Shoemaker and Noonan appealed to indeterminacy of denotation as a way to save LEM in light of vague identity statements. By the Shoemaker–Noonan position, if 'a' and 'b' determinately denote objects, then

(1) it is true that either $a = b$ or $a \neq b$; and
(2) it is either true that $a = b$ or true that $a \neq b$.

If this appeal had worked, we could have said that the objects weren't indeterminately identical, the statements were indeterminate due to an indeterminacy of denotation. In the present section I have tried to save LEM by distinguishing between LEM and PBV and arguing that (1) can be true even though (2) is false. The explanation of vague identity statements lies neither in the LEM-violating indeterminacy of the objects referred to in these statements nor in the indeterminacy of the denotation of certain components of these statements, but in our failure to draw a sharp line between what counts as part of the same object and what doesn't.[1]

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ON SETS AND WORLDS: A REPLY TO MENZEL

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IN [4] I offered a Cantorian argument that there can be no set of all truths, and noted one application: against an approach to possible worlds in terms of maximal consistent sets of propositions. Shortly thereafter Selmer Bringsjord offered a similar argument against set-theoretical worlds in [1].

In [7], a recent and important contribution to the discussion, Christopher Menzel has raised a number of critical points regarding Bringsjord's argument. But these points also apply against my earlier and more general argument. I want to address them directly here.

I

There cannot, it appears, be a set of all truths. For essentially the same Cantorian reasons, it seems, there can be neither world-stories nor worlds construed as maximal consistent sets of propositions.

But perhaps there *can* be a non-set *something-else* of all truths, or a non-set something-else appropriate to the propositions of a world-story or a world.

Menzel does ultimately have such a something-else to offer. He first considers and rejects, however, what may seem the obvious candidate: the proper classes of Von Neumann-Bernays-Gödel (VNBG) and similar systems.

In [7] as in [8] Menzel deftly distinguishes two conditions on proper classes: excessive size and unbounded rank.[1] It is the second he takes as the 'true conceptual boundary' between sets and proper classes. If propositions are admitted as urelements, however, a world-story S will be of *bounded* rank:

> since ... propositions for the world-story theorist are not sets, they themselves have no members and so have a rank of 0; hence, S itself has a rank of 1, i.e. S is bounded. ([7], p. 70)

There is then no need to consign world-stories (or a totality of truths) to the realm of proper classes. Or so the argument goes.

Note however that Menzel's 'ranks' here are confined exclusively to sets: it is sets alone which have ranks, determined exclusively by the ranks of their members. But this may be a dangerously provincial view of rank. Hierarchies of types have of course often been thought necessary for propositions, predicates, and properties as well as sets, and in the end an *integrated* theory of types may be required. We may find it necessary to rank sets of propositions above those sets mentioned in propositions they include, for example.[2] But given an integrated ranking of this sort Menzel's argument fails to go through. It is no longer clear, in particular, that world-stories will have a rank either of 1 or of any other bounded n.

Nonetheless there *are* strong reasons to think that Menzel's *conclusion* here is right: that proper classes won't do as a something-else appropriate to world-stories or a totality of truths.

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[1] Within VNBG proper classes are those too large to be sets. Menzel defines the *rank* of an object recursively as the least ordinal greater than the ranks of all its members. A class is *unbounded* if it contains members of arbitrarily high rank.

[2] This would seem the obvious way to deal with the following variant of Russell's paradox, for example:
Some sets of propositions contain propositions which mention those sets themselves. Some sets of propositions contain no such propositions. We will refer to the latter as 'non-self-mentioning' propositional sets.
Consider now the set S of all propositions which mention only non-self-mentioning sets. Is S self-mentioning or not?

Some of the objections to proper classes are technical. Within VNBG and similar systems, for example, proper classes are kept immune from Cantorian arguments only at the cost of sacrificing general provability for instances of the induction schema (see for example [6] p. 198).

Intuitive objections are even stronger, however. For in VNBG and similar systems proper classes are introduced precisely as classes which are not members of further (even finite) classes. That would clearly not seem to hold for either a 'class of all truths' or for classes appropriate to world-stories. For surely a class of all truths *would* be a member of further classes — of various classes of propositional classes and various classes of classes of truths. Surely a class of all truths would form a pair with a class of all falsehoods, for example. Worlds and their world-stories would seem if anything *less* amenable to treatment as proper classes. For consider the class of worlds in which I exist and the class of worlds in which this vase is hit with a hammer. Current philosophical use quite generally seems to demand *classes* of possible worlds.

II

The 'something-else' that Menzel ultimately suggests involves a recourse not to amplified but to amputated set theories. Perhaps the world-story theorist, though clearly a profligate realist regarding propositions, can follow an abstemiously constructivist line when it comes to sets:

> By adopting ZF-Power (or some similarly conservative set theory), then, and adjusting it appropriately to allow for the existence of large sets (and urelements), the world-story theorist is free to postulate the existence of his world-stories without fear of paradox, at least by way of [the Cantorian] argument; for that argument depends essentially on there being a power set of the world-story S; but there is simply no way of generating the full classical power set of an infinite set in ZF-Power ... ([7], p. 71)

It is far from clear, however, that large sets within ZF-Power will prove any more satisfactory here than proper classes.

Not too surprisingly, sacrifice of the power set axiom results in quite major technical limitations; at one blow we are effectively exiled from Cantor's paradise (some surprising technical limitations of ZF-Power appear in [10]). Nor does ZF-Power applied as Menzel envisages it really seem to satisfy constructivist scruples. Menzel relies on the fact that 'there is simply no way of generating the full classical power set of an infinite set in ZF-power ...' But of course there is no way of *generating* basic infinite sets within ZF-Power at all; they must be added by fiat of special axiom. The set of zero and its successors appears by axiom within ZF, and a set of all ordinals is added by special axiom in Menzel's system in [8]. If we are to

include the reals, and world-stories, and infinite sets of world-stories within ZF-Power we will need special axiomatic provision for these as well – a prodigal postulation of sets far from constructivist in spirit.

It is not clear, moreover, that sacrifice of the power set axiom alone will be enough to escape Cantorian difficulties regarding a set of all truths. For consider the following argument.

Let us suppose we *did* have a set T of all truths within ZF-Power. Then by the axiom schema of separation, carried over directly from ZF, we would have as a subset of T a set of all truths satisfying a particular condition $B(x)$. As long as our basic language is rich enough to express the notion that a truth t is about a topic c, it appears, one such condition will be 'x is about a set of truths.' By separation, then, we would have a set C of all truths about sets of truths.

More formally, using 'Axy' to indicate that x is about y, our condition 'x is about a set of truths' is:

$$\exists y \forall z (z \in y \supset z \in T \cdot \& \ Axy).$$

The axiom schema of separation, taken directly from ZF, is

$$\forall z_1 \ldots \forall z_n \forall a \exists y \forall x (x \in y \equiv x \in a \ \& \ B(x)),$$

where $z_1, \ldots, z_n$ are the free variables of $B(x)$ other than x, and the only restriction on $B(x)$ is that it does not contain y as a free variable (see for example [6] p. 175). Using T for a in this schema and our condition above for $B(x)$ would give us a set C of all truths about sets of truths:

$$\forall x (x \in C \equiv x \in T \ \& \ \exists y \forall z (z \in y \supset z \in T \cdot \& \ Axy)).$$

The existence of a set C, however, would give us a reductio. C would, in particular, be larger than T. For consider any one-to-one function f from T into C, mapping truths onto truths concerning sets of truths, and consider further a set c' of all truths which do not belong to the sets their assigned element is about. Here using '$a(f(x))$' to indicate the set (or union of sets) $f(x)$ is about,

$$x \in c' \equiv x \in T \ \& \ x \notin a(f(x)).$$

Clearly C will contain some truth about c' just as it contains some truth regarding any set of truths. But by familiar reasoning f can assign no element of T to any truth about c'. C is larger than T.

One branch of the reductio, then, is this: given a set T of all truths, it appears, there would be a subset C larger than the set T of which it is a subset.[3]

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[3] Note that this is not a problem for sets in ZF-Power in general; the argument above relies essentially on the supposition that T is a set of all *truths*.

Here and elsewhere I am obliged to an anonymous referee for helpful comments.

Another branch is this. Each element of C is a truth. There are thus more truths than elements of T, and thus T cannot, as assumed, be a set of *all* truths. A similar argument would of course apply to world-stories.

In order to escape the Cantorian difficulties of C, it appears, something more would have to be sacrificed — something above and beyond the power set axiom. Perhaps additional restrictions on the axiom schema of separation are called for. Perhaps the language of the system must exclude or restrict the notion of truths about sets of truths — though since there clearly *are* truths about any sets of truths this would seem far from satisfactory. Or perhaps here again we should simply conclude that there is no set T of all truths.

<p style="text-align:center">III</p>

Variations on the Cantorian argument can be used to show that there is no set of all necessary truths, no set of all falsehoods, no set of all propositions, and no set of all things known by an omniscient being (see also [3] and [5]). If extensions are construed standardly in terms of sets, 'true', 'false', 'necessary', and 'is a proposition' have no extensions. Since the conjunction of a necessary and a contingent proposition is itself contingent, moreover, the argument can also be adapted to show there is no set of all *contingent* truths or falsehoods.[4] Nor is there any need to interpret the argument as restricted to linguistic entities in any sense; the argument would be the same against a set of all facts or all states of affairs.

In a concluding caveat, however, Menzel appeals to two outlines of possible worlds which appear to escape Cantorian arguments. One of these is Plantinga's:

> a state of affairs S is *complete* or *maximal* if for every state of affairs S′, S includes S′ or precludes S′. And a possible world is simply a possible state of affairs that is maximal. ([9], p. 45)

The other is Menzel's own:

> A world-story [could] be a set S that is maximal in the sense that for any proposition q, S *entails* (but not necessarily *contains*) either q or its negation, but not both ... ([7], p. 72)

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[4] In the standard argument (as in [4]) we envisage a necessary truth corresponding to each element of the power set. Here we envisage the conjunction of each of these with a chosen contingent proposition.

It appears that even a set of all atomic propositions will fall victim. For suppose a set A of all atomic propositions and a set C of all connectives. Within classical atomism compounds from these will give us all propositions. If A and C are sets, however, by standard principles of set theory all permutations of elements of A sprinkled with permutations of elements of C will *also* form a set. We know independently that there can be no set of all propositions, however, and thus by modus tollens it appears there can be no set of all classical atomic propositions.

Do these approaches offer an escape from Cantorian difficulties of sets and worlds?

That is unclear. Menzel's outline demands universal quantification over propositions, Plantinga's calls for quantification over all states of affairs, and it appears that any genuine explication of the notion of 'maximality' crucial to possible worlds must be similarly quantificational in form. The lingering difficulty here is the following. By our Cantorian arguments there can be neither a set of all propositions nor a set of all states of affairs. But the only real semantics for *quantification* we have is in terms of sets.[5]

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[5] In [2], p. 262, Martin Davies suggests further difficulties: a Cantorian argument against a set of all worlds, which would seem to apply however worlds are conceived. Crucial premises of Davies' argument, however, involve principles both messy and intriguing regarding philosophy of mind and the conceivability of propositions.