REFERENTIAL AND SUBSTANTIAL LOGICS

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ABSTRACT. This article develops a logic with two fundamental components: objects and labels. We compare the properties of the two universes that can be constructed from these building blocks and show how they naturally resolve a class of linguistic paradoxes. We conclude with a application to modal logics involving context fields.

1. DEFINITIONS

Consider an interpretation $(O, L, []_1, \langle\rangle_1)$. O is a set of objects and L is a set of labels. We also have two operators:

For $l \in L$ and $o \in O$,

$[]_1 : L \to P(O), [l]_1 = \{\text{the set of all } o \in O \text{ referred to by } l\}$

$\langle\rangle_1 : O \to P(L), \langle o\rangle_1 = \{\text{the set of all l} \in L \text{ that refers to } o\}$.

From these we define the following:

For $m \in M \subseteq L, p \in P \subseteq O$

$[] : P(L) \to P(O), [M] = \bigcup_{m\in M} [m]_1,$

$\langle\rangle : P(O) \to P(L), \langle P\rangle = \bigcup_{p\in P} \langle p\rangle_1.$

where $P(X)$ is the powerset (set of all subsets) of X, and extend the definition so that

$[l] = [\{l\}]$ and

$\langle o\rangle = \langle\{o\}\rangle$.

2. CONSTRUCTIONS

The universe in which we wish to reason can be constructed in two ways:

A *referential* construction is obtained by starting with the set L and then defining O as follows:

Date: October 21, 2022.
2010 *Mathematics Subject Classification.* Primary: 03B65. Secondary: 91F20.
Key words and phrases. referential logic, substantial logic, objects, labels, paradox, context.

$$O = [L] \tag{1}$$

i.e. O consists of anything to which a label refers. We will refer to such a construction as R. A *substantial* construction is obtained by starting with the set O and then defining L as follows:

$$L = \langle O \rangle \tag{2}$$

i.e. L consists of anything referring to an object. We will refer to such a construction as S. Where necessary we will delineate specific labels with "" and specific objects with ''.

3. Properties common to R and S

For what follows, $l \in L, o \in O$. The following three properties hold for both R and S:

$$o \in [l] \iff l \in \langle o \rangle \tag{3}$$

Proof. $\implies$: Suppose $l \notin \langle o \rangle$. Then l is not in the set of objects referring to o. So o is not in the set of objects referred to by l, i.e. $o \notin [l]$, a contradiction.
$\impliedby$: Suppose $o \notin [l]$. Then o is not in the set of objects referred to by l. So l is not in the set of objects referring to o, i.e. $l \notin \langle o \rangle$, a contradiction. $\square$

$$l \in \langle [l] \rangle \tag{4}$$

Proof. $l \notin \langle [l] \rangle \iff [l] \notin [l]$ by (3), a contradiction. $\square$

$$o \in [\langle o \rangle] \tag{5}$$

Proof. $o \notin [\langle o \rangle] \iff \langle o \rangle \notin \langle o \rangle$ by (3), a contradiction. $\square$

For example consider $o = {}' \diamond {}'$. Now $\langle o \rangle$ contains labels like "diamond" and "lozenge", which are specific selectors, but also contains the general selector "quadrilateral", which refers to many other objects as well, such as $'\square'$, etc. So $[\langle o \rangle]$ is much larger than the single element o.

As another example take $l =$ "a rock". This can be a label for any individual rock, each of which may have many other specific labels, making $\langle [l] \rangle$ much larger than the single element l. The label l is acting as a general selector.

4. Additional properties of R

$$L \subseteq O \tag{6}$$

In R, every label can be referred to by another label e.g. if $l \in L, x =$"the label l", then $l \in [x]$, so by (1), $l \in O$.

Let us investigate $[x]$, where

$x =$"An object to which no label refers".

Assume o is an object to which no label refers. Then $\langle o \rangle = \emptyset$. But $o \in [x]$, so $x \in \langle o \rangle$ by (3). This is a contradiction, so $o \notin O$ by (1) and $[x] = \emptyset$.

Now consider

$y =$"A label that does not refer to any object"

We have just shown that $x \in [y]$. So there are redundant labels in R, but no inaccessible objects. Call the set of redundant labels $L_\emptyset = L \setminus \langle O \rangle$.

5. Additional properties of S

$$L \nsubseteq O \tag{7}$$

Whilst O may be declared to contain L, this is not a fundamental property for general O and we cannot a priori assume this. We will instead assume the opposite, (7). This means labels cannot be referred to in a sentence in S.

Consider
$y =$"A label that does not refer to any object"

Assume $y \in L$. Then $[y] \neq \emptyset$ so there exists some $z \in [y]$ such that $[z] = \emptyset$. But $z \in \langle O \rangle$ by (2) $\implies$ $[z] \neq \emptyset$, contradiction. So $y \notin L$ and there are no redundant labels in S.

Now consider $[x]$, where

$x =$"an object to which no label refers"

Assume $x \in L$. By (2), $[x] \neq \emptyset$ so there exists some $o \in [x]$ and $x \in \langle o \rangle$ by (3). But $o \in [x] \implies \langle o \rangle = \emptyset$, a contradiction. So $x \notin L$.

Call the set of inaccessible objects $O_\emptyset = O \setminus [L]$.

Open question: Is it possible to say anything about $O_\emptyset$ at all? Does $O_\emptyset = \emptyset$?

One thing we can try is referring to the set $O_\emptyset$:

$X =$ "The set of objects to which no label refers"

An attempted investigation might go as follows:

Assume $o_1, o_2, o_3, \cdots$ is an enumeration of all objects to which no label refers., i.e. $\forall o_i, \langle o_i \rangle = \emptyset$.

So $[X] = \cup o_i$.

But $X \in \langle [X] \rangle$ and (4) gives

$X \in \langle [X] \rangle = \langle \cup o_i \rangle = \cup \langle o_i \rangle_1$ (by the definition of $\langle\rangle$)

Now $\forall o_i, \langle o_i \rangle = \emptyset \implies \langle \{o_i\} \rangle = \cup \langle o_i \rangle_1 = \langle o_i \rangle_1 = \emptyset$,

So $\cup \langle o_i \rangle_1 = \cup \emptyset = \emptyset$, a contradiction.

The problem is that we cannot enumerate the objects o_i in this way. Their defining property is that they cannot be directly referred to, so we gain no real insight from this approach.

6. Limit points of the [] and $\langle\rangle$ operators

Use the following notation to indicate repeated application of the [] and $\langle\rangle$ operators:

$[\cdots [[l]] \cdots]$ n times $= [l]^n$

$\langle \cdots \langle \langle o \rangle \rangle \cdots \rangle n\text{times} = \langle o \rangle^n$

Define the limit of the operators [], $\langle\rangle$ as

$[l]_\infty = \lim_{n\to\infty} [l]^n$

$\langle o \rangle_\infty = \lim_{n\to\infty} \langle o \rangle^n$

If an application of the $[]$ or $\langle\rangle$ operators is undefined, for example $[o]$ where $o \in P(O \setminus L)$, then the limit has the value of its input (o in this case).

For a set X, we will call $[X]_\infty \setminus X$ the *object boundary* of X, and denote it $\partial_{[]} X$.

For a set X, we will call $\langle X\rangle_\infty \setminus X$ the *label boundary* of X, and denote it $\partial_{\langle\rangle} X$.

7. General properties of limit points

Since $[[l]_\infty]_\infty = [l]_\infty$, $\partial_{[]}(\partial_{[]} X) = [\partial_{[]} X]_\infty \setminus \partial_{[]} X = \partial_{[]} X \setminus \partial_{[]} X = \emptyset$.

Since $\langle\langle l\rangle_\infty\rangle_\infty = \langle l\rangle_\infty$, $\partial_{\langle\rangle}(\partial_{\langle\rangle} X) = \langle\partial_{\langle\rangle} X\rangle_\infty \setminus \partial_{\langle\rangle} X = \partial_{\langle\rangle} X \setminus \partial_{\langle\rangle} X = \emptyset$.

By (3), $[l]_\infty = \langle[l]_\infty\rangle$ and $\langle l\rangle_\infty = [\langle l\rangle_\infty]$

8. Limit points in R

There are three possibilities for $[l]_\infty$:

(1) $[l]^m \subseteq [l]^k$ where $k \leq m$, which would result in recursion. For example ["this label"]^m ="this label" for all m. $[l]_\infty$ can be defined as the set $\bigcup_{i\in[k,m]} [l]^i$.

(2) $[l]^m \in P(O \setminus L)$. In this case the sequence terminates since $[l]^{m+1}$ is undefined and the limit is $[l]^m$.

(3) $[l]_\infty \notin P(O)$. For example $l =$"1 the label whose first word is a number one greater than this label's first word", yields the infinite sequence $[l]^m =$"m the label whose first word is a number one greater than this label's first word", with "m" in the label replaced by the index m as a number.

The set of these type 3 limits is $\partial_{[]} O$ and $[L]_\infty = [O]_\infty$. $\partial_{[]} L = \partial_{[]} O \cup (O \setminus L)$. Note that the elements of $L_\emptyset$ map to $\emptyset$ under $[]$, which is absorbed by the union operation when calculating $[O]_\infty$ and $[L]_\infty$.

In contrast, there is only one possibility for $\langle o\rangle_\infty$:

All elements of L have type 3 limits, $\langle o\rangle_\infty \notin P(O)$, by the example given for (6), and the elements of $O \setminus L$ map to L under $\langle\rangle$. So $\langle o\rangle_\infty$ are all of type 3 and $\langle O\rangle_\infty \subset \langle L\rangle_\infty$. $\langle O\rangle_\infty \neq \langle L\rangle_\infty$ since we have previously shown in §4 that redundant labels can exist in L. $\partial_{\langle\rangle} L = \partial_{\langle\rangle} O \cup \langle L_\emptyset\rangle_\infty$.

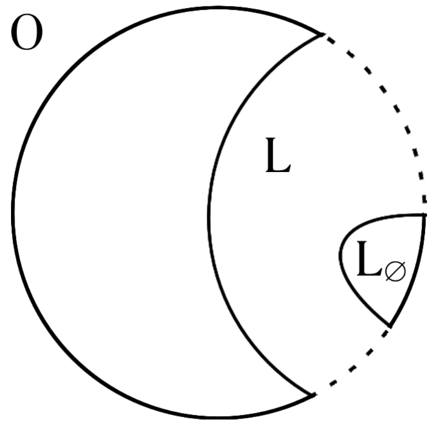

FIGURE 1. The boundary of $\partial_{\langle\rangle} O$ in R is represented by the dashed line.

It is tempting to think that all type 3 elements of $\partial_{[]} L$ and $\partial_{\langle\rangle} L$ are in O since they have a label referring to them, however what is that label? It is not one that can be constructed a priori and therefore cannot be declared to be in L in general.

9. Limit points in S

By (7), all elements of L have type 2 $[l]_\infty$ limits, $[l]^m \in P(O \setminus L)$. In fact $[L]_\infty = [L] \subseteq O = [O]_\infty$. This gives $\partial_{[]} O = \emptyset$, $\partial_{[]} L = O \setminus O_\emptyset$.

All elements o of $O \setminus O_\emptyset$ map to L under $\langle\rangle$, so by (7), all $\langle o\rangle_\infty$ are of type 3, $\langle o\rangle_\infty \notin P(O)$. All elements of $O_\emptyset$ map to $\emptyset$ under $\langle\rangle$, which is absorbed by the union operation when calculating $\langle O\rangle_\infty$. This gives $\langle O\rangle_\infty = \langle L\rangle_\infty = L = \partial_{\langle\rangle} O$ and $\partial_{\langle\rangle} L = \emptyset$.

10. Resolution of Paradoxes

Consider linguistic "paradoxes" of the form:

$a = 3$

therefore

a dog $=$ 3 dog,

which rely on slavish substitution of identicals. Reformulated in terms of labels and objects:

$[a] = 3$

therefore

$\langle 'a' \rangle$ dog $= [a]$ dog

is clearly false.

11. Applications

Define the context field to be the set of all possible worlds and times, and a context (w, t), to be a subset of this space, where we will use ws to indicate the worlds and ts to indicate the times. We will use subscripts to indicate the context in which a clause is being considered. If either w and/or t is omitted it is assumed that the clause is true for all worlds and/or times.

A worldly example:

$[2]_{w_1} + [3]_{w_1} = [7]_{w_1}$ is false,

$[2]_{w_2} + [3]_{w_2} = [7]_{w_2}$ is true,

$\langle [2]_{w_1} \rangle_{w_2} + \langle [3]_{w_1} \rangle_{w_2} = \langle [7]_{w_1} \rangle_{w_2}$ is false.

In w_2 different meanings have been assigned to the labels 2, 3, and/or 7 than in w_1 ("our" world). Operators $(+, =)$ are labels we are assuming here are equivalent in w_1 and w_2.

A temporal example:

$c =$ "lump of clay", something that can be moulded into multiple forms.

$\langle o \rangle_{t_0}$ ='lump of clay'	$[c]_{t_0} = o$
$\langle o \rangle_{t_1}$ ='pottery'	$[c]_{t_1} = \emptyset$
$\langle o \rangle_{t_2}$ ='mug'	$[c]_{t_2} = \emptyset$
$\langle o \rangle_{t_3}$ ='lump of clay'	$[c]_{t_3} = o$

As the clay changes shape it is no longer referred to by its original label.

12. Summary

We have developed referential and substantial logics and explored the topological and other properties of these constructions, highlighting their similarities and differences. We have also shown interesting applications in resolving paradoxes and to modal logic.