



A new paradox and the reconciliation of Lorentz and Galilean transformations

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Abstract

One of the most debated problems in the foundations of the special relativity theory is the role of conventionality. A common belief is that the Lorentz transformation is correct but the Galilean transformation is wrong (only approximately correct in low speed limit). It is another common belief that the Galilean transformation is incompatible with Maxwell equations. However, the “principle of general covariance” in general relativity makes any spacetime coordinate transformation equally valid. This includes the Galilean transformation as well. This renders a new paradox. This new paradox is resolved with the argument that the Galilean transformation is equivalent to the Lorentz transformation. The resolution of this new paradox also provides the most straightforward resolution of an older paradox which is due to Selleri in (Found Phys Lett 10:73–83, 1997). I also present a consistent electrodynamics formulation including Maxwell equations and electromagnetic wave equations under the Galilean transformation, in the exact form for any high speed, rather than in low speed approximation. Electrodynamics in rotating reference frames is rarely addressed in textbooks. The presented formulation of electrodynamics under the Galilean transformation even works well in rotating frames if we replace the constant velocity \mathbf{v} with $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. This provides a practical tool for applications of electrodynamics in rotating frames. When electrodynamics is concerned, between two inertial reference frames, both Galilean and Lorentz transformations are equally valid, but the Lorentz transformation is more convenient. In rotating frames, although the Galilean electrodynamics does not seem convenient, it could be the most convenient formulation compared with other transformations, due to the intrinsic complex nature of the problem.

Keywords New paradox · Selleri’s paradox · Sagnac effect · Maxwell equations · Conventionality of simultaneity · Equivalence · General relativity

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1 Introduction

Issac Newton holds an absolute view of space and time. Newtonian mechanics is based on the Galilean transformation between two inertial reference frames K and K' . Einstein's theory of special relativity is based on the Lorentz transformation, so that the speed of light in both reference frames is the same constant c in all directions.

Given the fact that we agree on that the Lorentz transformation is a *valid* transformation from K to K' , another question arises: Is the Lorentz transformation the *only valid* transformation? There are different opinions. A person with a “yes” answer should be called a strong relativist. A person with a “no” answer is usually called a conventionalist.

There are three different philosophical views in the spectrum: the absolutist, the relativist and the conventionalist. Each group is opposed to the *other two groups* in opinion. The following are a few examples in each group.

(1) Absolutist

Some examples include G. Sagnac, F. Selleri and P. Marmet. Selleri believes in the existence of absolute simultaneity, but not exactly the absolute time scale. He believes that reference frame K' has a different time scale from K but both K and K' must have the same simultaneity standard (two remote events simultaneous in K must also be simultaneous in K'). He investigated a transformation (Selleri 1996),

$$\begin{aligned}t' &= t/\gamma, \\x' &= \gamma(x - vt),\end{aligned}\tag{1}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Selleri calls this the “inertial transformation” and it has the following properties:

(i) If two events are simultaneous in K , then they are also simultaneous in any other frame K' .

Earlier, Tangherlini (1961) and Mansouri and Sexl (1977) also investigated this transformation in Eq. (1). Mansouri and Sexl call it “absolute simultaneity”. The difference is, Mansouri and Sexl have a conventionalist view, considering that Eq. (1) is equivalent to the Lorentz transformation, while Selleri takes an absolute view, regarding Eq. (1) as “nature’s choice” (Selleri 1996) and believes that special relativity is wrong. In Selleri’s words (Selleri 1997): “we must conclude that the famous synchronisation problem is solved by nature itself: it is not true that the synchronisation procedure can be chosen freely because Einstein’s convention leads to an unacceptable discontinuity in the physical theory.” This is known as Selleri’s paradox, which we shall discuss further in Sect. 5.

(ii) If the one-way speed of light in K is constant in all directions, then in reference frame K' , the light speed is variable in different directions,

$$c_\theta = \frac{c}{1 + \beta \cos \theta},\tag{2}$$

where $\beta = v/c$, and θ is the angle of the direction with respect to x' -axis. Among all the inertial frames with various parameter v , there is only one reference frame (where

$v = 0$) in which the one-way speed of light is constant c in all directions. It is called the “privileged frame” by Selleri (1996, 1997) or the “ether frame” by Mansouri and Sexl (1977).

(iii) The two-way speed (or round-trip speed) of light in any direction, in any reference frame, is the same constant c .

(2) Strong relativist

Some authors, for example, Ohanian, hold very strong anticonventionalist views. Ohanian (2009) published a book *Einstein’s Mistakes*. A number of the “mistakes” of Einstein that Ohanian criticized in his book are actually the conventionalist views of Einstein, which in my opinion are not Einstein’s mistakes.

(3) Weak conventionalist

Reichenbach (1958) believes that the “Einstein synchronization” is a convention. The one-way speeds of light from A to B and from B to A are not necessarily equal, as long as the two-way speed of light is a constant in all directions. The synchronization of the clock at B may depend on an arbitrary parameter $0 < \varepsilon < 1$, with Einstein synchronization being a special case with $\varepsilon = 1/2$.

Several authors (Edwards 1963; Robertson 1949; Mansouri and Sexl 1977) have developed this idea further quantitatively. These are known as “test theories of special relativity”. Zhang (1997) offered an analysis and comparison of these test theories. Edwards (1963) investigated a transformation (Zhang 1997, p. 80),

$$\begin{aligned} t' &= \gamma \left[\left(1 + \frac{b}{c}v \right) t - \left(\frac{v}{c^2} + \frac{b}{c} \right) x \right], \\ x' &= \gamma(x - vt), \end{aligned} \quad (3)$$

where v is the velocity of reference frame K' relative to K along x -axis, and b is an arbitrary but predefined constant, which is specific to frame K' . The light has a variable one-way speed depending on the direction θ in the Edwards transformation,

$$c_\theta = \frac{c}{1 - b \cos \theta}. \quad (4)$$

However, the two-way speed of light is always c , a constant. The Edwards transformation is the most general form of transformation which has anisotropic one-way light speed while keeping isotropic two-way light speed. Selleri’s inertial transformation has not been compared with these test theories in literature. It is easy to see, if $b = 0$, the Edwards transformation in Eq. (3) reduces to the Lorentz transformation; if $b = -v/c$, it reduces to Selleri’s “inertial transformation” as in Eq. (1).

Einstein had a limited conventionalist view. Einstein (1961) gave an interpretation of his special relativity: “That light requires the same time to traverse the path $A \rightarrow M$ as for the path $B \rightarrow M$ is in reality *neither a supposition nor a hypothesis* about the physical nature of light, but a *stipulation* which I can make of my own freewill in order to arrive at a definition of simultaneity.” I agree with Einstein on this, but this is considered one of the “Einstein’s mistakes” by Ohanian.

Ohanian has strong objections to this view. He writes (Ohanian 2009): “he was stipulating something that was not subject to. ... Einstein was entitled to make a

hypothesis about the speed of light, but not a stipulation. The speed of light is either constant or not, and only measurement can decide what it is. ... Einstein was lucky... What he had asserted by stipulation actually was confirmed by experiment. In the end, he turned out to have been right for the wrong reason.” I disagree with Ohanian on this. The reader is referred to a comprehensive review of the whole spectrum of views regarding the conventionality thesis by Anderson et al. (1998).

2 My paradox

It is a common understanding that Einstein’s theory of special relativity is a revolution against Newtonian mechanics. The theory of special relativity is a refutation and modification of the classical Newtonian mechanics. What is the relationship between the theory of special relativity (SR) and the theory of general relativity (GR)? Do SR and GR contradict each other, or is SR a special case of GR? The common understanding is that SR is consistent with GR and is a special case of GR under two conditions: (i) Only inertial reference frames are concerned. (ii) There is no existence of gravity. This means GR always applies where SR applies.

My paradox is this: the following two views of (A) from SR and (B) from GR contradict each other:

(A) In SR, this is taught in most of the textbooks: Michelson and Morley conducted an experiment (M-M experiment) to detect the motion of the earth relative to the ether. They applied the Galilean transformation and showed it would predict a fringe shift when the two perpendicular arms of the interferometer are slowly rotated. No fringe shift was detected in the experiment. Einstein then proposed the theory of relativity with the principle of relativity and the principle of constancy of the speed of light. So M-M experiment renders a verdict: Newtonian mechanics is wrong and SR is correct; the Galilean transformation is wrong and the Lorentz transformation is correct. Galilean transformation is only approximately correct in the low speed limit because it is an approximation of Lorentz transformation in the low speed limit.

(B) In GR, according to the “principle of general covariance” of Einstein (1916), any form of smooth (or diffeomorphic) coordinate transformation

$$\begin{aligned}x' &= x'(x, y, z, t), \\y' &= y'(x, y, z, t), \\z' &= z'(x, y, z, t), \\t' &= t'(x, y, z, t),\end{aligned}\tag{5}$$

is set to equal footing (see Sect. 3 for a quotation from Einstein’s 1916 paper). The Galilean transformation

$$\begin{aligned}x' &= x - vt, \\y' &= y, \\z' &= z,\end{aligned}$$

$$t' = t, \quad (6)$$

is certainly just a special case of Eq. (5), and should be as valid as the Lorentz transformation.

In the following sections, I shall provide a resolution to this paradox: the Galilean transformation is equivalent to the Lorentz transformation in describing physically observable phenomena. This view is in accordance with Einstein's principle of covariance in general relativity.

3 Equivalence of Galilean transformation and Lorentz transformation

In his paper *The Foundation of the General Theory of Relativity*, Einstein (1916) advocates for the principle of general co-variance:

“The general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any substitutions whatever (general co-variant).

... All our space-time verifications invariably amount to a determination of space-time coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meetings of two or more of these points. Moreover, the results of our measurements are nothing but verifications of such meetings of the material points of our measuring instruments with other material points, coincidences between the hands of a clock and points on the clock dial, and observed point-events happening at the same place at the same time.

The introduction of a system of reference serves no other purpose than to facilitate the description of the totality of such coincidences. We allot to the universe four space-time variables x_1, x_2, x_3, x_4 in such a way that for every point-event there is a corresponding system of values of the variables $x_1 \dots x_4$. To two coincident point-events there corresponds one system of values of the variables $x_1 \dots x_4$, i.e. coincidence is characterized by the identity of the co-ordinates. If, in place of the variables $x_1 \dots x_4$, we introduce functions of them, x'_1, x'_2, x'_3, x'_4 , as a new system of co-ordinates, so that the systems of values are made to correspond to one another without ambiguity, the equality of all four co-ordinates in the new system will also serve as an expression for the space-time coincidence of the two point-events. As all our physical experience can be ultimately reduced to such coincidences, there is no immediate reason for preferring certain systems of co-ordinates to others, that is to say, we arrive at the requirement of general co-variance.”

The “principle of general covariance” is straightforward enough to establish that the Galilean transformation is equivalent to the Lorentz transformation.

It is still a common belief that the Galilean transformation is wrong because it implies infinite signal speed. This is not true. What is true is its converse statement— infinite signal speed implies the possible implementation of the Galilean transformation, but not the other way around. When the clock at location A reads time t_A , we

send a signal to location B with infinite speed, and set the clock at B the same as t_A . The clocks at different locations coordinated in such a way will obey the Galilean transformation.

What is not immediately intuitive is that even with finite signal (light) speed, we can still coordinate clocks in different locations in such a way that they obey the Galilean transformation from one reference frame to another. I shall illustrate this in the following.

Given an arbitrary reference frame K, we adopt Einstein time coordination protocol in it so that the one-way light speed is a constant c in all directions. I shall call K the *primary Galilean reference frame* or simply the *primary frame*. Note the primary Galilean reference frame is by arbitrary human choice, instead of the unique choice dictated by nature as Newton believed.

Notation: We use upper case letters (X, Y, Z, T) to denote the space and time coordinates in the primary frame K.

Now consider another reference frame K' which has velocity v relative to K in the X direction. We can again perform the Einstein time coordination protocol in K' , the space and time coordinates are denoted by (x_E, y_E, z_E, t_E) . These clocks in K' measuring t_E are called E-clocks.

Now we define a coordinate transformation *inside* reference frame K' as follows.

$$\begin{aligned} t_N &\stackrel{\text{def}}{=} \gamma t_E + \frac{\gamma v}{c^2} x_E, \\ x_N &\stackrel{\text{def}}{=} \frac{1}{\gamma} x_E, \\ y_N &\stackrel{\text{def}}{=} y_E, \\ z_N &\stackrel{\text{def}}{=} z_E, \end{aligned} \tag{7}$$

where $\stackrel{\text{def}}{=}$ denotes equal by definition. Again notice, Eq. (7) is also just a special case of the most general coordinate transformation Eq. (5), which is allowed by Einstein. Note Eq. (7) is not a transformation between two reference frames, but rather the internal re-coordination of the space coordinates and time coordinate separately for the same reference frame K' .

Starting from the assumption that (x_E, y_E, z_E, t_E) and (X, Y, Z, T) are related by the Lorentz transformation, it just needs some straightforward calculation to prove that (x_N, y_N, z_N, t_N) and (X, Y, Z, T) are related by the Galilean transformation (see Fig. 1),

$$\begin{aligned} t_N &= T, \\ x_N &= X - vT, \\ y_N &= Y, \\ z_N &= Z. \end{aligned} \tag{8}$$

I shall call (x_N, y_N, z_N) the Newton coordinates, and t_N the N-time for frame K' . They do not correspond to what we would measure with physical instruments.

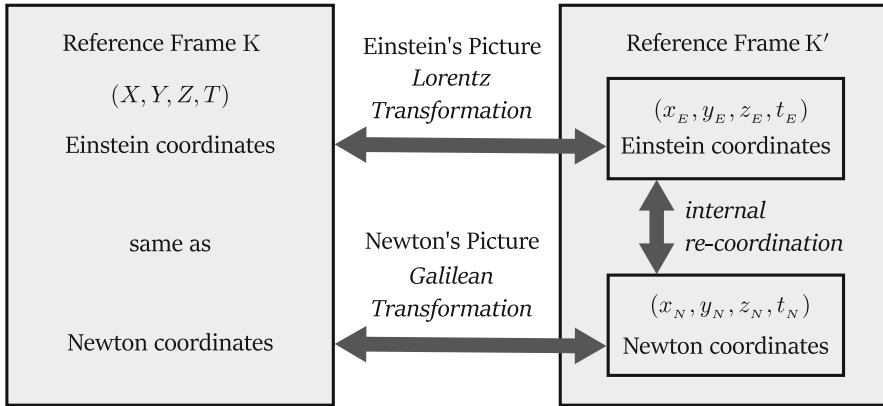
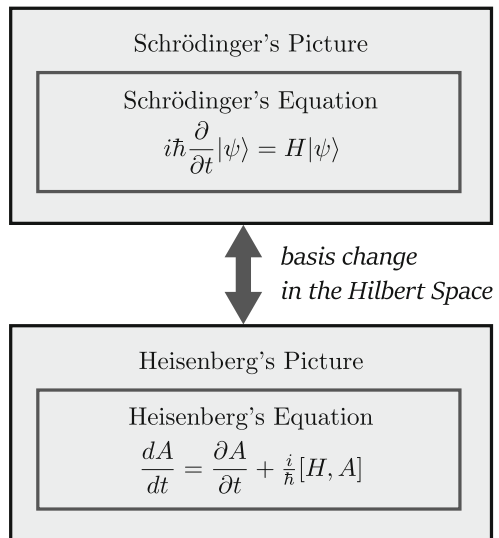


Fig. 1 Relationship between Lorentz transformation and Galilean transformation

Fig. 2 Analogy in quantum mechanics: relationship between Schrödinger's picture and Heisenberg's picture



This can be put in an analogy in quantum mechanics. It is similar to the relationship between the Schrödinger's picture and Heisenberg's picture. In the Schrödinger's picture, the operators representing physical observables are static while the quantum states evolve with time. Historically, Heisenberg took a very different approach. In the Heisenberg's picture, the state is static while the operators (or infinite dimensional matrices) representing physical observables evolve with time. If we just look at the appearances of these two theories, they seem to be totally different and unrelated. It was Dirac who revealed their relationship. They are related by a basis change in the Hilbert space. Hence Dirac united the two theories in the framework of the abstract Hilbert spaces. The Schrödinger's picture and the Heisenberg's picture are equivalent, and they describe the same physical phenomena (Fig. 2). Of course an analogy cannot substitute for a proof, but it helps us to understand the concepts.

Equation 7 suggests a practical method to implement a system of clocks called N-clocks for reference frame K' , which tell the N-time t_N . At each location in space we place an N-clock side by side with the E-clock. The N-clock can be implemented using an E-clock by embedding a computer chip in it. The computer takes the E-time t_E and its own coordinate x_E (and the parameter v) as input and then computes its output t_N according to Eq. (7). This should not be a novel idea, nor difficult in practice, as all the modern atomic clocks have utilized sophisticated electronic circuits in them.

Another note on the notation of the symbols: we use K and K' to denote two reference frames. However, for the simplicity of the appearance of symbols, we do not use the prime ($'$) to denote the coordinates in the reference frame K' , but rather, we use lower cases (without primes), (x_E, y_E, z_E, t_E) and (x_N, y_N, z_N, t_N) for the Einstein and Newton coordinates in frame K' respectively. The Einstein and Newton coordinates in frame K are identical, which are (X, Y, Z, T) .

There might be an objection to this idea of implementation of N-clocks: the computation on the computer may take a time delay, rather than giving an instant output from the input. In fact, the implementation of N-clocks can even be achieved in a much easier way, without using a computer doing the translation on the fly all the time. We realize that at any particular location with coordinate x_E in K' , t_N is linearly related to t_E in such a way $t_N = \gamma t_E + a$, where γ is a scaling factor depending on the speed v only, and $a = (\gamma v/c^2)x_E$ is an offset, which is a constant at each location x_E . We only need to re-calibrate the E-clock to obtain an N-clock by re-labeling the time unit on the clock by a factor of $1/\gamma$ and then adding a constant a once and for all.

Note that now we have two sets of coordinate and time systems for the *same reference frame* K' . E-time is no longer the unique God-given time standard for K' . When we speak of time, we must make clear whether it is E-time or N-time to avoid confusion, both of which are equally legitimate. Newton time coordination provides a different simultaneity standard from Einstein simultaneity. When we talk about distance, we must make clear whether it is E-distance or N-distance. When we talk about speed, we must make clear what coordinates and time we are using. If we use E-distance and E-time, we get E-speed. If we use N-distance and N-time, we get N-speed. Hence the N-speed of light is not a constant in K' , but this is just a different description. Both describe the same physical phenomena.

The inverse transformation of Eq. (7) can be easily obtained:

$$\begin{aligned} t_E &= \frac{1}{\gamma} t_N - \frac{\gamma v}{c^2} x_N, \\ x_E &= \gamma x_N, \\ y_E &= y_N, \\ z_E &= z_N. \end{aligned} \tag{9}$$

This formulation might seem the comeback of Newton's absolute space and absolute time, and the primary frame K might look like the absolute ether reference frame, but this is not the case. As we discussed earlier, the primary frame K is an arbitrarily choice by convention. Any inertial frame can be chosen as the primary frame K . It is "preferred by humans", but not "privileged by nature".

In reference frame K' we define the E-velocity to be

$$u_E \stackrel{\text{def}}{=} (u_{x_E}, u_{y_E}, u_{z_E}) \stackrel{\text{def}}{=} \left(\frac{dx_E}{dt_E}, \frac{dy_E}{dt_E}, \frac{dz_E}{dt_E} \right), \tag{10}$$

and N-velocity to be

$$u_N \stackrel{\text{def}}{=} (u_{x_N}, u_{y_N}, u_{z_N}) \stackrel{\text{def}}{=} \left(\frac{dx_N}{dt_N}, \frac{dy_N}{dt_N}, \frac{dz_N}{dt_N} \right). \tag{11}$$

It is straightforward to find that E-velocity and N-velocity are related by

$$\begin{aligned} u_{x_E} &= \frac{u_{x_N}}{1 - \frac{v(v+u_{x_N})}{c^2}}, \\ u_{y_E} &= \frac{u_{y_N}}{\gamma \left(1 - \frac{v(v+u_{x_N})}{c^2} \right)}, \\ u_{z_E} &= \frac{u_{z_N}}{\gamma \left(1 - \frac{v(v+u_{x_N})}{c^2} \right)}, \end{aligned} \tag{12}$$

and its inverse transformation is

$$\begin{aligned} u_{x_N} &= \frac{u_{x_E}}{\gamma^2 \left(1 + \frac{vu_{x_E}}{c^2} \right)}, \\ u_{y_N} &= \frac{u_{y_E}}{\gamma \left(1 + \frac{vu_{x_E}}{c^2} \right)}, \\ u_{z_N} &= \frac{u_{z_E}}{\gamma \left(1 + \frac{vu_{x_E}}{c^2} \right)}. \end{aligned} \tag{13}$$

Let us look at a few examples. For a light beam with E-velocity $u_{x_E} = c$, it translates to N-velocity $u_{x_N} = c - v$. For a light beam with E-velocity $u_{x_E} = -c$, it translates to N-velocity $u_{x_N} = -(c + v)$. Take another example. Let $v = 0.90c$. A mass particle with E-velocity $u_{x_E} = 0.99c$ translates to N-velocity $u_{x_N} = 0.10c$. E-velocity $u_{x_E} = -0.99c$ translates to N-velocity $u_{x_N} = -1.73c$. The magnitude of N-velocity can exceed c but the physics is the same. Newton coordination is completely as valid as Einstein coordination. Any physical phenomena which can be described by Einstein coordination can be described by (or translated to) Newton coordination as well.

4 Electrodynamics under Galilean transformation

It is also a common belief that the Galilean transformation is incompatible with the Maxwell equations. A quick rebuttal is that “non-invariant” and “incompatible” are

different concepts. The latter means “logically contradicting”, while the former does not.

Suppose in reference frame K , we adopt Einstein coordination $x^0 = t, x^1 = x, x^2 = y, x^3 = z$ and rationalized natural units (Heaviside-Lorentz units and $c = 1$). The Maxwell equations in vacuum are in the form

$$\begin{aligned}\nabla \times \mathbf{B} - \partial_t \mathbf{E} &= \mathbf{j}, \\ \nabla \cdot \mathbf{E} &= \rho, \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0, \\ \nabla \cdot \mathbf{B} &= 0.\end{aligned}\tag{14}$$

When the sources are zero, $\mathbf{j} = 0$ and $\rho = 0$, the electromagnetic wave equations are

$$\begin{aligned}(\partial_t^2 - \nabla^2) \mathbf{E} &= 0, \\ (\partial_t^2 - \nabla^2) \mathbf{B} &= 0.\end{aligned}\tag{15}$$

A particle with electric charge q and velocity \mathbf{u} is subject to Lorentz 3-force

$$\mathbf{f} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}).\tag{16}$$

We can obtain the Maxwell equations in reference frame K' using the Galilean transformation. These equations have a different *form* (and are more complex than Eq. (14)). This only means that they are not convenient but it does not mean they are wrong. I shall argue from the following two aspects.

(1) Even if the Maxwell equations in reference frame K' under Galilean transformation have a different *form* than that in K , they describe the same physical phenomena. They are the same physical law in different *forms*. Let us take an example in Newtonian mechanics. With the usual time standard, Newton's second law (in x -direction) takes the form of

$$f_x = m \frac{d^2 x}{dt^2}.\tag{17}$$

Now suppose we adopt a new time standard τ with $\tau = e^t$, Newton's second law will take the form

$$f_x = m \tau \frac{d}{d\tau} \left(\tau \frac{dx}{d\tau} \right).\tag{18}$$

Newton's law is the same but just in a different *form*. It is only a matter of convenience.

(2) The Maxwell equations are not invariant under Galilean transformation. This is only the case when the Maxwell equations are written in 3-vector form. If they are written in 4-dimensional tensor form

$$\begin{aligned}\partial_\alpha F^{\alpha\beta} &= j^\beta, \\ \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} &= 0,\end{aligned}\tag{19}$$

they are covariant under any linear transformations, which include the Lorentz transformation, and the Galilean transformation as well. The only difference is that the Lorentz transformation is pseudo-orthogonal while the Galilean transformation is not. But there is no requirement in the principle that a transformation has to be pseudo-orthogonal. In reference frame K , the spacetime quadratic form is

$$\begin{aligned}
 ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
 &= dt^2 - dx^2 - dy^2 - dz^2.
 \end{aligned}
 \tag{20}$$

The pseudo-metric tensor $g_{\mu\nu}$ is diagonal. Under Lorentz transformation, it remains diagonal. However, under Galilean transformation, it becomes non-diagonal. The Galilean transformation can be written as

$$(x')^\mu = \Lambda^\mu_\nu x^\nu,
 \tag{21}$$

where Λ is a matrix

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
 \tag{22}$$

In reference frame K' ,

$$\begin{aligned}
 ds^2 &= g'_{\mu\nu} (dx')^\mu (dx')^\nu \\
 &= (1 - v^2) (dt')^2 - 2vdt'dx' - (dx')^2 - (dy')^2 - (dz')^2,
 \end{aligned}
 \tag{23}$$

with the pseudo-metric tensor

$$g'_{\mu\nu} = \begin{bmatrix} 1/\gamma^2 - v & 0 & 0 \\ -v & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},
 \tag{24}$$

where $\gamma = 1/\sqrt{1 - v^2}$, and $c = 1$.

In the following subsections, we shall derive the Maxwell equations in 3-vector form to show they are just as valid under Galilean transformation. They take more complex forms but that is only a matter of convenience rather than validity.

In reference frame K , the contravariant and covariant field tensors $F^{\mu\nu}$, $F_{\mu\nu}$, and the 3-vector field strengths \mathbf{E} and \mathbf{B} are related by

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}, \quad F_{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}.
 \tag{25}$$

Under Galilean transformation, the charge and current transform according to

$$\begin{aligned}\rho' &= \rho, \\ \mathbf{j}' &= \mathbf{j} - \rho \mathbf{v}.\end{aligned}\quad (26)$$

We also notice the differential operators transform according to

$$\begin{aligned}\partial_{t'} &= \partial_t + \mathbf{v} \cdot \nabla, \\ \nabla' &= \nabla.\end{aligned}\quad (27)$$

In reference frame K' under Galilean transformation Eq. (6), the contravariant field tensor is

$$\begin{aligned}(F')^{\mu\nu} &= \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta} \\ &= \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -(B_3 - vE_2) & B_2 + vE_3 \\ E_2 & B_3 - vE_2 & 0 & -B_1 \\ E_3 & -(B_2 + vE_3) & B_1 & 0 \end{bmatrix}.\end{aligned}\quad (28)$$

The covariant field tensor is

$$\begin{aligned}F'_{\mu\nu} &= g'_{\mu\alpha} g'_{\nu\beta} F^{\alpha\beta} \\ &= \begin{bmatrix} 0 & E_1 & E_2 - vB_3 & E_3 + vB_2 \\ -E_1 & 0 & -B_3 & B_2 \\ -(E_2 - vB_3) & B_3 & 0 & -B_1 \\ -(E_3 + vB_2) & -B_2 & B_1 & 0 \end{bmatrix}.\end{aligned}\quad (29)$$

The contravariant 4-velocity in reference frame K' is

$$(u')^\mu = \frac{d(x')^\mu}{ds} = \gamma_{u'} [1, u'_1, u'_2, u'_3]^{tr}, \quad (30)$$

where the superscript tr denotes matrix transpose and

$$\gamma_{u'} = 1/\sqrt{1 - u'^2}. \quad (31)$$

The covariant 4-velocity in reference frame K' is

$$\begin{aligned}u'_\mu &= g'_{\mu\nu} u'^\nu \\ &= \gamma_{u'} [1/\gamma^2 - vu'_1, -v - u'_1, -u'_2, -u'_3]^{tr}.\end{aligned}\quad (32)$$

The Lorentz 4-force is

$$(f')^\mu = q (F')^{\mu\nu} u'_\nu. \quad (33)$$

When we consider the Maxwell equations under Galilean transformation, the Galilean transformation only describes how the space and time coordinates transform. There is still another question: what is the transformation law for the EM fields \mathbf{E} and \mathbf{B} ? In general, the fields \mathbf{E}' and \mathbf{B}' observed in reference frame K' are not necessarily the same as those \mathbf{E} and \mathbf{B} in frame K . We know this as this is the case of the field transformation in relativistic electrodynamics which employs the Lorentz transformation for the coordinate transformation. The field transformation in relativity is motivated by the wish that the Maxwell equations keep the same form in frame K' . With Galilean coordinate transformation, we know that the Maxwell equations cannot keep the same form. Then what will be the correct form of field transformation?

In my opinion, the field transformation is not the nature's law. It can be arbitrary by convention. Our consideration is again convenience rather than absolute truth. The EM field as a whole is described by the field tensor $F^{\mu\nu}$. We view this field tensor $F^{\mu\nu}$ more essential than the 3-vectors \mathbf{E} and \mathbf{B} , which are just some names of the components of $F^{\mu\nu}$. Because the Lorentz transformation is pseudo-orthogonal, the contravariant components $F^{\mu\nu}$ and the covariant components $F_{\mu\nu}$ are the same, or differ by a minus sign. It does not make a difference whether we use the contravariant or covariant tensor. However, since the Galilean transformation is not pseudo-orthogonal, the contravariant components $F^{\mu\nu}$ and the covariant components $F_{\mu\nu}$ are different. This leaves us freedom of either choosing the contravariant components $F^{\mu\nu}$ or the covariant components $F_{\mu\nu}$, or a mixture of them, to be the field \mathbf{E}' and \mathbf{B}' . This in turn leads to different field transformation laws.

The transformation of the field is not directly observable. What matters to the physical phenomenon is the field together with how the field interacts with matter (Lorentz force law). In the following subsections, we discuss the four different choices of using the contravariant or covariant tensors, or a mixture of both, which lead to four different field transformation laws, the form of Maxwell equations, the equation of EM waves, as well as the Lorentz force law. In the appendix, we show a proof that all these approaches are equivalent, meaning leading to the same numerical values of the Lorentz force, despite the different forms of equations.

In the following subsections, we use the symbols $\mathbf{E}' = (E'_1, E'_2, E'_3)$, $\mathbf{B}' = (B'_1, B'_2, B'_3)$ for electric and magnetic field strengths, and auxiliary variables $\mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}'_3, \mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$, which can be viewed as the duals of \mathbf{E}' and \mathbf{B}' . To avoid confusion, note in each subsection, these symbols have different definitions, reflecting different views. (It is an abuse of notation to use the same symbol for different meanings, but these symbols are used in different subsections so that we don't have a fear of confusion. The reason for doing so is to make the symbols less messy by omitting extra subscripts or superscripts to distinguish different versions of them.)

Note that some of the equations in the following formulations could be further simplified, given the fact that \mathbf{v} is a constant, but we choose not to. This way, these formulas will be applicable in a broader context, namely in rotating reference frames (Sect. 7), where we only have to replace \mathbf{v} with $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

4.1 EM field in the form of contravariant tensor $F^{\mu\nu}$

We choose to define the electric and magnetic fields in reference frame K' as

$$\begin{bmatrix} E'_1 \\ E'_2 \\ E'_3 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} (F')^{10} \\ (F')^{20} \\ (F')^{30} \end{bmatrix}, \quad \begin{bmatrix} B'_1 \\ B'_2 \\ B'_3 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} (F')^{32} \\ (F')^{13} \\ (F')^{21} \end{bmatrix}. \quad (34)$$

This means

$$(F')^{\mu\nu} = \begin{bmatrix} 0 & -E'_1 & -E'_2 & -E'_3 \\ E'_1 & 0 & -B'_3 & B'_2 \\ E'_2 & B'_3 & 0 & -B'_1 \\ E'_3 & -B'_2 & B'_1 & 0 \end{bmatrix}. \quad (35)$$

Comparing with Eq. (28), we find the field transformation in 3-vector form is

$$\begin{aligned} \mathbf{E}' &= \mathbf{E}, \\ \mathbf{B}' &= \mathbf{B} - \mathbf{v} \times \mathbf{E}. \end{aligned} \quad (36)$$

The inverse transformation of Eq. (36) is

$$\begin{aligned} \mathbf{E} &= \mathbf{E}', \\ \mathbf{B} &= \mathbf{B}' + \mathbf{v} \times \mathbf{E}'. \end{aligned} \quad (37)$$

The Maxwell equations in 3-vector form are

$$\begin{aligned} \nabla' \times (\mathbf{B}' + \mathbf{v} \times \mathbf{E}') - (\partial_{t'} - \mathbf{v} \cdot \nabla') \mathbf{E}' &= \mathbf{j}' + \rho' \mathbf{v}, \\ \nabla' \cdot \mathbf{E}' &= \rho', \\ \nabla' \times \mathbf{E}' + (\partial_{t'} - \mathbf{v} \cdot \nabla') (\mathbf{B}' + \mathbf{v} \times \mathbf{E}') &= 0, \\ \nabla' \cdot (\mathbf{B}' + \mathbf{v} \times \mathbf{E}') &= 0. \end{aligned} \quad (38)$$

This can be simplified to

$$\begin{aligned} \nabla' \times \mathbf{B}' - \partial_{t'} \mathbf{E}' &= \mathbf{j}', \\ \nabla \cdot \mathbf{E}' &= \rho', \\ \nabla' \times \mathbf{E}' + \partial_{t'} \mathbf{B}' &= (\mathbf{v} \cdot \nabla') (\mathbf{B}' + \mathbf{v} \times \mathbf{E}') - \mathbf{v} \times \partial_{t'} \mathbf{E}', \\ \nabla \cdot \mathbf{B}' &= -\nabla' \cdot (\mathbf{v} \times \mathbf{E}'). \end{aligned} \quad (39)$$

The first two equations are Galilean invariant. When the sources are zero, the equations of electromagnetic wave in vacuum are

$$\begin{aligned} [(\partial_{t'} - \mathbf{v} \cdot \nabla')^2 - \nabla'^2] \mathbf{E}' &= 0, \\ [(\partial_{t'} - \mathbf{v} \cdot \nabla')^2 - \nabla'^2] (\mathbf{B}' + \mathbf{v} \times \mathbf{E}') &= 0. \end{aligned} \quad (40)$$

The Lorentz 4-force is

$$\begin{aligned}
 (f')^\mu &= q (F')^{\mu\nu} u'_\nu \\
 &= q\gamma u' \begin{bmatrix} (u'_1 + v)E'_1 + u'_2E'_2 + u'_3E'_3 \\ (1/\gamma^2 - vu'_1)E'_1 + u'_2B'_3 - u'_3B'_2 \\ (1/\gamma^2 - vu'_1)E'_2 - (u'_1 + v)B'_3 + u'_3B'_1 \\ (1/\gamma^2 - vu'_1)E'_3 + (u'_1 + v)B'_2 - u'_2B'_1 \end{bmatrix}. \tag{41}
 \end{aligned}$$

The Lorentz 3-force is

$$\mathbf{f}' = q \{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] \mathbf{E}' + (\mathbf{u}' + \mathbf{v}) \times \mathbf{B}' \}. \tag{42}$$

4.2 EM field in the form of covariant field tensor $F_{\mu\nu}$

We choose to define the electric and magnetic fields in reference frame K' as

$$\begin{bmatrix} E'_1 \\ E'_2 \\ E'_3 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} F'_{01} \\ F'_{02} \\ F'_{03} \end{bmatrix}, \quad \begin{bmatrix} B'_1 \\ B'_2 \\ B'_3 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} F'_{32} \\ F'_{13} \\ F'_{21} \end{bmatrix}. \tag{43}$$

We also define auxiliary dual fields $\mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}'_3, \mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$ such that

$$(F')^{\mu\nu} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -\mathcal{E}'_1 & -\mathcal{E}'_2 & -\mathcal{E}'_3 \\ \mathcal{E}'_1 & 0 & -\mathcal{B}'_3 & \mathcal{B}'_2 \\ \mathcal{E}'_2 & \mathcal{B}'_3 & 0 & -\mathcal{B}'_1 \\ \mathcal{E}'_3 & -\mathcal{B}'_2 & \mathcal{B}'_1 & 0 \end{bmatrix}, \quad F'_{\mu\nu} = \begin{bmatrix} 0 & E'_1 & E'_2 & E'_3 \\ -E'_1 & 0 & -B'_3 & B'_2 \\ -E'_2 & B'_3 & 0 & -B'_1 \\ -E'_3 & -B'_2 & B'_1 & 0 \end{bmatrix}. \tag{44}$$

Comparing with Eq. (29), we find the field transformation in 3-vector form is

$$\begin{aligned}
 \mathbf{E}' &= \mathbf{E} + \mathbf{v} \times \mathbf{B}, \\
 \mathbf{B}' &= \mathbf{B}. \tag{45}
 \end{aligned}$$

The dual field vectors \mathcal{E}' and \mathcal{B}' are related to \mathbf{E}' and \mathbf{B}' through

$$\begin{aligned}
 \mathcal{E}' &= \mathbf{E}' - \mathbf{v} \times \mathbf{B}', \\
 \mathcal{B}' &= \mathbf{B}' - \mathbf{v} \times (\mathbf{E}' - \mathbf{v} \times \mathbf{B}'). \tag{46}
 \end{aligned}$$

The inverse transformation of Eq. (45) is

$$\begin{aligned}
 \mathbf{E} &= \mathbf{E}' - \mathbf{v} \times \mathbf{B}', \\
 \mathbf{B} &= \mathbf{B}'. \tag{47}
 \end{aligned}$$

The Maxwell equations in 3-vector form are

$$\begin{aligned}\nabla' \times \mathbf{B}' - (\partial_{t'} - \mathbf{v} \cdot \nabla') (\mathbf{E}' - \mathbf{v} \times \mathbf{B}') &= \mathbf{j}' + \rho' \mathbf{v}, \\ \nabla' \cdot (\mathbf{E}' - \mathbf{v} \times \mathbf{B}') &= \rho', \\ \nabla' \times (\mathbf{E}' - \mathbf{v} \times \mathbf{B}') + (\partial_{t'} - \mathbf{v} \cdot \nabla') \mathbf{B}' &= 0, \\ \nabla' \cdot \mathbf{B}' &= 0.\end{aligned}\quad (48)$$

This can be simplified to

$$\begin{aligned}\nabla' \times \mathbf{B}' - \partial_{t'} \mathbf{E}' &= \mathbf{j}' + \nabla' \times [\mathbf{v} \times (\mathbf{E}' - \mathbf{v} \times \mathbf{B}')] + \mathbf{v} \times (\nabla' \times \mathbf{E}'), \\ \nabla' \cdot \mathbf{E}' &= \rho' + \nabla' \cdot (\mathbf{v} \times \mathbf{B}'), \\ \nabla' \times \mathbf{E}' + \partial_{t'} \mathbf{B}' &= 0, \\ \nabla' \cdot \mathbf{B}' &= 0.\end{aligned}\quad (49)$$

The last two equations are Galilean invariant. When the sources are zero, the equations of electromagnetic wave in vacuum are

$$\begin{aligned}[(\partial_{t'} - \mathbf{v} \cdot \nabla')^2 - \nabla'^2] (\mathbf{E}' - \mathbf{v} \times \mathbf{B}') &= 0, \\ [(\partial_{t'} - \mathbf{v} \cdot \nabla')^2 - \nabla'^2] \mathbf{B}' &= 0.\end{aligned}\quad (50)$$

The Lorentz 4-force is

$$\begin{aligned}(f')^\mu &= q (F')^{\mu\nu} u'_\nu \\ &= q \gamma u' \begin{bmatrix} (u'_1 + v) \mathcal{E}'_1 + u'_2 \mathcal{E}'_2 + u'_3 \mathcal{E}'_3 \\ (1/\gamma^2 - v u'_1) \mathcal{E}'_1 + u'_2 \mathcal{B}'_3 - u'_3 \mathcal{B}'_2 \\ (1/\gamma^2 - v u'_1) \mathcal{E}'_2 - (u'_1 + v) \mathcal{B}'_3 + u'_3 \mathcal{B}'_1 \\ (1/\gamma^2 - v u'_1) \mathcal{E}'_3 + (u'_1 + v) \mathcal{B}'_2 - u'_2 \mathcal{B}'_1 \end{bmatrix}.\end{aligned}\quad (51)$$

The Lorentz 3-force is

$$\begin{aligned}\mathbf{f}' &= q \{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] \mathcal{E}' + (\mathbf{u}' + \mathbf{v}) \times \mathcal{B}' \} \\ &= q \{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] [\mathbf{E}' - \mathbf{v} \times \mathbf{B}'] + (\mathbf{u}' + \mathbf{v}) \times [\mathbf{B}' - \mathbf{v} \times (\mathbf{E}' - \mathbf{v} \times \mathbf{B}')] \}.\end{aligned}\quad (52)$$

4.3 EM field in the form of a mixture of $F_{\mu\nu}$ and $F^{\mu\nu}$

We choose to define the electric and magnetic fields in reference frame K' as

$$\begin{bmatrix} E'_1 \\ E'_2 \\ E'_3 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} F'_{01} \\ F'_{02} \\ F'_{03} \end{bmatrix}, \quad \begin{bmatrix} B'_1 \\ B'_2 \\ B'_3 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} (F')^{32} \\ (F')^{13} \\ (F')^{21} \end{bmatrix}.\quad (53)$$

We also define auxiliary dual fields $\mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}'_3, \mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$ such that

$$(F')^{\mu\nu} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -\mathcal{E}'_1 & -\mathcal{E}'_2 & -\mathcal{E}'_3 \\ \mathcal{E}'_1 & 0 & -\mathcal{B}'_3 & \mathcal{B}'_2 \\ \mathcal{E}'_2 & \mathcal{B}'_3 & 0 & -\mathcal{B}'_1 \\ \mathcal{E}'_3 & -\mathcal{B}'_2 & \mathcal{B}'_1 & 0 \end{bmatrix}, \quad F'_{\mu\nu} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & E'_1 & E'_2 & E'_3 \\ -E'_1 & 0 & \mathcal{B}'_3 & \mathcal{B}'_2 \\ -E'_2 & \mathcal{B}'_3 & 0 & -\mathcal{B}'_1 \\ -E'_3 & -\mathcal{B}'_2 & \mathcal{B}'_1 & 0 \end{bmatrix}. \quad (54)$$

Comparing with Eqs. (29) and (28), we find the field transformation in 3-vector form is

$$\begin{aligned} \mathbf{E}' &= \mathbf{E} + \mathbf{v} \times \mathbf{B}, \\ \mathbf{B}' &= \mathbf{B} - \mathbf{v} \times \mathbf{E}. \end{aligned} \quad (55)$$

In the component form, assuming \mathbf{v} is in x direction,

$$\begin{aligned} E'_1 &= E_1, & B'_1 &= B_1, \\ E'_2 &= E_2 - vB_3, & B'_2 &= B_2 + vE_3, \\ E'_3 &= E_3 + vB_2, & B'_3 &= B_3 - vE_2. \end{aligned} \quad (56)$$

The dual field vectors \mathcal{E}' and \mathcal{B}' are related to \mathbf{E} and \mathbf{B} through

$$\begin{aligned} \mathcal{E}' &= \mathbf{E}, \\ \mathcal{B}' &= \mathbf{B}. \end{aligned}$$

However, we need to find the relation between $\mathcal{E}', \mathcal{B}'$ and \mathbf{E}', \mathbf{B}' . To do this, we need to find the inverse transformation of Eq. (55). This is more complex than previous cases. By Eq. (55), we have

$$\begin{aligned} \left[\frac{\mathcal{I}}{\gamma^2} + \mathbf{v} \otimes \mathbf{v} \right] \cdot \mathbf{E} &= \mathbf{E}' - \mathbf{v} \times \mathbf{B}', \\ \left[\frac{\mathcal{I}}{\gamma^2} + \mathbf{v} \otimes \mathbf{v} \right] \cdot \mathbf{B} &= \mathbf{B}' + \mathbf{v} \times \mathbf{E}', \end{aligned}$$

where $\mathbf{v} \otimes \mathbf{v}$ is the tensor product of \mathbf{v} with itself, which can be viewed as a matrix or linear transformation acting on a vector, \mathcal{I} is the unit tensor or the unit matrix, and $\gamma = 1/\sqrt{1 - v^2}$. Solving these equations, we obtain the inverse field transformation

$$\begin{aligned} \mathbf{E} &= \mathfrak{F}(\mathbf{E}' - \mathbf{v} \times \mathbf{B}'), \\ \mathbf{B} &= \mathfrak{F}(\mathbf{B}' + \mathbf{v} \times \mathbf{E}'), \end{aligned} \quad (57)$$

where \mathfrak{F} is a matrix whose inverse is

$$\mathfrak{F}^{-1} = \frac{\mathcal{I}}{\gamma^2} + \mathbf{v} \otimes \mathbf{v}.$$

In the component form, assuming \mathbf{v} is in x direction,

$$\mathfrak{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma^2 & 0 \\ 0 & 0 & \gamma^2 \end{bmatrix},$$

and

$$\begin{aligned} E_1 &= E'_1, & B_1 &= B'_1, \\ E_2 &= \gamma^2 (E'_2 + vB'_3), & B_2 &= \gamma^2 (B'_2 - vE'_3), \\ E_3 &= \gamma^2 (E'_3 - vB'_2), & B_3 &= \gamma^2 (B'_3 + vE'_2). \end{aligned} \quad (58)$$

Compare Eqs. (56) and (58) with the relativistic field transformations, we find they are similar. The relativistic forward and inverse transformations are symmetric regarding γ , while Eq. (56) involves no γ factor but its inverse Eq. (58) involves a factor of γ^2 . The dual field vectors \mathcal{E}' and \mathcal{B}' are related to \mathbf{E} and \mathbf{B} through

$$\begin{aligned} \mathcal{E}' &= \mathfrak{F} (\mathbf{E}' - \mathbf{v} \times \mathbf{B}'), \\ \mathcal{B}' &= \mathfrak{F} (\mathbf{B}' + \mathbf{v} \times \mathbf{E}'). \end{aligned} \quad (59)$$

The Maxwell equations in 3-vector form are

$$\begin{aligned} \nabla' \times [\mathfrak{F} (\mathbf{B}' + \mathbf{v} \times \mathbf{E}')] - (\partial_{t'} - \mathbf{v} \cdot \nabla') [\mathfrak{F} (\mathbf{E}' - \mathbf{v} \times \mathbf{B}')] &= \mathbf{j}' + \rho' \mathbf{v}, \\ \nabla' \cdot [\mathfrak{F} (\mathbf{E}' - \mathbf{v} \times \mathbf{B}')] &= \rho', \\ \nabla' \times [\mathfrak{F} (\mathbf{E}' - \mathbf{v} \times \mathbf{B}')] + (\partial_{t'} - \mathbf{v} \cdot \nabla') [\mathfrak{F} (\mathbf{B}' + \mathbf{v} \times \mathbf{E}')] &= 0, \\ \nabla' \cdot [\mathfrak{F} (\mathbf{B}' + \mathbf{v} \times \mathbf{E}')] &= 0. \end{aligned} \quad (60)$$

The purpose of this is only to show that the Maxwell equations can indeed be written out explicitly in terms of the quantities in frame K' under Galilean transformation. To actually solve these equations, it is easier to solve them using variables \mathbf{E} and \mathbf{B} satisfying the following equations,

$$\begin{aligned} \nabla' \times \mathbf{B} - (\partial_{t'} - \mathbf{v} \cdot \nabla') \mathbf{E} &= \mathbf{j}' + \rho' \mathbf{v}, \\ \nabla' \cdot \mathbf{E} &= \rho', \\ \nabla' \times \mathbf{E} + (\partial_{t'} - \mathbf{v} \cdot \nabla') \mathbf{B} &= 0, \\ \nabla' \cdot \mathbf{B} &= 0. \end{aligned} \quad (61)$$

This is in fact the same as Eq. (69) in the next subsection. After solving this for \mathbf{E} and \mathbf{B} , \mathbf{E}' and \mathbf{B}' can be obtained via the transformation Eq. (55).

When the sources are zero, the equations of electromagnetic wave in vacuum are

$$\left[(\partial_{t'} - \mathbf{v} \cdot \nabla')^2 - \nabla'^2 \right] (\mathbf{E}' - \mathbf{v} \times \mathbf{B}') = 0,$$

$$\left[(\partial_{t'} - \mathbf{v} \cdot \nabla')^2 - \nabla'^2 \right] (\mathbf{B}' + \mathbf{v} \times \mathbf{E}') = 0. \tag{62}$$

This is because $\left[(\partial_{t'} - \mathbf{v} \cdot \nabla')^2 - \nabla'^2 \right]$ is a scalar differential operator while \mathfrak{F} is a constant matrix.

The Lorentz 4-force is

$$\begin{aligned} (f')^\mu &= q (F')^{\mu\nu} u'_\nu \\ &= q\gamma u' \begin{bmatrix} (u'_1 + v)\mathcal{E}'_1 + u'_2\mathcal{E}'_2 + u'_3\mathcal{E}'_3 \\ (1/\gamma^2 - vu'_1)\mathcal{E}'_1 + u'_2\mathcal{B}'_3 - u'_3\mathcal{B}'_2 \\ (1/\gamma^2 - vu'_1)\mathcal{E}'_2 - (u'_1 + v)\mathcal{B}'_3 + u'_3\mathcal{B}'_1 \\ (1/\gamma^2 - vu'_1)\mathcal{E}'_3 + (u'_1 + v)\mathcal{B}'_2 - u'_2\mathcal{B}'_1 \end{bmatrix}. \end{aligned} \tag{63}$$

The Lorentz 3-force is

$$\begin{aligned} \mathbf{f}' &= q \left\{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] \mathcal{E}' + (\mathbf{u}' + \mathbf{v}) \times \mathbf{B}' \right\} \\ &= q \left\{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] \mathfrak{F} (\mathbf{E}' - \mathbf{v} \times \mathbf{B}') + (\mathbf{u}' + \mathbf{v}) \times \mathbf{B}' \right\}. \end{aligned} \tag{64}$$

4.4 EM field in the form of a mixture of $F_{\mu\nu}$ and $F^{\mu\nu}$

We choose to define the electric and magnetic fields in reference frame K' as

$$\begin{bmatrix} E'_1 \\ E'_2 \\ E'_3 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} (F')^{10} \\ (F')^{20} \\ (F')^{30} \end{bmatrix}, \quad \begin{bmatrix} B'_1 \\ B'_2 \\ B'_3 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} F'_{32} \\ F'_{13} \\ F'_{21} \end{bmatrix}. \tag{65}$$

We also define auxiliary dual fields $\mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}'_3, \mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$ such that

$$(F')^{\mu\nu} = \begin{bmatrix} 0 & -E'_1 & -E'_2 & -E'_3 \\ E'_1 & 0 & -\mathcal{B}'_3 & \mathcal{B}'_2 \\ E'_2 & \mathcal{B}'_3 & 0 & -\mathcal{B}'_1 \\ E'_3 & -\mathcal{B}'_2 & \mathcal{B}'_1 & 0 \end{bmatrix}, \quad F'_{\mu\nu} = \begin{bmatrix} 0 & \mathcal{E}'_1 & \mathcal{E}'_2 & \mathcal{E}'_3 \\ -\mathcal{E}'_1 & 0 & -\mathcal{B}'_3 & \mathcal{B}'_2 \\ -\mathcal{E}'_2 & \mathcal{B}'_3 & 0 & -\mathcal{B}'_1 \\ -\mathcal{E}'_3 & -\mathcal{B}'_2 & \mathcal{B}'_1 & 0 \end{bmatrix}. \tag{66}$$

Comparing with Eqs. (28) and 29, we find the field transformation in 3-vector form is

$$\begin{aligned} \mathbf{E}' &= \mathbf{E}, \\ \mathbf{B}' &= \mathbf{B}. \end{aligned} \tag{67}$$

We find the dual field vectors \mathcal{E}' and \mathcal{B}' are related to \mathbf{E}' and \mathbf{B}' through

$$\begin{aligned} \mathcal{E}' &= \mathbf{E}' + \mathbf{v} \times \mathbf{B}', \\ \mathcal{B}' &= \mathbf{B}' - \mathbf{v} \times \mathbf{E}'. \end{aligned} \tag{68}$$

The Maxwell equations in 3-vector form are

$$\begin{aligned}\nabla' \times \mathbf{B}' - (\partial_{t'} - \mathbf{v} \cdot \nabla') \mathbf{E}' &= \mathbf{j}' + \rho' \mathbf{v}, \\ \nabla' \cdot \mathbf{E}' &= \rho', \\ \nabla' \times \mathbf{E}' + (\partial_{t'} - \mathbf{v} \cdot \nabla') \mathbf{B}' &= 0, \\ \nabla' \cdot \mathbf{B}' &= 0.\end{aligned}\quad (69)$$

This can be simplified to

$$\begin{aligned}\nabla' \times \mathbf{B}' - \partial_{t'} \mathbf{E}' &= \mathbf{j}' - (\mathbf{v} \cdot \nabla') \mathbf{E}', \\ \nabla' \cdot \mathbf{E}' &= \rho', \\ \nabla' \times \mathbf{E}' + \partial_{t'} \mathbf{B}' &= (\mathbf{v} \cdot \nabla') \mathbf{B}', \\ \nabla \cdot \mathbf{B}' &= 0.\end{aligned}\quad (70)$$

The second and the fourth equations are Galilean invariant. When the sources are zero, the equations of electromagnetic wave in vacuum are

$$\begin{aligned}\left[(\partial_{t'} - \mathbf{v} \cdot \nabla')^2 - \nabla'^2 \right] \mathbf{E}' &= 0, \\ \left[(\partial_{t'} - \mathbf{v} \cdot \nabla')^2 - \nabla'^2 \right] \mathbf{B}' &= 0.\end{aligned}\quad (71)$$

If \mathbf{v} is in the direction of x , Eq. (71) can be simplified,

$$\begin{aligned}\frac{1}{\gamma^2} \partial_{x'}^2 \mathbf{E}' + \partial_{y'}^2 \mathbf{E}' + \partial_{z'}^2 \mathbf{E}' + 2v \partial_{t'} \partial_{x'} \mathbf{E}' - \partial_{t'}^2 \mathbf{E}' &= 0, \\ \frac{1}{\gamma^2} \partial_{x'}^2 \mathbf{B}' + \partial_{y'}^2 \mathbf{B}' + \partial_{z'}^2 \mathbf{B}' + 2v \partial_{t'} \partial_{x'} \mathbf{B}' - \partial_{t'}^2 \mathbf{B}' &= 0.\end{aligned}\quad (72)$$

The Lorentz 4-force is

$$\begin{aligned}(f')^\mu &= q (F')^{\mu\nu} u'_\nu \\ &= q \gamma u' \begin{bmatrix} (u'_1 + v) E'_1 + u'_2 E'_2 + u'_3 E'_3 \\ (1/\gamma^2 - v u'_1) E'_1 + u'_2 \mathcal{B}'_3 - u'_3 \mathcal{B}'_2 \\ (1/\gamma^2 - v u'_1) E'_2 - (u'_1 + v) \mathcal{B}'_3 + u'_3 \mathcal{B}'_1 \\ (1/\gamma^2 - v u'_1) E'_3 + (u'_1 + v) \mathcal{B}'_2 - u'_2 \mathcal{B}'_1 \end{bmatrix}.\end{aligned}\quad (73)$$

The Lorentz 3-force is

$$\begin{aligned}\mathbf{f}' &= q \left\{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] \mathbf{E}' + (\mathbf{u}' + \mathbf{v}) \times \mathcal{B}' \right\} \\ &= q \left\{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] \mathbf{E}' + (\mathbf{u}' + \mathbf{v}) \times (\mathbf{B}' - \mathbf{v} \times \mathbf{E}') \right\}.\end{aligned}\quad (74)$$

4.5 Discussions

Let us summarize the four different approaches. These are distinguished by different field transformation rules (now switching to SI units):

$$(1) \quad \begin{aligned} \mathbf{E}' &= \mathbf{E}, \\ \mathbf{B}' &= \mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E}. \end{aligned} \quad (75)$$

$$(2) \quad \begin{aligned} \mathbf{E}' &= \mathbf{E} + \mathbf{v} \times \mathbf{B}, \\ \mathbf{B}' &= \mathbf{B}. \end{aligned} \quad (76)$$

$$(3) \quad \begin{aligned} \mathbf{E}' &= \mathbf{E} + \mathbf{v} \times \mathbf{B}, \\ \mathbf{B}' &= \mathbf{B} - \frac{1}{c^2} \mathbf{v} \times \mathbf{E}. \end{aligned} \quad (77)$$

$$(4) \quad \begin{aligned} \mathbf{E}' &= \mathbf{E}, \\ \mathbf{B}' &= \mathbf{B}. \end{aligned} \quad (78)$$

4.5.1 About the relativist view

Consider a solenoid with a magnet bar inside it. The magnet is at rest and the solenoid is moving. From the point of view of the magnet, there is magnetic field \mathbf{B} in space but $\mathbf{E} = 0$. The electron in the wire has a velocity \mathbf{u} and is subject to Lorentz force $\mathbf{f} = q(\mathbf{u} \times \mathbf{B})$. However, because motion is relative, from the point of view of the solenoid, the solenoid itself is at rest and the electron has zero velocity $\mathbf{u} = 0$. The force on the electron is $\mathbf{f} = q\mathbf{E}$ with $\mathbf{E} \neq 0$. This serves as part of the motivation of the special theory of relativity in Einstein's 1905 paper (Einstein 1905). With the Lorentz transformation, the relativistic field transformation is

$$\begin{aligned} \mathbf{E}' &= \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - (\gamma - 1)\hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \mathbf{E}), \\ \mathbf{B}' &= \gamma\left(\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right) - (\gamma - 1)\hat{\mathbf{v}}(\hat{\mathbf{v}} \cdot \mathbf{B}), \end{aligned} \quad (79)$$

where $\hat{\mathbf{v}}$ is the unit vector in the direction of \mathbf{v} .

However, this relativistic field transformation is not the only field transformation that explains the relativity. It is just a convention for the purpose of convenience. The Galilean transformation together with field transformation choice Eq. (77) can embody this principle of relativity as well.

4.5.2 About the Galilean approximations

Le Bellac and Lévy-Leblond (1973) published a paper with the title “Galilean electromagnetism”. There are two differences between their study and ours. First, they pursue a theory which is the approximation of the relativistic electrodynamics in the low speed limit. Second, they try to make this approximate theory in Galilean invariant form. By contrast, our formulation of electrodynamics is completely exact and

rigorous without any approximation, and furthermore, our formulation is not Galilean invariant, which is expected.

They studied two Galilean limits, the electric limit $|\mathbf{E}| \gg c|\mathbf{B}|$ and the magnetic limit $c|\mathbf{B}| \gg |\mathbf{E}|$. The field transformations under these two limits do correspond to our transformations Eqs. (75) and (76). However, they claim that transformation Eq. (77) does not correspond to any Galilean limit. This should not be the case. As can be seen, in the low speed approximation $\gamma \approx 1$, the Lorentz field transformation Eq. (79) is reduced to Eq. (77). With $|\mathbf{E}| \gg c|\mathbf{B}|$, Eq. (77) is further reduced to the electric limit Eq. 75. With $c|\mathbf{B}| \gg |\mathbf{E}|$, Eq. (77) is further reduced to the magnetic limit Eq. (76).

Rousseaux (2008) gave an example showing that using Eq. (77), the composition of transformations from K to K' and from K' to K'' is different from that from K directly to K'' . This issue can also be dismissed. We must bear in mind that the electrodynamics in 3-vector form is not Galilean invariant. Our derivation of all these field transformations Eqs. (75) through (78) is based on the assumption that K is the primary reference frame (which can be viewed as “preferred frame” with isotropic speed of light). Eq. (77) applies to transformations from K to K' , and from K to K'' , but it does not apply from K' to K'' , which must take a different form, because K' is not the primary reference frame.

5 Selleri’s paradox and a new resolution

Selleri’s paradox (1997) refers to the setting of Sagnac’s experiment on a rotating disk, which has angular velocity ω . At one point A on the rim of the disk, two beams of light are sent in opposite directions along the circle with radius r . Because the disk is rotating, the two beams of light take different times Δt_1 and Δt_2 to travel along the circle and come back to A . Note these time intervals are measured by a single clock at a single location. This implies anisotropic speed of light along the circle. This is confirmed by the fringe shift in the experiment and is known as Sagnac effect. Selleri applies the transformation in Eq. (1) to the circle on the disk and shows that the light speed in the direction of rotation and the opposite direction are

$$c_+ = \frac{c}{1 + v/c}, \quad c_- = \frac{c}{1 - v/c}, \quad (80)$$

where $v = \omega r$ and the ratio of the two is

$$\frac{c_-}{c_+} = \frac{c + v}{c - v} \neq 1. \quad (81)$$

Selleri suggests to take the limit of $r \rightarrow \infty$ and $\omega \rightarrow 0$ while keeping $v = \omega r$ constant. In this limit, acceleration is $v^2/r \rightarrow 0$, which locally is effectively an inertial reference frame. The ratio c_-/c_+ is a constant during this limit process, and in the limit, $c_-/c_+ \neq 1$ implies anisotropic light speed in the inertial reference frame. This contradicts the principle of constancy of the speed of light in special relativity (Fig. 3).

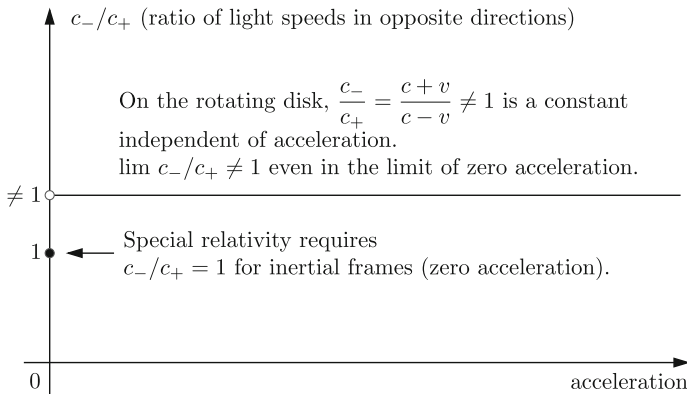


Fig. 3 Selleri’s paradox with discontinuity argument in the limit of zero acceleration on a large rotating disk

The M-M experiment is related to Selleri’s paradox. The earth is not exactly an inertial frame. It is orbiting the sun and has acceleration. We can imagine that the earth is fixed to the rim of a huge disk rotating around the sun. This is exactly the Selleri’s limit of large r but small ω on a rotating disk.

Several authors (Rizzi and Tartaglia 1998, 1999; Budden 1998; Weber 2004; Kassner 2012) have discussed this paradox, reflecting conflicting views. Selleri argues for the absolute simultaneity and the existence of a “privileged reference frame” and claims special relativity is wrong. A relativist contends to “circumvent” the paradox: the culprit that leads to this apparent contradiction is the “central synchronization”, which is different from “Einstein synchronization”. If we adopt “Einstein synchronization” on the circle, the “local speed” of light is still a constant c in both directions.

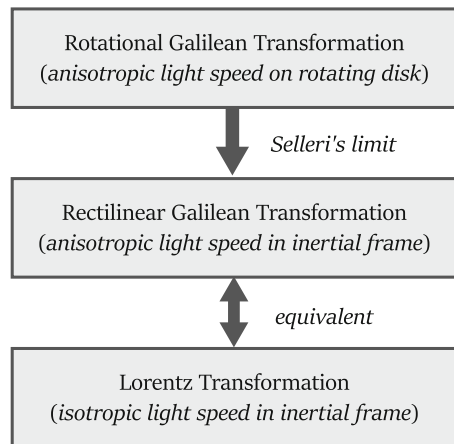
Rizzi et al. (2004) have more detailed discussions of Selleri’s transformation. The coordinate transformations are closely related to synchronization of clocks in a reference frame, as each coordinate transformation determines one synchronization standard. This paper discusses the validity of internal coordinates transformations and it is stressed that the observable quantities are dependent on the reference frame, but should be independent of the coordinates chosen within a given frame.

In an inertial reference frame K , let r, φ be the polar coordinates and t be the time coordinate. Suppose a disk is rotating with angular velocity ω relative to K . We also use polar coordinates r', φ' in the rotating reference frame K' , and let t' be the time in K' . The kinematics in the rotating reference frame has been well studied (Møller 1952) using the transformation

$$\begin{aligned}
 t' &= t, \\
 r' &= r, \\
 \varphi' &= \varphi - \omega t.
 \end{aligned}
 \tag{82}$$

(Selleri’s inertial transformation Eq. (1) works only on the circle (the rim) to have constant two-way speed of light, but not on the entire disk.)

Fig. 4 Resolution of Selleri's paradox



We may call Eq. (82) the “rotational Galilean transformation” or “Galilean-like transformation”. We may call any transformation “Galilean-like transformation” if it has the Galilean or Newtonian character $t' = t$. On the rim of the disk with radius r , light has different linear velocities, $c - \omega r$ in the direction of rotation, and $c + \omega r$ in the opposite direction. In Selleri's limit of $r \rightarrow \infty$, $\omega \rightarrow 0$ while keeping $v = \omega r$ constant, Eq. (82) transfers to Galilean transformation as in Eq. (6). In the limiting inertial reference frame, light has velocity $c - v$ in one direction and $c + v$ in the opposite direction. However, the speed of light depends on the convention of time coordination standard. We have concluded that the Galilean transformation is equivalent to the Lorentz transformation. This offers the most straightforward resolution of Selleri's paradox (see Fig. 4).

Selleri used the transformation in Eq. (1) to make his argument of the paradox, instead of the rotational Galilean transformation in Eq. (82). The cause for Selleri's paradox is his absolutist view. He believes the transformation in Eq. (1) is the only correct transformation of nature's choice and the Lorentz transformation is wrong. Mansouri and Sexl (1977) already argued that the transformation in Eq. (1) is equivalent to the Lorentz transformation. In fact, the one-way speed of light has no absolute meaning. All transformations including Eq. (1), the Galilean transformation and the Lorentz transformation, are equivalent.

6 Selleri's limit: Sagnac effect and Michelson-Morley experiment

Rizzi and Ruggiero (2004) edited a book *Relativity in Rotating Frames*, which includes 18 papers contributed by 23 authors and a round-table discussion with heated debate of these authors. The key issues are Ehrenfest paradox and Sagnac effect, the synchronization and coordinate transformation for the rotating disk. This book is stimulating. As I mentioned in Sect. 1, there are three different philosophical views: the absolutist, the relativist and the conventionalist. Each group is opposed to the *other two groups*

in opinion. The papers compiled in this book reflect all these three views. Especially the dialogues in the round-table are a debate of all these three views.

The M-M experiment and the Sagnac experiment were conducted in the late 1800s and early 1900s in order to detect the absolute motion of the earth. Sagnac observed fringe shift while M-M did not.

There is a debate what Sagnac effect proves. Absolutists, like Sagnac himself, believe it is the proof of the existence of the ether and anisotropic speed of light, while relativists believe it is the proof of relativity theory. Sagnac himself adopted the rotational Galilean transformation Eq. (82), which results in anisotropic light speed on the circle. The relativists adopt the “Einstein synchronization” on the circle, which results in the constant speed of light on the circle, but it also results in a discontinuity, or “time-lag” around the circle. In general, the “time-lag” around any closed path (Gourgoulhon 2013) is

$$\Delta t' = \frac{1}{c^2 \Gamma_0} \oint \Gamma^2 \mathbf{v} \cdot d\mathbf{l} \approx \frac{2}{c^2} \omega \cdot \mathcal{A}, \quad (83)$$

where $\Gamma = 1/\sqrt{1 - \omega^2 r^2/c^2}$, Γ_0 is the Γ where the path starts and ends, \mathbf{v} is the velocity of the point on the disk relative to \mathbf{K} , \mathcal{A} is the area vector enclosed by the closed path. The approximation holds when $\omega r \ll c$. In my opinion, these are just equivalent explanations, like the two sides of the same coin.

What will happen if we put an M-M interferometer on the rim of a big rotating disk? We can imagine that the earth is fixed to the rim of a huge disk rotating around the sun. The light making a round trip along one arm of the M-M interferometer is a closed path. In theory, the fringe shift should not be exactly zero. However, if the size of the arm is very small compared with the radius of the rotating disk (distance between the earth and the sun), its enclosed area is approximately zero. Hence the fringe shift should be very close to zero (much less than the experimental uncertainty) and no significant fringe shift can be detected. This is exactly the Selleri’s limit of large r but small ω on a rotating disk. The earth also has spin. This is also a rotating system. For the similar reason, on the surface of the earth, where M-M experiment was conducted, it is close to the Selleri’s limit. In the limit of inertial frame, the M-M experiment results in no fringe shift, regardless whether light speed is constant or not. The light speed is only the result of the convention of time coordination, rather than absolute truth.

7 Selleri’s limit: connection to electrodynamics in rotating frames

Schiff (1939) studied the Maxwell equations in rotating frames under the “rotational Galilean transformation”

$$\begin{aligned} t' &= t, \\ x' &= x \cos \omega t + y \sin \omega t, \\ y' &= -x \sin \omega t + y \cos \omega t, \end{aligned}$$

$$z' = z. \quad (84)$$

This is the Cartesian form of the same transformation as in Eq. (82). He used the covariant field tensor only, which corresponds to our case of subsection 4.2. He studied the Maxwell equations in the coordinate form. If those equations in component form are written in the 3-vector form, they should look identical to our Eq. (49). The difference is, in their equations in the rotational context, $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ is a variable, while in Eq. (49), \mathbf{v} is a constant. This should not be a surprise. For the rotational transformation Eq. (84), the charge and current transform as

$$\begin{aligned} \rho' &= \rho, \\ \mathbf{j}' &= \mathbf{j} - \rho \mathbf{v}, \quad \text{where } \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \end{aligned} \quad (85)$$

and the differential operators transform as

$$\begin{aligned} \partial_{t'} &= \partial_t + \mathbf{v} \cdot \nabla, \quad \text{where } \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \\ \nabla' &= \nabla, \end{aligned} \quad (86)$$

This is in the same apparent form as Eqs. (26 and (27, except that \mathbf{v} is no longer a constant.

The rotational Galilean transformation Eq. (84) and the rectilinear Galilean transformation Eq. (6) are different, but share certain similarities. Within a small neighborhood on the rotating disk, the rotational Galilean transformation can be approximated by the rectilinear Galilean transformation, and in the Selleri's limit, the rotational Galilean transformation transits to the rectilinear Galilean transformation.

Our formulation of electrodynamics under Galilean transformation in Sect. 4 works for rotating reference frame under rotational Galilean transformation Eq. (84) as well, if we replace the constant velocity \mathbf{v} with $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Care needs to be taken about whether \mathbf{v} is constant when we try to simplify the formulas, because a simplification involving the differential operator ∇ may be valid when \mathbf{v} is constant, but not valid when $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

8 Conclusion

A new paradox is rendered and resolved with the argument that the Galilean transformation and the Lorentz transformation are equivalent and equally valid. This leads to a straightforward resolution of Selleri's paradox. The Maxwell equations, equations of electromagnetic waves in vacuum and the Lorentz force formulas under Galilean transformations are investigated, which may find practical applications. The coordinate, vector component, and tensor component values found via the Galilean transformation do not have the values one would measure with instruments, but are generalized values. The author would like to thank the anonymous referees for their constructive suggestions.

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Appendix

In subsections 4.1 through 4.4, we discussed the Maxwell equations, equations of EM waves and the Lorentz force laws under different field transformations, all of which are under the assumption of Galilean coordinate transformation. These equations under different field transformations take very different forms. This may raise a question to some people: which field transformation leads to a definition of \mathbf{E}' and \mathbf{B}' which is the “physical EM field” in the reference frame K' ? We have commented in Sect. 4 that the numerical values of \mathbf{E}' and \mathbf{B}' themselves do not have absolute meanings. It is how the EM field interact with matter (Lorentz force law) that has a physical meaning. In subsections 4.1 through 4.4, the Maxwell equations and Lorentz force laws all appear in very different forms, however, when we put the field transformations and the Lorentz force law together, we can compare the numerical values of Lorentz force, and we find that all of which are exactly equal. In the following, we “prove” this is indeed the case. The proof is trivial, but might be illustrative for the readers. The reason that the proof is trivial is that to obtain the Lorentz force in all the subsections, we used the same definition of the Lorentz 4-force using the field tensor $F^{\mu\nu}$ (Eq. 33), which is the starting point in the first place. In the same reference frame, if the Lorentz 4-force is the same, so is the Lorentz 3-force and vice versa. The Lorentz force derived under the Galilean transformation even agrees with the Lorentz force in the relativistic case using Lorentz transformation.

When we discuss the formula for the Lorentz force law in frame K' , it should be expressed in the field \mathbf{E}' , \mathbf{B}' , and the velocity of the test charge \mathbf{u}' , all measured in frame K' . These are Eqs. (42), (52), (64) and (74). However, \mathbf{E}' and \mathbf{B}' have different definitions in these formulas and we cannot compare them directly. To compare the numerical value of these formulas, it is a good idea that we express \mathbf{E}' and \mathbf{B}' in terms of \mathbf{E} and \mathbf{B} . Since subsection 4.4 has the simplest field transformation, we start with subsection 4.4 first, and then compare all other subsections with it. At the end, we also compare with the Lorentz force in K' in relativity resulted from Lorentz transformation, and we find they are also the same.

(4.4) Using the field transformation (copied from Eq. 67)

$$\mathbf{E}' = \mathbf{E},$$

$$\mathbf{B}' = \mathbf{B},$$

and the Lorentz force law (copied from Eq. 74)

$$\mathbf{f}' = q \left\{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] \mathbf{E}' + (\mathbf{u}' + \mathbf{v}) \times (\mathbf{B}' - \mathbf{v} \times \mathbf{E}') \right\},$$

we obtain

$$\mathbf{f}' = q \left\{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] \mathbf{E} + (\mathbf{u}' + \mathbf{v}) \times (\mathbf{B} - \mathbf{v} \times \mathbf{E}) \right\}. \quad (87)$$

(4.1) Using the field transformation (copied from Eq. 36),

$$\begin{aligned} \mathbf{E}' &= \mathbf{E}, \\ \mathbf{B}' &= \mathbf{B} - \mathbf{v} \times \mathbf{E}, \end{aligned}$$

and the Lorentz force law (copied from Eq. (42))

$$\mathbf{f}' = q \left\{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] \mathbf{E}' + (\mathbf{u}' + \mathbf{v}) \times \mathbf{B}' \right\},$$

we obtain the same formula as Eq. (87).

(4.2) Using the field transformation (copied from Eq. 45)

$$\begin{aligned} \mathbf{E}' &= \mathbf{E} + \mathbf{v} \times \mathbf{B}, \\ \mathbf{B}' &= \mathbf{B}, \end{aligned}$$

and the Lorentz force law (copied from Eq. 52)

$$\mathbf{f}' = q \left\{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] [\mathbf{E}' - \mathbf{v} \times \mathbf{B}'] + (\mathbf{u}' + \mathbf{v}) \times [\mathbf{B}' - \mathbf{v} \times (\mathbf{E}' - \mathbf{v} \times \mathbf{B}')] \right\},$$

we obtain the same formula as Eq. (87).

(4.3) Using the field transformation (copied from Eq. 55)

$$\begin{aligned} \mathbf{E}' &= \mathbf{E} + \mathbf{v} \times \mathbf{B}, \\ \mathbf{B}' &= \mathbf{B} - \mathbf{v} \times \mathbf{E}, \end{aligned}$$

and the Lorentz force law (copied from Eq. 64)

$$\mathbf{f}' = q \left\{ [1 - \mathbf{v} \cdot (\mathbf{u}' + \mathbf{v})] \mathfrak{F} (\mathbf{E}' - \mathbf{v} \times \mathbf{B}') + (\mathbf{u}' + \mathbf{v}) \times \mathbf{B}' \right\},$$

and by Eq. (57), we obtain the same formula as Eq. (87).

(*) Equivalence to the relativistic case

The relativistic field transformation (copied from Eq. 79) is

$$\begin{aligned} \mathbf{E}' &= \gamma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - (\gamma - 1) \hat{\mathbf{v}} (\hat{\mathbf{v}} \cdot \mathbf{E}), \\ \mathbf{B}' &= \gamma (\mathbf{B} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}) - (\gamma - 1) \hat{\mathbf{v}} (\hat{\mathbf{v}} \cdot \mathbf{B}). \end{aligned}$$

It will be more convenient to use the component form in the calculations, which we omitted. The Lorentz force is invariant from frame K to frame K' :

$$\mathbf{f}' = q (\mathbf{E}' + \mathbf{u}' \times \mathbf{B}'). \quad (88)$$

However, we must be cautious in that the velocity \mathbf{u}' in this formula is different from \mathbf{u}' in Eq. (87). The quantity \mathbf{u}' in Eq. (88) is the E-velocity but \mathbf{u}' in Eq. (87) is the N-velocity (see Eq. (13) in Sect. 3). After the conversion between E-velocity and N-velocity, it can be shown that Eq. (88) is the same as Eq. (87). Once again, we find that the Galilean approach to electrodynamics is equivalent to the Lorentz-Minkowskian electrodynamics. The only difference is which is more convenient.

Recently, Speake and Ortolan (2020) published a review paper on the Maxwell equations in rotating frames. They compared the approaches of Schiff (1939) and Irvine (1964). Schiff and Irvine adopted different methods and the forms of the Maxwell equations are quite different. Schiff's result corresponds to our subsection 4.2. Irvine used a method of orthogonal tetrads (also known as moving frames). The method of moving frames is a very useful technique in differential geometry. Let (x', y', z', t') be a local coordinate chart in any open neighborhood of a pseudo-Riemannian manifold. The arena in which to discuss rotating reference frames is the Minkowski space (flat spacetime), which is a trivial example of pseudo-Riemannian manifolds. However, the general language and theory of pseudo-Riemannian manifold is useful when we apply general curvilinear coordinates. In the tangent space at each point (x', y', z', t') , we can use the natural basis $(\partial_{x'}, \partial_{y'}, \partial_{z'}, \partial_{t'})$. These basis vectors in general are not pseudo-orthogonal. Of course, the basis of any vector space is not unique. In the tangent space at any point, we can always adopt a pseudo-orthogonal basis. These pseudo-orthogonal basis vectors vary smoothly from point to point in the manifold. These frames at each point form a smooth frame field. The dual of the vector fields in the frame fields is a differential 1-form field. However, this 1-form field is not exact. This is the case of rotating reference frames. There does not exist any coordinate chart (x', y', z', t') such that the natural bases $(\partial_{x'}, \partial_{y'}, \partial_{z'}, \partial_{t'})$ are pseudo-orthogonal at all points in an open neighborhood of the manifold. However, this does not prevent us from using pseudo-orthogonal basis at each point, and the pseudo-orthogonal moving frames (orthogonal tetrads). We can make transformations to change from the natural basis $(\partial_{x'}, \partial_{y'}, \partial_{z'}, \partial_{t'})$ to the pseudo-orthogonal basis locally at any time.

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