A Logic of Justification and Truthmaking

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Abstract. In the present paper we propose a system of propositional logic for reasoning about justification, truthmaking, and the connection between justifiers and truthmakers. The logic of justification and truthmaking is developed according to the fundamental ideas introduced by Artemov. Justifiers and truthmakers are treated in a similar way, exploiting the intuition that justifiers provide *epistemic* grounds for propositions *to be considered true*, while truthmakers provide *ontological* grounds for propositions *to be true*. This system of logic is then applied both for interpreting the notorious definition of knowledge as justified true belief and for advancing a new solution to Gettier counterexamples to this standard definition.

Keywords: epistemic logic; justification logic; justifiers; truthmaking; truthmakers; knowledge definition; Gettier problems.

1. Introduction

In [4] Artemov shows the usefulness of the logic of justification for analysing fundamental issues concerning epistemic justification and epistemology. The present paper builds on his logical framework in order to develop a more comprehensive logic of justification and truthmaking and to introduce a new definition of knowledge on the basis of it. The paper is organized as follows. In sections 2 and 3 both the current definition of knowledge as justified true belief and the present debate concerning the impact of Gettier cases are briefly reviewed. In particular, a general schema for interpreting and constructing Gettier cases is proposed. In section 4 a logic of justification and truthmaking is developed and shown to be sound and strongly complete with respect to a suitable semantics. Finally, in section 5 a new definition of knowledge is introduced and the impact of Gettier cases on this definition is scrutinized.

2. The current definition of knowledge

The definition of knowledge we are going to assess stems from the classical definition of knowledge codified by Aristotle. According to Aristotle's seminal analysis, knowledge can be acquired by either *evidential* or *inferential* processes. Intuition, i.e. the basic evidential process, provides direct justification relative to the first principles that constitute a certain science, whereas deduction, i.e. the basic inferential process, provides indirect justification to the propositions that are inferred from the principles. In accordance with this conception, scientific knowledge is thought of as the outcome of the indicated processes. Hence, an epistemic subject knows that a proposition p is true precisely when he assumes the truth of *p* on the basis of either evidence or inference, provided both evidence and inference are dependable. As a consequence, knowledge is determined as justified true conviction, where justification ensures truth. In particular, in cases of inferential justification, two conditions are to be met: (i) the inferential process must be sound and (ii) the premises on which the inference is based must be true[1].

This classical definition of knowledge can be analyzed by using epistemic logic and its possible worlds semantics. In such framework, $\mathbf{K}(p)$, for *p* is known, is construed as *p* is true in every epistemic world that is possible from the point of view of the total knowledge of an epistemic subject. In a similar sense, $\mathbf{J}(p)$, for *p* is justified, can be construed as *p* is true in every epistemic world that is possible from the point of view of the total evidence that is available to an epistemic subject. In order to specify knowledge and justification, several axioms on **K** and **J** can be introduced. In particular, the concept of knowledge is typically determined by introducing the following axioms:

$\mathbf{T_K}$: $\mathbf{K}(p) \rightarrow p$.
Knowledge is reflexive: what is known is true.

$\mathbf{4_K}$: $\mathbf{K}(p) \rightarrow \mathbf{K}(\mathbf{K}(p))$.
Knowledge is introspective: what is known is known to be known.

$\mathbf{5_K}$: $\neg\mathbf{K}(p) \rightarrow \mathbf{K}(\neg\mathbf{K}(p))$.
Knowledge is perfectly introspective: what is unknown is known to be unknown[2].

The logic of justification is less established and different kinds of justification can be classified according to their connection with truth and belief. In particular, we propose and adopt the following classification.

JUSTIFICATION	*insufficient for truth*	*sufficient for truth*
insufficient for belief	moderate	objectively strong

[1] The structure of scientific knowledge is described by Aristotle in his *Posterior Analytics*. In 71 b 20-25 Aristotle says that "if knowledge is such as we have assumed, demonstrative knowledge must proceed from premises which are true, primary, immediate, better known than, prior to, and causative of the conclusion. On these conditions only will the first principles be properly applicable to the fact which is to be proved. Syllogism indeed will be possible without these conditions, but not demonstration; for the result will not be knowledge".

[2] The first two axioms are commonly accepted as axioms characterizing a standard notion of knowledge. In what follows we assume a certain familiarity with epistemic logic (see [6], ch.2 and ch.3, and [11], ch.1, for an introduction to the key concepts and systems).

sufficient for belief	subjectively strong	absolutely strong

In order to denote these kinds of justification we introduce the following notation:

- **J**(p) := the agent has moderate justifiers for p.
- $\mathbf{J_1}$(p) := the agent has objectively strong justifiers for p.
- $\mathbf{J_S}$(p) := the agent has subjectively strong justifiers for p.
- $\mathbf{J_{ABS}}$(p) := the agent has absolutely strong justifiers for p.

It follows from this definitions that:
1) $\mathbf{J_{ABS}}(p) \rightarrow \mathbf{J_S}(p) \wedge \mathbf{J_1}(p)$;
2) $\mathbf{J_1}(p) \rightarrow \mathbf{J}(p) \wedge p$;
3) $\mathbf{J_S}(p) \rightarrow \mathbf{J}(p)$.

Aristotle identified knowledge with the outcome of absolutely strong justification, so that to know p implies to believe p and p.

Definition 2.1: knowledge (*classical concept*).
$\mathbf{K}(p) := \mathbf{J_{ABS}}(p)$: to know is to have absolutely strong justifiers.

In the current debate about knowledge the concept of justification that is in use is the moderate one, according to which to possess a justifier for a proposition cannot be a sufficient condition either for an agent to believe the proposition or for the proposition to be true. In the subsequent discussion, we will intend as *current definition of knowledge* the definition of knowledge as justified true belief, where the belief is justified by a moderate justifier. Still, in order to avoid the explicit introduction of the belief operator, we will always make use of the concept of subjectively strong justification, so that knowledge can be identified with having a subjectively strong justifier for a true proposition.

Definition 2.2: knowledge (*current concept*)
$\mathbf{K}(p) := \mathbf{J_S}(p) \wedge p$: to know is to have subjectively strong justifiers for a truth.

In this definition two elements can be distinguished: the subjective element of justification, $\mathbf{J_S}(p)$, and the objective element of truth, p. Since these elements are not connected, the possibility of challenging the correctness of the definition is open.

3. The challenge to the current definition

As well-known, the current definition of knowledge was challenged by Gettier's paper [8], that opened a massive debate both on the appropriateness of the definition and on the possibility of analyzing knowledge as a conjunction of more basic conditions.[3]

In this section we will introduce and consider two schemas allowing the construction of Gettier-style counterexamples[4] to the current definition of knowledge: the *deceptive deduction schema* and *the deceptive classification schema*.

The basic idea underlying the deceptive deduction schema stems from the critical analysis of the definition of knowledge as true belief proposed by Russell ([15], ch. XIII). According to Russell, who is following Aristotle's analysis, a true belief cannot count as knowledge if one of the following conditions occurs: (i) the believed proposition is deduced from a false premise; (ii) the believed proposition is deduced by a fallacious inferential process. The critical cases proposed by Russell are instances of the following schemas, where $\Box$ is to be interpreted as logical necessity and is introduced in order to model the relation of logical consequence between the premises and the conclusions of inferences.

Russell Schema I:[5]

1)	$\mathbf{J}_S(\Box(\varphi_1 \rightarrow \varphi) \wedge \varphi_1)$
2)	$\neg\varphi_1$
3)	$\Box(\varphi_2 \rightarrow \varphi) \wedge \varphi_2$
4) conclusion:	$\mathbf{J}_S(\varphi) \wedge \varphi$

Russell Schema II:[6]

1)	$\mathbf{J}_S(\Box(\varphi_1 \rightarrow \varphi) \wedge \varphi_1)$
2)	$\neg\Box(\varphi_1 \rightarrow \varphi)$
3)	$\Box(\varphi_2 \rightarrow \varphi) \wedge \varphi_2$
4) conclusion:	$\mathbf{J}_S(\varphi) \wedge \varphi$

[3] See [9, Part III, ch.8] for a suitable introduction to the debate.

[4] It is not our intention to cover the mass of examples proposed in the literature. Still, a little thought suffices to conclude that almost all of them can be classified as instances of either of the schemas.

[5] Russell says: "If a man believes that the late Prime Minister's last name began with a B, he believes what is true, since the late Prime Minister was Sir Henry Campbell Bannerman. But if he believes that Mr. Balfour was the late Prime Minister, he will still believe that the late Prime Minister's last name began with a B, yet this belief, though true, would not be thought to constitute knowledge".

[6] Russell says: "If I know that all Greeks are men and that Socrates was a man, and I infer that Socrates was a Greek, I cannot be said to know that Socrates was a Greek, because, although my premisses and my conclusion are true, the conclusion does not follow from the premisses".

The previous schemas can be subsumed under the following general schema.

DDS (*deceptive deduction schema*):

1)	$\mathbf{J}_S(\Box(\varphi_1 \to \varphi) \wedge \varphi_1)$
2)	$\neg(\Box(\varphi_1 \to \varphi) \wedge \varphi_1)$
3)	$\Box(\varphi_2 \to \varphi) \wedge \varphi_2$
4) conclusion:	$\mathbf{J}_S(\varphi) \wedge \varphi$

In **DDS** cases, φ cannot be said to be known. Still, if we define knowledge as justified true belief, we are enforced to conclude that φ is known by the epistemic subject.

Remark 1: Gettier original cases can be subsumed under Russell schema I. To be sure, the simplest versions of Gettier cases are instances of the following two schemas[7].

Gettier Schema I:

1)	$\mathbf{J}_S(\Box(P(a_1) \to \exists x P(x)) \wedge P(a_1))$
2)	$\neg P(a_1)$
3)	$\Box(P(a_2) \to \exists x P(x)) \wedge P(a_2)$
4) conclusion:	$\mathbf{J}_S(\exists x P(x)) \wedge \exists x P(x)$

Gettier Schema II:

1)	$\mathbf{J}_S(\Box(P(a_1) \to P(a_1) \vee P(a_2)) \wedge P(a_1))$
2)	$\neg P(a_1)$
3)	$\Box(P(a_2) \to P(a_1) \vee P(a_2)) \wedge P(a_2)$
4) conclusion:	$\mathbf{J}_S(P(a_1) \vee P(a_2)) \wedge (P(a_1) \vee P(a_2))$

To subsume them under Russell schema I, put

$\varphi_1 = P(a_1)$
$\varphi_2 = P(a_2)$
$\varphi = \exists x P(x) \,/\, P(a_1) \vee P(a_2)$

[7] An extended analysis of the original Gettier case I is proposed in [4, §10].

Hence, $\Box(\varphi_1 \rightarrow \varphi)$ is a valid implication; condition 2) is simply φ_1; condition 3) is $\Box(\varphi_2 \rightarrow \varphi) \wedge \varphi_2$; the conclusion is $\mathbf{J_S}(\varphi) \wedge \varphi$. Hence, **DDS** captures the basic ideas underlying the simplest Gettier cases.

Let us now focus on the deceptive classification schema. Suppose that x_1 and x_2 are objects of types T_1 and T_2 and that both x_1 and x_2 have the typical aspect of an object of type T_1. Suppose I see x_1 and x_2 and correctly identify their aspects as the typical aspects of objects of type T_1, concluding that both x_1 and x_2 are objects of type T_1. Let φ_1 be the proposition that x_1 has the typical aspect of an object of type T_1 and φ be the proposition that x_1 is of type T. Let τ be the assumption that the context is typical, i.e. it is such that every object having the typical aspect of an object of type T is indeed an object of type T.

DCS (*deceptive classification schema*):

1)	$\mathbf{J_S}(\Box(\tau \wedge \varphi_1 \rightarrow \varphi) \wedge \tau \wedge \varphi_1)$
2)	$\neg\tau$
3)	φ
4) conclusion:	$\mathbf{J_S}(\varphi) \wedge \varphi$

In this case, φ cannot be said to be known, because the assumption that the context in which we are is typical is false and it is just by chance that our belief comes out to be true.[8] Now, **DCS** is a special case of Russell schema I, and so of **DDS**: indeed, it suffices to take $(\tau \wedge \varphi_1)$ as the antecedent of the boxed implication and to note that $\neg\tau$ implies $\neg(\tau \wedge \varphi_1)$.[9]

Remark 2. The Aristotelian definition is not affected by **DDS**. In effect, according to the Aristotelian conception, the kind of justification involved in knowledge is absolutely strong. As a consequence, the epistemic condition is $\mathbf{J_{ABS}}(\Box(\varphi_1 \rightarrow \varphi) \wedge \varphi_1)$ and implies $\Box(\varphi_1 \rightarrow \varphi) \wedge \varphi_1$, which excludes premise 2).

[8] Many well-known cases (the fake sheep case; the fake barn case; the sure-fire match case) are instances of the deceptive classification schema (see [10], §3, for a presentation of these cases). Take the fake sheep case as paradigm: Looking into a field, *a* sees an animal that looks just like a sheep; *a* forms a tacitly inferential belief that there is a sheep in the field; actually, what *a* sees is an animal of a different species, but there is an actual sheep not so far away. In this case, *a* is justified in believing that there is a sheep in the field and there is indeed a sheep in the field.

[9] I want to thank an anonymous referee for pointing out this fact, thus allowing me to concentrate on **DDS** only.

Now, our operative aim is to find a version of the current definition of knowledge that is not challenged by instances of **DDS**[10]. The idea is as follows. The deceptive deduction schema highlights that there is no knowledge when a justified and true conclusion is deduced from some justified, but untrue, premise. If we dig deeper, we observe that the conclusion is justified by a justifier that depends on the justifier of an untrue premise, but it is made true by a truthmaker for which the epistemic subject possesses no justifier. Indeed, the problem arises because of the combination of four conditions: (1) we possess a justifier for the truth of a certain proposition; (2) such justifier consists in the evidence that a certain state of affairs is actual; (3) such state of affairs is not a truthmaker for that proposition; (4) we possess no justifier which consists in the evidence that the right state of affairs is actual. Thus, if the combination of these conditions is sufficient for excluding knowledge, the definition of knowledge we are looking for has to disallow it. In which way? Intuitively, *if we know that a proposition φ is true, then the state of affairs we are justified to assume as truthmaker for φ has both to be manifest and to coincide with the state of affairs making φ true*. Hence, we have to be justified in assuming that φ is made true by a truthmaker to which we have access and such state of affairs has to be the actual truthmaker for φ.

The end of the following section is to provide a logical tool for making such intuition precise. In order to achieve this end, we introduce a system of logic of justification and truthmaking that extends the system **J4** of logic of justification proposed by Artemov ([4], §3) as a development of the logic of proof (see [2] and [3] for an extended description). This system will enable us to cope with both justification, explicit and implicit, and truthmaking, and will provide the resources to introduce the definition of knowledge we are looking for. In particular, we will be able both to explicitly refer to justifiers and truthmakers and to define the relations of dependence (between justifiers) and extension (between truthmakers). Such relations will play an important role in making precise the conditions under which the deceptive deduction schema can be applied.

4. A system of logic of justification and truthmaking.

The logic of justification provides an appropriate framework for characterizing the concept of explicit justification, where a proposition is said to be explicitly justified when the subject is aware of a justifier for it. To this framework we adjoin (1) the tools for treating the concept of implicit justification, where a proposition is said to be implicitly justified when it is a consequence of explicitly justified propositions[11], and (2) the tools

[10] We do not claim that the definition we will introduce is immune to every possible counterexample. In any case the comprehensive schema seems to be general enough to justify optimism.

[11] Systems of logic concerning both *explicit* and *implicit knowledge*, as well as systems concerning *explicit justification* and *implicit knowledge*, have already been studied (see [11], ch. 2, and [5] for instance). Still, systems con-

for treating the concept of truthmaker, where a truthmaker for a proposition is any actual state of affairs that makes such proposition true. The basic system of logic of justification and truthmaking will be denoted by **LJT**.

Let *L*(**LJT**) be the language of **LJT**.
The alphabet of *L*(**LJT**) contains the following signs:

- logical constants: ¬, ∧.
- a countable set of propositional variables.
- a countable set JTm_0 of justification constants.
- a countable set of justification variables.
- operators on justifiers: ×, +, ! .
- a countable set of truthmaking variables.
- the structural state of affairs constant: *1*.
- a composition operator: • .
- the access operator: **A**.
- explicit modality constructor: [ ].
- implicit modality constructor: [ ]*.

The other logical connectives are defined in the usual way.

Two sorts of terms occur.

1) the set of terms for justifiers (*JTm*) is defined by the following rules:
$j := c \mid x \mid j{\times}i \mid j{+}i \mid !j$ where c / x is a constant / variable for justifiers.

2) the set of terms for truthmakers (*TTm*), defined by the following rules:
$t := 1 \mid v \mid t{\bullet}s$, where v is a variable for truthmakers.

The set of formulas is defined by the following rules:
$\varphi := p \mid \neg\varphi \mid \varphi\wedge\varphi \mid [j]\varphi \mid [j]^*\varphi \mid [t]\varphi \mid \mathbf{A}(t)$

Here p is a propositional variable, j a term for justifiers and t a term for truthmakers.

The intended interpretation of the propositions of *L*(**LJT**) is the following one.

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cerning both *explicit* and *implicit justification*, where implicit justification is defined in terms of what is explicitly justified, have not yet been advanced.

1) *justification propositions*.

A justification proposition like [*j*]φ is interpreted in the present context as *j* is a subjectively strong explicit justifier for φ. Analogously, a justification proposition like [*j*]*φ is interpreted as *j* is a subjectively strong implicit justifier for φ.[12] The operators +, ×, and ! are used to construct new, complex justifiers. Intuitively, *j*+*i* provides justification for all the propositions that can be justified either by *j* or by *i*, while *j*×*i* provides justification to all the propositions that can be justified by applying *modus ponens* to premises justified by *j* and by *i*. Finally, ! is a justification checker: if *j* is a justifier for φ, !*j* justifies that *j* is such a justifier.

2) *truthmaking propositions*.

A truthmaking proposition like [*t*]φ is interpreted as state of affairs *t* is an actual truthmaker for φ, while a truthmaking proposition like [*t*•*s*]φ is interpreted as the composed state of affairs *t*•*s* is an actual truthmaker for φ. On this interpretation it is evident that [*t*]φ implies [*t*•*s*]φ, for any state of affairs *s*. In what follows it is also assumed that the truth of φ is a necessary condition for a state of affairs to be a truthmaker for φ, i.e. that no proposition possesses a truthmaker in a world in which it is not true.

3) *access propositions*.

A proposition like **A**(*t*) is to be interpreted as the epistemic subject has access to state of affairs *t*, or state of affairs *t* is manifest to the epistemic subject.[13] The access operator is introduced in order to make explicit an element that is common to all Gettier cases, i.e. the distinction between the state of affairs *t* that the subject assumes as a truthmaker for a proposition and the state of affairs *s* that actually makes that proposition true. Indeed, *t* is a state of affairs to which the epistemic subject has access, whereas *s* is typically hidden. In addition, the access operator allows us to model the difference between having a justifier for [*t*]φ and having a justifier for [*t*]φ in case *t* is accessible to the epistemic subject. In effect, an *explicit* justifier for [*t*]φ is always an *implicit* justifier for [*t*•*s*]φ, since the truth of the first proposition implies the truth of the second one, whereas an explicit justifier for **A**(*t*) ∧ [*t*]φ is not necessarily an implicit justifier for **A**(*t*•*s*) ∧ [*t*•*s*]φ, since to have an access to *t* cannot imply to have an access to any larger state of affairs. This difference is crucial, because the value of the definition of knowledge we are going to propose depends on it.

[12] According to the proof-theoretical interpretation an explicit justifier is a proof and justifies every line of that proof. Along similar lines, an implicit justifier can be taken to be an ideal proof where every sound inference has been made. Analogously, if an explicit justifier is construed as being a piece of evidence, then an implicit justifier can be taken to be an ideal proof based on that piece of evidence.

[13] Note that **A** is an operator that applies to states of affairs, not to justifiers. **A**(*t*) states that the epistemic subject has access to state of affairs *t*, not that she has access to a justifier for the actuality of *t*. In this sense, the introduction of **A** is neutral with respect to the internalism / externalism debate.

A key characteristic that **LJT** shares with the logic of justification is that logical axioms are *a priori* justified. Intuitively, the epistemic subject accepts logical axioms, including the ones concerning justification and truthmaking, as immediately justified. In order to make these assumptions precise the tool of *constant specifications* is introduced. In what follows we only consider constant specifications construed as functions from the set of justification constants to the set of logical axioms. In particular, we only consider *axiomatically appropriate constant specifications*, i.e. constant specifications that are onto the set of logical axioms (see [7], §8). As a consequence, to every logical axiom a justification constant is assigned, witnessing that the axiom is accepted by the subject.

Definition 4.1: constant specification.
CS is a constant specification if and only if

i) CS: $JTm_0 \rightarrow \{\varphi \mid \varphi \text{ is a logical axiom}\}$
ii) CS is onto $\{\varphi \mid \varphi \text{ is a logical axiom}\}$

4.1. The system LJT

The system **LJT**(*CS*) of logic for justification and truthmaking, with constant specification *CS*, is defined by the following groups of axioms and rules.

Group 1: propositional axioms and *modus ponens*.

Group 2: axioms and rules concerning explicit justification.

J1: $[j](\varphi\rightarrow\psi) \rightarrow ([i]\varphi \rightarrow [j\times i]\psi)$
J2: $[j]\varphi \vee [i]\varphi \rightarrow [j+i]\varphi$
J3: $[j]\varphi \rightarrow [!j][j]\varphi$
RJ: $[c]\varphi$, where φ is an axiom instance of **LJT** and $\varphi\in CS(c)$

Group 2 introduces the axioms characterizing the system **J4** of Artemov (see [4], §7). **J1** states that, given two justifiers, *j* and *i*, the complex justifier $j\times i$ provides justification to any proposition that can be deduced from implications justified by *j* and propositions justified by *i* by applying *modus ponens*. Thus, *modus ponens* is internalized and propositional deduction is accepted by the epistemic subject as providing justification. **J2** states that given two justifiers, *j* and *i*, the complex justifier $j+i$ provides justification to any proposition justified by either *j* or *i*. **J3** states that justification is accessible: all justified propositions can be acknowledged as such. Therefore, justification itself is completely internalized. Finally, **RJ** takes care of the fact that axioms are justified as such. Indeed, the role of an axiomatically appropriate constants specification is precisely to ensure that all the logical axioms are a priori justified.

Group 3: axioms and rules concerning implicit justification.
J*1: $[j]^*(\varphi\rightarrow\psi) \rightarrow ([i]^*\varphi \rightarrow [j\times i]^*\psi)$
J*2: $[j]^*\varphi \vee [i]^*\varphi \rightarrow [j+i]^*\varphi$
J*3: $[j]^*\varphi \rightarrow [!j]^*[j]^*\varphi$
J*4: $[j]\varphi \rightarrow [j]^*\varphi$
J*5: $[c]\varphi \rightarrow [j]^*\varphi$
J*6: $[j\times j]^*\varphi \leftrightarrow [j+j]^*\varphi \leftrightarrow [!j]^*\varphi \leftrightarrow [j]^*\varphi$

Group 3 is the group of axioms we propose to characterize the system of logic of implicit justification. The first three axioms state that implicit justification, as far as the basic operations are involved, works like explicit justification. **J*4** says that what is explicitly justified is implicitly justified by the same justifier. **J*5** says that what is a priori justified is implicitly justified by any justifier. **J*6** says that $j\times j$, $j+j$ and j provide implicit justification to the same propositions. **J*6** turns out to be intuitive once we reflect on the standard interpretation of ×, +, ! (see [3], §6) and consider the fact a proposition is implicitly justified by j when it is a consequence of what is explicitly justified by j.

Group 4: axioms and rules concerning truthmaking.
T1: $[t](\varphi\rightarrow\psi) \rightarrow ([s]\varphi \rightarrow [t\bullet s]\psi)$
T2: $[t]\varphi \rightarrow [t\bullet s]\varphi$
T3: $[t]\varphi \leftrightarrow [t\bullet t]\varphi$
T4: $[t\bullet s]\varphi \leftrightarrow [s\bullet t]\varphi$
T5: $[t\bullet(s\bullet r)]\varphi \leftrightarrow [(t\bullet s)\bullet r]\varphi$
T6: $[t]\varphi \leftrightarrow [1\bullet t]\varphi$
T7: $[t][t]\varphi \vee [t]\neg[t]\varphi$
T8: $[t]\varphi \rightarrow \varphi$
RT: φ / $[1]\varphi$

Group 4 provides a basic characterization of the logic of truthmakers.[14] The idea underlying the present selection of axioms is that truthmakers are states of affairs that can be

[14] See [1, ch.1] for an introduction to the general theory of truthmaking. A discussion concerning what axioms are to be assumed in order to characterize the truthmaking relation can be found in [13] and [14]. The assumptions we have made are consistent with the ideas exposed in these texts. Finally, we will assume that truthmakers are states of affairs, as characterized *e.g.* in [1, ch.4]. Nevertheless, any conception allowing both for an operation of composition of truthmakers and for the existence of a structural truthmaker could be adopted.

composed to give rise to larger states of affairs. A logically closed set of propositions (**T1** and **RT**) is associated to each state of affairs, in such a way that a more inclusive set is associated with a larger state of affairs (**T2**). Any proposition that is associated to a certain state of affairs is made true by it, and so is true (**T8**), while the set of all the propositions made true by a certain state of affairs t is determined by t itself (**T7**). As a consequence, the same sets of propositions are associated to the same states of affairs in every world that is possible from the point of view of a reference world. The composition operation is assumed to induce a semilattice structure on the set of states of affairs with unit element 1. The basic idea is that there is no difference between a state of affairs and the same state of affairs composed with itself and that the order of composition is not significant. (**T3** through **T6**). The unit element plays the role of a structural state of affairs, common to all possible worlds, and allows us to express the thesis that every logical truth is made true by the same structural state of affairs (**RT**), and so by any state of affairs (by **T3** and **T6**).

Group 5: linking axioms and rules.
JT1: $[c]\varphi \rightarrow [1]\varphi$
JT2: $[1]\varphi \rightarrow [j]^*\varphi$
JT3: $\mathbf{A}(t\bullet s) \leftrightarrow \mathbf{A}(t) \wedge \mathbf{A}(s)$

Since every proposition that is justified by a constant is a logical truth, **JT1** states that what is justified by a constant is made true by the structural state of affairs. Furthermore, since the structural state of affairs is introduced as a truthmaker for every logical truth and every logical truth is implicitly justified by any justifier, **JT2** states that what is made true by 1 is implicitly justified by j, for any j. Finally, **JT3** states that the epistemic subject has access to a composed state of affairs precisely when she has access to its components, thus highlighting that the operation of composition on states of affairs reflects the operation of conjunction on propositions.

Before introducing the semantics for L(**LJT**), let us prove some theorems.

Th1: $\vdash_{\mathbf{LJT}} [t](\varphi\rightarrow \psi) \rightarrow ([t]\varphi \rightarrow [t]\psi)$.
$\vdash_{\mathbf{LJT}} [t](\varphi\rightarrow\psi) \rightarrow ([t](\varphi \rightarrow [t\bullet t]\psi)$, by **T1**
$\vdash_{\mathbf{LJT}} [t](\varphi\rightarrow\psi) \rightarrow ([t](\varphi \rightarrow [t]\psi)$, by **T3**

Th2: $\vdash_{\mathbf{LJT}} [t]\varphi \rightarrow [t][t]\varphi$.
$\vdash_{\mathbf{LJT}} [t]\neg[t]\varphi \rightarrow \neg[t]\varphi$, by **T8**
$\vdash_{\mathbf{LJT}} [t]\varphi \rightarrow \neg[t]\neg[t]\varphi$

$\vdash_{LJT} [t]\varphi \rightarrow [t][t]\varphi$, by **T7**

Th3: $\vdash_{LJT} \neg[t]\varphi \rightarrow [t]\neg[t]\varphi$.
$\vdash_{LJT} [t][t]\varphi \rightarrow [t]\varphi$, by **T8**
$\vdash_{LJT} \neg[t]\varphi \rightarrow \neg[t][t]\varphi$
$\vdash_{LJT} \neg[t]\varphi \rightarrow [t]\neg[t]\varphi$, by **T7**

Th4: $\vdash_{LJT} [1]\varphi \rightarrow [t]\varphi$.
$\vdash_{LJT} [1]\varphi \rightarrow [1\bullet t]\varphi$, by **T2**
$\vdash_{LJT} [1]\varphi \rightarrow [t]\varphi$, by **T6**

Theorems **Th2**-**Th4** will turn out to be useful in proving completeness.

Th5: $\vdash_{LJT} [j][t]\varphi \rightarrow [i][t\bullet s]\varphi$, for a certain i.
$\vdash_{LJT} [t]\varphi \rightarrow [t\bullet s]\varphi$, by **T2**
$\vdash_{LJT} [c]([t]\varphi \rightarrow [t\bullet s]\varphi)$, by **RJ**
$\vdash_{LJT} [j][t]\varphi \rightarrow [c\times j][t\bullet s]\varphi)$, by **J1**

Th6: $\vdash_{LJT} [c]\varphi \rightarrow [j][1]\varphi$, for a certain j.
$\vdash_{LJT} [c]\varphi \rightarrow [1]\varphi$, by **JT1**
$\vdash_{LJT} [c']([c]\varphi \rightarrow [1]\varphi)$, by **RJ**
$\vdash_{LJT} [!c][c]\varphi \rightarrow [c'\times !c][1]\varphi$, by **J1**
$\vdash_{LJT} [c]\varphi \rightarrow [c'\times !c][1]\varphi$, by **J3**

Furthermore, given **JT1**, rule **RT** turns out to be derivable.

RT: $\vdash_{LJT} \varphi \Rightarrow \vdash_{LJT} [1]\varphi$.

Proof. By induction on the length of a derivation. 1) If φ is an axiom instance, then the conclusion follows from **RJ** and **JT1**. 2) If φ is derived by **RJ**, then φ has the form $[c]\psi$ for some c and ψ, and so $\vdash_{LJT} [!c]\varphi$, by **J3**, and $\vdash_{LJT} [1]\varphi$, by **JT1**. 3) If φ is derived from ψ and $\psi\rightarrow\varphi$ by *modus ponens*, then, by induction hypothesis, $\vdash_{LJT} [1]\psi$ and $\vdash_{LJT} [1](\psi\rightarrow\varphi)$, and the conclusion follows by **T1** and **T6**.

Finally, in **LJT** the following versions of the modal rule of necessitation are derivable.

RJ1 (*Rule of explicit justification*): $\vdash_{LJT} \varphi \Rightarrow \vdash_{LJT} [j]\varphi$, for *some* term j.

Proof. By induction on the length of a derivation. 1) If φ is an axiom instance, the conclusion follows from **RJ**. 2) If φ is derived by **RJ**, then φ has the form $[c]\psi$ for some c and ψ, and so $\vdash_{\mathbf{LJT}} [!c]\varphi$, by **J3**. 3) If φ is derived from ψ and ψ→φ by *modus ponens*, then, by induction hypothesis, $\vdash_{\mathbf{LJT}} [i]\psi$ and $\vdash_{\mathbf{LJT}} [j](\psi\rightarrow\varphi)$. Hence $\vdash_{\mathbf{LJT}} [j\times i]\varphi$, by **J1**.

RJ2 (*Rule of implicit justification*): $\vdash_{\mathbf{LJT}} \varphi \Rightarrow\ \vdash_{\mathbf{LJT}} [j]^*\varphi$, for *every* term *j*.

Proof. By induction on the length of a derivation. 1) If φ is an axiom instance, the conclusion follows from **RJ** and **J*5**. 2) If φ is derived by **RJ**, then φ has the form $[c]\psi$ for some c and ψ, and so $\vdash_{\mathbf{LJT}} [j]^*\varphi$, by **J3** and **J*4**. 3) If φ is derived from ψ and ψ→φ by *modus ponens*, then, by induction hypothesis, $\vdash_{\mathbf{LJT}} [j]^*\psi$ and $\vdash_{\mathbf{LJT}} [j]^*(\psi\rightarrow\varphi)$. Hence $\vdash_{\mathbf{LJT}} [j\times j]^*\varphi$, by **J*1**, and so $\vdash_{\mathbf{LJT}} [j]^*\varphi$, by **J*6**.

4.2. Semantics for LJT.

The semantics for **LJT** is an extension of the currently standard semantics for justification logic introduced by Fitting in [7][15]. Within the framework of Fitting semantics we can model explicit justification by introducing a function that, given a justifier *j* and a possible world *w*, selects the set of all formulas for which *j* provides explicit justification at *w*. In a similar way, we model implicit justification by introducing a function that, given a justifier *j* and a possible world *w*, selects the set of all formulas for which *j* provides implicit justification at *w*. In addition, as in the case of explicit and implicit justification, we can model actual truthmaking by introducing a function that, given a truthmaker *t* and a possible world *w*, selects the set of all formulas for which *t* provides truth at *w*. Intuitively, the formulas associated to *t* at *w* represent the propositions that are made true by the state of affairs that is represented by *t* at *w*.

Thus, a frame for $L(\mathbf{LJT})$ is a tuple = $(W,R_T,R_J,\mathcal{T},\mathcal{J},\mathcal{J}^*)$ where

– W is a non-empty set of worlds
– $R_T \subseteq W\times W$ is a reflexive relation
– $R_J \subseteq W\times W$ is a transitive relation
– $\mathcal{T}: W\times TTm \rightarrow \wp(L(\mathbf{LJT}))$ is the truthmaking selection function
– $\mathcal{J}: W\times JTm \rightarrow \wp(L(\mathbf{LJT}))$ is the explicit justification selection function
– $\mathcal{J}^*: W\times JTm \rightarrow \wp(L(\mathbf{LJT}))$ is the implicit justification selection function

[15] See also [4] and [12] for extensions.

The intended interpretation of the accessibility relations is the following one. R_J is the epistemic possibility relation: $R_J(w,v)$ states that v is consistent with the set of propositions for which the epistemic subject has justification at w. R_T is the truthmaking possibility relation: $R_T(w,v)$ states that v is qualitatively indistinguishable with respect to the states of affairs that are actual at w.[16] It is assumed that R_J is transitive, so that the propositions for which the epistemic subject has explicit justification at w are explicitly justified in any world that is accessible to w. Finally, since every world is qualitatively indistinguishable relative to itself, R_T is assumed to be reflexive.

The intended interpretation of the selection functions is the one proposed above. These functions are furthermore constrained by a set of conditions which reflect the meaning of the axioms that characterize the corresponding operators.

Conditions on J:
J1) $\varphi \rightarrow \psi \in J(w,j)$ and $\varphi \in J(w,i) \Rightarrow \psi \in J(w,j \times i)$
J2) $\varphi \in J(w,j)$ or $\varphi \in J(w,i) \Rightarrow \varphi \in J(w,j+i)$
J3) $\varphi \in J(w,j) \Rightarrow [j]\varphi \in J(w,!j)$
RJ) $R_J(w,v) \Rightarrow J(w,j) \subseteq J(v,j)$

Conditions on J^*:
*J*1*) $\varphi \rightarrow \psi \in J^*(w,j)$ and $\varphi \in J^*(w,i) \Rightarrow \psi \in J^*(w,j \times i)$
*J*2*) $\varphi \in J^*(w,j)$ or $\varphi \in J^*(w,i) \Rightarrow \varphi \in J^*(w,j+i)$
*J*3*) $\varphi \in J^*(w,j) \Rightarrow [j]^*\varphi \in J^*(w,!j)$
*J*4*) $J(w,j) \subseteq J^*(w,j)$
*J*5*) $J(w,c) \subseteq J^*(w,j)$
*J*6*) $J^*(w,j \times j) = J^*(w,j+j) = J^*(w,!j) = J^*(w,j)$
*RJ**) $R_J(w,v) \Rightarrow J^*(w,j) \subseteq J^*(v,j)$

Conditions on T:
T1) $\varphi \rightarrow \psi \in T(w,t)$ and $\varphi \in T(w,s) \Rightarrow \psi \in T(w,s \bullet t)$
T2) $\varphi \in T(w,t) \Rightarrow \varphi \in T(w,t \bullet s)$
T3) $T(w,t) = T(w,t \bullet t)$
T4) $T(w,t \bullet s) = T(w,s \bullet t)$
T5) $T(w,(t \bullet s) \bullet r) = T(w,t \bullet (s \bullet r))$

[16] Note that, depending on the ontology one accepts, qualitatively indistinguishable worlds could still be numerically distinguishable. This idea can be clarified through the following example (due to Armstrong). Let us consider two toy worlds consisting of two states of affairs: a world w_1, described by $Fa \wedge Gb$, and a world w_2, described by $Fb \wedge Ga$. Since a and b are not qualitatively different, these worlds are qualitatively indistinguishable. Still, since a and b are numerically different, the worlds are, or can be assumed to be, numerically distinguishable.

$T6)\ T(w,t) = T(w, 1\bullet t)$
$T7)\ [t]\varphi \in T(w,t)$ or $\neg[t]\varphi \in T(w,t)$
$RT)\ R_T(w,v) \Rightarrow T(w,t) = T(v,t)$

Conditions on J, J^* and T:
$JT1)\ J(w,c) \subseteq T(w,1)$
$JT2)\ T(w,1) \subseteq J^*(w,j)$

A model for $L(\mathbf{LJT})$ is a tuple $M = (W,R_T,R_J,T,J,J^*,V)$, where V is a modal valuation that assigns to every propositional variable and every basic access proposition a set of possible worlds.

Definition 4.2: $M,w \models \varphi$ (φ is true at w in M).
1) $M,w \models p \Leftrightarrow w \in V(p)$
2) $M,w \models \mathbf{A}(v) \Leftrightarrow w \in V(\mathbf{A}(v))$
3) $M,w \models \neg\varphi \Leftrightarrow$ not $M,w \models \varphi$
4) $M,w \models \varphi\wedge\psi \Leftrightarrow M,w \models \varphi$ and $M,w \models \psi$
5) $M,w \models [t]\varphi \Leftrightarrow \forall v \in W(R_T(w,v) \Rightarrow M,v \models \varphi)$ and $\varphi \in T(w,t)$
6) $M,w \models [j]\varphi \Leftrightarrow \forall v \in W(R_J(w,v) \Rightarrow M,v \models \varphi)$ and $\varphi \in J(w,j)$
7) $M,w \models [j]^*\varphi \Leftrightarrow \forall v \in W(R_J(w,v) \Rightarrow M,v \models \varphi)$ and $\varphi \in J^*(w,j)$

Note that $\forall v \in W(R_T(w,v) \Rightarrow M,v \models \varphi)$ states that φ is true in all the worlds that are qualitatively indistinguishable relative to w, i.e. that the truth of φ is fixed by the qualitative aspects of the actual states of affairs of w. Hence, $M,w \models [t]\varphi$ states that, given the aspects of the actual states of affairs of w, the truth of φ is fixed and, in particular, that t is a truthmaker for φ.

Definition 4.3: M is a model for $\mathbf{LJT}(CS)$.
M is a model for $\mathbf{LJT}(CS) \Leftrightarrow \forall c,w \in W(CS(c) \subseteq J(w,c))$.

Definition 4.4: M respects the justification condition (JC).
M respects JC: $\forall j,w \in W(M,w \models [j]^*\varphi \Leftrightarrow \varphi \in J^*(w,j))$.

Definition 4.5: M respects the truthmaking condition (TC).
M respects $TC \Leftrightarrow \forall t,w \in W(M,w \models [t]\varphi \Leftrightarrow \varphi \in T(w,t))$.

Definition 4.6: M respects the access condition (AC).
M respects $AC \Leftrightarrow \forall t,s,w \in W(M,w \models \mathbf{A}(t\bullet s) \Leftrightarrow M,w \models \mathbf{A}(t)$ and $M,w \models \mathbf{A}(s))$.

Theorem 4.1: Let *CS* be a fixed constant specification. Then **LJT**(*CS*) is sound with respect to the class of all models for **LJT**(*CS*) respecting *JC*, *TC* and *AC*.

The proof is by induction on the derivations in **LJT**(*CS*).
We only check axioms **T1**-**T2**, **T7**-**T8**, and **JT1**-**JT3**. Indeed, the validity of axioms **T3**-**T6** is straightforward given conditions *T3*-*T6* and the validity of axioms on explicit and implicit justification is proved in a similar way.

T1: $[t](\varphi\rightarrow\psi) \rightarrow ([s](\varphi \rightarrow [t\bullet s]\psi)$.
Suppose $M,w \models [t](\varphi\rightarrow\psi)$ and $M,w \models [s]\varphi$. Then
i) $\forall v\in W(R_T(w,v) \Rightarrow M,v \models \varphi\rightarrow\psi)$ and $\varphi\rightarrow\psi\in T(w,t)$
ii) $\forall v\in W(R_T(w,v) \Rightarrow M,v \models \varphi)$ and $\varphi\in T(w,s)$.
Thus, $\forall v\in W(R_T(w,v) \Rightarrow M,v \models \psi)$ and $\psi\in T(w,t\bullet s)$, by *T1*.

T2: $[t]\varphi \rightarrow [t\bullet s]\varphi$
Suppose $M,w \models [t]\varphi$. Then $\forall v\in W(R_T(w,v) \Rightarrow M,v \models \varphi)$ and $\varphi\in T(w,t)$.
Thus, $\forall v\in W(R_T(w,v) \Rightarrow M,v \models \varphi)$ and $\varphi\in T(w,t\bullet s)$, by *T2*.

T7: $[t][t]\varphi \vee [t]\neg[t]\varphi$
Suppose not $M,w \models [t][t]\varphi$. Then $[t]\varphi\notin T(w,t)$, by *TC*.
Thus, $\neg[t]\varphi\in T(w,t)$, by *T3* and $M,w \models [t]\neg[t]\varphi$, by *TC* again.

T8: $[t]\varphi \rightarrow \varphi$
Suppose $M,w \models [t]\varphi$. Then $\forall v\in W(R_T(w,v) \Rightarrow M,v \models \varphi)$ and $\varphi\in T(w,t)$.
Thus, $M,w \models \varphi$, since R_T is reflexive.

JT1: $[c]\varphi \rightarrow [1]\varphi$
Suppose $M,w \models [c]\varphi$. Then $\varphi\in CS(c)$.
Since $CS(c) \subseteq J(w,c) \subseteq T(w,1)$ by *CS* and *JT1*, $\varphi\in T(w,1)$.
Thus, $M,w \models [1]\varphi$, by *TC*.

JT2: $[1]\varphi \rightarrow [j]^*\varphi$
Suppose $M,w \models [1]\varphi$. Then $\varphi\in T(w,1)$, by *TC*.
Thus, $\varphi\in J^*(w,j)$, by *JT2*, and so $M,w \models [j]^*\varphi$, by *JC*.

JT3: $\mathbf{A}(t\bullet s) \leftrightarrow \mathbf{A}(t) \wedge \mathbf{A}(s)$
Straightforward, since *M* respects *AC*.

Theorem 4.2: Let CS be a fixed constant specification. Then **LJT**(CS) is strongly complete with respect to the class of all models for **LJT**(CS) respecting JC, TC and AC.

Completeness is proved by canonicity.

Definition 4.7: canonical model for **LJT**.
The canonical model $M^C = (W^C, R_T{}^C, R_J{}^C, \mathcal{T}^C, \mathcal{J}^C, \mathcal{J}^{*C}, V^C)$ for **LJT**(CS) is such that:

– W^C is the set of maximally **LJT**(CS) consistent sets (denoted by w, v, ...).
– $R_T{}^C$ is such that $R_T{}^C(w,v) \iff w_T \subseteq v$, where $w_T := \{\varphi \mid \exists t \in TTm([t]\varphi \in w)\}$.
– $R_J{}^C$ is such that $R_J{}^C(w,v) \iff w_J \subseteq v$, where $w_J := \{\varphi \mid \exists j \in JTm([j]^*\varphi \in w)\}$.
– $\mathcal{T}^C$ is such that $\mathcal{T}^C(w,t) = w/t$, where $w/t := \{\varphi \mid [t]\varphi \in w\}$.
– $\mathcal{J}^C$ is such that $\mathcal{J}^C(w,j) = w/j$, where $w/j := \{\varphi \mid [j]\varphi \in w\}$.
– $\mathcal{J}^{*C}$ is such that $\mathcal{J}^{*C}(w,t) = w/j^*$, where $w/j^* := \{\varphi \mid [j]^*\varphi \in w\}$.
– V^C is such that $V^C(p) = \{w \mid p \in w\}$ and $V^C(\mathbf{A}(t) = \{w \mid \mathbf{A}(t) \in w\}$.

Note that, for each t, $w/t \subseteq w_T \subseteq w$, by definition of w_T and **T8**. In addition, for each j, $w/j \subseteq w/j^*$, by **J*4**, and $w/j^* \subseteq w_J$, by definition of w_J.

Lemma 1: M^C is a model for **LJT**(CS) respecting JC, TC and AC.
We first show that M^C respects JC and TC.

M^C respects JC.
It suffices to show that $\varphi \in w/j^* \Rightarrow \forall v \in W(w_J \subseteq v \Rightarrow \varphi \in v)$.
Suppose $\varphi \in w/j^*$. Then $\varphi \in w_J$, and so $\forall v \in W(w_J \subseteq v \Rightarrow \varphi \in v)$.

M^C respects TC.
It suffices to show that $\varphi \in w/t \Rightarrow \forall v \in W^C(w_T \subseteq v \Rightarrow \varphi \in v)$.
Suppose $\varphi \in w/t$. Then $\varphi \in w_T$, and so $\forall v \in W^C(w_T \subseteq v \Rightarrow \varphi \in v)$.

We now check that $R_J{}^C$ is transitive and that $R_T{}^C$ is reflexive. The reflexivity of $R_T{}^C$ I a straightforward consequence of **T8**. The transitivity of $R_J{}^C$ is proved as follows.

Suppose $w_J \subseteq v$.
It suffices to show that $w_J \subseteq v_J$.
Suppose $\varphi \in w_J$, i.e. $\exists j \in JTm([j]^*\varphi \in w)$.
Then $\exists j \in JTm([j]^*[j]^*\varphi \in w)$, **by J*3**, and so $\exists j \in JTm([j]^*\varphi \in w/j^*)$.

Thus, $\exists j \in JTm([j]^*\varphi \in v)$, since M^C respects JC, and so $\varphi \in v_J$.

Finally, let us check that M meets the conditions on **LJT**(*CS*) frames. We check conditions *RJ*, *RJ** and *RT*, since the other ones follow without difficulty from the corresponding axioms on justifiers and truthmakers.

RJ: $R_J^C(w,v) \Rightarrow \mathcal{J}^C(w,j) \subseteq \mathcal{J}^C(v,j)$.
We have to show that $w_J \subseteq v \Rightarrow w/j \subseteq v/j$.
Suppose $\varphi \in w/j$, i.e. $[j]\varphi \in w$.
Then $[!j][j]\varphi \in w$, by **J3**; $[!j]^*[j]\varphi \in w$, by **J*4**; $[j]\varphi \in w_J$; $w_J \subseteq v$; $[j]\varphi \in v$; $\varphi \in v/j$.

*RJ**: $R_J^C(w,v) \Rightarrow \mathcal{J}^{*C}(w,j) \subseteq \mathcal{J}^{*C}(v,j)$.
We have to show that $w_J \subseteq v \Rightarrow w/j^* \subseteq v/j^*$.
Suppose $\varphi \in w/j^*$, i.e. $[j]^*\varphi \in w$.
Then $[!j]^*[j]^*\varphi \in w$, by **J*3**; thus $[j]^*\varphi \in w_J$ and $w_J \subseteq v$; $[j]^*\varphi \in v$; $\varphi \in v/j^*$.

RT: $R_T^C(w,v) \Rightarrow \mathcal{T}^C(w,t) = \mathcal{T}^C(v,t)$.
We have to show that $w_T \subseteq v \Rightarrow w/t = v/t$.
Suppose $\varphi \in w/t$, i.e. $[t]\varphi \in w$.
Thus $[t][t]\varphi \in w$, by **Th2**; $[t]\varphi \in w_T$; $[t]\varphi \in v$; $\varphi \in v/t$.
Suppose $\varphi \notin w/t$, i.e. $[t]\varphi \notin w$, i.e. $\neg[t]\varphi \in w$, since $w \in W^C$.
Thus $[t]\neg[t]\varphi \in w$, by **Th3**; $\neg[t]\varphi \in w_T$; $\neg[t]\varphi \in v$; $[t]\varphi \notin v$, since $w \in W^C$; $\varphi \notin v/t$.

Truth Lemma: $\forall w \in W^C(M,w \models \varphi \Leftrightarrow \varphi \in w)$.
The only interesting cases are the new ones.

$M^C,w \models \mathbf{A}(t) \Leftrightarrow \mathbf{A}(t) \in w$.
Straightforward, by the definition of V^C.

$M^C,w \models [j]\varphi \Leftrightarrow [j]\varphi \in w$.
$M^C,w \models [j]\varphi \Leftrightarrow \forall v \in W^C(R_J^C(w,v) \Rightarrow M^C,v \models \varphi)$ and $\varphi \in \mathcal{J}^C(w,j)$.
By induction hypothesis and the definition of M^C we have
$M^C,w \models [j]\varphi \Leftrightarrow \forall v \in W^C(w_J \subseteq v \Rightarrow \varphi \in v)$ and $\varphi \in w/j$.
Suppose $[j]\varphi \in w$. Then $\varphi \in w/j \subseteq w_J \subseteq v$, whence the conclusion.
Suppose $[j]\varphi \notin w$. Then $\varphi \notin w/j$, whence the conclusion.

$M^C,w \models [j]^*\varphi \Leftrightarrow [j]^*\varphi \in w$.
$M^C,w \models [j]^*\varphi \Leftrightarrow \varphi \in w/j^*$, since M^C respects JC.

$M^C,w \models [t]\varphi \Leftrightarrow [t]\varphi \in w$.
$M^C,w \models [t]\varphi \Leftrightarrow \varphi \in w/t$, since M^C respects *TC*.

Note that M^C respects *AC*:
$M^C,w \models \mathbf{A}(t\bullet s) \Leftrightarrow \mathbf{A}(t\bullet s) \in w$, by the definition of V^C
$M^C,w \models \mathbf{A}(t\bullet s) \Leftrightarrow \mathbf{A}(t) \wedge \mathbf{A}(s) \in w$, by **JT3**
$M^C,w \models \mathbf{A}(t\bullet s) \Leftrightarrow \mathbf{A}(t) \in w$ and $\mathbf{A}(s) \in w$, by the definition of *w*
$M^C,w \models \mathbf{A}(t\bullet s) \Leftrightarrow M^C,w \models \mathbf{A}(t)$ and $M^C,w \models \mathbf{A}(s)$, by the definition of V^C

This concludes our proof.

5. A new definition of knowledge
We are now equipped for introducing our definition of knowledge.

Definition 5.1: Let *j* be a justifier.
i) *j* is *correct* if and only if $[j]^*\varphi \rightarrow \varphi$, for every φ.
ii) *j* is *consistent* if and only if $[j]^*\varphi \rightarrow \neg[j]^*\neg\varphi$, for every φ.

As noted in section 3, if we define knowledge as possession of a correct justifier, then we get a simple solution to problematic instances of **DDS**. Yet, the concept of knowledge we obtain becomes too powerful to find a general application within the current epistemological debate. Hence, we consider three less powerful notions of knowledge.

Definition 5.2: consistent knowledge.
Let *j* be a subjectively strong consistent justifier.

i) $\mathbf{K^{**}}(j,\varphi) := [j]\varphi \wedge \varphi$
(possession of a consistent justifier for a true proposition)

ii) $\mathbf{K^{*}}(j,\varphi) := [j][t]\varphi \wedge [t]\varphi$, for a certain *t*.
(possession of a consistent justifier for a proposition made true by *t*)

iii) $\mathbf{K}(j,\varphi) := [j](\mathbf{A}(t) \wedge [t]\varphi) \wedge [t]\varphi$, for a certain *t*.
(possession of a consistent justifier supporting the actuality of a truthmaker of φ)

Unconditional definitions of knowledge can be obtained by existential quantification:

$\mathbf{K^{**}}(\varphi) := [j]\varphi \wedge \varphi$, for some j;
$\mathbf{K^{*}}(\varphi) := [j][t]\varphi \wedge [t]\varphi$, for some t and j.
$\mathbf{K}(\varphi) := [j](\mathbf{A}(t) \wedge [t]\varphi) \wedge [t]\varphi$, for some t and j.

$\mathbf{K^{**}}(j,\varphi)$ codifies an explicit version of the current definition of knowledge, where the justifier on which knowledge is based is displayed. $\mathbf{K^{*}}(j,\varphi)$ codifies the idea according to which the justification must be based on the right ontological grounds. Finally, $\mathbf{K}(j,\varphi)$ codifies the idea according to which the justification must be based on the actual access to the state of affairs that makes true the justified proposition (note that there exist constants c' and c such that $\mathbf{K^{*}}(j,\varphi) \rightarrow \mathbf{K^{**}}(c'\times j,\varphi)$ and $\mathbf{K}(j,\varphi) \rightarrow \mathbf{K^{*}}(c\times j,\varphi)$).

The definition of knowledge with which we will work is iii):
to know φ *is to possess a consistent justifier for the appearing of a truthmaker for* φ.

Theorem 5.1: if j is consistent, then $[j][1]\varphi \rightarrow [1]\varphi$.

$\vdash_{\mathbf{LJT}} \neg[1]\varphi \rightarrow [1]\neg[1]\varphi$	by **Th.3**
$\vdash_{\mathbf{LJT}} \neg[1]\varphi \rightarrow [j]^{*}\neg[1]\varphi$	by **JT2**
$\vdash_{\mathbf{LJT}} \neg[j]^{*}\neg[1]\varphi \rightarrow [1]\varphi$	by p.l.
$\vdash_{\mathbf{LJT}} [j]^{*}[1]\varphi \rightarrow [1]\varphi$	by consistency
$\vdash_{\mathbf{LJT}} [j][1]\varphi \rightarrow [1]\varphi$	by **J*4**

Theorem 5.1 shows that assuming consistency takes care of Russell's warning concerning the necessity that knowledge must not be based on a fallacious inference, since any consistent justifier is correct with respect to logical truths.

5.1. A solution to Gettier problem

In order to consider whether the above definitions are subjected to Gettier problem, let us translate the premises of Russell schema 1 in terms of justifiers and truthmakers, assuming that the justifier concerning the inferential part is correct.

Russell Schema I:

1) $[c](\varphi_1 \rightarrow \varphi) \wedge [j_1][t_1]\varphi_1$
2) $\neg\varphi_1$
3) $[1](\varphi_2 \rightarrow \varphi) \wedge [t_2]\varphi_2$

Since the language of **LJT** is more expressive, we are now in a position to make explicit some constraints that are implicitly introduced when the schema is used in order to construct counterexamples to a candidate definition of knowledge.

4) constraint on the premises:
4.1) j_1 is an *explicit justifier* for $\mathbf{A}(t_1)$
4.2) j_1 is not an *implicit justifier* for $\mathbf{A}(t_2)$ or $[t_2]\varphi$

This constraint states that j_1 provides justification neither for the proposition that t_2 is manifest nor for the proposition that t_2 is a truthmaker for φ. In this way, we make explicit the crucial assumption that the justifier of φ is in no way connected with the actual truthmaker of φ. Note that it is precisely this assumption that allows us to conclude that there is no knowledge in cases in which Russell schema is instantiated.

5) constraint on the conclusion:
5.1) constraint on j: if $[j]\varphi$ is true, then j depends on j_1
5.2) constraint on t: if $[t]\varphi$ is true, then t extends t_2

Constraint 5.1 makes explicit the assumption that j_1 is the key justifier for φ by stating that every other justifier is dependent on j_1. We define this kind of dependence by stating that j depends on i when, for every φ, $[j]^*\varphi \rightarrow [i]^*\varphi$, i.e. every proposition justifiable by j is a logical consequence of what is justified by i. The underlying idea is that j depends on i when it is obtained from i by iterating the application of **RJ** and **J1**.

Definition 5.3: *J*-dependence.
j depends on i <=> for every φ, $[j]^*\varphi \rightarrow [i]^*\varphi$.

Constraint 5.2 makes explicit the assumption that t_2 is the key truthmaker for φ by stating that every other truthmaker is an extension of t_2. Having at our disposal an operation of truthmaker composition, we are able to characterize this notion by stating that t extends s when t includes s as a component.

Definition 5.4: *T*-extension.
t extends s <=> t coincides with $s \bullet r$, for some r.

Both constraint 5.1 and 5.2 are essential in order to see why a candidate definition of knowledge is unsuccessful. In fact, in constructing a counterexample, we typically start with a basic justifier and a basic unrelated truthmaker and derive the existence of both a (possibly complex) justifier and a (possibly composed) unrelated truthmaker together implying that the subject knows that a certain proposition is true.

We can now pose the question as to whether the above definitions of knowledge are subject to Gettier problem. The following theorems give us the answers.

Theorem 5.2: **K**** is subject to counterexamples derived from Russell schema 1.

$[c](\varphi_1 \rightarrow \varphi) \wedge [j_1][t_1]\varphi_1$, where j_1 is not related to t_2
$\neg\varphi_1$
$[1](\varphi_2 \rightarrow \varphi) \wedge [t_2]\varphi_2$

$[j]\varphi \wedge \varphi$, i.e. **K****$(j,\varphi)$, for a certain j

It is a straightforward corollary of the following crucial theorem.

Theorem 5.3: **K*** is subject to counterexamples derived from Russell schema 1.

$[c](\varphi_1 \rightarrow \varphi) \wedge [j_1][t_1]\varphi_1$, where j_1 is not related to t_2
$\neg\varphi_1$
$[1](\varphi_2 \rightarrow \varphi) \wedge [t_2]\varphi_2$

$[j][t_1{\bullet}t_2]\varphi \wedge [t_1{\bullet}t_2]\varphi$, i.e. **K***$(j,\varphi)$ for a certain j

Proof:

$[t_2]\varphi_2 \vdash_{\mathbf{LJT}} [t_2{\bullet}t_1]\varphi_2$	by **T2**
$[t_2]\varphi_2 \vdash_{\mathbf{LJT}} [t_1{\bullet}t_2]\varphi_2$	by **T4**
$[t_1{\bullet}t_2]\varphi_2, [1](\varphi_2 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [1{\bullet}(t_1{\bullet}t_2)]\varphi$	by **T1**
$[t_1{\bullet}t_2]\varphi_2, [1](\varphi_2 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [t_1{\bullet}t_2]\varphi$	by **T6**
$[t_2]\varphi_2, [1](\varphi_2 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [t_1{\bullet}t_2]\varphi$	by p.l.

Thus, from premise 3, we conclude $[t_1{\bullet}t_2]\varphi$.

$[t_1]\varphi_1 \vdash_{\mathbf{LJT}} [t_1{\bullet}t_2]\varphi_1$	by **T2**
$[t_1{\bullet}t_2]\varphi_1, [1](\varphi_1 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [1{\bullet}(t_1{\bullet}t_2)]\varphi$	by **T1**
$[t_1{\bullet}t_2]\varphi_1, [1](\varphi_1 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [t_1{\bullet}t_2]\varphi$	by **T6**
$[t_1]\varphi_1, [1](\varphi_1 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [t_1{\bullet}t_2]\varphi$	by p.l.
$[c](\varphi_1 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [1](\varphi_1 \rightarrow \varphi)$	by **JT1**
$[t_1]\varphi_1, [c](\varphi_1 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [t_1{\bullet}t_2]\varphi$	by p.l.
$\vdash_{\mathbf{LJT}} [c](\varphi_1 \rightarrow \varphi) \rightarrow ([t_1]\varphi_1 \rightarrow [t_1{\bullet}t_2]\varphi)$	by p.l.
$\vdash_{\mathbf{LJT}} [i]([c](\varphi_1 \rightarrow \varphi) \rightarrow ([t_1]\varphi_1 \rightarrow [t_1{\bullet}t_2]\varphi))$	by **RJ1**

$[c](\varphi_1 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [!c][c](\varphi_1 \rightarrow \varphi)$ by **J3**

$[c](\varphi_1 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [i{\times}!c]([t_1]\varphi_1 \rightarrow [t_1{\bullet}t_2]\varphi)$ by **J1**

$[j_1][t_1]\varphi_1, [c](\varphi_1 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [i{\times}!c{\times}j_1][t_1{\bullet}t_2]\varphi$ by **J1**

Thus, from premise 1, we conclude $[i{\times}!c{\times}j_1][t_1{\bullet}t_2]\varphi$.

Therefore, from premises 1 and 3, we get $[i{\times}!c{\times}j_1][t_1{\bullet}t_2]\varphi$ and $[t_1{\bullet}t_2]\varphi$. Note that constraints 5.1 and 5.2 are met. Indeed, $i{\times}!c{\times}j_1$ is obtained by multiplying j_1 times a number of constants, so that $[j]\varphi \rightarrow [j_1]^*\varphi$ by **J*1**, **J*5**, **J*6**, while t_2 is a component of $t_1{\bullet}t_2$.

The final diagnosis is well-known: the epistemic subject cannot be said to know φ, even if φ is both justified by means of a consistent justifier and true. Still, if we analyse the structure of the second proof, we note that, in order to find a state of affairs that is both a truthmaker for φ and a justified truthmaker for φ, it is necessary to compose the truthmaker for φ_2 with the justified truthmaker for φ_1. Thus, in order to find the right truthmaker for φ, it is necessary to compose a state of affairs that is manifest to the epistemic subject with a state of affairs to which she has no actual access.

Theorem 5.4: **K** is not affected by Gettier problem.
Let us translate the schema **DDS** in terms of justifiers and truthmakers.

DDS (premises):
1) $\mathbf{J}_S(\Box(\varphi_1 \rightarrow \varphi) \wedge \varphi_1)$
2) $\neg\Box(\varphi_1 \rightarrow \varphi) \vee \neg\varphi_1$
3) $\Box(\varphi_2 \rightarrow \varphi) \wedge \varphi_2$

Condition 1) states that the subject is in possession of a justifier both relative to the necessity of the implication and relative to the truth of its antecedent. Thus, an appropriate translation of the premises in terms of justifiers is the following one.

DDS (premises):
1) $[j_1][1](\varphi_1 \rightarrow \varphi) \wedge [j_1][t_1]\varphi_1$
2) $\neg[1](\varphi_1 \rightarrow \varphi) \vee \neg\varphi_1$
3) $[1](\varphi_2 \rightarrow \varphi) \wedge [t_2]\varphi_2$

We have to show that $\mathbf{K}(j,\varphi)$ is not derivable from these premises, if the constraints 4 and 5 are satisfied, i.e. if

4.1. j_1 is an explicit justifier for $\mathbf{A}(t_1)$.
4.2. j_1 is not an implicit justifier for $\mathbf{A}(t_2)$ or $[t_2]\varphi$.
5.1. for any j, if $[j]\varphi$ is true, then j depends on j_1.
5.2. for any t, if $[t]\varphi$ is true, then t extends t_2.

Suppose $\mathbf{K}(j,\varphi)$ is true, for some consistent j.

$\mathbf{K}(j,\varphi) \vdash_{\mathbf{LJT}} [j](\mathbf{A}(t) \wedge [t]\varphi)$	def. **K**
$\mathbf{K}(j,\varphi) \vdash_{\mathbf{LJT}} [t]\varphi$	def. **K**
$\vdash_{\mathbf{LJT}} [c_1](\mathbf{A}(t) \wedge [t]\varphi \rightarrow [t]\varphi)$	by **RJ**
$\mathbf{K}(j,\varphi) \vdash_{\mathbf{LJT}} [c_1{\times}j][t]\varphi$	by **J1**
$\vdash_{\mathbf{LJT}} [c_2]([t]\varphi \rightarrow \varphi)$	by **RJ**
$\mathbf{K}(j,\varphi) \vdash_{\mathbf{LJT}} [c_2{\times}c_1{\times}j]\varphi$	by **J1**
$\mathbf{K}(j,\varphi) \vdash_{\mathbf{LJT}} [c_2{\times}c_1{\times}j]\varphi \wedge [t]\varphi$	by p.l.

Since $[c_2{\times}c_1{\times}j]\varphi$, $c_2{\times}c_1{\times}j$ depends on j_1, by constraint 5.1, and so

i) $[c_2{\times}c_1{\times}j]^*\varphi \rightarrow [j_1]^*\varphi$, for every φ.
ii) $[j]^*\varphi \rightarrow [j_1]^*\varphi$, for every φ, given the axioms on $[j]^*$.

Thus, $[j_1]^*(\mathbf{A}(t) \wedge [t]\varphi)$. In addition, since $[t]\varphi$, t extends t_2, by constraint 5.2, and so $t = t_2{\bullet}t_0$ for some t_0. Thus, $[j_1]^*(\mathbf{A}(t_2{\bullet}t_0) \wedge [t_2{\bullet}t_0]\varphi)$, for some t_0, and so, $[j_1]^*\mathbf{A}(t_2{\bullet}t_0)$, by the axioms on $[j_1]^*$, and $[j_1]^*\mathbf{A}(t_2)$, by **JT2** and the axioms on $[j_1]^*$. Therefore, we get that j_1 is not an implicit justifier for $\mathbf{A}(t_2)$, in contradiction with constraint 4.2.

A final point. Suppose the premises are the right ones:

$[j_1][I](\varphi_1 \rightarrow \varphi) \wedge [j_1](\mathbf{A}(t_1) \wedge [t_1]\varphi_1) \wedge [t_1]\varphi_1$, where j_1 is consistent.

We can ask whether it is possible to derive $\mathbf{K}(j,\varphi)$, for a certain j.

Theorem 5.5: derivation of correct knowledge.
$[j_1][I](\varphi_1 \rightarrow \varphi) \wedge [j_1](\mathbf{A}(t_1) \wedge [t_1]\varphi_1) \wedge [t_1]\varphi_1 \rightarrow \mathbf{K}(j,\varphi)$.

Let H = $[j_1][I](\varphi_1 \rightarrow \varphi) \wedge [j_1](\mathbf{A}(t_1) \wedge [t_1]\varphi_1) \wedge [t_1]\varphi_1$.

j_1 is consistent.

$[j_1][I](\varphi_1 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [I](\varphi_1 \rightarrow \varphi)$	by theorem 5.1

$[1](\varphi_1 \rightarrow \varphi) \wedge [t_1]\varphi_1 \vdash_{\mathbf{LJT}} [1 \bullet t_1]\varphi$ by **T1**
$[1](\varphi_1 \rightarrow \varphi) \wedge [t_1]\varphi_1 \vdash_{\mathbf{LJT}} [t_1]\varphi$ by **T6**
$[1](\varphi_1 \rightarrow \varphi) \wedge \mathbf{A}(t_1) \wedge [t_1]\varphi_1 \vdash_{\mathbf{LJT}} \mathbf{A}(t_1) \wedge [t_1]\varphi$ by p.l.
$\vdash_{\mathbf{LJT}} [1](\varphi_1 \rightarrow \varphi) \rightarrow (\mathbf{A}(t_1) \wedge [t_1]\varphi_1 \rightarrow \mathbf{A}(t_1) \wedge [t_1]\varphi)$ by p.l.
$\vdash_{\mathbf{LJT}} [j]([1](\varphi_1 \rightarrow \varphi) \rightarrow (\mathbf{A}(t_1) \wedge [t_1]\varphi_1 \rightarrow \mathbf{A}(t_1) \wedge [t_1]\varphi))$ by **RJ1**
$[j_1][1](\varphi_1 \rightarrow \varphi) \vdash_{\mathbf{LJT}} [j \times j_1](\mathbf{A}(t_1) \wedge [t_1]\varphi_1 \rightarrow \mathbf{A}(t_1) \wedge [t_1]\varphi))$ by **J1**
$[j_1][1](\varphi_1 \rightarrow \varphi) \wedge [j_1](\mathbf{A}(t_1) \wedge [t_1]\varphi_1) \vdash_{\mathbf{LJT}} [j \times j_1 \times j_1](\mathbf{A}(t_1) \wedge [t_1]\varphi)$ by **J1**
$H \vdash_{\mathbf{LJT}} [j \times j_1 \times j_1](\mathbf{A}(t_1) \wedge [t_1]\varphi) \wedge [t_1]\varphi_1$ by p.l.

Conclusion

The logic of explicit justification extends the standard epistemic logic enabling us to analyse the structure of basic epistemic states with more accuracy. The logic of implicit and explicit justification represents a further step in this direction, by adding the possibility of distinguishing propositions that are explicitly justified and propositions justified as consequences of the former ones. Finally, the logic of justification and truthmaking allows us to explore the connections between justifiers and truthmakers and to provide a new framework for interpreting classical epistemological problems. In particular, in studying Gettier problem within this framework, it was shown that the language of **LJT** provides the tools to accomplish a more insightful analysis of both the premises and the constraints on which a Gettier argument is based. As a consequence, we were able to isolate the element that is responsible of the problem and to find a solution built on the identification of a connection between the epistemic and the ontological ground for the truth of a proposition.

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