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Updating data semantics

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Abstract This paper has three main goals. First, to motivate a puzzle about how ignorance-expressing language like *maybe* and *if* interact: they (surprisingly) iterate and when they do they exhibit scopelessness. Second, to argue that there is an ambiguity in our theoretical toolbox and that resolving that opens the door to a solution to the puzzle. And third, to explore the reach of that solution (it turns out to do work in unexpected places). Along the way, the paper highlights a number of pleasing properties of two elegant semantic theories (data semantics and update semantics), explores some meta-theoretic properties of dynamic notions of meaning, dips its toe into some hazardous waters (epistemic contradictions and presupposition projection), and offers characterization theorems for the space of meanings an indicative conditional can have.

1 Ignorance and information

I am ignorant about a great many things. And so are you. Our language, equipped as it is with modals and conditionals, is well suited to express some of this.

(1) a. Maybe the picnic is a success.
b. The weather's gotta be fine by now.
c. If the weather held, the picnic is going as planned.

This is good: by sharing our ignorance we can winnow away at it.

Modal claims like these are equally quantificational claims: that there is a possibility compatible with the relevant information in which the picnic is a success, that all of the possibilities compatible with the relevant information are fine weather possibilities, that none of the weather-holding possibilities compatible with the relevant information are also plan-disrupted picnic-wise possibilities. But when we exchange information about our ignorance, we also

exchange information about how that ignorance might get resolved. That is why we can gloss the information at stake in the examples in (1) by these:

(2) a. There are ways of extending the relevant information that include the information that the picnic is a success.
b. Every way of extending the relevant information includes the information that the weather is fine by now.
c. Every way of extending the relevant information that includes the information that the weather held includes the information that the picnic is going on as planned.

These ignorance-resolving glosses are also quantificational claims. But rather than quantifying (pointwise) over possibilities they quantify (setwise) over states our partial information can grow into.

I want to focus on this ignorance-resolving aspect of our modal talk and look at a puzzle about expressions of iterated ignorance from this perspective. The puzzle will be (in part) about sentences that give voice to conditional ignorance like:

(3) a. Maybe he told Tom, if he didn't tell Harry.
b. Maybe if he didn't tell Harry, he told Tom.

First observation: these are fine and pointful things to say. Second observation: they seem to say the same thing. Hence, when *maybe* co-occurs with conditionals, it seems scopeless.

Now the puzzle: (i) while it is tough to iterate *maybe* and *must*, (ii) *maybe* mixes with conditionals with ease and is scopeless when it does; but (iii) conditionals are sufficiently *must*-like. So something's got to give. Each step will get a defense below.

Though the puzzle is general, its bite is best felt from the perspective of theories that emphasize the ignorance-resolving aspect of our modal talk. That will be the starting point (and serve as a defense of (iii)): a sketch of data semantics (in Section 3) and an update semantics for indicative conditionals (in Section 4) and a look at some of their (shared and not shared) properties (Section 5).[1] The next task: sharpen the puzzle about expressions of iterated ignorance (thereby defending (i) and (ii) in Section 6). Puzzles demand solutions. A place to look: there is an ambiguity in our theoretical toolkit

1 Data semantics is developed in Veltman 1985 and update semantics in Veltman 1996. The dynamic strict conditional is from Gillies 2004 (and then Gillies 2009, 2010).

and exposing it offers a new perspective on epistemic possibility modals (Section 7). Embracing that perspective makes a (positive) difference in some unexpected places (Sections 8 and 9). But it also raises some hard questions, the answering of which (surprisingly) leads to characterizing in a precise sense what indicative conditionals can and can't mean (Section 10).

2 Preliminaries

Assume that sentences of natural language can be represented by sentences in an intermediate logical language. We will use a basic modal propositional language for this.

Definition 1. L_1 is the smallest set including a (finite) set $A = \{p, q, \ldots\}$ of atomic sentences, the designated atomic sentence $\bot$, and that is closed under the boolean connectives $\neg$, $\wedge$ and the modal operators $\Diamond$,$\rightarrow$. (L_0 is the non-modal part of L_1.)

Expressions of epistemic possibility — we will mostly be concerned with *maybe* — will be represented by $\Diamond$, expressions of epistemic necessity (*must* and (epistemic) *gotta*) by $\Box$, and the indicative conditional by $\rightarrow$. The designated atom $\bot$, which will always be false, is a useful way of abbreviating an arbitrary contradiction.

Our job is to assign meanings for, and characterize entailment relations between, sentences of L_1 (and, later, some extensions of it) thereby assigning meanings for, and characterizing entailment relations between, the relevant sentences of natural language that they represent.[2] Throughout I will assume that we only care about non-vacuous uses of the indicative: that is, uses of $\phi \rightarrow \psi$ in situations which do not rule out ϕ.[3]

2 The theories sketched below have something to say about all sentences of the L languages but sometimes we focus on special cases: those with no modal operators (descriptive sentences), those with modal operators that do not take scope over other modal operators (non-iterated modal sentences), and those with modal operators that do take such scope (iterated modal sentences). Some conventions: $p, q, r, \ldots$ range over atomic sentences and $\phi, \psi, \ldots$ over arbitrary sentences; the various theories specify compositional semantic values and entailment relations, the notation for which will bear identifying subscripts (which will be ommitted when this won't lead to confusion). And, finally, some shorthand: an iterated construction where (say) a necessity modal takes scope over a possibility modal is a $\Box > \Diamond$ construction, possibility-over-necessity is $\Diamond > \Box$, and so on.

3 For defenses of this see (among others) Stalnaker 1975, von Fintel 1998. Whether this stipulation is pragmatically encoded or semantically encoded (as sketched in Gillies 2009) makes no difference for our current purposes.

This choice of regimenting intermediate language is defensible but not entirely innocent: we have ruled out from the start the restrictor view of conditionals that treats indicative *if*-clauses exclusively as devices for restricting our epistemic modals.[4] To illustrate:

(4) a. Maybe he told Tom, if he didn't tell Harry.
$\Diamond$(*he didn't tell Harry*)(*he told Tom*)
b. It's gotta be a diamond if it's a red face card.
$\Box$(*it's a red face card*)(*it's a diamond*)
c. If the butler isn't the culprit, the driver is.
$\Box$(*the butler isn't the culprit*)(*the driver is the culprit*)

The restrictor view treats apparently conditional constructions as two-place modal constructions. What you might be tempted to call a conditional's antecedent is, in fact, the restrictor (first argument) to the modal; what you might be tempted to call a conditional's consequent is the nuclear scope of the modal.[5] So in (4a) the modal is the two-place *maybe*: assuming it is an existential modal, (4a) says that within the worlds in which he didn't tell Harry, some are worlds where he told Tom. Mutatis mutandis for (4b), swapping in the universal all for what the two-place *gotta be* expresses.

In a bare conditional (that is, one in which there is no modal scoping over the whole conditional and no modal scoping over the consequent) like (4c) there is nothing for the *if* to restrict so, since the idea is that that is what/all *if*s do, the restrictor view posits a (covert) necessity modal. This is a principled way of getting things right for this case. But the recipe goes wrong in other cases. For instance:

(5) a. If Yellow is in the box, then Red might be and Blue must be.
b. If Lenny is at the party, then Carl might be but Monty isn't.

Following the basic recipe gets things wrong in cases like this where we have a bare conditional with an interesting nuclear scope (the nuclear scope is a conjunction, not a modal, and the conjuncts have different modal force).[6] That may or may not be a decisive reason against the restrictor view but it is

4 The restrictor view is inspired by Lewis (1975) and defended and extended in Kratzer 1979, 1986, 2012.

5 The terminology comes from thinking of modals semantically as generalized quantifiers.

6 See Gillies 2010: §9 for a more careful statement of the choices examples like these force for the restrictor view.

a defense for setting it aside since it is this kind of conditional ignorance that is our main focus. Feel free to read what follows as a reductio of that stance.

3 Data semantics

In data semantics, sentences aren't true or false full-stop but only true or false with respect to an information state. States play a role very much like possible worlds in the standard semantics for modal logic: we won't say what they are but will carve out a job for them to do. That job is that they act as indices: the things at which sentences are true or false and the things that modals quantify over and shift.[7] An expression of relative ignorance at an information state quantifies over certain states, depending on how the information in that state can and can't grow.

Definition 2 (Information states, growth)**.** Let I be a non-empty set of information states. For each $s \in I$ let v_s be a (partial) function from $A \cup \{\bot\}$ to the set $\{0, 1\}$ of truth-values such that $v_s(\bot) = 0$. Finally, let $\langle I, \leq \rangle$ be a partial order such that:

i. If $s \leq s'$ then $v_s \subset v_{s'}$.

ii. Every maximal chain in $\langle I, \leq \rangle$ has a maximal element. If s is a maximal element then v_s is total.

If $s \leq s'$ say that s can grow into state s'.

When s can grow into s' this represents one way our knowledge can grow and our ignorance can shrink. But not all ways of gaining information will always be comparable: if we are ignorant about (the truth of) ϕ then one possibility is that we learn ϕ (is true) and another is that we learn $\neg\phi$ (is true). This is a fork in the road of how our ignorance can be resolved. (Once we learn the facts about an atomic sentence though that is settled from there on out.) The assumption that there is a maximal chain and a maximal element in it means that no matter what information you have it is in principle possible to have all your questions answered.[8] Our current information is compatible

7 The set-up for data semantics, though not the specific clauses, is somewhat like the semantics for intuitionistic logic in Kripke 1965. See Muskens 2013 for how to embed data semantics in the three-valued three-sorted functional type theory TY_2^3.

8 A subset I^* of I is a chain iff the restriction of $\leq$ to I^* is a linear order. A chain I^* is maximal iff if I' is a chain containing I^* then $I^* = I'$. A maximal element s^* in such a maximal chain I^* is a state such that there is no $s \in I$ where $s^* < s$.

with lots of complete pictures of the ways things are. One of these, though we don't yet know which one, represents the actual situation.

Since we are interested in whether ϕ holds on the basis of partial information characterizing s, the relation true-in-s is partial: it has to be possible that there are states s and sentences ϕ such that the information we have in s doesn't settle whether ϕ. We will put this in terms of a (partial) denotation function $[\![\cdot]\!]^s$ taking sentences to truth values.

Definition 3 (Data semantics)**.** Let s be any state. Then $[\![\cdot]\!]^s$ is the partial function from L_1 to truth-values such that $[\![\bot]\!]^s = 0$ and:

i. atoms

   a. $[\![p]\!]^s = 1$ iff $v_s(p) = 1$

   b. $[\![p]\!]^s = 0$ iff $v_s(p) = 0$

ii. *not*

   a. $[\![\neg\phi]\!]^s = 1$ iff $[\![\phi]\!]^s = 0$

   b. $[\![\neg\phi]\!]^s = 0$ iff $[\![\phi]\!]^s = 1$

iii. *and*

   a. $[\![\phi \wedge \psi]\!]^s = 1$ iff $[\![\phi]\!]^s = 1$ and $[\![\psi]\!]^s = 1$

   b. $[\![\phi \wedge \psi]\!]^s = 0$ iff $[\![\phi]\!]^s = 0$ or $[\![\psi]\!]^s = 0$

iv. *maybe*

   a. $[\![\Diamond\phi]\!]^s = 1$ iff $[\![\phi]\!]^{s'} = 1$ for some s' such that $s \leq s'$

   b. $[\![\Diamond\phi]\!]^s = 0$ iff $[\![\phi]\!]^{s'} = 1$ for no s' such that $s \leq s'$

v. *if*

   a. $[\![\phi \rightarrow \psi]\!]^s = 1$ iff $[\![\phi]\!]^{s'} = 1$ and $[\![\psi]\!]^{s'} = 0$ for no s' such that $s \leq s'$

   b. $[\![\phi \rightarrow \psi]\!]^s = 0$ iff $[\![\phi]\!]^{s'} = 1$ and $[\![\psi]\!]^{s'} = 0$ for some s' such that $s \leq s'$

Entailment: $\phi_1, \ldots, \phi_n \models_{\mathrm{DS}} \psi$ iff for every s: if $[\![\phi_1]\!]^s = 1$ and … and $[\![\phi_n]\!]^s = 1$ then $[\![\psi]\!]^s = 1$.

Introducing $\supset$ and $\Box$ has the expected results.[9]

Fact 1. Let $\phi \supset \psi$ abbreviate $\neg(\phi \wedge \neg\psi)$ and let $\Box\phi$ abbreviate $\top \rightarrow \phi$ (where $\top =_{\text{df}} \neg\bot$). Then:

i. $[\![\phi \supset \psi]\!]^s = 1$ iff $[\![\phi]\!]^s = 0$ or $[\![\psi]\!]^s = 1$ and
$[\![\phi \supset \psi]\!]^s = 0$ iff $[\![\phi]\!]^s = 1$ and $[\![\psi]\!]^s = 0$.

ii. $[\![\Box\phi]\!]^s = 1$ iff $[\![\phi]\!]^{s'} = 0$ for no s' such that $s \leq s'$ and
$[\![\Box\phi]\!]^s = 0$ iff $[\![\phi]\!]^{s'} = 1$ for some s' such that $s \leq s'$.

We'll look at some basic properties of data semantics, and how neatly it explains some otherwise tricky data, in Section 5.[10]

4 Update semantics

In update semantics information states aren't unanalyzed primitives. They are sets of possible worlds: those compatible with the information gathered so far.

Definition 4 (Information states)**.** Let $W = 2^{A \cup \{\bot\}}$ be the set of possible worlds (where $w(\bot) = 0$ for every $w \in W$). s is an information state iff $s \subseteq W$. $\emptyset$ is the absurd state and W is the state of (total) ignorance.

Before we took compositional semantic values to be (partial) truth conditions. Here context change potentials — functions from information states to information states — play that role. Thus the meaning of sentences is

9 And some surprises: $\Box\phi \not\models \phi$ since it's possible for an atom p to be undefined at a state s and yet in every state s' that it can grow into p is true. This can make sense out of Karttunen's Problem (Karttunen 1972):

(i) [Seeing the pouring rain outside.]
a. It is raining.
b. ??It must be raining.

This seems backwards from the standard semantics, but there is an explanation in data semantics: ϕ asymmetrically entails $\Box\phi$ and so the modal is unnecessarily weak in this situation and so weird to use. There are, however, arguments against 'weak *must*' solutions to Karttunen's Problem (see von Fintel & Gillies 2010 where there is also a strong alternative solution).

10 Proofs, some of which are more fun than you might predict, can be found in the appendix.

identified with instructions or recipes for changing information states: $s[\phi]$ tells us how to incorporate ϕ's meaning $[\phi]$ in a state s.[11]

Definition 5 (Update semantics w/ indicatives)**.** Let s be any state. Define $[\cdot]_{\text{US}}$ as follows:

i. $s[p] = \{w \in s\colon w(p) = 1\}$

ii. $s[\neg\phi] = s \setminus s[\phi]$

iii. $s[\phi \wedge \psi] = s[\phi][\psi]$

iv. $s[\Diamond\phi] = \{w \in s\colon s[\phi] \neq \emptyset\}$

v. $s[\phi \rightarrow \psi] = \{w \in s\colon s[\phi][\psi] = s[\phi]\}$

Say that ϕ is <u>true</u> or <u>supported</u> in s, $s \models\!\!\!\models_{\text{US}} \phi$, iff $s[\phi] = s$. And entailment: $\phi_1, \ldots, \phi_n \models\!\!\!\models_{\text{US}} \psi$ iff for any s: $s[\phi_1]\ldots[\phi_n] \models\!\!\!\models_{\text{US}} \psi$.

The clauses for (i.)–(iv.) represent the conservative core of the semantics: later, we will consider possible departures for $\Diamond$ (and $\rightarrow$ and the derived $\Box$) but will insist that any candidate update function agrees about (i.)–(iv.).

Again, the derived meanings for $\supset$ and $\Box$ are what we want.[12]

Fact 2. Let $\phi \supset \psi$ abbreviate $\neg(\phi \wedge \neg\psi)$ and let $\Box\phi$ abbreviate $\top \rightarrow \phi$. Then: (i) $s[\phi \supset \psi] = s[\neg\phi] \cup s[\psi]$, and (ii) $s[\Box\phi] = \{w \in s\colon s[\phi] = s\}$.

There is a difference between the type of change induced by a descriptive sentence and the type induced by a sentence with a modality like *maybe* or *if*. Declarative programs eliminate possibilities: in fact, for any declarative ϕ, $W[\phi]$ behaves like a classical proposition and hence $s[\phi] = s \cap W[\phi]$. The modalities are test programs: they check whether a given state has a certain property.[13]

11 In dynamic logic the semantic type of programs is relational: program π expresses the set of ordered pairs $\langle s, t\rangle$ such that executing π in s (possibly) terminates in t. In update semantics all sentences are of this type (where, in fact, the denoted relations are functions (written in post-fix)): they express constraints on what a state must look like in order to comply with their instructions.

12 Epistemic necessity is strong in update semantics: $\Box\phi \models\!\!\!\models_{\text{US}} \phi$.

13 An operator $\bigcirc$ is a test operator iff for any ϕ and state s: $s[\bigcirc\phi] = s$ or $s[\bigcirc\phi] = \emptyset$. Doing a routine induction on declarative ϕ and ψ — which you should totally do — you will see that $W[\neg\phi] = W \setminus W[\phi]$ and $W[\phi \wedge \psi] = W[\phi] \cap W[\psi]$. But $W[\bigcirc\phi]$ is either W or $\emptyset$ and so $s[\bigcirc\phi]$ won't in general be the same as $s \cap W[\bigcirc\phi]$.

5 Some basic properties

The two theories aren't equivalent but they share some nice properties. Here we will just look at a few of them.[14]

First: since both treat indicatives as strict conditionals over partial information and treat epistemic possibility modals as quantifiers over this same domain, the two operators are linked.[15]

Suppose that either the gardener did it (p), the driver did it (q), or the butler did it (r) and that whoever did it acted alone (so $r \equiv \neg(p \vee q)$).

(6) a. #The driver might be the culprit, and moreover, if the gardener isn't the culprit, the butler is. $\Diamond q \wedge (\neg p \rightarrow r)$

b. It's not so that if the gardener didn't do it, the driver did. Maybe it was the butler. $\neg(\neg p \rightarrow q) \wedge \Diamond r$

This pattern is easily explained by a theory which takes *if* to be an (epistemic) strict conditional where *must* and *maybe* are duals.

Definition 6 (Equivalence)**.** For any ϕ and ψ:

i. ϕ and ψ are equivalent$_{DS}$ ($\phi \cong_{DS} \psi$) iff $\phi \models_{DS} \psi$, $\psi \models_{DS} \phi$, $\neg\phi \models_{DS} \neg\psi$, and $\neg\psi \models_{DS} \neg\phi$.

ii. ϕ and ψ are equivalent$_{US}$ ($\phi \cong_{US} \psi$) iff for any state s: $s[\phi] = s[\psi]$.

Fact 3. For any ϕ, ψ:

i. $\phi \rightarrow \psi \cong_{DS} \Box(\phi \supset \psi)$ and $\phi \rightarrow \psi \cong_{US} \Box(\phi \supset \psi)$.

ii. $\neg(\phi \rightarrow \psi) \cong_{DS} \Diamond(\phi \wedge \neg\psi)$ and $\neg(\phi \rightarrow \psi) \cong_{US} \Diamond(\phi \wedge \neg\psi)$.

iii. $\Box\phi \cong_{DS} \neg\Diamond\neg\phi$ and $\Box\phi \cong_{US} \neg\Diamond\neg\phi$.

Both theories say that *if* is a strict conditional and that (except in states of total information) $\Box$ is non-trivial, and hence they both say that *if* and $\supset$ are different. But there are differences to the difference. For instance: in data

14 A systematic look at data semantics and its logic is in Veltman 1985. See van der Does, Groeneveld & Veltman 1997 for the structural properties of update semantics (without indicatives and without embedding) and Gillies 2004 for some of the features of indicatives.

15 Both theories also, as they should, predict that in states of complete information — maximal states in data semantics and singletons in update semantics — *must* and *maybe* aren't useful ($\Box\phi$ and $\Diamond\phi$ are then equivalent to ϕ) and indicatives are material conditionals.

semantics an indicative plus its antecedent need not entail its consequent but in update semantics a conditional plus its antecedent does entail its consequent. And neither says that modus tollens always works.[16]

This is not without motivation. An example (from Veltman 1985): there are three missing marbles (red, blue, and yellow) and two boxes (box #1 and box #2), with at least one marble in each box.

(7) a. If red is in box #2, then if blue is in box #2 then yellow is in box #1. $p \rightarrow (q \rightarrow r)$

b. It's not so that if blue is in box #2 then yellow is in box #1. $\neg(q \rightarrow r)$

Applying modus tollens on (7a) and (7b) would seem to entail that the red marble isn't in box #2. But that is too hasty.

(8) a. ??Red isn't in box #2. $\neg p$

b. Maybe red isn't in box #2. $\Diamond \neg p$

The observation is that the weaker (8b) is preferred to (8a). Jumping straight to the bare prejacent is not what your information supports.

Similarly: we know the culprit (who acted alone) was either the gardener or the butler, but we don't know which. At least these two things are true:

(9) a. If the gardener isn't the culprit, then it must be butler. $\neg p \rightarrow \Box q$

b. Maybe the butler isn't the culprit. $\Diamond \neg q$

16 Sometimes it is alleged that the dynamic treatment of conditionals requires something called a "revisionary logic" (two recent examples: Dorr & Hawthorne 2013, Stojnić 2017). This is meant as a weighty objection (revising things, I guess, being a priori bad). But I confess that I do not understand it. The allegation doesn't come with a specific example that the dynamic account wrongly says is an entailment that isn't or a specific example that the dynamic account wrongly says isn't an entailment that is.

The irony is that there is a natural sense in which the dynamic strict conditional is rather classical. The collection of entailment patterns that indicative conditionals in natural language seem to go in for is a bundle that threatens to restrict what they could mean to just one thing, the material conditional (think: modus ponens plus import/export). And yet the indicative conditional seems to have a meaning richer than the material conditional. There is a dynamic payoff: $(\phi \wedge \psi) \rightarrow \chi \cong_{\text{US}} \phi \rightarrow (\psi \rightarrow \chi)$ even though $\phi \rightarrow \psi \not\cong_{\text{US}} \phi \supset \psi$. To see this let $\{w_1, w_2\}$ where $w_1(p) = 1$ and $w_1(q) = 0$ and $w_2(p) = w_2(q) = 1$. Note that $s[\neg(p \rightarrow q)] = s$ and $s[\neg(p \supset q)] = \{w_1\}$. Hence $p \rightarrow q$ is stronger than $p \supset q$.

Note that (9b) is equivalent to $\neg\Box q$ and using this to tollens (9a)'s modus is too much. While your information supports the weaker (10b), not so the bare prejacent (10b).

(10) a. ??The gardener is the culprit. p
b. Maybe the gardener is the culprit. $\Diamond p$

Mere reflection on our ignorance about the butler is not enough to condemn the gardener.[17] Both of the theories we're looking at get this right.

Fact 4. While $\phi \rightarrow \psi, \neg\psi \not\models \neg\phi$ it does hold that $\phi \rightarrow \psi, \neg\psi \models \Diamond\neg\phi$.

To be clear: it's not that modus tollens (inexplicably) fails or that no instances of it are good. There are principled boundaries, and one of those boundaries is whether the consequent of the conditional contains material that itself expresses something about our ignorance.[18] Whether such a sentence is true or false isn't a stable or persistent fact, and lack of persistence lines up exactly with when modus tollens goes wrong.

Definition 7 (Persistence)**.** For any ϕ:

i. In data semantics ϕ t-persistent iff if $[\![\phi]\!]^s = 1$ then $[\![\phi]\!]^{s'} = 1$ for every $s' \geq s$ and ϕ f-persistent iff if $[\![\phi]\!]^s = 0$ then $[\![\phi]\!]^{s'} = 0$ for every $s' \geq s$.

ii. In update semantics ϕ persistent iff if $s[\phi] = s$ then $s'[\phi] = s'$ for every $s' \subseteq s$.

Fact 5. In data semantics $\Diamond\phi$ isn't t-persistent and $\Box\phi$ and $\phi \rightarrow \psi$ are not f-persistent. In update semantics existential modal claims ($\Diamond\phi$) and negations of universal modal claims ($\neg\Box$, $\neg(\phi \rightarrow \psi)$) aren't persistent.

17 If you prefer to not have any context setting, the same point can be made another way. Consider this argument:

(i) a. Either the gardener is the culprit or the butler is the culprit and not both.
b. If the gardener isn't the culprit, then it must be butler.
c. It's not the case that butler must be the culprit. (= Maybe the butler isn't the culprit.)
d. ??So: the gardener is the culprit.

18 What goes for the embedded modals and conditionals we have been considering goes for probabilistic ones like *likely*, too (Yalcin 2012).

Maybe there are other replies to apparent counterexamples like (7) and (9), and maybe those other replies are convincing. What is relevant here is that both data semantics and update semantics: (i) classify instances of modus tollens as generally not entailments, (ii) shed light exactly on what instances invalidate the pattern (and so where the boundary is), and (iii) do this without any special pleading about the 'real' logical form of the sentences involved.

6 Iterated ignorance

Now to sharpen our puzzle: there are problems when it comes to iterated ignorance. Iterating *must* and *maybe* is hard and requires special set-ups and forces higher-order readings. None of that holds for *if* and *maybe*: it's easy, requires no special set-ups, and the readings are (scopeless) first-order expressions of ignorance. This gets at something pretty fundamental about how ignorance-expressing language behaves.

So the first bit: epistemic modals do not naturally go in for iterated readings. Before you get too busy trying to conjure counterexamples, let me say just what the claim is. The claim is not that there are no well-formed strings in which ignorance expressing language embeds other ignorance expressing language. Here are some examples of just that:

(11) a. Maybe the yellow marble must be in box #2. $\Diamond\Box p$
b. It must be that maybe the butler is the culprit. $\Box\Diamond p$

These are well-formed.[19]

The claim also is not that well-formed strings in which ignorance expressing language embeds other ignorance expressing language can't be interpreted or have coherent readings. Again: (11a) can say something substantive. Suppose Alex has been investigating where the marbles are and we have conclusive evidence that she has determined their respective locations but we don't know what she has concluded. We can report where things stand with (11a): it's compatible with our information that she has concluded that the yellow marble is in box #2. Similarly: (11b), awkward as it is, can say something substantive. Suppose we have conclusive evidence that the

19 Examples (11a) and (11b) are related to examples in Moss 2015, the relevant difference being that we have dropped *probably*s for *must*s. Some speakers do find (11b) hard to parse. Two things. First: with some priming (which we'll get to in a bit) I think this improves. Second: if $\Box > \Diamond$ iterations are gibberish in your dialect, that makes the puzzle about how *if*s and *maybe*s iterate more puzzling not less.

detective thinks the butler's alibi doesn't hold water. We can report what we have concluded with (11b): it follows from our information that the detective hasn't yet ruled out the butler.

Now consider these simple non-iterated modal claims:

(12) a. The yellow marble must be in box #2. $\Box p$
b. Maybe the butler is the culprit. $\Diamond p$

Non-iterated modals don't always entail their iterated counterparts.[20] For example: from the fact that the detective hasn't ruled out the butler it doesn't follow that our information settles that fact and so nor does it follow that our information settles that the butler did it.

Those are some of the things that the claim is not. What the claim is: such sentences require for their coherent interpretations that there is some multiplicity of bodies of information and that the different modals target those different sources. In a context where there is no such multiplicity (11a) and (11b) are weird. One way to bring that out: (13a) and (13b) are not available as respective glosses of them.

(13) a. ??It is compatible with the relevant information that it follows from the relevant information that yellow is in box #2.
b. ??It follows from the relevant information that it is compatible with the relevant information that butler is the culprit.

Different bodies of information, different modals, and so iteration can make sense; same body of information, and so same family of modals, leads to iterating weirdness.[21]

Both data semantics and update semantics have something to say here, and that is because both frameworks assume that even though the partial information being modeled reflects our ignorance about a great many things,

20 The exception here is that if $\Box$ is factive then $\Box p$ does entail $\Diamond\Box p$.

21 Some have suggested that pointful, non-collapsing iteration of modals in contexts with a multiplicity of sources of information spells doom for update semantics (for instance, Moss 2015). But it should be clear that, given such multiplicty (indeed, the multiplicity being obligatory), the existence of such pointful iterated modals in (11) and their not being entailed by the non-iterated counterparts in (12) has no bearing — regardless of theoretical framework — at all as to whether *maybe* and *must* do or do not telescope, go in for negative introspection, or have other alleged bad-making features of S5. Unless, of course, the fact that *I know that it is raining* doesn't entail *You know that I know that it is raining* equally counterexamples KK.

the information about that ignorance is nevertheless complete. In both frameworks our ignorance-expressing language freely quantifies over all ways that ignorance can be resolved. So even though your information at a state can be partial, 'one's knowledge of the changes which one's partial knowledge could yet undergo is complete' (Veltman 1985: 216).

Fact 6. Let $\bigcirc$ be any string of 0 or more of the 1-place operators $\neg, \Diamond, \Box$ and let ϕ be any persistent sentence. Then $\bigcirc\phi \cong \psi$ where ψ is one of $\phi, \neg\phi, \Diamond\phi, \Box\phi, \neg\Diamond\phi, \neg\Box\phi$.

In seeing why this is true you will have seen that while both theories predict that iterated modals telescope, they do so in different directions. Still, even though the two systems say different things about when iterated modals collapse and when they do what they collapse to, they agree that $\Box\Diamond p$ and $\Diamond\Box p$ try to get their point across in a way that is more complicated than necessary and thus suggests unsettledness about how ignorance can get resolved. And that is weird, since they are both built around the idea that there isn't ignorance of that sort.

Now to the second bit of the puzzle: indicatives have no problem at all mixing with *maybe*s and when they do they are scopeless expressions of first-order ignorance.

(14) a. Maybe if the gardener didn't do it, the butler did. $\Diamond > \rightarrow$
b. Maybe the butler did it, if the gardener didn't. $\rightarrow > \Diamond$
c. Maybe the gardener didn't do it and the butler did. $\Diamond > \wedge$

There's the iterated part: in both data semantics and update semantics indicative conditionals are $\Box$-like. So the iterated ignorance in both (14a) and (14a) should be surprising. But it's not weird or hard to parse or higher-order ignorance that requires a multiplicity of sources of information. Just plain, first-order ignorance will do.

Further, there is the scoplessness part: even though the conditionals in (14a) and (14b) have different operator-plus-scope packages they nevertheless hang together and relate to the simple non-iterated conjunctive uncertainty in (14c).[22] How can it be that a $\Diamond > \Box$ iterated construction and a $\Box > \Diamond$ iterated

22 It seems like indicatives and *maybe*s go in for scopelessness: (14a) $\Leftrightarrow$ (14b). For instance:

(i) a. Alex: Maybe if the gardener didn't do it, the butler did.
b. Billy: Right. Maybe the butler did it, if the gardener didn't.
c. Alex: ??What? No, I wouldn't say that.

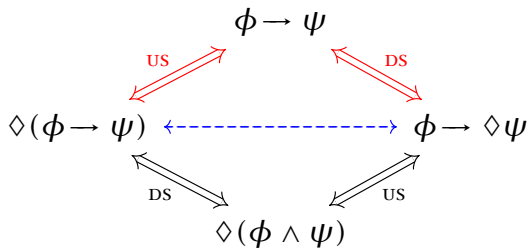


Figure 1 Conditional ignorance

construction say the same thing (and, given our focus on non-vacuous uses, be equivalent to the corresponding non-iterated $\Diamond > \wedge$)?

This behavior is not accounted for in data semantics and not accounted for in update semantics either. (Or, as far as I am aware, in any theory that takes indicatives to be $\Box$-like in the relevant way.) Worse: in each theory, exactly one of the conditional ignorance constructions comes out so strong as to render them pointless. This is all true even when the ϕ's and ψ's involved are descriptive (the case of principle interest).

Fact 7. For any ϕ, ψ:

i. $\phi \rightarrow \Diamond\psi \not\cong_{\text{DS}} \Diamond(\phi \rightarrow \psi)$ but $\phi \rightarrow \Diamond\psi \cong_{\text{DS}} \phi \rightarrow \psi$.

ii. $\phi \rightarrow \Diamond\psi \not\cong_{\text{US}} \Diamond(\phi \rightarrow \psi)$ but $\Diamond(\phi \rightarrow \psi) \cong_{\text{US}} \phi \rightarrow \psi$.

Neither theory makes the prediction we want (that relative scope of *maybe* and *if* doesn't matter) and there is a complementary distribution of the predictions we don't want (one version or another of trivializing).[23] The

Similarly it seems that (14b) $\Leftrightarrow$ (14c):

(ii) a. #Maybe the butler did it, if the gardener didn't and, moreover, it can't be that the gardener didn't do it and the butler did.
b. #It can't be that the gardener didn't do it and the butler did, and, moreover, maybe the butler did it, if the gardener didn't.

23 The trivializing equivalence of $\phi \rightarrow \Diamond\psi$ to $\phi \rightarrow \psi$ might remind you of Lewis's (1973) argument that on pain of a similar triviality counterfactuals can't obey conditional excluded middle and duality between *might*- and *would*-counterfactuals. The trivializing problem we are facing is more robust since it doesn't rely on duality (and since we also want to predict that $\phi \rightarrow \Diamond\psi$ is equivalent to $\Diamond(\phi \rightarrow \psi)$).

situation can be seen graphically in Figure 1. While each theory trades on incomplete or partial states of information, each also assumes that the paths forward from any such state are transparent. Since it seems like that is where the problem comes from, this makes the prospects of a minimally-altering solution seem dim.[24]

7 Compatibility and possibility

I want to explore the prospects of solving our puzzle in the framework of update semantics. So take that basic set-up as given.

Possibility modals like *maybe* are vehicles for expressing compatibility: *maybe it is raining* says that *it is raining* is compatible with the relevant information. There are two routes to compatibility and in a classical set-up they coincide.

Route one ϕ is compatible with a body of information s (a set of worlds) iff the information carried by ϕ can be added to s without absurdity.

On the usual way of understanding the relevant bits in this gloss (information is propositional, adding is intersection): ϕ is compatible with s iff $s \cap [\![\phi]\!] \neq \emptyset$.

Route two ϕ is compatible with s iff there is some non-trivial part of s in which ϕ is true.

On the usual way of understanding the relevant bits in this gloss (parthood is subsethood, truth is propositional inclusion): ϕ is compatible with s iff $s' \subseteq [\![\phi]\!]$ for some $s' \subseteq s$ such that $s' \neq \emptyset$.

Since $s \cap [\![\phi]\!] \neq \emptyset$ iff there is some $s' \neq \emptyset$ such that $s' \subseteq s$ and $s' \subseteq [\![\phi]\!]$ it follows that the two routes end up at the same place. Since they end up in the same place it makes no difference which route we take *maybe* to go by.

In update semantics there is logical space for the routes to compatibility to come apart. Definition 5 offers a route one semantics for *maybe*. But there is room for another, non-equivalent, route two expression of possibility.

24 I won't have anything to say about things like $\Box(\phi \rightarrow \psi)$ or $\Diamond\neg(\phi \rightarrow \psi)$: both data semantics and update semantics have the same predictions about $\Box\Box$- and $\Diamond\Diamond$-iterations. The trouble-making feature really is mixing the existential $\Diamond$ with the $\Box$-like $\rightarrow$ and it seems we are destined to have one entailment we want and one we don't. can have at most one of the right entailments. To make this vivid: Veltman (1985: 215–216) offers an alternative definition that blocks the entailment from $\phi \rightarrow \Diamond\psi$ to $\phi \rightarrow \psi$ in data semantics but notes that then the widescoped $\Diamond(\phi \rightarrow \psi)$ entails $\phi \rightarrow \psi$.

Definition 8 (Compatibility, two ways)**.** A state s' is a (non-trivial) substate of s iff $s' \subseteq s$ and $s' \neq \emptyset$. (In that case, write: $s' \sqsubseteq s$.) A sentence ϕ is consistent with s iff $s[\phi] \neq \emptyset$. ϕ is coherent with respect to s iff $s'[\phi] = s'$ for some $s' \sqsubseteq s$. A sentence ϕ is consistent (coherent) full-stop iff ϕ is consistent (coherent) with respect to some s.

The labels are unimportant. What is important is that consistency (route one compatibility) and coherence (route two) are both reasonable ways of getting at compatibility and that in update semantics they can come apart because coherence (full-stop) asymmetrically entails consistency (full-stop). Coherence entails consistency: if for some s there is an $s' \sqsubseteq s'$ such that $s'[\phi] = s'$ then there is a state consistent with ϕ (namely, s'). But consistency does not entail coherence: take any state s with some mix of p and $\neg p$ worlds and note that (i) $s[\Diamond p \wedge \neg p]$ will return just the $\neg p$ worlds in s, but (ii) for no $s' \sqsubseteq s$ is it the case that $s'[\Diamond p \wedge \neg p] = s'$.

These different ways of being compatible with a state have different properties. Here is one: coherence commutes with conjunction while consistency does not (there's a counterexample in the previous paragraph).

Fact 8. For any state s and sentences ϕ, ψ: if $\phi \wedge \psi$ is coherent with s then $\psi \wedge \phi$ is coherent with s. However, there are states s and sentences ϕ, ψ such that: $\phi \wedge \psi$ is consistent with s but $\psi \wedge \phi$ is inconsistent with s.

Coherent sentences express information that can hang together all at once. Conjunctions that are consistent but not coherent don't do that: the downstream conjunct destroys the information needed for the upstream conjunct to make its contribution. As a result, such conjunctions don't commute and updates with such sentences aren't (in the jargon) idempotent.

Definition 9 (Idempotence properties)**.** For any ϕ:

i. ϕ is idempotent iff for any state s: if $s[\phi] \neq \emptyset$ then $s[\phi][\phi] = s[\phi]$.

ii. ϕ is anti-idempotent iff for any state s: if $s[\phi] \neq \emptyset$ then $s[\phi][\phi] \neq s[\phi]$.

Idempotence properties are equally success properties: ϕ is idempotent iff updating with it is always successful, landing you in a state in which ϕ is true: $s[\phi][\phi] = s[\phi]$ iff $s[\phi] \models \phi$. And ϕ is anti-idempotent iff

Figure 2 Idempotence (left) and anti-idempotence (right)

updating with it is anti-successful, landing you in a state in which ϕ isn't true: $s[\phi][\phi] \neq s[\phi]$ iff $s[\phi] \not\models \phi$. This is all in Figure 2.[25]

You might suspect that coherence goes hand in hand with idempotence and thus lack of coherence with lack of idempotence. That is not quite right. Anti-idempotence is logically stronger than non-idempotence: the anti-idempotent sentences are (always) destructive. It is this destruction that is the mark of non-coherence.

Fact 9. For any sentence ϕ: ϕ is not coherent iff ϕ is anti-idempotent.

Take conjunctions like $\Diamond p \wedge \neg p$ as handy examples of the sorts of sentences that mark the boundary where consistency and coherence come apart. It is consistent, but not coherent. So it's anti-idempotent (anti-successful) and its reverse-order commutation is not consistent.

Here, then, is a hypothesis. Epistemic possibility is a language's mechanism of expressing epistemic compatibility. But there are two distinct sorts of compatibility. It would therefore be unsurprising if natural language didn't find a way of expressing both sorts. So I want to explore what mileage can be got out of an alternative *maybe* that takes the coherence-route to compatibility.

For now let's simply add an operator $\Diamond_d$ to our language (d because the interpretation has data semantics roots). For comparison, we will want to keep this separate from the update semantics $\Diamond$, which we will now for clarity write $\Diamond_u$. Officially, we have to change the language just a bit.

25 A $[\phi]$-labeled arrow from state s to t represents that $s[\phi] = t$; double-circled states represent fixed-points, so that a $[\phi]$–labeled arrow from s to (double-circled) s represents that $s[\phi] = s$ (which is to say that $s \models \phi$).

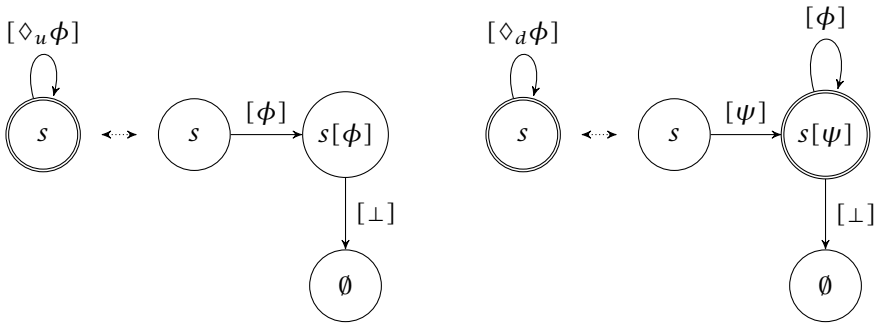


Figure 3 Supporting *maybe*: $\Diamond_u\phi$ (left) and $\Diamond_d\phi$ (right)

Definition 10. For any sentence ϕ of L_1 let $u(\phi)$ be the result of replacing every occurrence of $\Diamond$ in ϕ (if any) with $\Diamond_u$ and for any set of sentences $A \subseteq L_1$ let $u(A) = \{u(\phi)\colon\ \phi \in A\}$. For any sentence ϕ of L_1 let $d(\phi)$ be the result of replacing every occurrence of $\Diamond$ in ϕ (if any) with $\Diamond_d$ and for any set of sentences $A \subseteq L_1$ let $d(A) = \{d(\phi)\colon\ \phi \in A\}$. Finally, let $L_2 = u(L_1) \cup d(L_1)$.

To interpret the new *maybe*: add an additional clause for updating a state with $[\Diamond_d\phi]$, re-producing the clause for $[\Diamond_u\phi]$.

Definition 11. Amend Definition 5 by adding clauses as follows:

iv. $s[\Diamond_u\phi] = \{w \in s\colon\ s[\phi] \neq \emptyset\}$

iv′. $s[\Diamond_d\phi] = \{w \in s\colon\ s'[\phi] = s' \text{ for some } s' \sqsubseteq s\}$

Both modals are compatibility tests, one a consistency test and one a (stronger) coherence test.[26] Figure 3 illustrates the structural difference this makes.[27]

The difference between these two *maybe*s makes no difference if their prejacent is non-modal.

Fact 10. For any descriptive ϕ, ψ and any state s: $s[\Diamond_u\phi] = s[\Diamond_d\phi]$. Thus $\neg(\phi \rightarrow \psi) \cong \Diamond_d(\phi \wedge \neg\psi)$.

26 A suggestive but somewhat misleading alternative name for $\Diamond_d$ is substate possibility. That makes it seem like it is the explicit quantification over successor states (i.e., substates) that makes a difference, but that isn't right since the consistency testing $\Diamond_u$ is also equivalent to a substate formulation: $s[\Diamond_u\phi] = \{w \in s\colon\ s[\phi] = s' \text{ for some } s' \sqsubseteq s\}$.

27 To see what's going on notice two things. First: $s[\phi]$ is consistent iff $s[\phi] \not\models \bot$, and so iff updating $s[\phi]$ with $\bot$ moves you from $s[\phi]$ to $\emptyset$. Second: our language is plentiful in that whenever s' is a non-trivial substate of s, there is a sentence ψ such that $s[\psi] = s'$.

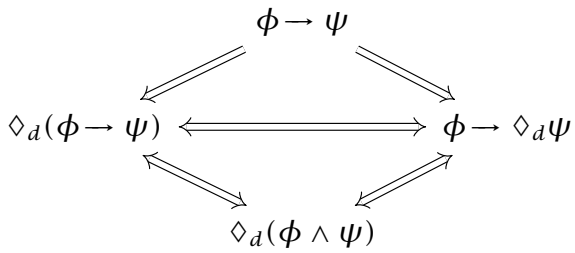


Figure 4 Conditional ignorance, again

But checking for some supporting successor state (as in data semantics) solves the puzzle about iterated ignorance in conditional constructions.

Fact 11. For descriptive ϕ, ψ:

i. $\phi \rightarrow \Diamond_d\psi \cong \Diamond_d(\phi \wedge \psi) \cong \Diamond_d(\phi \rightarrow \psi)$.

ii. $\Diamond_d(\phi \rightarrow \psi) \ncong \phi \rightarrow \psi$.

There is still omniscience about how ignorance can get resolved, but coherence-testing allows for meaningful (non-higher-order) iterated ignorance.[28] And that predicts the pattern of entailments and non-entailments we wanted. This is recorded in Figure 4.

I'm not arguing that we throw out consistency-testing $\Diamond_u$ in favor of cohenrence-testing $\Diamond_d$. I am arguing that we should explore what it might do for us. A solution to the scopelessness puzzle is a start.

8 One surprising upshot

But wait, there's more. Substate support is a higher bar for compatibility than consistency. It would be nice if there were independent reasons for thinking this higher bar of compatibility gets exploited elsewhere by our modal talk.

28 In working through the proofs you will see that $\Diamond_d\Box\phi \models \Diamond\phi$ (as in data semantics) and that $\Box\Diamond_d\phi \not\models \Box\phi$ and $\Box\Diamond_d\phi \models \Diamond_d\phi$ (as in update semantics). While the earlier pragmatic explanation still applies for stacked modals — get your point across in a simpler, non-iterated way — that does not apply to *if*s with *maybe*s (the things they are equivalent to aren't appreciably simpler).

Since we know when consistency and compatibility come apart we know one place to look.

(15) a. It might be raining. It isn't raining.
$\Diamond p \wedge \neg p$
b. ??Maybe./Maybe it might be raining and isn't raining.
$\Diamond(\Diamond p \wedge \neg p)$

There are non-trivial states which do not crash on updating with (15a) even though no states support it. Order of course matters: no non-trivial state can be successfully updated with $\neg p \wedge \Diamond p$.[29] What matters for us isn't whether (15a) should be predicted to be consistent. The issue is: given that it is consistent in a system like update semantics, what should we say about (15b)?

And here's the thing: if the *maybe*s are the consistency-testing $\Diamond_u$ then the iterated (15b) must be true in any state witnessing the consistency of (15a). But that is not so if the *maybe*s are the coherence-testing $\Diamond_d$.

Fact 12. For any state s and descriptive ϕ:

i. If $\Diamond_u \phi \wedge \neg \phi$ is consistent with s then $s \models \Diamond_u(\Diamond_u \phi \wedge \neg \phi)$.

ii. For no (non-absurd) state s is it the case that $s \models \Diamond_d(\Diamond_d \phi \wedge \neg \phi)$.

The reason is simply that $\Diamond_d$ requires that its prejacent is coherent and $\Diamond \phi \wedge \neg \phi$ is not.

This is, I think, interesting. But I grant that there is not much empirical motivation here: iterated things like (15b) are just too wonkily expressed.

Still, the structural features at work here do have an empirical upshot.

(16) a. #If it isn't raining but it might be raining, then the picnic is on.
$(\neg \phi \wedge \Diamond \phi) \rightarrow \psi$
b. #If it might be raining and it isn't raining, then the picnic is on.
$(\Diamond \phi \wedge \neg \phi) \rightarrow \psi$

Both of these are terrible. So the terribleness doesn't seem to care about the order of the conjuncts in the antecedents. (And the things that might be said to help us hear the order sensitivity between the consistent but not

29 Sometimes this asymmetry is taken to be problematic for updates semantics, but there are plenty of things to be said in defense. (See, for instance, the discussion in Dorr & Hawthorne 2013.)

coherent $\Diamond\phi \wedge \neg\phi$ and the inconsistent $\neg\phi \wedge \Diamond\phi$ — for instance: focus only on monotonic information change between conjuncts! — don't help here.) This is not great news for the simple update semantics above: with just the assumption that *if*s presuppose the compatibility of their antecedents, our original update semantics predicts the terribleness of (16a) but not the terribleness of (16b).[30] And the reason is exactly the reason operative in Fact 12: $\neg\phi \wedge \Diamond_u\phi$ is inconsistent and so $\Diamond_u(\neg\phi \wedge \Diamond_u\phi)$, which is presupposed by (16a), is bound to crash any state. So (16a) is terrible. But $\Diamond_u(\Diamond_u\phi \wedge \neg\phi)$, which is presupposed by (16b), is true in any state witnessing the consistency of $\Diamond_u\phi \wedge \neg\phi$. So the prediction is that it's possible for (16b) to be true, which is the opposite of predicting terribleness.

Taking the possibility claims at stake to be coherence-testing *maybe*s, on the other hand, predicts symmetric terribleness between (16a) and (16b). The case for (16a): $\neg\phi \wedge \Diamond_d\phi$ is inconsistent and so $\Diamond_d(\neg\phi \wedge \Diamond_d\phi)$ is too, and so (16a) presupposes something that will wreck any context. No wonder it's terrible. The case for (16b), reflecting what is in Fact 12, goes the same way: even though $\Diamond_d\phi \wedge \neg\phi$ is consistent, it is not coherent and so the (16b)'s presupposition $\Diamond_d(\Diamond_d\phi \wedge \neg\phi)$ is not consistent. The two conditionals pattern alike not because they have inconsistent antecedents (though one does) but because they both presuppose something inconsistent.[31]

9 Another upshot

But wait, there's still more. Two (not at all related) properties of epistemic possibility are on a collision course. One common test in the standard battery for whether ϕ presupposes π is to see whether the presupposition π projects when ϕ is embedded under expressions of epistemic possibility.[32] Adding the

30 See Yalcin 2007.

31 Of course what can be said for conditional antecedents can be said for various attitudes. Take, for instance, supposition.

(i) a. #Suppose it isn't raining but it might be raining. Supp($\neg\phi \wedge \Diamond\phi$)
b. #Suppose it might be raining but it isn't raining. Supp($\Diamond\phi \wedge \neg\phi$)

Grant me two assumptions: (i) that Supp presupposes that its compliment may be; and (ii) that the *maybe* involved is the coherence-testing $\Diamond_d$. Even though the complements to Supp are different (one consistent, one not), the symmetric terribleness gets explained since the suppositions both presuppose something inconsistent.

32 Example:

standard dynamic presupposition operator into our set-up, true to form, predicts this projection behavior.[33] But this does not play nicely with epistemic counterparts of free choice entailments, that (surprisingly!) disjunctions of epistemic possibility claims entail both disjuncts. The coherence-testing $\Diamond_d$ provides a clean solution, predicting weaker projection behavior that is compatible with the free choice entailments. That is the sketch. I owe an explanation of the parts and how they fit together.

Start with free choice entailments.[34] Epistemic possibility interacting with disjunction gives rise to some surprising behavior.[35]

(17) a. Maybe Alex is in Chicago or in LA.
b. Maybe Alex is in Chicago or maybe Alex is in LA.
c. Alex is in Chicago or Alex is in LA.

(18) a. Maybe Alex is in Chicago.

(i) a. Sophie might realize there is no more ice cream.
b. Sophie realizes there is no more ice cream.
c. There is no more ice cream.

Since (ia) entails (ic) this is evidence that the non-modal (ib) presupposes (ic).

33 That is, as we will see, the mixing of the presupposition operator ∂ with $\Diamond_u$. The presupposition operator in dynamic semantics first appears in Beaver 1992.

34 It is, in fact, a matter of debate whether these are entailments or implicatures, where 'implicatures' is understood in the neo-gricean way in which implicatures arise from conventionally encoded mechanisms such as exhaustivity operators in logical forms rather than post-semantic rational reconstruction between cooperative speakers (see, for instance, Fox 2007). In fact I find pro-entailment arguments, for example those in Barker 2010, convincing. Other entailment-accounts of (epistemic) free choice include Kamp 1973, Aloni 2007, Zimmerman 2000, Geurts 2005. It is enough for present purposes that free choice as entailment is (surprisingly!) at odds with the standard projection behavior of presuppositions under epistemic possibility.

35 There are counterparts for deontic *may*:

(i) a. Alex may go to Chicago or LA.
b. Alex may go to Chicago or may go to LA.
c. Alex will go to Chicago or Alex will go to LA.

(ii) a. Alex may go to Chicago.
b. Alex may go to LA.
c. Alex may go to Chicago and Alex may go to LA.

As with epistemic *maybe*: (ia) entails both (iia) and (iib) (though, and what sets the deontic version apart from the epistemic version, it doesn't entail (iic)). Same goes for the widescope disjunction (ib): it entails its disjuncts. Finally: the non-modal (ic) doesn't entail its disjuncts, showing that the *and*-reading of *or* isn't generally available.

b. Maybe Alex is in LA.

First observation: (surprisingly!) the narrowscope (17a) seems to entail both (18a) and (18b). Second observation: the widescope disjunction (17b) also (surprisingly!) entails both disjuncts. It seems that *or* is behaving in very *and*-like ways. That makes it tempting to conclude that *or* can express conjunction. Third observation: (17c) does not entail its disjuncts. The unavailability of an *and*-like reading without a nearby modal suggests that the explanation, whatever it is, won't be simple.[36]

To summarize: the behavior is robust. The contours of free choice entailment can be thought of as a constraint on the entailment relation.

Definition 12. A possibility operator $\Diamond$ satisfies free choice iff $\Diamond\phi \vee \Diamond\psi \models\!\!\!| \Diamond\phi$ and $\Diamond\phi \vee \Diamond\psi \models\!\!\!| \Diamond\psi$.

This constrains but doesn't determine what we can say about what *maybe* and *or* mean.

Now to the the projection constraint. Let's write $\phi \gg \pi$ to indicate (in the meta-language) that ϕ presupposes π. The constraint is usually that expressions of epistemic possibility are a hole to presuppositions: if $\phi \gg \pi$ then $\Diamond\phi \gg \pi$. We will use something weaker.

Definition 13. A possibility operator $\Diamond$ satisfies strong projection iff if $\phi \gg \pi$ then for every s: if $s \models\!\!\!| \Diamond\phi$ then $s \models\!\!\!| \pi$.

Strong projection falls out as a for-free prediction of combining the consistency-testing $\Diamond_u$ with a dynamic presupposition operator ∂.

Definition 14. Extend our language L_2 by adding a presupposition operator: if ϕ is in L_2 then so is $\partial\phi$. And add an update clause as follows:

v. $s[\partial\phi] = s$ if $s[\phi] = s$ and undefined otherwise

36 I am not going to solve this problem. But I am going to show how its presence causes unexpected trouble elsewhere and how the coherence-testing *maybe* cleanly avoids it. In what follows, I won't be giving a semantics of *or* or *maybe* (or anything else) which delivers the free choice entailments. Instead, I will be arguing for a way to open up space for such a semantics by showing how we can weaken projection behavior under *maybe* so that the two phenomena are compatible.

If π is a basic presupposition of ϕ then to interpret ϕ in s, update s with $\partial\pi \wedge \phi$.[37]

This is enough to derive strong projection.

Fact 13. If $\phi \gg \pi$ then $s \models\!\!\!\mid \Diamond_u \phi$ only if $s \models\!\!\!\mid \pi$.

Here's the gist of the proof: for $\Diamond_u \phi$ to be true in s, $\Diamond_u(\partial\pi \wedge \phi)$ must be true in s. And so, since $\Diamond_u$ is the consistency-testing *maybe*, applying $[\partial\pi \wedge \phi]$ to s must land us in some non-absurd state, in which case $s[\partial\pi]$ must be defined and hence π true in s.

The bad news is that strong projection is not compatible with the free choice behavior of *maybe*. Together they imply that true and felicitous utterances of the form $\Diamond\phi \vee \Diamond\psi$ where $\phi \gg \pi_1$ and $\psi \gg \pi_2$ for incompatible π_1 and π_2 can only occur in absurd contexts. But there are such utterances in non-absurd contexts.

An example: we know that Alex is looking for her keys, but we only know that she has it narrowed down to them being in her office or in her car. She gets up, and bolts out.

(19) Maybe Alex realizes her keys are in her office or maybe she realizes they are in her car. $\Diamond\phi \vee \Diamond\psi$ $(\phi \gg \pi_1, \psi \gg \pi_2)$

This can be a true and felicitous thing to say. But our two principles together entail (on pain of triviality) that (19) should be unusable.

Fact 14. If $\phi \gg \pi_1$ and $\psi \gg \pi_2$ and π_1 and π_2 are not compatible then $s \models\!\!\!\mid \Diamond\phi \vee \Diamond\psi$ only if $s \models\!\!\!\mid \bot$.

The key assumptions in proving this are the free choice entailments and strong projection. Since things like (19) aren't useable only in absurd contexts, something's got to give.

Enter coherence-testing *maybe*. Pairing it with the presupposition operator makes room for projection behavior which is, in a precise sense, weaker

37 Since $[\cdot]$ is now a partial function on information states, officially we have to understand the test clauses this way:

$$s[\Diamond_u\phi] = \begin{cases} s & \text{iff } s[\phi] = t \neq \emptyset \\ \emptyset & \text{iff } s[\phi] = \emptyset \end{cases}$$

And similarly for $\Diamond_d$. If $[\phi]$ is everywhere defined, this is a purely stylistic difference.

than strong projection and compatible with the free choice entailment pattern:

Fact 15. If $\phi \gg \pi$ then $s \models \Diamond_d \phi$ only if $s \models \Diamond_d \pi$ (and not, in general, only if $s \models \pi$).

The reason: for a state s to be a fixed-point of the update with $\Diamond_d(\partial\pi \wedge \phi)$ some (non-absurd) substate of s must support π. That guarantees that s must support $\Diamond_d \pi$. However, since this can happen without s itself supporting π, the truth of ϕ at s doesn't require the truth of π at s.

This is enough to show that Fact 14 doesn't hold for $\Diamond_d$: even assuming the free choice entailments as a constraint, what (19) requires is that the context supports the possibility that Alex's keys are in her office and the possibility that they are in her car. That package is something a non-absurd context can do.

The derived weak presupposition here makes predictions beyond cases that implicates the free choice behavior of *maybe*: we would expect there to be situations in which $\phi \gg \pi$ and (apparently) $\Diamond\phi \not\gg \pi$ but instead $\Diamond\phi \gg \Diamond\pi$. And in fact that's the way things are. Some examples (based on some in Fauconnier 1985 and Kay 1992):

(20) a. Alex: Hey, why's that guy so glum?
b. Billy: Maybe his partner left him.

(21) a. Alex: Why's that guy chewing so much gum?
b. Billy: Maybe he quit smoking.

The observation is that Billy's replies here don't seem to presuppose that the guy had a partner or that he did smoke. Just that those are possibilities.

10 How many maybes?

We have seen there is value to a coherence-testing *maybe*. Now it's time to face up to some nagging questions. Is this a better analysis of *maybe*? Is *maybe* ambiguous? If so, what sort of ambiguity?

I don't think coherence-testing *maybe*, for all it's pluses, is the only *maybe* in town. I will give three reasons.

One reason: the motivating idea is that epistemic compatibility is ambiguous and so it would be natural if modals that express compatibility were similarly ambiguous. Another reason (from a suitably high altitude) is struc-

tural: one of the jobs that *maybe* has is maintaining a healthy relationship with *must* and one of the jobs *must* has is maintaining a healthy relationship with *if*. These relationships can't be guaranteed without a consistency-testing *maybe*. We will shortly get a view of that from a lower-altitude vantage point. The third reason: the dual of coherence-testing *maybe* is not what *must* means. This, as we will see, is related to the second reason.

Let's look at the second reason. We have been assuming that $\Box$, the operator expressions of epistemic necessity give voice to, has a meaning that itself can be expressed in terms of $\rightarrow$ plus $\top$: $\Box\phi$ just in case ϕ, conditional on the trivial supposition (and so conditional on every supposition).[38] We have taken this to be definitional, but it is what you would expect and want if the conditional is a strict conditional.

Definition 15. A conditional $\rightarrow$ is a strict conditional with respect to a (normal) necessity operator $\Box$ iff $\phi \rightarrow \psi \cong \Box(\phi \supset \psi)$.

Fact 16. If $\rightarrow$ is a strict conditional then $\Box\phi \cong \top \rightarrow \phi$.

This is a robust interdefinability constraint: we can take either $\rightarrow$ or $\Box$ as basic and introduce the other in terms of it. Let's continue to assume that the indicative conditional is a strict conditional and thus that preserving this connection between $\rightarrow$ and $\Box$ is not up for grabs. We will see that in an update semantics framework, and in the presence of some simple and pleasing properties, this jointly constrains the space of options for possible meanings that $\rightarrow$ and $\Box$ can have. Start with $\Box$:

Definition 16. $\Box$ is introspective iff for any state s and sentence ϕ:

i. If $s \models \phi$ then $s \models \Box\phi$; and

ii. If $s \not\models \phi$ then $s \models \neg\Box\phi$.

This embodies a kind of equilibrium: the information in a state is partial, but what is and isn't supported by that partial information is settled and that in turn determines what is and isn't entailed by that partial information.[39] A $\Box$ without these properties can't really lay claim to being what strong epistemic necessity modals in natural language express.

38 Modulo *must*'s evidential bit since that dimension isn't relevant for what we are up to here.

39 Introspectiveness of $\Box$ is part of what it takes for a belief state to be stable in an autoepistemic theory (Moore 1985, Stalnaker 1993).

Introspectiveness is simple and intuitive and, it turns out, powerful.

Fact 17. Let $[\cdot]$ be any candidate update function for our language and $\models$ its corresponding fixed-point supports/true-in relation. Assume that $\rightarrow$ is a strict conditional with respect to $\Box$. Then the following are equivalent:

i. $\Box$ is introspective;

ii. $s[\phi \rightarrow \psi] = \{w \in s\colon\ s[\phi][\psi] = s[\phi]\}$;

iii. $s[\Box\phi] = \{w \in s\colon\ s[\phi] = s\}$.

Thus the (Ramsey) test profile and the fixed-point test for *must* are completely characterized by interdefinability and introspectiveness.

The kind of stability that introspectiveness gets at can be got at in other ways with the same result.

Definition 17. A conditional connective $\rightarrow$ satisfies free deduction iff for any s: $s[\phi] \models \psi$ iff $s \models \phi \rightarrow \psi$.

Free deduction is nothing more than the semantic counterpart to conditional proof plus modus ponens. A good thing. You can verify that the dynamic Ramsey test conditional in update semantics satisfies it.[40] In the current set-up where $\rightarrow$ is a test, this completely characterizes the Ramsey test fixed-point conditional.

Fact 18. Let $[\cdot]$ be any candidate update function for our language and $\models$ its corresponding fixed-point supports/true-in relation. Then $\rightarrow$ is a test operator that satisfies free deduction iff $s[\phi \rightarrow \psi] = \{w \in s\colon\ s[\phi][\psi] = s[\phi]\}$. As a corollary: given interdefinability, iff $\Box$ is the fixed-point test modal.

Part of why this is true: $\bigcirc$ is a test operator iff it is bivalent. That is: $[\bigcirc\phi]$ is a test iff for every state s either $s \models \bigcirc\phi$ or $s \models \neg \bigcirc \phi$.[41] So the test-behavior and bivalence of $\rightarrow$ stand and fall together.

40 Data semantics almost does, too. You have to replace the modus ponens direction of free deduction with this: if $\Gamma \models \phi \rightarrow \psi$ then $\Gamma, \phi \models \Box\psi$.

41 *Proof*: Left-to-right direction: clearly $s \models \bigcirc\phi$ if $s[\bigcirc\phi] = s$. So suppose $s[\bigcirc\phi] \neq s$. Since the function $[\bigcirc\phi]$ is a test, it follows that $s[\bigcirc\phi] = \emptyset$ and so $s[\neg \bigcirc \phi] = s \setminus \emptyset = s$ and so $s \models \neg \bigcirc \phi$. The right-to-left direction: if $s \models \bigcirc\phi$ then $s[\bigcirc\phi] = s$. And if $s \models \neg \bigcirc \phi$ then $s[\neg \bigcirc \phi] = s$ and so $s \setminus s[\bigcirc\phi] = s$ and so $s[\bigcirc\phi] = \emptyset$.

Bivalence in general is a bad idea in the presence of partial information, but it is a good idea for modals and conditionals whose point is to say how different resolutions of that ignorance can and can't interact. In its presence, free deduction amounts to conditional introspectivness: in a state s, hypothetically adding ϕ will either resolve our ignorance in favor of ψ or it won't. If it does, the conditional $\phi \rightarrow \psi$ is supported. If it doesn't, then the counterexample $\phi \wedge \neg\psi$ is still possible and so the negation of the conditional $\neg(\phi \rightarrow \psi)$ is supported.[42]

Intermediate upshot: in this framework, any non-Ramsey test conditional either doesn't have a robust and healthy relationship with *must* or the implicated *must* isn't introspective.[43]

Now to the third reason: what about a substate-based $\Box$? There are two non-equivalent options — write them $\Box_1$ and $\Box_2$ — and neither does what the fixed-point $\Box$ does.

Definition 18. For any state s and sentence ϕ:

i. $s[\Box_1\phi] = \{w \in s\colon s'[\phi] = s' \text{ for every } s' \sqsubseteq s\}$

ii. $s[\Box_2\phi] = \{w \in s\colon s'[\neg\phi] = s' \text{ for no } s' \sqsubseteq s\}$

Since these aren't equivalent to the substate $\Box$ they can't be introspective (Fact 17). And their associated strict conditionals can't be the Ramsey test conditional and hence can't satisfy free deduction (Fact 18 and footnote 42). They have other disqualifying properties as well.

42 Another characterization in this neighborhood: in the presence of introspectiveness for $\Box$, $\rightarrow$ satisfies

i. LTR deduction: if $s[\phi] \models \psi$ then $s \models \phi \rightarrow \psi$; and

ii. Lower boundedness: if $s \models \phi \rightarrow \psi$ then $s \models \neg(\phi \wedge \neg\psi)$

iff $\rightarrow$ is a strict conditional with respect to $\Box$ ($\phi \rightarrow \psi \cong \Box(\phi \supset \psi)$). Thus any non-introspective $\Box$ will have an associated strict conditional that is either not bounded from below by $\supset$ or doesn't go in for LTR deduction.

43 Take, for example, the conditional defended in Russell & Hawthorne 2016. It is almost test-like: applying $[\phi \rightarrow \psi]$ to s returns s if $s[\phi] \models \psi$, returns $\emptyset$ if $s[\phi] \models \neg\psi$, and returns $s[\phi \supset \psi]$ otherwise. Such a profile for $\rightarrow$ is, I think, approximately two-thirds correct. One cost of its less than full-throated commitment to the dynamic strict conditional is that (assuming $\Box\phi \cong \top \rightarrow \phi$) from $s \not\models \phi$ it does not follow that $s \models \neg\Box\phi$. Countermodel: let $s = \{w_1, w_2\}$ where $w_1(p) = 1$ and $w_2(p) = 0$. Where $\Box$ is derived from this $\rightarrow$, such a state does not support $\neg\Box p$ and (assuming that $\Diamond$ is dual to this $\rightarrow$-derived $\Box$) does not support $\Diamond\neg p$. These are not the predictions I want.

Fact 19. In general: $\Box_1\phi \not\cong \neg\Diamond_d\neg\phi$. While it does hold that $\Box_2\phi \cong \neg\Diamond_d\neg\phi$, in general: $\Box_2\phi \not\models \phi$.

In the proof you'll see that there are sentences like $\Box_2\neg(\Diamond_d p \wedge \neg p)$ that are (surprisingly!) supported in every state even though the prejacent $\neg(\Diamond_d p \wedge \neg p)$ isn't. Summing up: in addition to not satisfying introspection and thus its associated strict conditional not satisfying the good-making features of the Ramsey test conditional, $\Box_1$ is not a suitable because it is not the dual of $\Diamond_d$ and $\Box_2$ is not suitable because it is not factive.

The final upshot for *maybe*: what goes for $\Box$ goes for its dual and so we need, want, and (of necessity) have a consistency-testing *maybe*. That plus our earlier discussion leaves us sitting under a looming ambiguity: does *maybe* express $\Diamond_u$ or $\Diamond_d$? That is not a great place, theoretically, to sit.

But there is ambiguity and then there is ambiguity. Since the consistency-testing $\Diamond_u$ serves us so well, and since the possibility of its marginally stronger data semantic inspired counterpart $\Diamond_d$ only emerges in embedded environments of various sorts, it is natural to hope that $\Diamond_d$ is expressible in terms of $\Diamond_u$ and so natural to hope that the sort of ambiguity might be structural rather than lexical. Happily, that is the case.

Fact 20. To our language L_2 add an operator A and intepret it in this way:

vi. $s[\mathsf{A}\,\phi] = \{w \in s' \colon s'[\phi] = s' \text{ for some } s' \sqsubseteq s\}$

Then $\Diamond_d\phi \cong \Diamond_u \mathsf{A}\,\phi$.

The A operator is a bit like a meta-assertion operator.[44] It offers up the information its prejacent carries, and in addition it says that its prejacent is a coherent thing: it is a thing you can, in principle, get behind. Intuitively speaking, that is what bridges the gap between a consistency test and a coherence test. That means that the kind of ambiguity *maybe* may force on us is a resource conscious ambiguity: the strong meaning is derivable on-demand from the weak meaning. So a language equipped with a silent A operator that wants a $\Diamond_u$ would look a lot like our actual language does.

44 There is a similar proposal in Beaver 2001: §2.2, where a meta-assertion operator is used to add flexibility to trivalent accounts of presupposition projection. Don't lean to heavily on the name, though: this is a posited covert modality that is assigned a semantic value that has the same type as the other operators in the language.

11 Summing up

We got started with an unexpected way that our modal and conditional language interact: *maybe* and *if* iterate and when they do they seem to be scopeless with respect to each other. This is hard to predict if *maybe* expresses compatibility and *if* is a strict conditional. But there are two routes to compatibility. And while those two routes collapse in a classical, static set-up they can come apart in update semantics. What work would an alternative coherence-compatibility possibility operator do for us? A surprising amount it turns out. That is evidence for the existence of the stronger meaning of *maybe*. The structural dependencies between *maybe-must-if*, and how the coherence-testing *maybe* would disrupt them, is evidence for the weaker *maybe*. But things are not as bad as they maybe could be: the stronger meaning is expressible in terms of the weaker, leaving open that the structural ambiguity generally gets resolved in favor of the weaker *maybe* unless the embedding environment calls for its stronger counterpart.*

Appendix: Under the hood

Here are the proofs of the results reported in the main body of the paper. Some are interesting!

Fact 1. Let $\phi \supset \psi$ abbreviate $\neg(\phi \wedge \neg\psi)$ and let $\Box\phi$ abbreviate $\top \rightarrow \phi$ (where $\top =_{\text{df}} \neg\bot$). Then:

i. $[\![\phi \supset \psi]\!]^s = 1$ iff $[\![\phi]\!]^s = 0$ or $[\![\psi]\!]^s = 1$ and
$[\![\phi \supset \psi]\!]^s = 0$ iff $[\![\phi]\!]^s = 1$ and $[\![\psi]\!]^s = 0$.

ii. $[\![\Box\phi]\!]^s = 1$ iff $[\![\phi]\!]^{s'} = 0$ for no s' such that $s \leq s'$ and
$[\![\Box\phi]\!]^s = 0$ iff $[\![\phi]\!]^{s'} = 1$ for some s' such that $s \leq s'$.

Proof. Since this is a straightforward application of Definition 3, here are the details for half of cases (i) and (ii).

* This paper started life as a comment in a bar in Amsterdam (2006), and grew into a remark on a plane to Nebraska (2010), and then a note on a train from Osnabrück (2015) and now finally into a paper. Along the way it has had the good fortune of getting help from Nicholas Asher, Sam Carter, Kai von Fintel, (especially) Simon Goldstein, Jeff King, Ernie Lepore, Frank Veltman, Malte Willer, and two anonymous reviewers and an associate editor at *Mind*.

(i): consider any state s and note that

$$\begin{aligned}[\![\phi \supset \psi]\!]^s = 1 \text{ iff } &[\![\neg(\phi \wedge \neg\psi)]\!]^s = 1\\ &\text{iff } [\![(\phi \wedge \neg\psi)]\!]^s = 0\\ &\text{iff } [\![\phi]\!]^s = 0 \text{ or } [\![\neg\psi]\!]^s = 0.\end{aligned}$$

Since $[\![\neg\psi]\!]^s = 0$ iff $[\![\psi]\!]^s = 1$ and $[\![\phi]\!]^s = 0$ iff $[\![\neg\phi]\!]^s = 1$, it follows that $[\![\phi \supset \psi]\!]^s = 1$ iff $[\![\phi]\!]^s = 0$ or $[\![\psi]\!]^s = 1$, as required.

(ii): consider any state s and note that

$$\begin{aligned}[\![\Box\phi]\!]^s = 0 \text{ iff } &[\![\top \rightarrow \phi]\!]^s = 0\\ &\text{iff } [\![\top]\!]^{s'} = 1 \text{ and } [\![\phi]\!]^{s'} = 0 \text{ (some } s' \text{ s.t. } s \leq s')\\ &\text{iff } [\![\phi]\!]^{s'} = 0 \text{ (some } s' \text{ s.t. } s \leq s').\end{aligned}$$

□

Fact 2. Let $\phi \supset \psi$ abbreviate $\neg(\phi \wedge \neg\psi)$ and let $\Box\phi$ abbreviate $\top \rightarrow \phi$. Then: (i) $s[\phi \supset \psi] = s[\neg\phi] \cup s[\psi]$, and (ii) $s[\Box\phi] = \{w \in s\colon s[\phi] = s\}$.

Proof. Again, this follows directly from Definition 5 and so we will just cover case (ii). Consider any s and note that:

$$\begin{aligned}s[\Box\phi] &= s[\top \rightarrow \phi]\\ &= \{w \in s\colon s[\top][\phi] = s[\top]\}\\ &= \{w \in s\colon s[\phi] = s\}\end{aligned}$$

□

Fact 3. For any ϕ, ψ:

i. $\phi \rightarrow \psi \cong_{\text{DS}} \Box(\phi \supset \psi)$ and $\phi \rightarrow \psi \cong_{\text{US}} \Box(\phi \supset \psi)$.

ii. $\neg(\phi \rightarrow \psi) \cong_{\text{DS}} \Diamond(\phi \wedge \neg\psi)$ and $\neg(\phi \rightarrow \psi) \cong_{\text{US}} \Diamond(\phi \wedge \neg\psi)$.

iii. $\Box\phi \cong_{\text{DS}} \neg\Diamond\neg\phi$ and $\Box\phi \cong_{\text{US}} \neg\Diamond\neg\phi$.

Proof. Here we just cover (parts of) case i.

Consider any data semantic information state s and suppose $[\![\phi \rightarrow \psi]\!]^s = 1$. Take any s' such that $s \leq s'$. Since $[\![\phi \rightarrow \psi]\!]^s = 1$ it follows that if $[\![\phi]\!]^{s'} = 1$ then $[\![\psi]\!]^{s'} = 1$ and so $[\![\phi \supset \psi]\!]^{s'} = 1$ and hence $[\![\Box(\phi \supset \psi)]\!]^s = 1$. And so $\phi \rightarrow \psi \models\!\!\models_{\text{DS}} \Box(\phi \supset \psi)$. The other direction and the other pair of entailments are similar.

Consider any update semantic information state s. Note that $s[\phi \rightarrow \psi] = s$ if $s[\phi][\psi] = s[\phi]$ and $s[\phi \rightarrow \psi] = \emptyset$ otherwise. We need to see that $s[\Box(\phi \supset \psi)] = s$ if $s[\phi][\psi] = s[\phi]$ and $s[\Box(\phi \supset \psi)] = \emptyset$ otherwise. So suppose $s[\phi][\psi] = s[\phi]$. Note that $s[\phi] = s$ iff $s[\Box\phi] = s$. Now:

$$\begin{aligned} s[\phi][\psi] = s[\phi] \text{ iff } & s[\phi] \setminus s[\phi][\psi] = \emptyset \\ \text{iff } & s[\phi][\neg\psi] = \emptyset \\ \text{iff } & s[\phi \wedge \neg\psi] = \emptyset \\ \text{iff } & (s \setminus s[\phi \wedge \neg\psi]) = s \\ \text{iff } & s[\neg(\phi \wedge \neg\psi)] = s \\ \text{iff } & s[\phi \supset \psi] = s \\ \text{iff } & s[\Box(\phi \supset \psi)] = s. \end{aligned}$$

Similarly for the other case.

□

Fact 4. While $\phi \rightarrow \psi, \neg\psi \not\models \neg\phi$ it does hold that $\phi \rightarrow \psi, \neg\psi \models \Diamond\neg\phi$.

It is straightforward to turn the examples in Section 5 into countermodels for both data semantics and update semantics. So we focus on showing the weaker property that $\phi \rightarrow \psi, \neg\psi \models \Diamond\neg\phi$. For the case of update semantics it is useful to first have in hand the following simple properties (which, really, any useful way of assigning meanings to expressions of epistemic possibility should validate):

Lemma 1. For any information state s and sentence ϕ:

i. If $s \not\models \phi$ then $s \models \Diamond\neg\phi$.

ii. If $s \models \Diamond\phi$ and $s \neq \emptyset$ then $s \not\models \neg\phi$.

Proof of lemma. (i): suppose $s \not\models \phi$. So $s[\phi] \neq s$. Thus, since $s[\neg\phi] = s \setminus s[\phi]$, it follows that $(s \setminus s[\phi]) \neq \emptyset$ and so $s[\neg\phi] \neq \emptyset$. Hence $s[\Diamond\neg\phi] = s$ and so $s \models \Diamond\neg\phi$.

(ii): Suppose $s \models \Diamond\phi$ and $s \neq \emptyset$. Since $s \models \Diamond\phi$ it follows that $s[\phi] \neq \emptyset$ and so since $s \neq \emptyset$ we have that $(s \setminus s[\phi]) \neq s$. Hence $s[\neg\phi] \neq s$ and so $s \not\models \neg\phi$. □

Now to (the basic outline of) the proof of Fact 4:

Proof. For $\models_{\mathrm{DS}}$: the key part is seeing that if ψ is f-persistent then since in that case for no $s' \geq s$ is it the case that $[\![\phi]\!]^{s'} = 1$ (since ψ will likewise be false at s') and so $[\![\Box\neg\phi]\!]^{s} = 1$. (Note that $\Box\neg\phi \models \Diamond\neg\phi$ but $\Box\neg\phi \not\models \neg\phi$.) However, if ψ isn't f-persistent, then we can only conclude this if s' is such that $[\![\psi]\!]^{s'} = 0$.

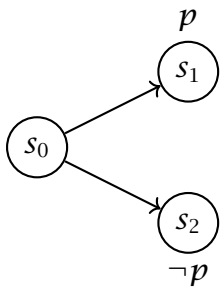
For $\models_{\mathrm{US}}$: let s be any information state and consider $s[\phi \rightarrow \psi][\neg\psi]$ such that $s[\phi \rightarrow \psi][\neg\psi] \neq \emptyset$. Notice that it follows that $s[\phi \rightarrow \psi] = s$ and so, since $s[\neg\psi] \neq \emptyset$, that $s[\neg\psi][\phi] \neq \emptyset$ and hence $s[\neg\psi] \models \Diamond\phi$. Hence by our lemma, $s \not\models \neg\phi$ and so $s[\neg\psi] \models \Diamond\neg\phi$.

□

Fact 6. Let $\bigcirc$ be any string of 0 or more of the 1-place operators $\neg, \Diamond, \Box$ and let ϕ be any persistent sentence. Then $\bigcirc\phi \cong \psi$ where ψ is one of $\phi, \neg\phi, \Diamond\phi, \Box\phi, \neg\Diamond\phi, \neg\Box\phi$.

Proof. Consider any (data semantic) information state and suppose $[\![\Diamond\Box\phi]\!]^{s} = 1$. Hence there is some $s' \geq s$ such that $[\![\Box\phi]\!]^{s'} = 1$. Take any maximal element s'' such that $s'' \geq s'$. It follows that $[\![\phi]\!]^{s''} \neq 0$ and since s'' is maximal that $[\![\phi]\!]^{s''} = 1$. Hence, since $\geq$ is transitive, there is a state $\geq$-reachable from s such that ϕ is true at that state and so $[\![\Diamond\phi]\!]^{s} = 1$. The other direction is similar, as are the other cases.

The assumption that ϕ is persistent is essential (in data semantics). Here is a countermodel (from Veltman 1985), exploiting the fact that $\Box p \vee \Box\neg p$ isn't f-persistent:



Note that $[\![\Box\Diamond(\Box p \vee \Box\neg p)]\!]^{s_0} = 1$ but that $[\![\Box(\Box p \vee \Box\neg p)]\!]^{s_0} = 0$.

Now consider any (update semantic) information state s and notice that:

$$\begin{aligned} s \not\models_{\text{US}} \Diamond\Box\phi \text{ iff } & s[\Diamond\Box\phi] = s \\ \text{iff } & s[\Box\phi] \neq \emptyset \\ \text{iff } & s[\Box\phi] = s \\ \text{iff } & s \not\models \Box\phi \end{aligned}$$

The other cases are similar. (In update semantics the persistence of ϕ is not relevant.) □

Fact 8. For any state s and sentences ϕ, ψ: if $\phi \wedge \psi$ is coherent with s then $\psi \wedge \phi$ is coherent with s. However, there are states s and sentences ϕ, ψ such that: $\phi \wedge \psi$ is consistent with s but $\psi \wedge \phi$ is inconsistent with s.

Proof. Suppose $\phi \wedge \psi$ is coherent with s: so $s'[\phi \wedge \psi] = s'$ for some $s' \sqsubseteq s$. Hence: $s'[\phi][\psi] = s'$. Since $s'[\phi] \subseteq s$ and $s'[\phi][\psi] = s'$, it follows that $s'[\phi] = s'$. Thus:

$$\begin{aligned} s' &= s'[\phi][\psi] \\ &= s'[\psi] \\ &= s'[\psi][\phi] \\ &= s'[\psi \wedge \phi] \end{aligned}$$

And so $\psi \wedge \phi$ is also coherent with respect to s.

To see that consistency doesn't commute with conjunction: consider any (non-absurd) state s such that $s[\phi] \neq \emptyset$ and note that $s[\Diamond\phi \wedge \neg\phi] \neq \emptyset$ but that $s[\neg\phi \wedge \Diamond\phi] = \emptyset$. □

Fact 9. For any sentence ϕ: ϕ is not coherent iff ϕ is anti-idempotent.

Proof. Suppose ϕ is coherent: so there is an s such that $s \neq \emptyset$ and $s[\phi] = s$. Hence $s \not\models \phi$. But since $s[\phi] = s$, we have that $s[\phi] \not\models \phi$ and so $s[\phi][\phi] = s[\phi]$. And hence ϕ isn't anti-idempotent.

Suppose ϕ is not anti-idempotent. Hence there is an s such that $s[\phi] \neq \emptyset$ and $s[\phi] \not\models \phi$. Thus ϕ is coherent. □

Fact 10. For any descriptive ϕ, ψ and any state s: $s[\Diamond_u\phi] = s[\Diamond_d\phi]$. Thus $\neg(\phi \rightarrow \psi) \cong \Diamond_d(\phi \wedge \neg\psi)$.

Proof. Since ϕ is descriptive $s[\phi] = s \cap W[\phi]$ (by a routine induction). So, first, suppose that $s[\phi] = \emptyset$. Hence: $s[\phi] = s \cap W[\phi] = \emptyset$ and so

for no $s' \sqsubseteq s$ is it the case that $s' \cap W[\phi] = s'[\phi] = s'$. In which case: $s[\Diamond_u \phi] = s[\Diamond_d \phi] = \emptyset$. Now suppose that $s[\phi] \neq \emptyset$. Consider $s[\phi]$:

$$\begin{aligned} s[\phi][\phi] &= s[\phi] \cap W[\phi] \\ &= (s \cap W[\phi]) \cap W[\phi] \\ &= s \cap W[\phi] \\ &= s[\phi] \end{aligned}$$

Since $s[\phi] \neq \emptyset$, we have that $s[\Diamond_u \phi] = s$. And since $s[\phi] \sqsubseteq s$ and $s[\phi][\phi] = s[\phi]$, we have that $s[\Diamond_d \phi] = s$. □

Fact 11. For descriptive ϕ, ψ:

i. $\phi \rightarrow \Diamond_d \psi \cong \Diamond_d(\phi \wedge \psi) \cong \Diamond_d(\phi \rightarrow \psi)$.

ii. $\Diamond_d(\phi \rightarrow \psi) \ncong \phi \rightarrow \psi$.

Proof. (i): We need to see that $\phi \rightarrow \Diamond_d \psi \cong \Diamond_d(\phi \wedge \psi)$ and then that $\Diamond_d(\phi \wedge \psi) \cong \Diamond_d(\phi \rightarrow \psi)$. Here we just sketch some of the relevant equivalences.

For $\phi \rightarrow \Diamond_d \psi \cong \Diamond_d(\phi \wedge \psi)$: to see this, suppose $s[\phi] \models \Diamond_d \psi$. Then note that it follows that there is some $s' \sqsubseteq s[\phi]$ such that $s'[\psi] = s'$ and so $s'[\phi \wedge \psi] = s'$. If, on the other hand, $s[\phi] \not\models \Diamond_d \psi$, then $s[\phi] \models \neg \Diamond_d \psi$. And so if $s' \sqsubseteq s[\phi]$ then $s' \not\models \psi$ and hence $s[\Diamond_d(\phi \wedge \psi)] = \emptyset$.

For $\Diamond_d(\phi \rightarrow \psi) \cong \Diamond_d(\phi \wedge \psi)$: suppose $s' \models \phi \rightarrow \psi$ for some $s' \sqsubseteq s$. Here we make substantive use of the assumption that a conditional $\phi \rightarrow \psi$ in a state presupposes that ϕ is compatible with that state to get that $s'[\Diamond_d \phi] = s'$ and $s'[\phi] \models \psi$. From here the reasoning should be familiar. Suppose, on the other hand, that every state $s' \sqsubseteq s$ is such that $s' \not\models \phi \rightarrow \psi$ and suppose $s'[\Diamond_d \phi] = s'$. Pick any witnessing substate s^* is s': it must be that $s^*[\psi] \neq s^*$ else $s^*[\phi \rightarrow \psi] = s^*$. Hence $s[\Diamond_d(\phi \wedge \psi)] = \emptyset$.

(ii): To see that $\Diamond_d(\phi \rightarrow \psi) \ncong \phi \rightarrow \psi$ consider a state s that contains a single counterexampling possibility w: $w(p) = 1$ and $w(q) = 0$ and all other possibilities confirming possibilities. □

Fact 12. For any state s and descriptive ϕ:

i. If $\Diamond_u \phi \wedge \neg \phi$ is consistent with s then $s \not\models \Diamond_u(\Diamond_u \phi \wedge \neg \phi)$.

ii. For no (non-absurd) state s is it the case that $s \not\models \Diamond_d(\Diamond_d \phi \wedge \neg \phi)$.

Proof. This one is easy, y'all! □

Fact 14. If $\phi \gg \pi_1$ and $\psi \gg \pi_2$ and π_1 and π_2 are not compatible then $s \models \Diamond\phi \vee \Diamond\psi$ only if $s \models \bot$.

Proof. Suppose $s \models \Diamond\phi \vee \Diamond\psi$. Hence by free choice $s \models \Diamond\phi$ and $s \models \Diamond\psi$. Since $\phi \gg \pi_1$ and $\psi \gg \pi_2$ by strong projection it follows that $s \models \pi_1$ and $s \models \pi_2$. But π_1 and π_2 are not compatible, and so $s \models \bot$. □

Fact 13. If $\phi \gg \pi$ then $s \models \Diamond_u\phi$ only if $s \models \pi$.

Proof. Suppose $s \models \Diamond_u\phi$. Thus $s[\Diamond_u(\partial\pi \wedge \phi)] = s$. (Since $\phi \gg \pi$ updating s with ϕ requires first updating s with $\partial\pi$.) So $s[\partial\pi \wedge \phi] = s'$ for some $s' \sqsubseteq s$. Hence, $s[\partial\pi] = s$ and so $s \models \pi$. □

Fact 15. If $\phi \gg \pi$ then $s \models \Diamond_d\phi$ only if $s \models \Diamond_d\pi$ (and not, in general, only if $s \models \pi$).

Proof. As in Fact 13, except that $s[\Diamond_d(\partial\pi \wedge \phi)] = s$ and so $s'[\partial\pi \wedge \phi] = s'$ for some $s' \sqsubseteq s$. Hence $s' \models \pi$ and so $s \models \Diamond_d\pi$.

To see that it need not be the case that $s \models \pi$ even when $\phi \gg \pi$ and $s \models \Diamond_d\phi$: note that $s[\partial\pi] = s$ asymmetrically entails that $s'[\partial\pi] = s'$ for some $s' \sqsubseteq s$. □

Fact 16. If $\rightarrow$ is a strict conditional then $\Box\phi \cong \top \rightarrow \phi$.

Proof. Note the following equivalences:

$$\begin{aligned} \top \rightarrow \phi &\cong \Box(\top \supset \phi) \\ &\cong \Box\top \supset \Box\phi \\ &\cong \Box\phi \end{aligned}$$

□

Fact 17. Let $[\cdot]$ be any candidate update function for our language and $\models$ its corresponding fixed-point supports/true-in relation. Assume that $\rightarrow$ is a strict conditional with respect to $\Box$. Then the following are equivalent:

i. $\Box$ is introspective;

ii. $s[\phi \rightarrow \psi] = \{w \in s\colon s[\phi][\psi] = s[\phi]\}$;

iii. $s[\Box\phi] = \{w \in s\colon s[\phi] = s\}$.

Proof. We show that (i) ⇒ (ii) ⇒ (iii) ⇒ (i).

(i) ⇒ (ii): There are two cases to consider: either $s[\phi][\psi] = s[\phi]$ or $s[\phi][\psi] \neq s[\phi]$. We show that in the first case $s[\phi \rightarrow \psi] = s$ and in the second case that $s[\phi \rightarrow \psi] = \emptyset$ and hence that (in general) $s[\phi \rightarrow \psi] = \{w \in s\colon\ s[\phi][\psi] = s[\phi]\}$.

So suppose $s[\phi][\psi] = s[\phi]$:

$$\begin{aligned} s[\phi][\psi] = s[\phi] \text{ iff } & s[\phi] \setminus s[\phi][\psi] = \emptyset \\ \text{iff } & s[\phi][\neg\psi] = \emptyset \\ \text{iff } & (s \setminus s[\phi][\neg\psi]) = s \\ \text{iff } & (s \setminus s[\phi \wedge \neg\psi]) = s \\ \text{iff } & s[\neg(\phi \wedge \neg\psi)] = s \end{aligned}$$

Thus (since support is fixed-point updating) $s \models \neg(\phi \wedge \neg\psi)$ and so (since $\Box$ is introspective) $s \models \Box\neg(\phi \wedge \neg\psi)$, which is to say $s \models \Box(\phi \supset \psi)$ and so $s \models \phi \rightarrow \psi$. Thus: $s[\phi \rightarrow \psi] = s$.

Now suppose $s[\phi][\psi] \neq s[\phi]$:

$$\begin{aligned} s[\phi][\psi] \neq s[\phi] \text{ iff } & s[\phi] \setminus s[\phi][\psi] \neq \emptyset \\ \text{iff } & s[\phi][\neg\psi] \neq \emptyset \\ \text{iff } & (s \setminus s[\phi][\neg\psi]) \neq s \\ \text{iff } & (s \setminus s[\phi \wedge \neg\psi]) \neq s \\ \text{iff } & s[\neg(\phi \wedge \neg\psi)] \neq s \end{aligned}$$

Thus (since support is fixed-point updating) $s \not\models \neg(\phi \wedge \neg\psi)$ and so (since $\Box$ is introspective) $s \models \neg\Box\neg(\phi \wedge \neg\psi)$, which is to say $s \models \neg\Box(\phi \supset \psi)$ and so $s[\neg\Box(\phi \rightarrow \psi)] = s$. Thus: $(s \setminus s[\Box(\phi \rightarrow \psi)]) = s$ and so $(s \setminus s[\phi \rightarrow \psi]) = s$. Hence $s[\phi \rightarrow \psi] = \emptyset$.

(ii) ⇒ (iii): this is routine since $\Box\phi \cong \top \rightarrow \phi$.

(iii) ⇒ (i): $s \models \phi$ iff $s[\phi] = s$ iff $s[\Box\phi] = s$ iff $s \models \Box\phi$. Similarly: $s \not\models \phi$ iff $s[\phi] \neq s$ iff $s[\Box\phi] = \emptyset$ iff $s \setminus s[\Box\phi] = s$ iff $s[\neg\Box\phi] = s$ iff $s \models \neg\Box\phi$. □

Fact 18. Let $[\cdot]$ be any candidate update function for our language and $\models$ its corresponding fixed-point supports/true-in relation. Then $\rightarrow$ is a test operator that satisfies free deduction iff $s[\phi \rightarrow \psi] = \{w \in s\colon\ s[\phi][\psi] = s[\phi]\}$. As a corollary: given interdefinability, iff $\Box$ is the fixed-point test modal.

The proof has the same structure as the proof for Fact 17. The right-to-left direction is (again) unsurprising and left as an exercise.

Proof. Suppose, first, that $s[\phi][\psi] = s[\phi]$. Hence $s[\phi] \models \psi$ and so by free deduction $s \models \phi \rightarrow \psi$. Since truth is fixed-point updating, it follows that $s[\phi \rightarrow \psi] = s$.

So now suppose that $s[\phi][\psi] \neq s[\phi]$. Hence $s[\phi] \not\models \psi$ and so $s \not\models \phi \rightarrow \psi$. Since $\rightarrow$ is a test, it is bivalent and so it follows that $s \models \neg(\phi \rightarrow \psi)$ and so $s[\neg(\phi \rightarrow \psi)] = s$. Thus $(s \setminus s[\phi \rightarrow \psi]) = s$ and so $s[\phi \rightarrow \psi] = \emptyset$. □

Fact 19. In general: $\Box_1\phi \not\cong \neg\Diamond_d\neg\phi$. While it does hold that $\Box_2\phi \cong \neg\Diamond_d\neg\phi$, in general: $\Box_2\phi \not\models \phi$.

The proof that $\Box_2$ is, and $\Box_2$ isn't, the dual of $\Diamond_d$ is left as homework.

Proof. Let $\phi = \neg(\Diamond_d p \wedge \neg p)$ and consider any s. We will show that $s \models \Box\phi$ and then show that not every s is such that $s \models \phi$. It then follows that $\Box_2$ isn't factive.

To see that $s \models \Box_2(\neg(\Diamond_d p \wedge \neg p)$ we need to see that there is no $s' \sqsubseteq s$ such that $s' \models \Diamond_d p \wedge \neg p$. Since $\Diamond_d p \wedge \neg p$ isn't coherent, there is no such s'. Hence: $s \models \Box_2(\neg(\Diamond_d p \wedge \neg p)$.

Now let s be any state such that $s \not\models p$ and $s \not\models \neg p$. Thus $s[\Diamond_d p \wedge \neg p] \neq s$ and $s[\Diamond_d p \wedge \neg p] \neq \emptyset$. Whence it follows that $(s \setminus s[\Diamond_d p \wedge \neg p]) \neq s$ and so that $s \not\models \neg(\Diamond_d p \wedge \neg p)$. □

Finally, we have come to an/the end:

Fact 20. To our language L_2 add an operator A and intepret it in this way:

vi. $s[\mathsf{A}\phi] = \{w \in s' \colon s'[\phi] = s' \text{ for some } s' \sqsubseteq s\}$

Then $\Diamond_d\phi \cong \Diamond_u \mathsf{A}\phi$.

Proof. Notice that $s[\mathsf{A}\phi] \neq \emptyset$ iff $s'[\phi] = s'$ for some $s' \sqsubseteq s$. Thus $s[\Diamond_u \mathsf{A}\phi] = s$ if $s'[\phi] = s'$ for some $s' \sqsubseteq s$. And $s[\Diamond_u \mathsf{A}\phi] = \emptyset$ if $s[\mathsf{A}\phi] = \emptyset$. □

References

Aloni, Maria. 2007. Free choice, modals, and imperatives. Natural Language Semantics 15(1). 65–94. http://dx.doi.org/10.1007/s11050-007-9010-2. URL http://dx.doi.org/10.1007/s11050-007-9010-2.

Barker, Chris. 2010. Free choice permission as resource-sensitive reasoning. Semantics & Pragmatics 3. 1–38. http://dx.doi.org/10.3765/sp.3.10.

Beaver, David I. 1992. The kinematics of presupposition. In Paul Dekker & Martin Stokhof (eds.), Proceedings of the eighth amsterdam colloquium. ILLC, University of Amsterdam.

Beaver, David I. 2001. Presupposition and assertion in dynamic semantics. CSLI Press.

van der Does, Jaap, Willem Groeneveld & Frank Veltman. 1997. An update on might. Journal of Logic, Language, and Information 6(4). 361–380. http://dx.doi.org/10.1023/A:1008219821036.

Dorr, Cian & John Hawthorne. 2013. Embedding epistemic modals. Mind 122(488). 867–913. http://dx.doi.org/10.1093/mind/fzt091.

Fauconnier, Gilles. 1985. Mental spaces: Aspects of meaning construction in natural language. MIT Press.

von Fintel, Kai. 1998. The presupposition of subjunctive conditionals. In Uli Sauerland & Orin Percus (eds.), MIT working papers in linguistics: The interpretive tract, vol. 25, 29–44.

von Fintel, Kai & Anthony S. Gillies. 2010. Must...stay...strong! Natural Language Semantics 1–33. http://dx.doi.org/10.1007/s11050-010-9058-2.

Fox, Danny. 2007. Free choice and the theory of scalar implicatures. In Uli Sauerland & Penka Stateva (eds.), Presupposition and implicature in compositional semantics, 71–120. Springer.

Geurts, Bart. 2005. Entertaining alternatives: Disjunctions as modals. Journal of Semantics 13. 383–410. http://dx.doi.org/10.1007/s11050-005-2052-4.

Gillies, Anthony S. 2004. Epistemic conditionals and conditional epistemics. Noûs 38. 585–616. http://dx.doi.org/10.1111/j.0029-4624.2004.00485.x.

Gillies, Anthony S. 2009. On truth-conditions for *if* (but not quite only *if*). The Philosophical Review 118(3). 325–348. http://dx.doi.org/10.1215/00318108-2009-002.

Gillies, Anthony S. 2010. Iffiness. Semantics and Pragmatics 3(4). 1–42. http://dx.doi.org/10.3765/sp.3.4.

Kamp, Hans. 1973. Free choice permission. Proceedings of the Aristotelian Society, New Series 74. 57–74.

Karttunen, Lauri. 1972. Possible and must. In John P. Kimball (ed.), Syntax and semantics, vol. 1. Academic Press.
Kay, Paul. 1992. The inheritance of presuppositions. Linguistics and Philosophy 15(4). 333–379.
Kratzer, Angelika. 1979. Conditional necessity and possibility. In Arnim von Stechow Rainer Bäuerle, Urs Egli (ed.), Semantics from different points of view, 117—147. Springer.
Kratzer, Angelika. 1986. Conditionals. Chicago Linguistics Society 22(2). 1–15.
Kratzer, Angelika. 2012. Modals and conditionals. Oxford University Press.
Kripke, Saul A. 1965. Semantical analysis of intuitionistic logic I. Studies in Logic and the Foundations of Mathematics 40. 92–130.
Lewis, David. 1973. Counterfactuals. Blackwell.
Lewis, David. 1975. Adverbs of quantification. In E. Keenan (ed.), Formal semantics of natural language. Cambridge University Press.
Moore, Robert C. 1985. Semantical considerations on nonmonotonic logic. Artificial Intelligence 25. 75–94.
Moss, Sarah. 2015. On the semantics and pragmatics of epistemic vocabulary. Semantics and Pragmatics 8(5). 1–81. http://dx.doi.org/10.3765/sp.
Muskens, Reinhard. 2013. Data semantics and linguistic semantics. In Maria Aloni, Michael Franke & Floris Roelofsen (eds.), The dynamic, inquisitive, and visionary life of ϕ, ?ϕ, and $\diamond\phi$. ILLC. URL http://www.illc.uva.nl/Festschrift-JMF/papers/23_Muskens.pdf.
Russell, Jeffrey Sanford & John Hawthorne. 2016. General dynamic triviality theorems. The Philosophical Review 125(3). 307–339. http://dx.doi.org/10.1215/00318108-3516936.
Stalnaker, Robert. 1975. Indicative conditionals. Philosophia 5. 269–286.
Stalnaker, Robert. 1993. A note on non-monotonic modal logic. Artificial Intelligence 64. 183–196.
Stojnić, Una. 2017. One's modus ponens: Modality, coherence and logic. Philosophy and Phenomenological Research 95(1). 167–214. http://dx.doi.org/10.1111/phpr.12307.
Veltman, Frank. 1985. Logics for conditionals. Ph.D. thesis, University of Amsterdam.
Veltman, Frank. 1996. Defaults in update semantics. Journal of Philosophical Logic 25. 221–261. http://dx.doi.org/10.1007/BF00248150.
Yalcin, Seth. 2007. Epistemic modals. Mind 116(464). 983–1026. http://dx.doi.org/10.1093/mind/fzm983.

Yalcin, Seth. 2012. A counterexample to modus tollens. Journal of Philosophical Logic 41. 1001–1024. http://dx.doi.org/10.1007/s10992-012-9228-4.

Zimmerman, Thomas Ede. 2000. Free choice disjunction and epistemic possibility. Natural Language Semantics 8. 255–290.