \tag{13}$$

2.2. *Tangle*

The *squared concurrence* $\mathcal{C}^2\left(\Psi\right)$ is a distinct entanglement measure referred to as *tangle*.[1, 36] From (12), the normalized tangle is given by

$$\tilde{\mathcal{T}}\left(\Psi\right) = \tilde{\mathcal{C}}^2\left(\Psi\right) = \frac{n}{n-1}\left(1 - \sum_i \lambda_i^4\right) = \frac{n}{n-1}\sum_{i \neq j} \lambda_i^2 \lambda_j^2. \tag{14}$$

2.3. *Robustness of entanglement*

For pure bipartite states, the *robustness of entanglement*[19, 37] can be computed from the squared sum of the Schmidt coefficients as follows

$$\mathcal{R}\left(\Psi\right) = \left(\sum_{i=1}^n \lambda_i\right)^2 - 1, \tag{15}$$

where $n = \min\left[\dim\left(\mathcal{H}_A\right), \dim\left(\mathcal{H}_B\right)\right]$.

The range of the robustness of entanglement is

$$0 \leq \mathcal{R}(\Psi) \leq n - 1. \tag{16}$$

Therefore, the normalized robustness $0 \leq \tilde{\mathcal{R}}(\Psi) \leq 1$ is given by

$$\tilde{\mathcal{R}}(\Psi) = \frac{1}{n-1}\left[\left(\sum_i \lambda_i\right)^2 - 1\right]. \tag{17}$$

Straightforward algebraic calculation shows that the robustness of entanglement is the same as the non-normalized coherence[38] of the density matrix in the Schmidt basis

$$
\begin{aligned}
\mathcal{R}\left(\Psi\right) &= \left(\sum_i \lambda_i\right)^2 - 1 = \left(\sum_i \lambda_i\right)^2 - \sum_i \lambda_i^2 \\
&= \left(\sum_i \lambda_i^2 + 2\sum_{i<j} \lambda_i\lambda_j\right) - \sum_i \lambda_i^2 \\
&= 2\sum_{i<j} \lambda_i\lambda_j = \sum_{i\neq j} \lambda_i\lambda_j.
\end{aligned}
\tag{18}
$$

The normalized robustness of entanglement is then the same as the *entanglement coherence* C_E defined in the Schmidt basis[39]

$$
\tilde{\mathcal{R}}\left(\Psi\right) = C_E = \frac{1}{n-1}\sum_{i\neq j} \lambda_i\lambda_j.
\tag{19}
$$

The entanglement coherence could be also expressed in terms of the reduced density matrices $\hat{\rho}_A = \mathrm{Tr}_B\left(\hat{\rho}_{AB}\right)$ and $\hat{\rho}_B = \mathrm{Tr}_A\left(\hat{\rho}_{AB}\right)$ as follows

$$
C_E = \frac{1}{n-1}\left[\left(\mathrm{Tr}\sqrt{\hat{\rho}_A}\right)^2 - 1\right] = \frac{1}{n-1}\left[\left(\mathrm{Tr}\sqrt{\hat{\rho}_B}\right)^2 - 1\right].
\tag{20}
$$

In the special case of two qubits, the robustness of entanglement reduces to concurrence

$$
\tilde{\mathcal{C}}\left(\Psi\right) = 2\lambda_1\lambda_2 = \tilde{\mathcal{R}}\left(\Psi\right),
\tag{21}
$$

but for higher dimensional systems that admit more than two non-zero Schmidt coefficients, those two entanglement measures are different.

2.4. *Schmidt number*

The *Schmidt number*,[40–43] also referred to as *degree of correlation*,[18] is another entanglement measure that uses the sum of the fourth powers of the Schmidt coefficients

$$
\mathcal{K}\left(\Psi\right) = \frac{1}{\sum_i \lambda_i^4}.
\tag{22}
$$

The Schmidt number could be interpreted as counting the average number of Schmidt modes actively involved in entanglement.[40] The range of the Schmidt number is within the interval $1 \leq \mathcal{K}\left(\Psi\right) \leq n$.

Therefore, the normalized Schmidt number $0 \leq \tilde{\mathcal{K}}\left(\Psi\right) \leq 1$ is given by

$$
\tilde{\mathcal{K}}\left(\Psi\right) = \frac{1}{n-1}\left(\frac{1}{\sum_i \lambda_i^4} - 1\right).
\tag{23}
$$

3. Relative sensitivity to quantum entanglement

Definition 3.1. (Relative sensitivity to quantum entanglement) Given two normalized entanglement measures $0 \leq \tilde{\mathcal{A}}(\Psi) \leq 1$ and $0 \leq \tilde{\mathcal{B}}(\Psi) \leq 1$, we say that $\tilde{\mathcal{B}}(\Psi)$ is more sensitive to quantum entanglement compared with $\tilde{\mathcal{A}}(\Psi)$ if and only if the ordering $\tilde{\mathcal{A}}(\Psi) \leq \tilde{\mathcal{B}}(\Psi)$ holds for any state $|\Psi\rangle$. Otherwise, we say that the two measures are unordered and their relative sensitivity is undefined.

Theorem 3.1. *The normalized versions of the Schmidt number $\tilde{\mathcal{K}}(\Psi)$, tangle $\tilde{\mathcal{C}}^2(\Psi)$ and concurrence $\tilde{\mathcal{C}}(\Psi)$ are ordered in an increasing order of sensitivity to quantum entanglement, namely, for any state $|\Psi\rangle$ whose singular value decomposition is described by a set of Schmidt coefficients $\{\lambda_i\}_{i=1}^n$, we have*

$$\tilde{\mathcal{K}}(\Psi) \leq \tilde{\mathcal{C}}^2(\Psi) \leq \tilde{\mathcal{C}}(\Psi). \tag{24}$$

Proof. The relationship $\tilde{\mathcal{C}}(\Psi) \geq \tilde{\mathcal{C}}^2(\Psi)$ between concurrence and tangle is straightforward and follows from the fact that the concurrence is bounded within $[0,1]$, namely, $\tilde{\mathcal{C}}(\Psi) \leq 1$. Therefore, $\tilde{\mathcal{C}}(\Psi) \times \tilde{\mathcal{C}}(\Psi) \leq \tilde{\mathcal{C}}(\Psi) \times 1$.

To show that $\tilde{\mathcal{C}}^2(\Psi) \geq \tilde{\mathcal{K}}(\Psi)$, we factor the normalized Schmidt number (23) as follows

$$\tilde{\mathcal{K}}(\Psi) = \frac{1}{n-1}\left(\frac{1-\sum_i \lambda_i^4}{\sum_i \lambda_i^4}\right) = \frac{1}{n-1}\left(1 - \sum_i \lambda_i^4\right)\left(\frac{1}{\sum_i \lambda_i^4}\right). \tag{25}$$

Because the quantity $\sum_i \lambda_i^4$ has a minimum only when there are n equal Schmidt coefficients, each with value of $\frac{1}{\sqrt{n}}$, we obtain

$$\sum_i \lambda_i^4 \geq n\left(\frac{1}{\sqrt{n}}\right)^4 = \frac{1}{n}. \tag{26}$$

Taking the reciprocal values gives

$$\frac{1}{\sum_i \lambda_i^4} \leq n. \tag{27}$$

This inequality can also be directly proved using Sedrakyan's inequality[44] as follows

$$\frac{1^2}{\sum_i \lambda_i^4} = \frac{\left(\sum_i \lambda_i^2\right)^2}{\sum_i \lambda_i^4} = \frac{\left(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2\right)^2}{\lambda_1^4 + \lambda_2^4 + \ldots + \lambda_n^4} \leq \frac{\left(\lambda_1^2\right)^2}{\lambda_1^4} + \frac{\left(\lambda_2^2\right)^2}{\lambda_2^4} + \ldots + \frac{\left(\lambda_n^2\right)^2}{\lambda_n^4} = 1 \times n.$$

After substitution of (27) in the Schmidt number (25), we conclude

$$\tilde{\mathcal{K}}(\Psi) = \frac{1}{n-1}\left(1 - \sum_i \lambda_i^4\right)\left(\frac{1}{\sum_i \lambda_i^4}\right) \leq \frac{1}{n-1}\left(1 - \sum_i \lambda_i^4\right) \times n = \tilde{\mathcal{C}}^2(\Psi). \tag{29}$$

This establishes the chain of inequalities as stated in the theorem. □

Theorem 3.2. *The normalized versions of the Schmidt number $\tilde{\mathcal{K}}(\Psi)$, robustness $\tilde{\mathcal{R}}(\Psi)$ and concurrence $\tilde{\mathcal{C}}(\Psi)$ are ordered in an increasing order of sensitivity to entanglement, namely, for the same set of Schmidt coefficients we have*

$$\tilde{\mathcal{K}}(\Psi) \le \tilde{\mathcal{R}}(\Psi) \le \tilde{\mathcal{C}}(\Psi).$$

Proof. To show that $\tilde{\mathcal{C}}(\Psi) \ge \tilde{\mathcal{R}}(\Psi)$, we use (19) to compute the square of the robustness

$$\tilde{\mathcal{R}}^2(\Psi) = \frac{1}{(n-1)^2}\left(2\sum_{i<j}\lambda_i\lambda_j\right)^2$$

and then compare it to the tangle

$$\tilde{\mathcal{C}}^2(\Psi) = \frac{n}{n-1} \times 2\sum_{i<j}(\lambda_i\lambda_j)^2.$$

For $n=2$, we have $\tilde{\mathcal{R}}(\Psi) = \tilde{\mathcal{C}}(\Psi) = 2\lambda_1\lambda_2$.

For $n \ge 3$, we will have n Schmidt coefficients, $\lambda_1, \lambda_2, \ldots, \lambda_n \ge 0$, which will generate $N = \binom{n}{2} = \frac{n(n-1)}{2}$ pairs $\lambda_i\lambda_j$ for which $i < j$. After re-indexing all N pairs using the single index

$$k = (i-1)n - \binom{i}{2} + j - i = \frac{1}{2}(i-1)(2n-i) + j - i \qquad (33)$$

and introducing the combined variable $x_k = \lambda_i\lambda_j \ge 0$, we have

$$\frac{(n-1)^2}{4}\tilde{\mathcal{R}}^2(\Psi) = \left(\sum_{i<j}\lambda_i\lambda_j\right)^2 = \left(\sum_{k=1}^{N}x_k\right)^2 = \sum_k x_k^2 + 2\sum_{k<l}x_k x_l.$$

Similarly, we obtain

$$\frac{(n-1)^2}{4}\tilde{\mathcal{C}}^2(\Psi) = \frac{n(n-1)}{2}\sum_{i<j}(\lambda_i\lambda_j)^2 = N \times \sum_{k=1}^{N}x_k^2 = \sum_k x_k^2 + \sum_{k<l}\left(x_k^2 + x_l^2\right).$$

Now, we take the difference

$$\frac{(n-1)^2}{4}\left[\tilde{\mathcal{C}}^2(\Psi) - \tilde{\mathcal{R}}^2(\Psi)\right] = \sum_k x_k^2 + \sum_{k<l}\left(x_k^2 + x_l^2\right) - \sum_k x_k^2 - 2\sum_{k<l}x_k x_l$$

$$= \sum_{k<l}(x_k - x_l)^2 \ge 0.$$

Since $\frac{(n-1)^2}{4} > 0$, it follows that $\tilde{\mathcal{C}}(\Psi) \ge \tilde{\mathcal{R}}(\Psi)$.

To show that $\tilde{\mathcal{R}}(\Psi) \ge \tilde{\mathcal{K}}(\Psi)$, we subtract (23) from (17) and observe that the relation $\left[\tilde{\mathcal{R}}(\Psi) - \tilde{\mathcal{K}}(\Psi)\right] \ge 0$ is equivalent to

$$\left(\sum_i \lambda_i\right)^2 \ge \frac{1}{\sum_i \lambda_i^4}. \qquad (36)$$

The general proof for $n \geq 2$ of the later relation is quite intricate due to the presence of the sum of fourth powers in the denominator of the fraction. Combining the normalization of the Schmidt coefficients $\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2 = 1$ with $1^3 = 1$ transforms (36) into

$$A \equiv \left(\sum_i \lambda_i \right)^2 \left(\sum_i \lambda_i^4 \right) \geq \left(\sum_i \lambda_i^2 \right)^3 \equiv B. \tag{37}$$

To proceed, we will need the following factorization

$$\lambda_i^4 + \lambda_j^4 - \lambda_i \lambda_j \left(\lambda_i^2 + \lambda_j^2 \right) = (\lambda_i - \lambda_j)^2 \left(\lambda_i^2 + \lambda_i \lambda_j + \lambda_j^2 \right) \geq 0 \tag{38}$$

and the inequality between arithmetic mean and geometric mean for a triple of non-negative numbers rearranged in the form

$$\lambda_i^3 + \lambda_j^3 + \lambda_k^3 - 3\lambda_i \lambda_j \lambda_k \geq 0. \tag{39}$$

Next, we expand the two sides of (37) into groups of similar terms

$$A = \sum_i \lambda_i^6 + 2 \sum_{i<j} \lambda_i \lambda_j \left(\lambda_i^4 + \lambda_j^4 \right) + \sum_{i<j} \lambda_i^2 \lambda_j^2 \left(\lambda_i^2 + \lambda_j^2 \right)$$
$$+ 2 \sum_{i<j<k} \lambda_i \lambda_j \lambda_k \left(\lambda_i^3 + \lambda_j^3 + \lambda_k^3 \right), \tag{40}$$

$$B = \sum_i \lambda_i^6 + 3 \sum_{i<j} \lambda_i^2 \lambda_j^2 \left(\lambda_i^2 + \lambda_j^2 \right) + 6 \sum_{i<j<k} \lambda_i^2 \lambda_j^2 \lambda_k^2. \tag{41}$$

Taking the difference between (40) and (41) gives

$$\frac{A - B}{2} = \sum_{i<j} \lambda_i \lambda_j \left(\lambda_i^4 + \lambda_j^4 \right) - \sum_{i<j} \lambda_i^2 \lambda_j^2 \left(\lambda_i^2 + \lambda_j^2 \right)$$
$$+ \sum_{i<j<k} \lambda_i \lambda_j \lambda_k \left(\lambda_i^3 + \lambda_j^3 + \lambda_k^3 \right) - 3 \sum_{i<j<k} \lambda_i^2 \lambda_j^2 \lambda_k^2$$
$$= \sum_{i<j} \lambda_i \lambda_j (\lambda_i - \lambda_j)^2 \left(\lambda_i^2 + \lambda_i \lambda_j + \lambda_j^2 \right)$$
$$+ \sum_{i<j<k} \lambda_i \lambda_j \lambda_k \left(\lambda_i^3 + \lambda_j^3 + \lambda_k^3 - 3\lambda_i \lambda_j \lambda_k \right)$$
$$\geq 0. \tag{42}$$

In the last step, we have used the fact that the Schmidt coefficients are non-negative, $\lambda_i \geq 0$. The minimal dimensional case, $n = 2$, is obtained trivially from (42) by plugging in $\lambda_3 = 0$. Thus, the Schmidt number is a lower bound on robustness, namely, $\left[\tilde{\mathcal{R}} \left(\Psi \right) - \tilde{\mathcal{K}} \left(\Psi \right) \right] \geq 0$. □

4. Quantum toy model

The relative sensitivity to quantum entanglement could be exploited in quantum applications, which require sharp resolution of either maximal entanglement or

complete separability of bipartite quantum systems. In such cases, the choice of suitable entanglement measure could depend on the actual practical task. For example, to achieve the sharpest possible resolution of maximally entangled states, one could use the Schmidt number $\tilde{\mathcal{K}}(\Psi)$, which has lowest sensitivity to quantum entanglement. Conversely, to achieve the sharpest possible resolution of completely separable states, one could use the concurrence $\tilde{\mathcal{C}}(\Psi)$, which has highest sensitivity to quantum entanglement. To illustrate the content of the latter two statements, next we construct and employ a minimal quantum toy model.

A major simplification of the toy model can be accomplished by noting that quantum dynamics due to internal Hamiltonians does not have an impact on the entanglement contained in a composite system.[31] Indeed, suppose that the initial state of a bipartite quantum system is expressed in the Schmidt basis as

$$|\Psi(0)\rangle = \sum_s \lambda_s |i\rangle_A \otimes |j\rangle_B. \tag{43}$$

If $\hat{H}_A$ is the internal Hamiltonian of subsystem A and $\hat{H}_B$ is the internal Hamiltonian of subsystem B, the overall unitary action with the Hamiltonian $\hat{H}_A \otimes \hat{I}_B + \hat{I}_A \otimes \hat{H}_B$ results in

$$\begin{aligned}
|\Psi(t)\rangle &= e^{-\frac{i}{\hbar}\left(\hat{H}_A \otimes \hat{I}_B + \hat{I}_A \otimes \hat{H}_B\right)t}|\Psi(0)\rangle \\
&= \sum_s \lambda_s \left(e^{-\frac{i}{\hbar}\hat{H}_A t}|i\rangle_A\right) \otimes \left(e^{-\frac{i}{\hbar}\hat{H}_B t}|j\rangle_B\right) \\
&= \sum_s \lambda_s |i(t)\rangle_A \otimes |j(t)\rangle_B. \tag{44}
\end{aligned}$$

Because the individual operators $e^{-\frac{i}{\hbar}\hat{H}_A t}$ and $e^{-\frac{i}{\hbar}\hat{H}_B t}$ are also unitary, they preserve the orthonormality of the basis sets $\{|i(t)\rangle_A\}$ and $\{|j(t)\rangle_B\}$ at all time t. This means that the Schmidt coefficients $\{\lambda_s\}$ remain constant (do not evolve in time) and any entanglement measure dependent on the Schmidt coefficients remains constant too.[31] The latter result allows us to set without loss of generality, $\hat{H}_A = 0$ and $\hat{H}_B = 0$, investigating the quantum dynamics resulting solely due to a non-zero interaction Hamiltonian $\hat{H}_{\text{int}} \neq 0$. In fact, it is exactly the dynamics due to the non-zero interaction Hamiltonian that has the capacity to entangle or disentangle the composite system.[31]

The minimal bipartite quantum model that prevents reduction of concurrence to the robustness of entanglement requires the composition of at least three-level subsystems (qutrits). Therefore, for the construction of our quantum toy model we choose two qutrits governed by the spin-1 Heisenberg interaction Hamiltonian[45]

$$\hat{H}_{\text{int}} = \hbar\omega \left(\hat{\sigma}_x \otimes \hat{\sigma}_x + \hat{\sigma}_y \otimes \hat{\sigma}_y + \hat{\sigma}_z \otimes \hat{\sigma}_z\right), \tag{45}$$

where the spin-1 matrices are given by

$$\hat{\sigma}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{46}$$

$$\hat{\sigma}_y = \frac{1}{\sqrt{2}} \imath \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{47}$$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{48}$$

The eigenvectors of the observable $\hat{\sigma}_z$ form the basis set $\{|\uparrow\rangle, |\circ\rangle, |\downarrow\rangle\}$, respectively with eigenvalues $\{1, 0, -1\}$.

To solve the Schrödinger equation for any initial state $|\Psi(0)\rangle$, it would be convenient to follow the standard procedure utilizing the energy eigenbasis. The interaction Hamiltonian $\hat{H}_{\text{int}}$ has five eigenstates with eigenvalue $\hbar\omega$:

$$|E_1\rangle = |\uparrow\uparrow\rangle, \tag{49}$$

$$|E_2\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\circ\rangle + |\circ\uparrow\rangle \right), \tag{50}$$

$$|E_3\rangle = \frac{1}{\sqrt{6}} \left(|\uparrow\downarrow\rangle + 2|\circ\circ\rangle + |\downarrow\uparrow\rangle \right), \tag{51}$$

$$|E_4\rangle = \frac{1}{\sqrt{2}} \left(|\circ\downarrow\rangle + |\downarrow\circ\rangle \right), \tag{52}$$

$$|E_5\rangle = |\downarrow\downarrow\rangle, \tag{53}$$

three eigenstates with eigenvalue $-\hbar\omega$:

$$|E_6\rangle = -\frac{1}{\sqrt{2}} \left(|\uparrow\circ\rangle - |\circ\uparrow\rangle \right), \tag{54}$$

$$|E_7\rangle = -\frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right), \tag{55}$$

$$|E_8\rangle = -\frac{1}{\sqrt{2}} \left(|\circ\downarrow\rangle - |\downarrow\circ\rangle \right), \tag{56}$$

and a single eigenstate with eigenvalue $-2\hbar\omega$:

$$|E_9\rangle = \frac{1}{\sqrt{3}} \left(|\uparrow\downarrow\rangle - |\circ\circ\rangle + |\downarrow\uparrow\rangle \right). \tag{57}$$

Throughout this work, we will set $\omega = 1$ rad/ps in the interaction Hamiltonian between the two qutrits.

The general solution of the Schrödinger equation in the energy eigenbasis is then

$$\imath\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle = \hat{H} \sum_n \alpha_n |E_n\rangle = \sum_n E_n \alpha_n |E_n\rangle, \tag{58}$$

which can be explicitly written as

$$|\Psi(t)\rangle = \sum_n \alpha_n e^{-\frac{i}{\hbar}E_n t}|E_n\rangle$$
$$= e^{-i\omega t}\left(\alpha_1|E_1\rangle + \alpha_2|E_2\rangle + \alpha_3|E_3\rangle + \alpha_4|E_4\rangle + \alpha_5|E_5\rangle\right)$$
$$+ e^{i\omega t}\left(\alpha_6|E_6\rangle + \alpha_7|E_7\rangle + \alpha_8|E_8\rangle\right) + e^{2i\omega t}\alpha_9|E_9\rangle, \quad (59)$$

where α_n is the initial quantum probability amplitude of the state $|E_n\rangle$ at $t = 0$.

5. Quantum dynamics of entanglement measures

Dynamics of the expectation value of any quantum observable $\hat{A}$ could be obtained from the solution (59) together with the Born rule

$$\langle\hat{A}\rangle = \langle\Psi(t)|\hat{A}|\Psi(t)\rangle \quad (60)$$

Here, we have chosen to track as quantum observables the individual projectors onto the eigenstates of $\hat{\sigma}_z \otimes \hat{\sigma}_z$, namely, $\hat{\mathcal{P}}(\uparrow\uparrow) = |\uparrow\uparrow\rangle\langle\uparrow\uparrow|$, $\hat{\mathcal{P}}(\uparrow\bigcirc) = |\uparrow\bigcirc\rangle\langle\uparrow\bigcirc|$, $\hat{\mathcal{P}}(\uparrow\downarrow) = |\uparrow\downarrow\rangle\langle\uparrow\downarrow|$, $\hat{\mathcal{P}}(\bigcirc\uparrow) = |\bigcirc\uparrow\rangle\langle\bigcirc\uparrow|$, $\hat{\mathcal{P}}(\bigcirc\bigcirc) = |\bigcirc\bigcirc\rangle\langle\bigcirc\bigcirc|$, $\hat{\mathcal{P}}(\bigcirc\downarrow) = |\bigcirc\downarrow\rangle\langle\bigcirc\downarrow|$, $\hat{\mathcal{P}}(\downarrow\uparrow) = |\downarrow\uparrow\rangle\langle\downarrow\uparrow|$, $\hat{\mathcal{P}}(\downarrow\bigcirc) = |\downarrow\bigcirc\rangle\langle\downarrow\bigcirc|$ and $\hat{\mathcal{P}}(\downarrow\downarrow) = |\downarrow\downarrow\rangle\langle\downarrow\downarrow|$.

For the initial state $|\Psi(0)\rangle$ we considered each of the nine eigenstates of $\hat{\sigma}_z \otimes \hat{\sigma}_z$. Projecting each of these initial states onto the energy eigenbasis using (49)-(57) gives the initial values for the energy quantum probability amplitudes α_n for the corresponding simulations. Due to the existing freedom of choice for the $\uparrow$ vs $\downarrow$ direction in space, there are mirror symmetries in the obtained solutions, which can be grouped into 4 cases.

5.1. *Case 0*

Trivial quantum dynamics, in which the initial state is also an energy eigenstate, occurs when $|\Psi(0)\rangle = |\uparrow\uparrow\rangle = |E_1\rangle$ or $|\Psi(0)\rangle = |\downarrow\downarrow\rangle = |E_5\rangle$. In those situations, the expectation values for the respective projectors remain unitary at all times, namely $\langle\hat{\mathcal{P}}(\uparrow\uparrow)\rangle = 1$ or $\langle\hat{\mathcal{P}}(\downarrow\downarrow)\rangle = 1$. The composite state vector $|\Psi(t)\rangle$ remains in a separable state at all times with constant Schmidt coefficients

$$\begin{cases} \lambda_1 &= 1 \\ \lambda_2 &= 0 \\ \lambda_3 &= 0. \end{cases} \quad (61)$$

All four entanglement measures remain zero at all times.

5.2. *Case 1*

Non-trivial quantum dynamics is obtained when the initial state is not an energy eigenstate, but is a superposition of several energy eigenstates with different eigenvalues. For $|\Psi(0)\rangle = |\uparrow\bigcirc\rangle = \frac{1}{\sqrt{2}}(|E_2\rangle - |E_6\rangle)$ or $|\Psi(0)\rangle = |\bigcirc\uparrow\rangle =$
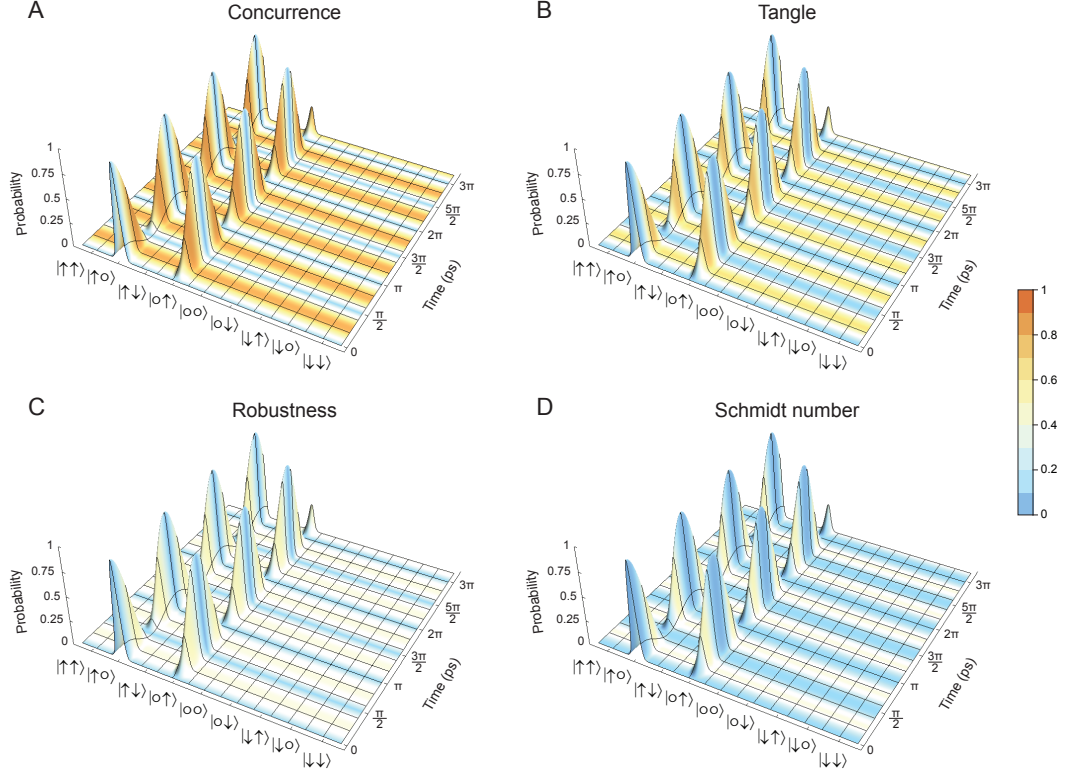
Fig. 1. Dynamics of the expectation values for the respective projectors onto the eigenstates of the quantum observable $\hat{\sigma}_z \otimes \hat{\sigma}_z$ simulated with the initial state $|\Psi(0)\rangle = |\uparrow \bigcirc\rangle$. The amount of quantum entanglement is measured with the use of concurrence in panel (A), tangle in panel (B), robustness of entanglement in panel (C) and the Schmidt number in panel (D). The coupling strength $\hbar\omega$ in the interaction Hamiltonian between the two qutrits was set by $\omega = 1$ rad/ps.

$\frac{1}{\sqrt{2}}(|E_2\rangle + |E_6\rangle)$, the quantum state vector oscillates forth-and-back between the states $|\uparrow \bigcirc\rangle$ and $|\bigcirc \uparrow\rangle$. Similarly, for $|\Psi(0)\rangle = |\bigcirc \downarrow\rangle = \frac{1}{\sqrt{2}}(|E_4\rangle - |E_8\rangle)$ or $|\Psi(0)\rangle = |\downarrow \bigcirc\rangle = \frac{1}{\sqrt{2}}(|E_4\rangle + |E_8\rangle)$, the quantum state vector oscillates forth-and-back between the states $|\bigcirc \downarrow\rangle$ and $|\downarrow \bigcirc\rangle$. In those four situations, the state vector $|\Psi(t)\rangle$ remains confined within a two-dimensional subspace of the composite Hilbert space, which implies that the entanglement measures cannot reach their absolute maximum. The composite state vector has dynamic Schmidt coefficients that depend on time

$$\begin{cases} \lambda_1 & = |\cos(\omega t)| \\ \lambda_2 & = |\sin(\omega t)| \\ \lambda_3 & = 0. \end{cases} \quad (62)$$

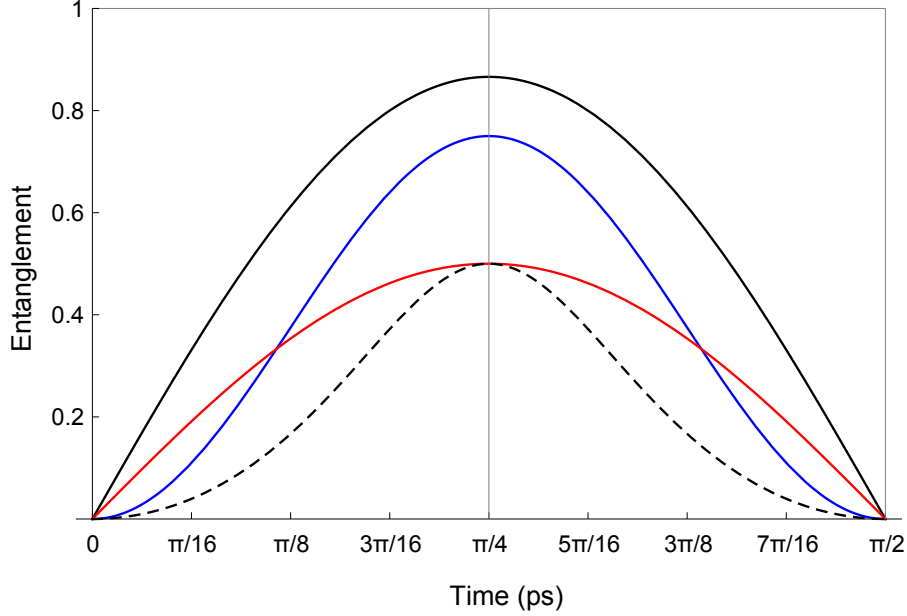All four entanglement measures undergo cyclic dynamics (Fig. 1). Each cycle, be-

Fig. 2. Dynamics of the entanglement measures for the simulation with initial state $|\Psi(0)\rangle = |\uparrow\bigcirc\rangle$. Concurrence $\tilde{\mathcal{C}}(\Psi)$ is denoted with solid black line, tangle $\tilde{\mathcal{T}}(\Psi)$ with solid blue line, robustness of entanglement $\tilde{\mathcal{R}}(\Psi)$ with solid red line and the Schmidt number $\tilde{\mathcal{K}}(\Psi)$ with dashed black line. The coupling strength $\hbar\omega$ in the interaction Hamiltonian between the two qutrits was set by $\omega = 1$ rad/ps. Each cycle, between two consecutive separable states, lasts $\frac{\pi}{2}$ ps.

tween two consecutive separable states, lasts $\frac{\pi}{2}$ ps. Concurrence $\tilde{\mathcal{C}}(\Psi)$ is most sensitive to the presence of quantum entanglement and forms an upper bound on all four entanglement measures (Fig. 2). Conversely, the Schmidt number $\tilde{\mathcal{K}}(\Psi)$ is least sensitive to the presence of quantum entanglement and forms a lower bound on all four entanglement measures (Fig. 2).

Theorem 5.1. *The tangle $\tilde{\mathcal{T}}(\Psi)$ and robustness of entanglement $\tilde{\mathcal{R}}(\Psi)$ are not partially ordered.*

Proof. The lack of partial order is demonstrated numerically (Fig. 2). For example, consider the Schmidt coefficients (62) with the following assignments: $\tilde{\mathcal{R}}\left[\Psi\left(\omega t = \frac{\pi}{16}\right)\right] > \tilde{\mathcal{T}}\left[\Psi\left(\omega t = \frac{\pi}{16}\right)\right]$ and $\tilde{\mathcal{R}}\left[\Psi\left(\omega t = \frac{\pi}{4}\right)\right] < \tilde{\mathcal{T}}\left[\Psi\left(\omega t = \frac{\pi}{4}\right)\right]$.   □

5.3. *Case 2*

Complicated quantum dynamics, manifesting varying quantum interference effects, occurs when $|\Psi(0)\rangle = |\uparrow\downarrow\rangle = \frac{1}{\sqrt{6}}|E_3\rangle - \frac{1}{\sqrt{2}}|E_7\rangle + \frac{1}{\sqrt{3}}|E_9\rangle$ or $|\Psi(0)\rangle = |\downarrow\uparrow\rangle = \frac{1}{\sqrt{6}}|E_3\rangle + \frac{1}{\sqrt{2}}|E_7\rangle + \frac{1}{\sqrt{3}}|E_9\rangle$. In those situations, the quantum state vector $|\Psi(t)\rangle$
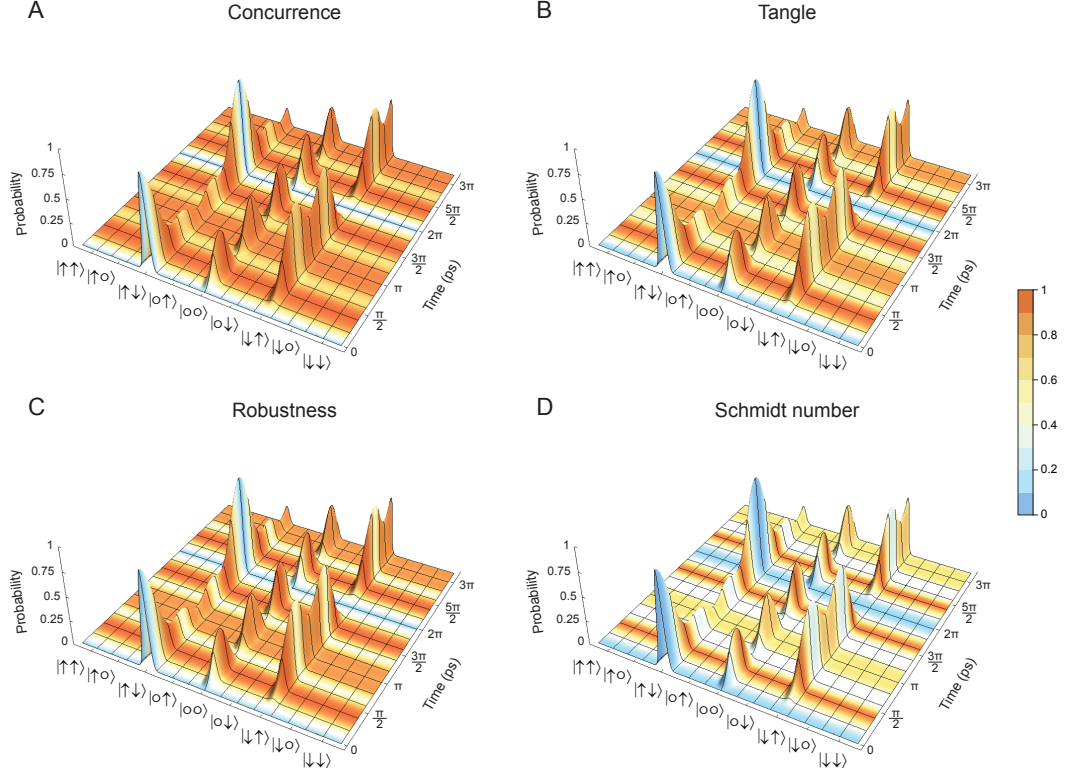
Fig. 3. Dynamics of the expectation values for the respective projectors onto the eigenstates of the quantum observable $\hat{\sigma}_z \otimes \hat{\sigma}_z$ simulated with the initial state $|\Psi(0)\rangle = |\uparrow\downarrow\rangle$. The amount of quantum entanglement is measured with the use of concurrence in panel (A), tangle in panel (B), robustness of entanglement in panel (C) and the Schmidt number in panel (D). The coupling strength $\hbar\omega$ in the interaction Hamiltonian between the two qutrits was set by $\omega = 1$ rad/ps.

oscillates with somewhat irregular pattern between the states $|\uparrow\downarrow\rangle$, $|\bigcirc\bigcirc\rangle$ and $|\downarrow\uparrow\rangle$. The composite state vector has dynamic Schmidt coefficients

$$
\begin{cases}
\lambda_1 & = \frac{1}{3}\sqrt{\frac{7}{2} + 3\cos\left(\omega t\right) + \frac{3}{2}\cos\left(2\omega t\right) + \cos\left(3\omega t\right)} \\
\lambda_2 & = \frac{2}{3}\sqrt{5 + 4\cos\left(\omega t\right)}\sin^2\left(\frac{1}{2}\omega t\right) \\
\lambda_3 & = \frac{2}{3}\left|\sin\left(\frac{3}{2}\omega t\right)\right|.
\end{cases}
\tag{63}
$$

All four entanglement measures undergo cyclic dynamics (Fig. 3). Each cycle, between two consecutive separable states, lasts 2π ps. Again, concurrence $\tilde{\mathcal{C}}\left(\Psi\right)$ and the Schmidt number $\tilde{\mathcal{K}}\left(\Psi\right)$ form bounds on the four entanglement measures from above and below, respectively (Fig. 4). Because, in this simulation it is possible for all 3 Schmidt coefficients to be non-zero, the entanglement measures explore almost the full range from separable to maximally entangled state (Fig. 4). Again, it is clearly seen in the vicinity of local minima that the tangle $\tilde{\mathcal{T}}\left(\Psi\right)$ and robustness
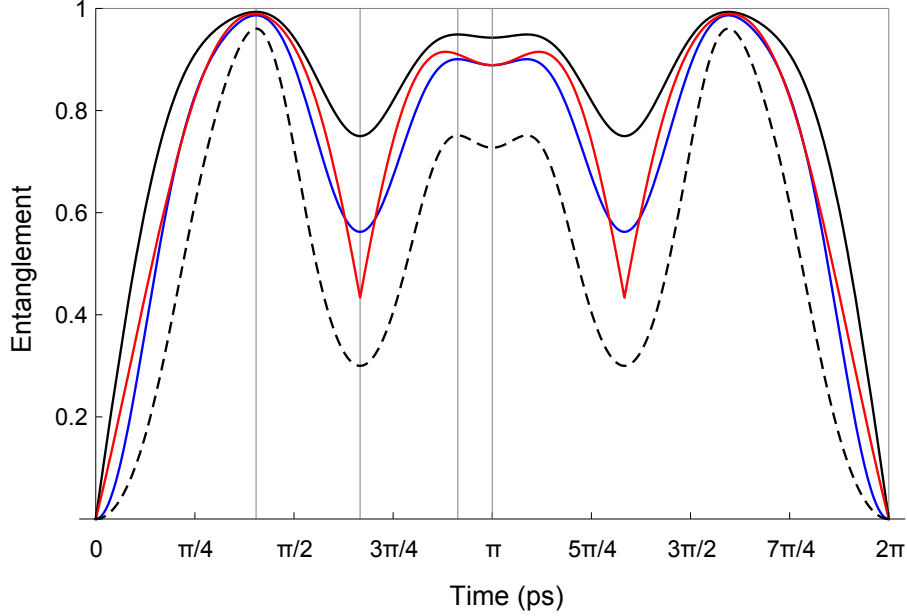
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Fig. 4. Dynamics of the entanglement measures for the simulation with initial state $|\Psi(0)\rangle = |\uparrow\downarrow\rangle$. Concurrence $\tilde{\mathcal{C}}(\Psi)$ is denoted with solid black line, tangle $\tilde{\mathcal{T}}(\Psi)$ with solid blue line, robustness of entanglement $\tilde{\mathcal{R}}(\Psi)$ with solid red line and the Schmidt number $\tilde{\mathcal{K}}(\Psi)$ with dashed black line. The coupling strength $\hbar\omega$ in the interaction Hamiltonian between the two qutrits was set by $\omega = 1$ rad/ps. Each cycle, between two consecutive separable states, lasts 2π ps.

of entanglement $\tilde{\mathcal{R}}(\Psi)$ and are not partially ordered (Fig. 4).

5.4. *Case 3*

Quantum dynamics with regular quantum interference pattern occurs when $|\Psi(0)\rangle = |\bigcirc\bigcirc\rangle = \sqrt{\frac{2}{3}}|E_3\rangle - \frac{1}{\sqrt{3}}|E_9\rangle$. In this case, the state vector $|\Psi(t)\rangle$ explores the complete spectrum from separable to maximally entangled states. The composite state vector has dynamic Schmidt coefficients

$$\begin{cases} \lambda_1 &= \frac{1}{3}\sqrt{5 + 4\cos(3\omega t)} \\ \lambda_2 &= \frac{2}{3}\left|\sin\left(\frac{3}{2}\omega t\right)\right| \\ \lambda_3 &= \frac{2}{3}\left|\sin\left(\frac{3}{2}\omega t\right)\right|. \end{cases} \tag{64}$$

All four entanglement measures undergo cyclic dynamics (Fig. 5). Each cycle, between two consecutive separable states, lasts $\frac{2\pi}{3}$ ps. The maximal entanglement reaches 1 at $\omega t = \frac{2\pi}{9}$ and $\omega t = \frac{4\pi}{9}$ (Fig. 6). Due to its high sensitivity to quantum entanglement, the concurrence $\tilde{\mathcal{C}}(\Psi)$ is able to resolve quite sharply the separable states (narrow blue bands in Fig. 5A). Conversely, due to its low sensitivity to quantum entanglement, the Schmidt number $\tilde{\mathcal{K}}(\Psi)$ is able to resolve quite sharply
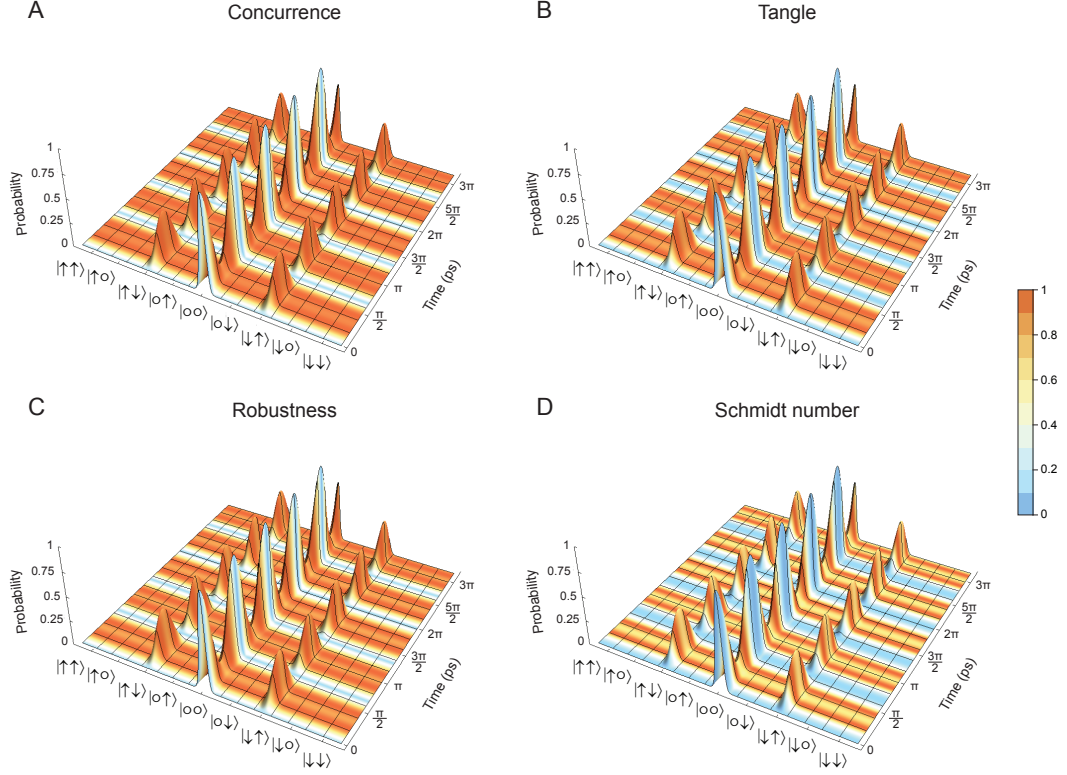
Fig. 5. Dynamics of the expectation values for the respective projectors onto the eigenstates of the quantum observable $\hat{\sigma}_z \otimes \hat{\sigma}_z$ simulated with the initial state $|\Psi(0)\rangle = |\bigcirc\,\bigcirc\rangle$. The amount of quantum entanglement is measured with the use of concurrence in panel (A), tangle in panel (B), robustness of entanglement in panel (C) and the Schmidt number in panel (D). The coupling strength $\hbar\omega$ in the interaction Hamiltonian between the two qutrits was set by $\omega = 1$ rad/ps.

the maximally entangled states (narrow red bands in Fig. 5D).

The sharpness of the bands produced in contour plots depends on the temporal rate of change of the entanglement measures. By construction, all normalized entanglement measures coincide at the two extreme cases: at separable states, where they have zero value, and at maximally entangled states, where they have unit value.[11, 12, 46, 47] The existence of partial order, $\mathcal{A}(t) \geq \mathcal{B}(t)$, then implies that in the neighborhood of a separable state, the quantum dynamics is described by U- or V-shaped cups such that the cup for $\mathcal{A}(t)$ is inside the cup for $\mathcal{B}(t)$ (Fig. 6). Consequently, the time derivative for $\mathcal{A}(t)$ decreases towards and raises from the separable state more steeply compared to the time derivative for $\mathcal{B}(t)$, namely, $\left|\frac{d\mathcal{A}(t)}{dt}\right| \geq \left|\frac{d\mathcal{B}(t)}{dt}\right|$. The roles played by the two entanglement measures are reversed in the neighborhood of a maximally entangled state, where the quantum dynamics is described by U- or V-shaped caps such that the cap for $\mathcal{B}(t)$ is inside the cap
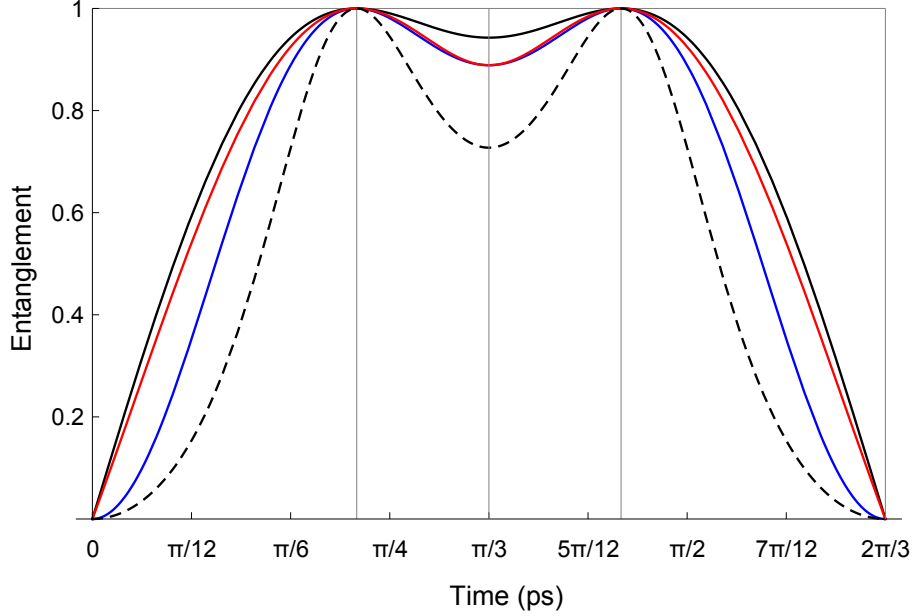
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Fig. 6. Dynamics of the entanglement measures for the simulation with initial state $|\Psi(0)\rangle = |\circ\circ\rangle$. Concurrence $\tilde{\mathcal{C}}(\Psi)$ is denoted with solid black line, tangle $\tilde{\mathcal{T}}(\Psi)$ with solid blue line, robustness of entanglement $\tilde{\mathcal{R}}(\Psi)$ with solid red line and the Schmidt number $\tilde{\mathcal{K}}(\Psi)$ with dashed black line. The coupling strength $\hbar\omega$ in the interaction Hamiltonian between the two qutrits was set by $\omega = 1$ rad/ps. Each cycle, between two consecutive separable states, lasts $\frac{2\pi}{3}$ ps.

for $\mathcal{A}(t)$ (Fig. 6). This means that the time derivative for $\mathcal{B}(t)$ raises towards and decreases from the maximally entangled state more steeply compared to the time derivative for $\mathcal{A}(t)$, namely, $\left|\frac{d\mathcal{B}(t)}{dt}\right| \geq \left|\frac{d\mathcal{A}(t)}{dt}\right|$. Different time derivatives result in different durations of the time windows for which the entanglement measures stay inside some fixed small region $\varepsilon > 0$ near 0 or 1. In general, steeper time derivatives imply shorter time windows and sharper resolution of quantum dynamics.

6. Entanglement statistics

To further illustrate how the Schmidt decomposition features in the calculation of measurable correlations between the outcomes of quantum observables, next we introduce the concept of state-dependent *entanglement statistics*. We also derive an alternative, but equivalent, characterization of separable/entangled states in terms of absence/presence of correlations between all/some local quantum observables.

We begin with the study of quantum operator statistics. Let $\mathcal{L}_S(H)$ be the set of self-adjoint (Hermitian) operators[48, 49] on a finite-dimensional complex Hilbert space H. We call the elements of $\mathcal{L}_S(H)$ *observable operators*. If $|\phi\rangle \in H$ is a vector state and $\hat{A} \in \mathcal{L}_S(H)$ is an operator, the *expectation* (or *average*) of $\hat{A}$ in the state

$|\phi\rangle$ is

$$\langle\hat{A}\rangle_{\phi} = \langle\phi|\hat{A}|\phi\rangle. \tag{65}$$

The *deviation* of $\hat{A}$ in the state $|\phi\rangle$ is

$$\hat{D}_{\phi}(\hat{A}) = \hat{A} - \langle\hat{A}\rangle_{\phi}\hat{I} \tag{66}$$

where $\hat{I} \in \mathcal{L}_{S}(H)$ is the identity operator. An interaction between any two observable operators $\hat{A}, \hat{B} \in \mathcal{L}_{S}(H)$ is described by the *ϕ-correlation* given by

$$\mathrm{Cor}_{\phi}(\hat{A}, \hat{B}) = \langle\phi|\hat{D}_{\phi}(\hat{A})\hat{D}_{\phi}(\hat{B})|\phi\rangle. \tag{67}$$

In general, $\mathrm{Cor}_{\phi}(\hat{A}, \hat{B})$ is not a real number and we have

$$\overline{\mathrm{Cor}_{\phi}(\hat{A}, \hat{B})} = \mathrm{Cor}_{\phi}(\hat{B}, \hat{A}). \tag{68}$$

We define the *ϕ-covariance* of $\hat{A}$ and $\hat{B}$ as

$$\Delta_{\phi}(\hat{A}, \hat{B}) = \mathrm{Re}\left[\mathrm{Cor}_{\phi}(\hat{A}, \hat{B})\right] \tag{69}$$

and the *ϕ-variance* of $\hat{A}$ as

$$\Delta_{\phi}(\hat{A}) = \Delta_{\phi}(\hat{A}, \hat{A}) = \mathrm{Cor}_{\phi}(\hat{A}, \hat{A}). \tag{70}$$

We have that

$$\mathrm{Cor}_{\phi}(\hat{A}, \hat{B}) = \langle\phi|(\hat{A} - \langle\hat{A}\rangle_{\phi}\hat{I})(\hat{B} - \langle\hat{B}\rangle_{\phi}\hat{I})|\phi\rangle = \langle\hat{A}\hat{B}\rangle_{\phi} - \langle\hat{A}\rangle_{\phi}\langle\hat{B}\rangle_{\phi}, \tag{71}$$

$$\Delta_{\phi}(\hat{A}, \hat{B}) = \mathrm{Re}\left[\langle\hat{A}\hat{B}\rangle_{\phi} - \langle\hat{A}\rangle_{\phi}\langle\hat{B}\rangle_{\phi}\right], \tag{72}$$

$$\Delta_{\phi}(\hat{A}) = \langle\hat{A}^{2}\rangle_{\phi} - \langle\hat{A}\rangle_{\phi}^{2}. \tag{73}$$

If $\mathrm{Cor}_{\phi}(\hat{A}, \hat{B}) = 0$, we say that $\hat{A}$ and $\hat{B}$ are *ϕ-uncorrelated*. Of course, from (71) it follows that $\mathrm{Cor}_{\phi}(\hat{A}, \hat{B}) = 0$ if and only if $\langle\phi|\hat{A}\hat{B}|\phi\rangle = \langle\hat{A}\rangle_{\phi}\langle\hat{B}\rangle_{\phi}$. Since quantum observables are represented by self-adjoint (Hermitian) operators[48, 49]

$$\overline{\langle\phi|\hat{A}\hat{B}|\phi\rangle} = \langle\phi|(\hat{A}\hat{B})^{\dagger}|\phi\rangle = \langle\phi|\hat{B}^{\dagger}\hat{A}^{\dagger}|\phi\rangle = \langle\phi|\hat{B}\hat{A}|\phi\rangle, \tag{74}$$

it follows that if $\hat{A}$ and $\hat{B}$ are ϕ-uncorrelated then $\hat{B}$ and $\hat{A}$ are ϕ-uncorrelated. For $\hat{A}, \hat{B} \in \mathcal{L}_{S}(H)$, their *commutator* is given by $\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$. An important connection between these concepts is the following *uncertainty principle*:[50]

$$\frac{1}{4}\left|\langle\phi|\left[\hat{A}, \hat{B}\right]|\phi\rangle\right|^{2} + \left[\Delta_{\phi}(\hat{A}, \hat{B})\right]^{2} = \left|\mathrm{Cor}_{\phi}(\hat{A}, \hat{B})\right|^{2} \leq \Delta_{\phi}(\hat{A})\Delta_{\phi}(\hat{B}). \tag{75}$$

As a special case, we have the Heisenberg–Robertson uncertainty principle[51, 52]

$$\frac{1}{4}\left|\langle\phi|\left[\hat{A}, \hat{B}\right]|\phi\rangle\right|^{2} \leq \Delta_{\phi}(\hat{A})\Delta_{\phi}(\hat{B}). \tag{76}$$

It follows from (75) that $\hat{A}$ and $\hat{B}$ are uncorrelated if and only if

$$\langle\phi|\left[\hat{A}, \hat{B}\right]|\phi\rangle = \Delta_{\phi}(\hat{A}, \hat{B}) = 0. \tag{77}$$

In this case, (75) gives no information.

An operator $\hat{A} \in \mathcal{L}_S(H)$ satisfying $0 \leq \hat{A} \leq \hat{I}$ is called an *effect*.[53] Effects correspond to quantum events or yes-no measurements. A *real-valued observable* is a set of effects $\mathcal{A} = \left\{ \hat{A}_x \colon x \in \Omega_{\mathcal{A}} \right\}$ where $\Omega_{\mathcal{A}}$ is a finite subset of $\mathbb{R}$ called the *outcome set* of $\mathcal{A}$ and we have $\sum_{x \in \Omega_{\mathcal{A}}} \hat{A}_x = \hat{I}$. We consider $\hat{A}_x$ to be the event that occurs when a measurement of $\mathcal{A}$ results in the outcome x. The *probability* that $\mathcal{A}$ has the outcome x when the system is in the state $|\phi\rangle \in H$ is $\langle \hat{A}_x \rangle_\phi = \langle \phi | \hat{A}_x | \phi \rangle$. Notice that $x \mapsto \langle \phi | \hat{A}_x | \phi \rangle$ is a probability measure because

$$\sum_{x \in \Omega_{\mathcal{A}}} \langle \phi | \hat{A}_x | \phi \rangle = \langle \phi | \sum_{x \in \Omega_{\mathcal{A}}} \hat{A}_x | \phi \rangle = \langle \phi | \hat{I} | \phi \rangle = \langle \phi | \phi \rangle = 1. \tag{78}$$

Corresponding to the observable $\mathcal{A}$ we have its *stochastic operator* $\widetilde{\mathcal{A}} = \sum_{x \in \Omega_{\mathcal{A}}} x \hat{A}_x$. Then $\widetilde{\mathcal{A}}$ is an observable operator and the *expectation* (or *average*) of $\mathcal{A}$ in the state $|\phi\rangle$ is

$$\langle \mathcal{A} \rangle_\phi = \langle \widetilde{\mathcal{A}} \rangle_\phi = \langle \phi | \widetilde{\mathcal{A}} | \phi \rangle = \langle \phi | \sum_{x \in \Omega_{\mathcal{A}}} x \hat{A}_x | \phi \rangle = \sum_{x \in \Omega_{\mathcal{A}}} x \langle \hat{A}_x \rangle_\phi. \tag{79}$$

Thus, $\langle \mathcal{A} \rangle_\phi$ is the sum of the outcomes of $\mathcal{A}$ times the probabilities that these outcomes occur. If $\mathcal{B} = \left\{ \hat{B}_y \colon y \in \Omega_{\mathcal{B}} \right\}$ is another observable, the *ϕ-correlation* of $\mathcal{A}$ and $\mathcal{B}$ is $\mathrm{Cor}_\phi(\mathcal{A}, \mathcal{B}) = \mathrm{Cor}_\phi(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$, the *$\phi$-covariance* of $\mathcal{A}$ and $\mathcal{B}$ is $\Delta_\phi(\mathcal{A}, \mathcal{B}) = \Delta_\phi(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ and the *ϕ-variance* of $\mathcal{A}$ is $\Delta_\phi(\mathcal{A}) = \Delta_\phi(\widetilde{\mathcal{A}})$. We conclude that

$$\mathrm{Cor}_\phi(\mathcal{A}, \mathcal{B}) = \sum_{x,y} xy \left(\langle \phi | \hat{A}_x \hat{B}_y | \phi \rangle - \langle \hat{A}_x \rangle_\phi \langle \hat{B}_y \rangle_\phi \right), \tag{80}$$

$$\Delta_\phi(\mathcal{A}, \mathcal{B}) = \sum_{xy} xy \, \mathrm{Re} \left[\langle \phi | \hat{A}_x \hat{B}_y | \phi \rangle - \langle \hat{A}_x \rangle_\phi \langle \hat{B}_y \rangle_\phi \right], \tag{81}$$

$$\Delta_\phi(\mathcal{A}) = \sum_{x,y} xy \left(\langle \phi | \hat{A}_x \hat{A}_y | \phi \rangle - \langle \hat{A}_x \rangle_\phi \langle \hat{A}_y \rangle_\phi \right). \tag{82}$$

We say that $\mathcal{A}, \mathcal{B}$ are *ϕ-uncorrelated* if $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}$ are ϕ-uncorrelated. Also, $\mathcal{A}, \mathcal{B}$ are *ϕ-independent* if

$$\langle \phi | \hat{A}_x \hat{B}_y | \phi \rangle = \langle \hat{A}_x \rangle_\phi \langle \hat{B}_y \rangle_\phi \tag{83}$$

for all $x \in \Omega_{\mathcal{A}}$, $y \in \Omega_{\mathcal{B}}$. Of course, if $\mathcal{A}, \mathcal{B}$ are ϕ-independent, then $\mathcal{B}, \mathcal{A}$ are ϕ-independent. It follows from (80) that if $\mathcal{A}, \mathcal{B}$ are ϕ-independent, then $\mathcal{A}, \mathcal{B}$ are ϕ-uncorrelated. The converse does not hold because there are examples of uncorrelated random variables that are not independent. Two observables $\mathcal{A}, \mathcal{B}$ *commute* if $\left[\hat{A}_x, \hat{B}_y \right] = 0$ for all $x \in \Omega_{\mathcal{A}}$, $y \in \Omega_{\mathcal{B}}$. If $\mathcal{A}, \mathcal{B}$ commute then so do $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}$ and the uncertainty principle reduces to

$$[\Delta_\phi(\mathcal{A}, \mathcal{B})]^2 = |\mathrm{Cor}_\phi(\mathcal{A}, \mathcal{B})|^2 \leq \Delta_\phi(\mathcal{A}) \Delta_\phi(\mathcal{B}). \tag{84}$$

We say that $\mathcal{A}, \mathcal{B}$ are *compatible* (or *jointly measurable*) if there exists an observable $\mathcal{C} = \left\{ \hat{C}_{xy} \colon x \in \Omega_{\mathcal{A}}, y \in \Omega_{\mathcal{B}} \right\}$ such that $\hat{A}_x = \sum_{y \in \Omega_{\mathcal{B}}} \hat{C}_{xy}$, $\hat{B}_y = \sum_{x \in \Omega_{\mathcal{A}}} \hat{C}_{xy}$. We

then call $\mathcal{C}$ a *joint observable* for $\mathcal{A}$, $\mathcal{B}$. Although $\mathcal{C}$ is an observable, it is not real-valued because its outcome space is $\Omega_{\mathcal{A}} \times \Omega_{\mathcal{B}}$. If $\mathcal{A}$, $\mathcal{B}$ commute, they are compatible with joint observable $\hat{C}_{xy} = \hat{A}_x \hat{B}_y$. If $\mathcal{A}$, $\mathcal{B}$ are compatible, they need not commute.[50]

We now apply the previous discussion to *entanglement statistics*. Let H_1, H_2 be finite-dimensional Hilbert spaces for two quantum systems. The combined system is described by the Hilbert space $H_1 \otimes H_2$. If $\hat{A} \in \mathcal{L}_S(H_1)$ is an observable operator for system 1, then in the combined system this operator is represented by $\hat{A} \otimes \hat{I}_2$ where $\hat{I}_2$ is the identity operator on H_2. Similarly, if $\hat{B} \in \mathcal{L}_S(H_2)$, then in the combined system $\hat{B}$ is represented by $\hat{I}_1 \otimes \hat{B}$. Since $\hat{A} \otimes \hat{I}_2$ and $\hat{I}_1 \otimes \hat{B}$ commute, they are jointly measured by the observable

$$(\hat{A} \otimes \hat{I}_2)(\hat{I}_1 \otimes \hat{B}) = \hat{A} \otimes \hat{B} \tag{85}$$

in the combined system. Similarly, if $\mathcal{A} = \left\{ \hat{A}_x \colon x \in \Omega_{\mathcal{A}} \right\}$, $\mathcal{B} = \left\{ \hat{B}_y \colon y \in \Omega_{\mathcal{B}} \right\}$ are real-valued observables on H_1, H_2, respectively, then

$$\mathcal{A} \otimes \hat{I}_2 = \left\{ \hat{A}_x \otimes \hat{I}_2 \colon x \in \Omega_{\mathcal{A}} \right\}, \hat{I}_1 \otimes \mathcal{B} = \left\{ \hat{I}_1 \otimes \hat{B}_y \colon y \in \Omega_{\mathcal{B}} \right\}$$

are real-valued observables on $H_1 \otimes H_2$. These observables commute and have joint observable

$$\mathcal{C} = \mathcal{A} \otimes \mathcal{B} = \left\{ \hat{C}_{xy} = \hat{A}_x \otimes \hat{B}_y \colon (x, y) \in \Omega_{\mathcal{A}} \times \Omega_{\mathcal{B}} \right\}. \tag{86}$$

Although $\mathcal{C}$ is an observable on $H_1 \otimes H_2$, it is not real-valued because $\Omega_{\mathcal{C}} = \Omega_{\mathcal{A}} \times \Omega_{\mathcal{B}}$. The corresponding stochastic operators become

$$\widetilde{(\mathcal{A} \otimes \hat{I}_2)} = \widetilde{\mathcal{A}} \otimes \hat{I}_2 = \sum_{x \in \Omega_{\mathcal{A}}} x \hat{A}_x \otimes \hat{I}_2, \tag{87}$$

$$\widetilde{(\hat{I}_1 \otimes \mathcal{B})} = \hat{I}_1 \otimes \widetilde{\mathcal{B}} = \sum_{y \in \Omega_{\mathcal{B}}} y \hat{I}_1 \otimes \hat{B}_y. \tag{88}$$

We define the *stochastic operator* for $\mathcal{C}$ to be

$$\widetilde{\mathcal{C}} = \widetilde{(\mathcal{A} \otimes \hat{I}_2)}\widetilde{(\hat{I}_1 \otimes \mathcal{B})} = \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{B}} = \sum_{x,y} xy \hat{A}_x \otimes \hat{B}_y. \tag{89}$$

Theorem 6.1. *The following statements are equivalent:*
(i) The vector state $|\alpha\rangle \in H_1 \otimes H_2$ is separable.
(ii) $\hat{A} \otimes \hat{I}_2$, $\hat{I}_1 \otimes \hat{B}$ are α-uncorrelated for all $\hat{A} \in \mathcal{L}_S(H_1)$, $\hat{B} \in \mathcal{L}_S(H_2)$.
(iii) $\mathcal{A} \otimes \hat{I}_2$, $\hat{I}_1 \otimes \mathcal{B}$ are α-independent for all observables $\mathcal{A}$ on H_1 and $\mathcal{B}$ on H_2.

Proof. (i)$\Rightarrow$(ii) Suppose $|\alpha\rangle \in H_1 \otimes H_2$ is separable with $|\alpha\rangle = |\phi\rangle \otimes |\psi\rangle$. We then have

$$\langle\alpha|(\hat{A} \otimes \hat{I}_2)(\hat{I}_1 \otimes \hat{B})|\alpha\rangle = \langle\phi| \otimes \langle\psi|\hat{A} \otimes \hat{B}|\phi\rangle \otimes |\psi\rangle = \langle\phi| \otimes \langle\psi| \left(\hat{A}|\phi\rangle \otimes \hat{B}|\psi\rangle \right)$$

$$= \langle\phi|\hat{A}|\phi\rangle\langle\psi|\hat{B}|\psi\rangle = \langle\alpha|\hat{A} \otimes \hat{I}_2|\alpha\rangle\langle\alpha|\hat{I}_1 \otimes \hat{B}|\alpha\rangle.$$

It follows that $\hat{A} \otimes \hat{I}_2$, $\hat{I}_1 \otimes \hat{B}$ are α-uncorrelated.

(ii)$\Rightarrow$(iii) Suppose $\hat{A} \otimes \hat{I}_2$, $\hat{I}_1 \otimes \hat{B}$ are α-uncorrelated for all $\hat{A} \in \mathcal{L}_S(H)$, $\hat{B} \in \mathcal{L}_S(H)$. Since $\hat{A}_x \in \mathcal{L}_S(H)$, $\hat{B}_y \in \mathcal{L}_S(H)$ we have that $\hat{A}_x \otimes \hat{I}_2$, $\hat{I}_1 \otimes \hat{B}_y$ are α-uncorrelated for all $x \in \Omega_{\mathcal{A}}$, $y \in \Omega_{\mathcal{B}}$. Hence,

$$\langle \alpha | (\hat{A}_x \otimes \hat{I}_2)(\hat{I}_1 \otimes \hat{B}_y) | \alpha \rangle = \langle \hat{A} \otimes \hat{I}_2 \rangle_\alpha \langle \hat{I}_1 \otimes \hat{B} \rangle_\alpha$$

so the observables $\mathcal{A} \otimes \hat{I}_2$, $\hat{I}_1 \otimes \mathcal{B}$ are α-independent.

(iii)$\Rightarrow$(i) Suppose $\mathcal{A} \otimes \hat{I}_2$, $\hat{I}_1 \otimes \mathcal{B}$ are α-independent for all observables $\mathcal{A}$ on H_1 and $\mathcal{B}$ on H_2. Let $|\alpha\rangle$ have Schmidt decomposition $|\alpha\rangle = \sum \lambda_i |\phi\rangle_i \otimes |\psi\rangle_i$. Since

$$\langle \alpha | \hat{A}_x \otimes \hat{B}_y | \alpha \rangle = \langle \hat{A}_x \otimes \hat{I}_2 \rangle_\alpha \langle \hat{I}_1 \otimes \hat{B}_y \rangle_\alpha \tag{90}$$

for all $x \in \Omega_{\mathcal{A}}$, $y \in \Omega_\beta$ and

$$\langle \hat{A}_x \otimes \hat{I}_2 \rangle_\alpha = \langle \alpha | \hat{A}_x \otimes \hat{I}_2 | \alpha \rangle = \sum_i \lambda_i \langle \phi_i | \otimes \langle \psi_i | (\hat{A}_x \otimes \hat{I}_2) \sum_j \lambda_j | \phi_j \rangle \otimes | \psi_j \rangle$$

$$= \sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \otimes \langle \psi_i | (\hat{A}_x \otimes \hat{I}_2) | \phi_j \rangle \otimes | \psi_j \rangle = \sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \hat{A}_x | \phi_j \rangle \langle \psi_i | \psi_j \rangle$$

$$= \sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \hat{A}_x | \phi_j \rangle \delta_{ij} = \sum_i \lambda_i^2 \langle \phi_i | \hat{A}_x | \phi_i \rangle \tag{91}$$

we conclude that

$$\sum_i \lambda_i^2 \langle \phi_i | \hat{A}_x | \phi_i \rangle \sum_j \lambda_j^2 \langle \psi_j | \hat{B}_y | \psi_j \rangle$$

$$= \langle \alpha | \hat{A}_x \otimes \hat{B}_y | \alpha \rangle = \sum_i \lambda_i \langle \phi_i | \otimes \langle \psi_i | (\hat{A}_x \otimes \hat{B}_y) \sum_j \lambda_j | \phi_j \rangle \otimes | \psi_j \rangle$$

$$= \sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \otimes \langle \psi_i | \left(\hat{A}_x | \phi_j \rangle \otimes \hat{B}_y | \psi_j \rangle \right)$$

$$= \sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \hat{A}_x | \phi_j \rangle \langle \psi_i | \hat{B}_y | \psi_j \rangle. \tag{92}$$

Since (92) holds for all $\mathcal{A}$, $\mathcal{B}$, we can let $\hat{A}_x = |\phi_1\rangle\langle\phi_1|$, $\hat{B}_y = |\psi_1\rangle\langle\psi_1|$ to obtain $\lambda_1^4 = \lambda_1^2$. Although λ_1 could be either 0 or 1 to satisfy the latter relation, we can also use the assumption that the Schmidt coefficients are sorted in descending order to fix $\lambda_1 = 1$. Therefore, $|\alpha\rangle = |\phi_1\rangle \otimes |\psi_1\rangle$ is separable. $\qquad\square$

If $|\alpha\rangle$ is entangled with Schmidt decomposition $|\alpha\rangle = \sum \lambda_i |\phi_i\rangle \otimes |\psi_i\rangle$, we have seen in (92) that for all $\hat{A} \in \mathcal{L}_S(H)$, $\hat{B} \in \mathcal{L}_S(H)$ we have

$$\langle \hat{A} \otimes \hat{B} \rangle_\alpha = \sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \hat{A} | \phi_j \rangle \langle \psi_i | \hat{B} | \psi_j \rangle. \tag{93}$$

If $|\alpha\rangle = |\phi\rangle \otimes |\psi\rangle$ is separable, then

$$\langle \hat{A} \otimes \hat{B} \rangle_\alpha = \langle \hat{A} \rangle_\phi \langle \hat{B} \rangle_\psi = \langle \hat{A} \otimes \hat{I}_2 \rangle_\alpha \langle \hat{I}_1 \otimes \hat{B} \rangle_\alpha. \tag{94}$$

At the other extreme, if the normalized entanglement number is $\tilde{e}(\alpha) = 1$, then $|\alpha\rangle$ is maximally entangled with Schmidt decomposition $|\alpha\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |\phi_i\rangle \otimes |\psi_i\rangle$ where $n = \min\left(\dim(H_1), \dim(H_2)\right)$. In this case, we have

$$\langle \hat{A} \otimes \hat{B} \rangle_\alpha = \frac{1}{n} \sum_{i,j} \langle \phi_i | \hat{A} | \phi_j \rangle \langle \psi_i | \hat{B} | \psi_j \rangle. \tag{95}$$

For a general state $|\alpha\rangle \in H_1 \otimes H_2$ the α-variance of $\hat{A} \otimes \hat{B}$ becomes

$$\Delta_\alpha(\hat{A} \otimes \hat{B}) = \langle \alpha | \hat{A}^2 \otimes \hat{B}^2 | \alpha \rangle - \langle \hat{A} \otimes \hat{B} \rangle_\alpha^2. \tag{96}$$

If $|\alpha\rangle = |\phi\rangle \otimes |\psi\rangle$ is separable, we obtain

$$\begin{aligned}
\Delta_\alpha(\hat{A} \otimes \hat{B}) &= \langle \phi | \otimes \langle \psi | \hat{A}^2 \otimes \hat{B}^2 | \phi \rangle \otimes | \psi \rangle - \langle \hat{A} \rangle_\phi^2 \langle \hat{B} \rangle_\psi^2 \\
&= \langle \phi | \otimes \langle \psi | \left(\hat{A}^2 | \phi \rangle \otimes \hat{B}^2 | \psi \rangle \right) - \left(\langle \hat{A} \rangle_\phi \langle \hat{B} \rangle_\psi \right)^2 \\
&= \langle \hat{A}^2 \rangle_\phi \langle \hat{B}^2 \rangle_\psi - \left(\langle \hat{A} \rangle_\phi \langle \hat{B} \rangle_\psi \right)^2.
\end{aligned} \tag{97}$$

If $|\alpha\rangle$ is entangled with Schmidt decomposition $|\alpha\rangle = \sum \lambda_i |\phi_i\rangle \otimes |\psi_i\rangle$, we obtain from (93) that

$$\begin{aligned}
\Delta_\alpha(\hat{A} \otimes \hat{B}) &= \langle \hat{A}^2 \otimes \hat{B}^2 \rangle_\alpha - \langle \hat{A} \otimes \hat{B} \rangle_\alpha^2 \\
&= \sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \hat{A}^2 | \phi_j \rangle \langle \psi_i | \hat{B}^2 | \psi_j \rangle - \left(\sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \hat{A} | \phi_j \rangle \langle \psi_i | \hat{B} | \psi_j \rangle \right)^2.
\end{aligned} \tag{98}$$

If $|\alpha\rangle$ is maximally entangled, this reduces to

$$\Delta_\alpha(\hat{A} \otimes \hat{B}) = \frac{1}{n} \sum_{i,j} \langle \phi_i | \hat{A}^2 | \phi_j \rangle \langle \psi_i | \hat{B}^2 | \psi_j \rangle - \frac{1}{n^2} \left(\sum_{i,j} \langle \phi_i | \hat{A} | \phi_j \rangle \langle \psi_i | \hat{B} | \psi_j \rangle \right)^2. \tag{99}$$

As expected, in the separable case we have $\langle \hat{A} \otimes \hat{I}_2 \rangle_\alpha = \langle \hat{A} \rangle_\phi$. However, when $|\alpha\rangle$ is entangled, we have by (93) that

$$\langle \hat{A} \otimes \hat{I}_2 \rangle_\alpha = \sum_i \lambda_i^2 \langle \hat{A} \rangle_{\phi_i} \tag{100}$$

and when $|\alpha\rangle$ is maximally entangled, $\langle \hat{A} \otimes \hat{I}_2 \rangle_\alpha = \frac{1}{n} \sum_i \langle \hat{A} \rangle_{\phi_i}$. Similarly, when $|\alpha\rangle$ is separable, we have

$$\Delta_\alpha(\hat{A} \otimes \hat{I}_2) = \Delta_\phi(\hat{A}) \tag{101}$$

and when $|\alpha\rangle$ is entangled, we have by (98) that

$$\Delta_\alpha(\hat{A} \otimes \hat{I}_2) = \sum_i \lambda_i^2 \langle \hat{A}^2 \rangle_{\phi_i} - \left(\sum \lambda_i^2 \langle \hat{A} \rangle_{\phi_i} \right)^2. \tag{102}$$

If $|\alpha\rangle$ is maximally entangled, this becomes

$$\Delta_\alpha(\hat{A} \otimes \hat{I}_2) = \frac{1}{n} \sum_i \langle \hat{A}^2 \rangle_{\phi_i} - \frac{1}{n^2} \left(\sum \langle \hat{A} \rangle_{\phi_i} \right)^2. \tag{103}$$

Similar equations hold for $\hat{I}_1 \otimes \hat{B}$.

For separable $|\alpha\rangle \in H_1 \otimes H_2$, from (94) it follows that $\mathrm{Cor}_\alpha(\hat{A} \otimes \hat{I}_2, \hat{I}_1 \otimes \hat{B}) = 0$, hence $\hat{A} \otimes \hat{I}_2, \hat{I}_1 \otimes \hat{B}$ are uncorrelated. Also, $\left[\hat{A} \otimes \hat{I}_2, \hat{I}_1 \otimes \hat{B}\right] = 0$ so no information is given by the uncertainty principle.

For the observable $\mathcal{C} = \left\{ \hat{A}_x \otimes \hat{B}_y \colon (x, y) \in \Omega_\mathcal{A} \times \Omega_\mathcal{B} \right\}$ with stochastic operator $\widetilde{\mathcal{C}} = \sum_{x,y} xy \hat{A}_x \otimes \hat{B}_y$ we have

$$\langle \mathcal{C} \rangle_\alpha = \langle \widetilde{\mathcal{C}} \rangle_\alpha = \langle \alpha | \sum_{x,y} xy \hat{A}_x \otimes \hat{B}_y | \alpha \rangle = \sum_{x,y} xy \langle \alpha | \hat{A}_x \otimes \hat{B}_y | \alpha \rangle. \tag{104}$$

If $|\alpha\rangle = \sum \lambda_i |\phi_i\rangle \otimes |\psi_i\rangle$ is entangled, we obtain from (93) that

$$\langle \mathcal{C} \rangle_\alpha = \langle \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{B}} \rangle_\alpha = \sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \widetilde{\mathcal{A}} | \phi_j \rangle \langle \psi_i | \widetilde{\mathcal{B}} | \psi_j \rangle. \tag{105}$$

As before, when $|\alpha\rangle = |\phi\rangle \otimes |\psi\rangle$ is separable this becomes $\langle \mathcal{C} \rangle_\alpha = \langle \mathcal{A} \rangle_\phi \langle \mathcal{B} \rangle_\psi$ and when $|\alpha\rangle$ is maximally entangled

$$\langle \mathcal{C} \rangle_\alpha = \frac{1}{n} \sum_{i,j} \langle \phi_i | \widetilde{\mathcal{A}} | \phi_j \rangle \langle \psi_i | \widetilde{\mathcal{B}} | \psi_j \rangle. \tag{106}$$

So far, we did not consider interactions because $\left[\hat{A} \otimes \hat{I}_2, \hat{I}_1 \otimes \hat{B}\right] = 0$. We now consider interactions on $H_1 \otimes H_2$. Let $\hat{A}, \hat{C} \in \mathcal{L}_S(H_1)$, $\hat{B}, \hat{D} \in \mathcal{L}_S(H_2)$ so that $\hat{A} \otimes \hat{B}, \hat{C} \otimes \hat{D} \in \mathcal{L}_S(H_1 \otimes H_2)$. The interaction statistics are given by

$$\mathrm{Cor}_\alpha(\hat{A} \otimes \hat{B}, \hat{C} \otimes \hat{D}) = \langle \alpha | (\hat{A} \otimes \hat{B})(\hat{C} \otimes \hat{D}) | \alpha \rangle - \langle \alpha | \hat{A} \otimes \hat{B} | \alpha \rangle \langle \alpha | \hat{C} \otimes \hat{D} | \alpha \rangle$$
$$= \langle \alpha | \hat{A}\hat{C} \otimes \hat{B}\hat{D} | \alpha \rangle - \langle \hat{A} \otimes \hat{B} \rangle_\alpha \langle \hat{C} \otimes \hat{D} \rangle_\alpha. \tag{107}$$

If $|\alpha\rangle$ has Schmidt decomposition $|\alpha\rangle = \sum_i \lambda_i |\phi_i\rangle \otimes |\psi_i\rangle$, then

$$\mathrm{Cor}_\alpha(\hat{A} \otimes \hat{B}, \hat{C} \otimes \hat{D}) = \sum_i \lambda_i \langle \phi_i | \otimes \langle \psi_i | (\hat{A}\hat{C} \otimes \hat{B}\hat{D}) \sum_j \lambda_j |\phi_j\rangle \otimes |\psi_j\rangle$$

$$- \left(\sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \hat{A} | \phi_j \rangle \langle \psi_i | \hat{B} | \psi_j \rangle \right) \left(\sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \hat{C} | \phi_j \rangle \langle \psi_i | \hat{D} | \psi_j \rangle \right)$$

$$= \sum_{i,j} \lambda_i \lambda_j \langle \phi_i | \hat{A}\hat{C} | \phi_j \rangle \langle \psi_i | \hat{B}\hat{D} | \psi_j \rangle$$

$$- \sum_{i,j,r,s} \lambda_i \lambda_j \lambda_r \lambda_s \langle \phi_i | \hat{A} | \phi_j \rangle \langle \psi_i | \hat{B} | \psi_j \rangle \langle \phi_r | \hat{C} | \phi_s \rangle \langle \psi_r | \hat{D} | \psi_s \rangle. \tag{108}$$

In the two extreme cases, when $|\alpha\rangle = |\phi\rangle \otimes |\psi\rangle$ is separable, we obtain

$$\mathrm{Cor}_\alpha(\hat{A} \otimes \hat{B}, \hat{C} \otimes \hat{D}) = \langle \phi | \hat{A}\hat{C} | \phi \rangle \langle \psi | \hat{B}\hat{D} | \psi \rangle - \langle \hat{A} \rangle_\phi \langle \hat{C} \rangle_\phi \langle \hat{B} \rangle_\psi \langle \hat{D} \rangle_\psi \tag{109}$$

and when $|\alpha\rangle$ is maximally entangled, we have

$$\mathrm{Cor}_{\alpha}(\hat{A} \otimes \hat{B}, \hat{C} \otimes \hat{D}) = \frac{1}{n} \sum_{i,j} \langle\phi_i|\hat{A}\hat{C}|\phi_j\rangle\langle\psi_i|\hat{B}\hat{D}|\psi_j\rangle$$

$$- \frac{1}{n^2} \sum_{i,j,r,s} \langle\phi_i|\hat{A}|\phi_j\rangle\langle\psi_i|\hat{B}|\psi_j\rangle\langle\phi_r|\hat{C}|\phi_s\rangle\langle\psi_r|\hat{D}|\psi_s\rangle. \quad (110)$$

A particularly simple case is $\hat{B} = \hat{D} = \hat{I}_2$, we have observable operators $\hat{A}\otimes\hat{I}_2$, $\hat{C}\otimes\hat{I}_2$ and we still have interaction if $\left[\hat{A}, \hat{C}\right] \neq 0$. In this case, if $|\alpha\rangle = \sum \lambda_i|\phi_i\rangle \otimes |\psi_i\rangle$, then

$$\mathrm{Cor}_{\alpha}(\hat{A} \otimes \hat{I}_2, \hat{C} \otimes \hat{I}_2) = \sum_i \lambda_i^2 \langle\phi_i|\hat{A}\hat{C}|\phi_i\rangle - \sum_{i,r} \lambda_i^2\lambda_r^2 \langle\hat{A}\rangle_{\phi_i}\langle\hat{C}\rangle_{\phi_r}. \quad (111)$$

For the two extreme cases, when $|\alpha\rangle = |\phi\rangle \otimes |\psi\rangle$ is separable, we have

$$\mathrm{Cor}_{\alpha}(\hat{A} \otimes \hat{I}_2, \hat{C} \otimes \hat{I}_2) = \langle\phi|\hat{A}\hat{C}|\phi\rangle - \langle\hat{A}\rangle_{\phi}\langle\hat{C}\rangle_{\phi} \quad (112)$$

and when $|\alpha\rangle$ is maximally entangled, we obtain

$$\mathrm{Cor}_{\alpha}(\hat{A} \otimes \hat{I}_2, \hat{C} \otimes \hat{I}_2) = \frac{1}{n} \sum_i \langle\phi_i|\hat{A}\hat{C}|\phi_i\rangle - \frac{1}{n^2} \sum_{i,r} \langle\hat{A}\rangle_{\phi_i}\langle\hat{C}\rangle_{\phi_r}. \quad (113)$$

It is straightforward to continue this discussion for interacting observables, $\hat{A}_x\otimes\hat{B}_y$, $\hat{C}_u \otimes \hat{D}_v$.

7. Concluding remarks

In this work, we have investigated four quantum entanglement measures that are based on Schmidt decomposition. After normalization of the measures, we have shown that partial order is possible between some entanglement measures. In particular, we have rigorously proved that the concurrence forms an upper bound on the tangle and entanglement robustness, whereas the Schmidt number forms a lower bound. The existing partial order was then utilized to introduce the concept of relative sensitivity to quantum entanglement, and with a minimal quantum toy model we have demonstrated how concurrence can be used to sharply demarcate separable states and Schmidt number to sharply demarcate maximally entangled states.

Each of the four entanglement measures could be computed using an explicit formula that is based on the Schmidt coefficients. Performing singular value decomposition, however, is a computationally expensive task.[54] Fortunately, a mathematical workaround proposed by Gudder[24, 25] could be utilized for those entanglement measures that require the sum of the fourth powers of the Schmidt coefficients, $\sum_i \lambda_i^4$. In the latter case, one can just compute the trace $\mathrm{Tr}(\hat{C}\hat{C}^{\dagger})^2$ obtained from the complex coefficient matrix $\hat{C}$, whose reshaping gives the bipartite quantum state vector in the basis $|i\rangle \otimes |j\rangle$, namely, $|\psi\rangle = \mathrm{res}(\hat{C})$.[32] This allows for fast evaluation of the concurrence and the Schmidt number, which provide the upper and lower bounds on the amount of available quantum entanglement, respectively.

State-dependent entanglement statistics provides an alternative, but equivalent, theoretical characterization of separable/entangled states in terms of absence/presence of measurable correlations between all/some local quantum observables. Noteworthy, Schmidt decomposition features prominently in the calculation of expectation values, variances, covariances and correlations between quantum observables, which obey the uncertainty principle.

Quantum entanglement is a precious physical resource that allows quantum devices to outperform their classical counterparts in terms of speed and efficiency. Therefore, the presented theorems with regard to the four entanglement measures: concurrence, tangle, entanglement robustness and Schmidt number, could be useful in practical quantum applications that require careful monitoring and utilization of the available quantum resources.

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