Do Abstract Mathematical Axioms About Infinite Sets Apply To The Real, Physical Universe?

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Abstract

Suppose one has a system, the infinite set of positive integers, P, and one wants to study the characteristics of a subset (or subsystem) of that system, the infinite subset of odd positives, O, relative to the overall system. In mathematics, this is done by pairing off each odd with a positive, using a function such as O=2P+1. This puts the odds in a one-to-one correspondence with the positives, thereby, showing that the subset of odds and the original set of positives are the same size, or have the same cardinality. This counter-intuitive result ignores the "natural" relationship of one odd for every two positives in the sequence of positive integers, which would suggest that O is one-half the size of P. However, in the set of axioms that constitute mathematics, it is considered valid. Fair enough. In the physical universe (i.e., the starting system), though, relationships between entities matter. In biochemistry, if you start with an organism, say a rat, and you want to study its heart, you can do this by removing some heart cells and studying them in isolation in a cell culture system. But, the results often differ compared to what occurs in the intact animal because studying the isolated cultured cells ignores the relationships in the intact body between the cells, the rest of the heart tissue and the rest of the rat. In chemistry, if a copper atom were studied in isolation, it would never be known that copper atoms in bulk can conduct electricity because the atoms share their electrons. In physics, the relationships between inertial reference frames in relativity, and observer and observed in quantum physics can't be ignored. Relationships matter in the physical world, but the mathematics of infinite sets is still used to describe it. Does this matter? It seems to, at least in physics. Infinities cause numerous problems in theoretical physics such as non-renormalizability and problems in unifying quantum mechanics and general relativity . This suggests that the pairing off method and the mathematics of infinite sets based on it are analogous to a cell culture system or studying a copper atom in isolation if they are used in studying the physical universe because they ignore the inherent relationships between entities. In the real, physical world, the natural, inherent, relationships between entities can't be ignored. Said another way, the set of axioms that constitute abstract mathematics may be similar but not identical to the set of physical axioms by which the real, physical universe runs. This suggests that the results from abstract mathematics about infinities may not apply to or should be modified for use in physics.

Introduction

Set theory is considered by many to be the theoretical foundation of mathematics (Bagaria, 2020), and the pairing-off method first developed by Cantor (1878) plays a major role in set theory. The pairing-off method uses a bijective function that lets the elements of one set be put into a one-to-one correspondence with the elements of another set. When the two sets whose elements form a one-to-one correspondence are infinite, the sets are said to have the same cardinality. For example, consider the sequential set of positive integers, P={1,2,3...} to be the starting system to be studied, and a subset, O, of P of the odd positive integers {1,3,5..} as the subsystem. We want to study the characteristics of subsystem O in its "natural milieu", P. If one starts with set P, it can be shown that subset O can be put into a one-to-one correspondence with P using a function such as f(O)=2P+1, which means that these two sets have the same size, or cardinality. The pairing-off method, however, ignores the natural relationship in set P of every odd being accompanied by an even, which suggests that subset O is one-half the size of P. But, the pairing-off method is accepted in mathematics, and I do not question this result here. However, the question discussed in this paper is whether or not this result and the mathematics of infinity in general are applicable to the physical sciences, specifically physics, where the natural relationships between components (or subsystems) are of major importance in determining the behavior of the system. This question should not be brushed off lightly because infinities have indeed caused major problems in physics such as in uniting quantum mechanics with general relativity. One physicist has even suggested that the field look for ways of doing away with infinite amounts (Tegmark, 2015).

To explore this question, this paper is structured as follows. The first section illustrates the pairing-off method in mathematics and describes in more detail how its use ignores the natural relationships between numbers in a sequence. The second section describes the importance of relationships between systems and subsystems in the behavior of physical systems. The third section discusses some specific problems with infinite amounts in physics and suggests that Cantor's pairing-off method and the abstract mathematics of infinities may need to be modified for use in this field. Finally, these results will be summarized in a short conclusion.

Infinite set (system)-infinite subset (subsystem) size comparison in abstract mathematics

As described above, Cantor's pairing-off method uses a bijective function that puts the elements of one set into a one-to-one correspondence with the elements of another set. If a one-to-one correspondence can be established between two infinite sets, they are said to have the same cardinality. As an example, suppose one starts with the system that is the infinite, sequential set of all positive integers, P {1,2,3,4,5...}, and you want to compare the size of the infinite subset, or subsystem, of odd positive integers, O {1,3,5...}, with P. Furthermore, you want this size comparison to reflect the situation within the "natural milieu" of the sequential set P. This can be be done by using a bijective function such as O=2P+1. Use of this function shows that there is a one-to-one correspondence between each odd and each positive integer as follows:

If P=1, the odd integer is 3 (gap between P and O = 2)
If P=2, the odd integer is 5 (gap between P and O = 3)
If P=3, the odd integer is 7 (gap between P and O = 4)
If P=4, the odd integer is 9 (gap between P and O = 5)
...

As can be seen, the gap between the positive integer and its corresponding odd integer keeps getting larger. However, because these are both countably infinite sets, this one-to-one correspondence can keep going forever, thereby showing that the sets are of the same size or cardinality. One might object and say that this thought experiment-derived result ignores the inherent, “natural” relationship in the starting set of positive integers of one odd positive integer for every two integers (i.e., (1,2), (3,4), (5,6), (7,8), (9,10)...) and therefore may create experimental artifacts. While the author agrees, it is considered by all mathematicians to be valid within the set of axioms that constitute abstract mathematics. Fair enough. But, as shown in the next section, ignoring relationships naturally and inherently present between components of a system in the sciences is a recipe for creating results that don't reflect reality. That is, it is a recipe for producing experimental artifacts.

Subsystem-system and system-system relationships in the real, physical universe

In the physical universe, system-subsystem and system-system relationships are of critical importance, and ignoring them in experiments can cause incorrect results, or experimental artifacts. This is true in all the sciences. Two common sense examples are as follows. In chemistry, if one were to study a single copper metal atom in isolation, one might think that copper cannot conduct electricity. However, this would be an experimental artifact because it is well known that copper in bulk, such as in a wire, does conduct electricity. This is because relationships between neighboring copper atoms let them share their electrons, which allows the electrons to flow more freely when a voltage is applied to the metal. In botany, if someone interested in the growth of a plant in a jungle grew this plant in the laboratory, the results would likely be different than the growth of that plant in its native forest environment where interactions with other plants, the weather, etc. affect plant growth. Thus, system-subsystem relationships cannot be overlooked in these chemistry and botany examples.

This also goes for fundamental physics. In special relativity, for instance, the length of an object or the rate of time's passage in inertial reference frame A as seen by an observer in inertial reference frame B will change depending on the velocity of A relative to B (Elani, 2020). Obviously, the relationship of one physical reference frame to another is of physical importance, and ignoring it can produce incorrect results. Furthermore, in the commonly accepted Copenhagen interpretation of quantum mechanics, the observer-observed relationship is of great importance. The observer or measuring apparatus interacts with and can affect the observed (Faye, 2019) which suggests that these two parts of the universe are physically related. Finally, the demonstration of seemingly instantaneous, or non-local, interactions between entities that once were close together but are now far apart (reviewed in Berkovitz, 2016) suggests an intimate and instantaneous relationship between these entangled systems even if they are separated.

The importance of system-subsystem and system-system relationships in the physical universe are also demonstrated by the presence of experimental artifacts or biases when these relationships are ignored. For instance, physics employs extensive use of advanced mathematics and computer simulations, which often require that a part of the system being studied is simplified, or “idealized”, to simplify the mathematics or reduce computational demands. It is argued that this idealization and simplification is analogous to ignoring significant aspects of the subsystem-system interaction, just as the cell culture system does in biology, and can lead to artifacts. Indeed, such idealization-related artifacts are widely known in general relativity, cosmology and astrophysics (Barausse et al., 2021; Gueguen, 2019), condensed matter physics (Langmann, 2019; Savin, 2016, Walet, N. R. and Moore, M.A., 2013) and many other specialties in physics.

Beyond chemistry and physics, relationships between subsystems and their parent systems are of fundamental importance in biology as well. For instance, if one wants to study the function of heart cells, a common experimental method for doing this is to remove the heart from the organism, and then separate out the different kinds of cells from the heart. The cells of interest are then cultured in Petri dishes and studied in this in vitro, or "non-physiological" (outside the whole organism) setting. However, while very valuable, it is widely known that the results in cell culture are often different than what occurs in the intact organism (Vertrees, 2009) because studying the cells in culture ignores the relationships in the intact body between the heart cells, the rest of the heart tissue, the rest of the organism and the environment. It also introduces conditions not present in the natural system (Pamies and Hartung, 2017). This results in experimental artifacts, which are experimental results that don't occur in the original system but are instead due to the experimental method used. Biologists always know that the experimental results found from studying something in vitro in cell culture are not necessarily representative of what occurs in the whole organism and must therefore be confirmed in the whole organism. In sum, the physical relationships between one part of a system and the overall system from which it came cannot be ignored in biology.

System-system relationships are also of critical importance in biology. For example, as one of the main centers of metabolism in the body, the liver affects and interacts with many other organ systems. Insulin, produced by the pancreas, travels through the blood and affects glucose uptake by muscle and fat tissue. Pheromones produced by one animal affect the behavior of other animals. Plants, animals and microbes all interact with each other and the environment within an ecosystem. To ignore relationships between neighboring or communicating systems in biology is the same as not fully understanding how biological systems function.

In sum, it seems clear that system-subsystem and system-system relationships are of key importance and cannot be overlooked in the real, physical universe and that removal of a subsystem from a system drastically alters the results compared to the overall system-subsystem whole.

Do the abstract mathematical axioms about infinite sets and infinities cause problems in studies of the real, physical universe?

As described above, system-subsystem relationships can be ignored in the pairing-off method in the mathematics of infinities, but they cannot be ignored in the real, physical universe without the risk of producing incorrect results. Nor can the relationships between neighboring systems be ignored. Physics is the study of the real, physical universe, and mathematics is used extensively in physics. So, should the abstract mathematical results about infinite sets and infinities, with their built-in ignoring of the relationships in a sequence of numbers, really be used without change in physics? I suggest the answer is no and that their use may need to be modified for studies in physics.

That theoretical physics has problems with infinities and infinitesimals is widely recognized. One of the earliest of these problems is, of course, Zeno's Paradoxes of motion. One of these, the Dichotomy Paradox, suggests that motion is impossible because for someone to move from point A to B requires that they always first have to move halfway from where they were to where they're going. This process continues forever, making movement of any distance impossible (reviewed in Huggett, 2019). The most common solution is that motion only occurs over unit-sized, or finite, distances (Cote, 2013) and times (Huggett, 2019). But, this seems to be contrary to the requirement for continuous space in relativity (Dowker, 2014). So, solutions to Zeno's Paradoxes seem to conflict with relativity. Problems with infinities in classical physics have been discussed by Van Bendegem (1992). More relevant to modern physics is that infinities are abundant in quantum field theory and necessitate the troublesome renormalization technique to remove them (Nicolai, 2009). While this is now accepted, general relativity and, therefore, quantum gravity are non-renormalizable (Doboszewski and Linnemann, 2018; Hossenfelder, 2013). Another issue with infinities is the non-renormalizability of inflationary cosmology (Fumagalli, J. et al., 2020). The problems with infinities in physics is such that Tegmark even suggests that physicists look for ways of doing away with them (Tegmark, 2015).

Mathematics is the language of physics. However, the above results make it seem possible that the use of the mathematics of infinities may cause problems and incorrect results and that their use should be modified in fundamental physics.

Conclusions

System-subsystem relationships may be ignorable in the set of axioms that constitute abstract mathematics as described above for the pairing-off method applied to the positive integers, but the evidence presented here makes it clear that neither they nor system-system relationships can be overlooked in the real, physical universe., and if they are, this can cause incorrect results. Indeed, the mathematics of infinities are widely known to cause problems in modern theoretical physics. This suggests that at least in the most fundamental of sciences, physics, the use of the pairing-off method and the mathematics of infinities may need to be modified.

An objection to this argument might be that physics has made great progress while using the mathematics of infinities, so why change? While this is true, theoretical physics seems to be at a standstill of late, and this is partly due to problems with infinities. The renormalization technique to remove infinities from equations in quantum physics was problematic for many years, and, as described, some fundamental physics theories such as general relativity, quantum gravity and inflationary cosmology are now plagued by non-renormalizable infinities so much so that some physicists wish to remove infinities from physics altogether (Tegmark, 2015).

It might also be said that the sequence of positive integers (1, 2, 3, 4, 5...) is nothing like the real universe, so the pairing-off method and the mathematics of infinities should not be a problem in physics. However, as shown above, infinities do cause problems in physics. Also, the pairing-off method is at the heart of set theory, which is at the heart of mathematics, and mathematics is used extensively in physics and thus, the problems with infinities cannot be ignored. Second, locations in space and moments in time have built-in spatial and temporal relationships to their neighboring locations and moments, respectively, in a very analogous way to the relationship between neighboring odd and even integers in the sequence of positive integers.

There is precedent for saying that although things can be mathematically valid, they may not occur in nature. For example, changing the sign of the time variable in physics equations is often mathematically fine, and the equation still works. But, time does not seem to run backwards in the real, physical universe.

While it is beyond the scope of this paper, there are other methods besides pairing-off for measuring the relative sizes of sets that may be of use in physics. One method is calculating the “measure” of a set. This is more of an intuitive, and physically meaningful, way of comparing the sizes of sets and subsets. Specifically, the Lebesgue measure can be used to compare the sizes of the sets in one-, two-, three- and n-dimensional spaces (Knapp, 2005) such as those found in the physical universe. Another way may be by using the concept of natural density, which is the density of a set within the set of natural numbers. The natural density of odd numbers relative to all the natural numbers is ½, for instance.

In conclusion, suppose a physicist submits a paper about the relationships found in a physical system but studies this system by destroying these built-in, inherent relationships, and then ignores the possibility that the results of this experimental processing may have been different than those that would have been obtained in the original system. I suggest that this paper would be, or should be, questioned. And, yet the mathematics of infinities is based on doing just this and is used extensively in physics. Given the known problems of infinities in physics, it seems reasonable that the mathematics of infinities should be modified for use in fundamental physics.

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