The Julius Caesar Objection

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1. Opening

Recent research has revealed three important points about Frege's philosophy of arithmetic. First, his attempt to derive axioms for arithmetic from principles of logic does not require Frege to appeal to his Axiom V, the axiom which gives rise to Russell's Paradox. The proofs sketched in *Die Grundlagen der Arithmetik* depend only upon what (alluding to Frege's method of introducing it) may be called *Hume's Principle*: The number of Fs is the same as the number of Gs just in case there is a one-one correspondence between the Fs and the Gs.[1] Formally, the relevant result is that, if a formalization of this Principle is added as an axiom to standard, axiomatic second-order logic, second-order arithmetic can be interpreted in the resulting theory.[2] Secondly, this theory—which may be called *Fregean Arithmetic*—is itself interpretable in second-order arithmetic and so, presumably, is consistent.[3] And thirdly, Frege's own formal proofs of axioms for arithmetic, given in his *Grundgesetze der Arithmetik*, do not depend *essentially* upon Axiom V.[4] Indeed, Frege himself knew that he did not require any more than Hume's Principle, this being essential if he is to draw certain of the philosophical conclusions he wishes to base upon his formal results.[5]

All of this having been said, the question arises why, upon receiving Russell's famous letter, Frege did not simply drop Axiom V, install Hume's Principle as an axiom, and claim himself to have established logicism anyway. The question is not only of historical interest. Though Frege did not himself adopt it, this position has seemed to some a worthy heir to Frege's logicism: On one version of it, Hume's Principle is thought of as embodying an *explanation* of the concept of number, whence, even though it is not a principle of logic, perhaps it has a similarly privileged epistemological position. In attempting to evaluate this contemporary view, I for one would very much like to know *why* Frege did not adopt it.

The historical question is made pressing by the fact that, in a letter to Russell, Frege explicitly considers adopting Hume's Principle as an axiom, remarking only that the "difficulties here" are not the same as those plaguing Axiom V.[6] Frege says nothing else about these "difficulties", but he must surely have had in mind the "third doubt" discussed in *Gl* §§66ff. It is this which forces Frege to abandon

Hume's Principle, understood as a free-standing explanation of names of numbers, and to replace it with an explicit definition from which Hume's Principle may be derived. Transposed from the context of his discussion of an analogous principle governing names of directions, the worry is this:

> In the proposition
> ["the number of Fs is the same as the number of Gs"]
> [the number of Fs] plays the part of an object, and our definition affords us a means of recognizing this object as the same again, in case it should happen to crop up in some other guise, say as [the number of Gs]. But this means does not provide for all cases. It will not, for instance, decide for us whether [Julius Caesar] is the same as [the number zero]—if I may be forgiven an example which looks nonsensical. Naturally, no-one is going to confuse [Julius Caesar] with [the number zero]; but that is no thanks to our definition of [number]. That says nothing as to whether the proposition
> ["the number of Fs is identical with *q*"]
> is to be affirmed or denied, except for the one case where *q* is given in the form of ["the number of Gs"]. What we lack is the concept of [number]; for if we had that, then we could lay it down that, if *q* is not a [number], our proposition is to be denied, while if it is a [number], our original definition will decide whether it is to be affirmed or denied. (*Gl* §66)

This is what I shall call *the Caesar objection*. As said, it is his inability to answer this objection that forces Frege to give an explicit definition in *Gl* §68, which definition requires reference to extensions and so requires (something like) the disastrous Axiom V.

To understand why Frege could not treat Hume's Principle as an axiom, we must understand the Caesar objection. But this has proved far from easy. I myself have come to the conclusion that the Caesar objection does not pose any *single* problem, but at least three different, though related, ones. I can not discuss all of these here, so let me simply indicate two of them, if only to set them aside.

The first problem is epistemological. Frege raises the Caesar objection against a proposed answer to the famous question of *Gl* §62, "How, then, are numbers to be given to us, if we cannot have any ideas or intuitions of them?" Frege takes it that, to answer this question, it is necessary and sufficient to explain the senses of identity-statements in which number-words occur (this claim being underwritten by the context principle—see *Gl* p. x and §107). The suggestion Frege is considering when he raises the Caesar objection is that this may be done by means of Hume's Principle. So the view against which the Caesar objection is offered is this: We recognize numbers as the referents of names of the form "the number of Fs", and our understanding of these names consists (wholly) in our grasp of Hume's Principle. Frege's objection to this view is, once again, that Hume's Principle "will not, for instance, decide for us whether [Caesar] is the same as the [number zero]..." (*Gl* §66). He concludes, since he is unable to answer the objection, that Hume's Principle fails as an explanation of the senses of identity-statements containing names of numbers. Now, the Caesar objection does seem to show that Hume's Principle, on its own, does not provide a sense for *all* identity-statements containing names of numbers.[7] But why should that be thought a difficulty?

It is rarely mentioned that Frege takes for granted we *do* recognize that Caesar is not a number. If, as is often said, his objection were that Hume's Principle does not decide the truth-values of *all* 'mixed' identity-statements, then our intuitions about the truth-value of this particular 'mixed' identity-statement would be quite irrelevant. But it is important to remember that Frege raises the Caesar objection in the context of a particular argument: One should not be so distracted by its apparent generality that one imagines it raised in a vacuum, so that it could only depend upon some general requirement that every well-formed sentence must have a truth-value. The specific objection Frege raises is not e.g. that Hume's Principle does not decide whether the singleton of the null set is the number zero (which, on Frege's explicit definition, it happens to be). The example Frege chooses is one about which he takes us to have strong intuitions: Whatever numbers may be, Caesar is not among them. Thus, one might think, there must be *more* to our apprehension of numbers than a mere recognition that they are the references of expressions governed by Hume's Principle. Any complete account of our apprehension of numbers as objects must include an account of what distinguishes people from numbers. But Hume's Principle alone yields no such explanation. That is why Frege writes: "Naturally, no one is going to confuse [Caesar] with the [number zero]; but that is no thanks to our definition of [number]" (*Gl* §62).[8]

The second problem raised by the Caesar objection is semantical, though it has obvious epistemological overtones. Hume's Principle is supposed to explain *names* of numbers, expressions which must be treated, semantically, as purporting to refer to objects, the numbers. Only if expressions of the form "the number of Fs" are so understood, as purporting to refer to objects, can our capacity to refer to numbers be explained in terms of our grasp of Hume's Principle. But why is "The number of Fs is the number of Gs", as explained by Hume's Principle, not just an idiomatic rendition of "There is a one-one correlation between the Fs and the Gs"? On what ground is it claimed that the former sentence has the sort of *semantic*, as opposed to *orthographic*, structure it needs to have? If this explanation of 'identity-statements' involving 'names of numbers' really does license us so to treat those statements, then the understanding conveyed by Hume's Principle must enable us to understand such predicates as "ξ is the number of Gs", these being true or false *of objects*. These predicates are formed, after all, merely by omitting a semantic constituent from the sentences so explained. To put the point differently, if "the number of Fs" is a semantic constituent of such sentences, it must be replaceable by a variable: There must be an intelligible question, as it were, whether the open sentence "*x* is the number of Gs" is true or false of any particular object, *independently of how it might be given to us* (cf. *Gl* §67). But Hume's Principle does not even appear to explain sentences of the form "*x* is the number of Gs", but only statements of the form "the number of Fs is the number of Gs".[9] The best we seem able to do is to understand such questions as whether the open sentence is true when a term of the form "the number of Fs" is substituted for the variable. But that is to invite the question whether our understanding of quantification over numbers is not merely *substitutional*; and if so, it would seem that our capacity to refer to numbers (at least, as objects independent of our ways of thinking of them) has not been explained.[10]

To summarize: To meet the Caesar objection, one must meet at least two challenges. The first is to show how, on the basis of the understanding of names of numbers captured by Hume's Principle, one can come to understand questions of 'trans-sortal identification' and, in particular, to know that numbers are of a sort different from people and other such objects.[11] The second is to explain how, on the same basis, one can arrive at an understanding of such predicates as "ξ is the number of Gs". These two challenges apply not only to the explanation of names of numbers embodied in Hume's Principle, but to *any* 'contextual' explanation of names, e.g., to the analogous explanation of names of directions considered in *Die Grundlagen*. Since Frege raises the Caesar objection both against this explanation of names of directions and against that of number in terms of Hume's Principle, some quite general problems must be raised by the Caesar objection. One should not conclude, however, that the Caesar objection does not also raise quite *specific* problems in the case of numbers. Such an aspect of the Caesar objection is what I wish to discuss here.

The Caesar objection is first raised in quite a different context. In *Gl* §55, Frege considers an 'inductive' definition of cardinal numbers, the two important clauses of which are:

> the number 0 belongs to a concept, if the proposition that *a* does not fall under that concept is true universally, whatever *a* may be.

> the number (n+1) belongs to a concept F, if there is an object *a* falling under F and such that the number *n* belongs to the concept "falling under F, but not *a*".

Among the objections Frege makes to these definitions is that "we can never—to take a crude example—decide by means of our definitions whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or not" (*Gl* §56). It is utterly implausible that this occurrence of the Caesar objection should not be closely related to that in *Gl* §66. Part of my goal here is to explain how these two objections are related and so to throw light on them both.

The remainder of this paper consists of four sections. In section 2, I shall discuss a version of the Caesar objection which Frege raises in *Grundgesetze*. Explaining why Frege is compelled to answer the question whether the True is a 'value-range' will motivate my treatment of the Caesar objection, as it arises in *Die Grundlagen*. In section 3, I shall argue that the Caesar objection arises, in *Gl* §56, as a manifestation of a technical obstacle to the development of arithmetic on the basis of the inductive definitions considered in §55. Furthermore, the later occurence of it, in §66, is connected to a similar technical obstacle to the development of arithmetic on the basis of Hume's Priniciple, the most obvious way of overcoming which forces Frege to provide a sense for such sentences as "Caesar is the number zero". In section 4, I shall argue that the technical obstacle can be overcome in an *un*obvious way, and so that there is a way of founding our knowledge of arithmetic on (an analogue of) Hume's Principle which, to some extent, sidesteps the Caesar objection. The resulting conception of the genesis of our knowledge of arithmetic has, I think, independent virtues, some of which I shall mention in the closing section 5.

2. Why the Caesar Objection Has To Be Taken Seriously

One common view about the Caesar objection is that the demand that a sense be provided for "Caesar is the number zero" is a consequence of Frege's general "requirement as regards concepts that, for any argument, they shall have a truth-value as their value...".[12] It follows from this that, if "F(0)" has a truth-value, every sentence resulting from the replacement of "0" by some other proper name must also have a truth-value. But there is no indication that, at the time of writing *Die Grundlagen*, Frege subscribed to this 'requirement of complete determination'; this interpretation of the Caesar objection thus reads post-1891 doctrines back into *Die Grundlagen* without independent justification.

Indeed, even though Frege does subscribe to the requirement of complete determination in his mature period, and even though it does lead to the Caesar objection, Frege does not raise analogues of the Caesar objection, even in his mature period, solely because he subscribes to this requirement. In *Gg* I §10, Frege considers the question whether truth-values—which, for him, are the referents of sentences—are value-ranges, and if so, which value-ranges they are; that is, whether such sentences as "The True is the value-range of the concept *non-self-identical*" are true or false. Frege shows, by means of the so-called 'permutation argument' presented in *Gg* I §10, that the semantical stipulations he has made up to that point do not decide such questions.[13] Frege eventually stipulates that the True is to be its own unit-class, and the False, its own unit-class, and argues, in *Gg* I §31, that this stipulation, together with those made earlier, suffices to provide every expression of the theory with a unique reference.[14]

Obviously, however, the requirement of complete determination will force Frege to provide the sentence "The True is the value-range of the concept *non-self-identical*" with a sense *only if it is well-formed*. Once made, the point is obvious: The version of the Caesar objection discussed in *Gg* I §10 would not arise if Frege did not treat truth-values as objects, sentences as singular terms; and so, his insistence that every well-formed sentence must have a truth-value can not, *on its own*, explain why this *or any other* instance of the Caesar objection should arise. The question is *why* Frege insists upon treating sentences as singular terms when doing so causes so much trouble. Answering this question will help us to understand the answer to the analogous question concerning the instance of the Caesar objection discussed in *Die Grundlagen*.

It is almost *cliché* to remark that Frege's texts are conspicuously thin on argument for the claim that sentences are proper names. His argument, such as it is, is that sentences are 'saturated', like proper names; hence, sentences can not be functional expressions, and so must be proper names. But this argument is, to put it kindly, unpersuasive: Why shouldn't there be more than one *kind* of saturated expression, just as there are different kinds of 'unsaturated' expressions? Given just *how* poor this argument is, I strongly suspect that Frege is not concerned to establish that truth-values 'really are' objects at all, but rather *that they can be so treated*, should that be convenient.[15] So perhaps it would be best to inquire why one might *want* to treat truth-values as objects.

Dummett has suggested that doing so simplifies Frege's formal system by simplifying its ontology. If we treat truth-values as objects, then we need not distinguish between concepts (that is, functions whose values are always truth-values) and functions more generally. Nor need we distinguish between one-place functions whose arguments are always truth-values and others.[16] Now, I have no quarrel with Dummett's claim that Frege was motivated to treat truth-values as objects because doing so simplifies his system somehow, but I am not sure he has identified the nature of the simplification the identification effects.

Frege's Axiom V governs terms which (purport to) stand for what he calls value-ranges. As it is often formulated, Axiom V is:

(Vc) $\grave{\varepsilon}.F\epsilon = \grave{\varepsilon}.G\epsilon$ iff $\forall x(Fx \equiv Gx)$

Thus, the *extension* of the concept $F\xi$ is the same as that of the concept $G\xi$ just in case these concepts are co-extensive. As Frege formulates Axiom V, however, it reads:

(Vf) $\dot{\varepsilon}.F\epsilon = \dot{\varepsilon}.G\epsilon$ iff $\forall x(Fx = Gx)$

Thus, the *value-range* of the function $F\xi$ is the same as that of the function $G\xi$ just in case these functions have the same value for every argument. Now, in the sort of theory with which we are most familiar—namely, one which distinguishes the logical types of sentences from those of proper names—one might well want to take both Vc and Vf as axioms.[17] In Frege's theory, however, since sentences are of the same logical type as proper names, one-place (first-level) functions are of the same logical type as one-place (first-level) concepts. So, for Frege, Axiom Vf includes Axiom Vc as a kind of special case.

Frege uses Axiom Vf, in large part, so that he may speak of the value-ranges of one-place functions, instead of speaking of the functions themselves.[18] But it also allows him to speak of the value-ranges of *two-place* functions, which he calls *double* value-ranges. Consider, for example, the function '$\xi+\eta$'. Fix one of its arguments, say the second, and consider the function '$\xi+2$'. The value-range of this function, $\dot{\varepsilon}.\epsilon+2$, is the graph of the function whose value, for a given argument *x*, is *x*+2. Suppose the second argument is now allowed to vary; the resulting value-range, $\dot{\varepsilon}.\epsilon+n$, will be the graph of the function whose value, for given argument *x*, is *x*+*n*. What then is the value-range of the function $\dot{\varepsilon}.\epsilon+\xi$? It is the *double* value-range $\dot{\alpha}[\dot{\varepsilon}.\epsilon+\alpha]$, the graph of the function whose value, for argument *y*, is the value-range $\dot{\varepsilon}.\epsilon+y$.[19] This double value-range Frege uses as if it were the value-range of the two-place function itself. And it is easy to see that $\dot{\alpha}\dot{\varepsilon}.f\epsilon\alpha = \dot{\alpha}\dot{\varepsilon}.g\epsilon\alpha$ if, and only if, $\forall x\forall y(fxy = gxy)$, this being Theorem 2 of *Grundgesetze*.

By means of this lovely construction,[20] Frege manages to do without special axioms governing the value-ranges of two-place (and similarly, many-place) functions. Not that it would have been a difficult matter to formulate such axioms, but the use of double value-ranges certainly does simplify Frege’s system. The same construction enables Frege to utilize the double value-range $\dot{\alpha}\dot{\varepsilon}.\epsilon R\alpha$ of the *relation* $\xi R\eta$ as if it were its extension. And it is at this point that the utility of Frege’s identification of truth-values as objects becomes apparent: One can see this by considering what would happen were we to try to mimic Frege’s use of double-value-ranges in a theory which did *not* treat truth-values as objects. Consider, first, a theory containing only Axiom Vc. The ‘double extension’ term “$\grave{\alpha}[\grave{\varepsilon}.\epsilon<\alpha]$”, which one might have supposed could serve as a term denoting the extension of the relation $\xi<\eta$, *is not even well-formed*. Extension terms are formed by prefixing ‘$\grave{\alpha}$’, say, to a one-place *predicate*, the argument-place of which is then filled by ‘α’. But “$\grave{\varepsilon}.\epsilon<\eta$” is not a predicate at all; it is a functional expression. That this expression is not of the correct type for the formation of extension terms is a consequence of the fact that truth-values are of a different logical type from extensions: Only if truth-values are of the same logical type as extensions, will concepts—i.e., functions from objects to truth-values—be of the same logical type as functions from objects to extensions.

Now, it is true that one does not have to treat truth-values as objects to effect something like Frege’s reduction of the extensions of relations to those of concepts. One remedy is to add Axiom Vf to the theory and write “$\dot{\alpha}[\grave{\varepsilon}.\epsilon<\alpha]$” instead of “$\grave{\alpha}[\grave{\varepsilon}.\epsilon<\alpha]$”; the extension of $\xi<\eta$ is then the *value-range* of the function $\grave{\varepsilon}.\epsilon<\eta$. Really to see to what Frege’s treating truth-values as objects amounts, however, one needs to consider a different remedy, which involves neither identifying truth-values as objects nor taking both Vc and Vf as axioms.[21]

We continue to suppose that we working in a language which distinguishes the logical types of names and sentences. Suppose, now, that we limit ourselves to Axiom Vf. We can then speak, easily enough, of the single and multiple value-ranges of functions. Can we also be speak of the extensions of concepts? A familiar trick will enable us to do that, too: We can employ *characteristic functions*.[22] To do so, we require a description-operator ‘ι’[23] and two arbitrary objects—which we denote by ‘F’ and ‘T’. We then *define* the extension of the concept Φξ as follows:

$$\grave{\varepsilon}.\Phi\epsilon = \dot{\varepsilon}.\iota x.[(\Phi\epsilon \,\&\, x{=}T) \vee (\neg\Phi\epsilon \,\&\, x{=}F)]$$

Thus, the extension of Φξ is the value-range of the function whose value, for given argument x, is T if x is Φ; F, if x is not Φ. It is easy enough to show that, so defined, ‘$\grave{\varepsilon}.\Phi\epsilon$’ satisfies Axiom Vc.

In fact, one could go yet further and refuse to make any serious use of anything *but* characteristic functions. Instead of introducing primitive relations into the system, for example, one would introduce their characteristic functions. One could even introduce the logical constants in terms of *their* characteristic functions: Expressions such as “2+2=4 ∨ 1+1=3” would then be names, not sentences. To be able to form sentences (and so make any assertions), one would need to have at least one real predicate in the language, and the most natural choice for such a predicate would be one which meant: ξ is identical with T. This predicate would be remarkably like Frege’s horizontal, which means: ξ is identical with the True; indeed, one might wonder whether he had something like this construction in mind when he wrote, in *Begriffsschrift*, that the system has only one predicate.[24] Thus, although Frege could have distinguished truth-values from objects and still been able to make do with Axiom Vf, he does not take this course: What he effectively does is to *identify* concepts with their characteristic functions. Once the identification is made, the truth-values become the objects T and F, in terms of which the characteristic functions were defined. Nothing could be more natural, mathematically speaking.[25]

Treating sentences as of the same logical type as proper names thus has substantial technical advantages in the context of Frege’s system. Once one sees that it amounts to identifying concepts with their characteristic functions, it should not seem all that perplexing. Nonetheless, Frege’s making this move imposes certain obligations on him. As he understands the notion of logical type, two expressions are of the same logical type only if they are intersubstitutable *salva significatione*. Thus, if there is any sentence of the form ‘...*t*...’ which has a sense, and a corresponding sentence of the form ‘...*u*...’ which does not, then *t* and *u* can not be of the same logical ‘Sort’.[26] Hence, if truth-values are of the same Sort as value-ranges, identity-statements such as “The True = $\dot{\varepsilon}.\epsilon \neq \epsilon$” simply *must* have a sense, since “$\dot{\varepsilon}.\epsilon \neq \epsilon = \dot{\varepsilon}.\epsilon \neq \epsilon$” most certainly does.

3. The Caesar Objection and the Feasibility of the Logicist Project

We have seen that the version of the Caesar objection Frege considers in *Grundgesetze* arises because he takes value-ranges and truth-values to be of the same logical Sort. More generally, a version of the Caesar objection will arise whenever one makes such suppositions about the Sorts of objects of apparently different kinds. For example, if one supposed that (names of) directions had to be of the same logical Sort as (names of) the lines whose directions they were, "ℓ = dir(ℓ)" would have to have a sense. For to say that 'ℓ' and 'dir(ℓ)' are of the same logical Sort is just to say that they are intersubstitutable *salva significatione*, whence this sentence must have a sense, since "dir(ℓ) = dir(ℓ)" does.

Why, then, does Frege raise the question whether Caesar is a number in *Die Grundlagen*? One might suggest that it is because he is assuming that all objects are of a single logical Sort: That would certainly give rise to the Caesar objection. But Frege does not argue for this claim, which is not so much as mentioned. A similar answer could, of course, be given to the question why Frege raises the Caesar objection in *Grundgesetze*. But, even though Frege did then hold that all objects belong to a single Sort, he had a specific, technical reason to suppose that truth-values and value-ranges were of the same Sort. One might wonder, therefore, whether there is not, in the case of Hume's Principle, too, a similarly specific reason Frege needed to suppose that numbers were of the same Sort as objects of other kinds.

Imagine a language, devoid of mechanisms for reference to numbers, into which names of numbers are to be introduced by means of Hume's Principle. Prior to the definition of names of numbers, the speakers of this language may be supposed to understand names of, and predicates true or false of, objects of various other kinds; among these *basic objects*, one might suppose, will be such things as people and trees and rocks and rivers.[27] Hume's Principle will, in the first instance, explain (or define) terms of the form "the number of Fs", where "Fξ" is a predicate true or false of basic objects. Once speakers have understood the contextual explanation of these terms, they will understand terms which refer to numbers, such as "the number of Roman emperors", and predicates true or false of numbers, such as "ξ is a number less than 5".[28] Now, if names of numbers, so explained, were of the same Sort as names of basic objects, a version of the Caesar objection would arise: The question whether Caesar is a number, and, if so, which one he is, would have to be provided with an answer. But *why* does Frege suppose that numbers are of the same Sort as basic objects?

This supposition is not gratuitous; nor is it based upon some prior assumption that all objects are of the same Sort. The need for the supposition is connected with Frege’s oft-repeated insistence that numbers can themselves be counted.[29] One way to explain the force of this idea would be to observe that, once names of numbers have been introduced by means of Hume’s Principle, no *further* explanation of such expressions as “the number of numbers less than 5” would appear to be required. One would expect speakers immediately to understand such expressions and to know, for example, that “the number of Roman emperors is the same as the number of numbers less than 5” is true if, and only if, there is a one-one correlation between the Roman emperors and the numbers less than 5. It would appear to follow that the predicates “ξ was a Roman emperor” and “ξ is a number less than 5” must be of the same Sort, for the names explained by Hume’s Principle contain predicates of a single Sort, namely, those of the same Sort as “ξ was a Roman emperor”. But predicates of this Sort are predicates true or false of objects of the same Sort as basic objects (for the Sorts of predicates are *determined* by the Sorts of their acceptable arguments). Hence numbers must be of the same Sort as basic objects; hence the question what to make of “Caesar is the number zero”.[30]

Frege’s insistence that numbers can be counted is also important within the context of his logicism. The counting of numbers—that is, the use of terms of the form “the number of numbers less than 5”—is essential to the execution of the logicist project, as Frege envisions it. Frege proves that every number has a successor by showing that every natural number *n* is succeeded by the number of numbers less than or equal to *n*; the proof thus makes essential use of numerical terms which contain predicates true or false of numbers. Hence—to reprise the above—if expressions such as “the number of numbers less than 5” are supposed to have been explained by Hume’s Principle, names of numbers apparently must be of the same Sort as names of persons.

There is a deep misunderstanding of what has just been argued which must be avoided here. One might well remark that the use of terms of the form “the number of numbers less than or equal to *n*” is not essential to Frege’s proofs, since he could equally well have used terms formed from predicates true or false of objects of other sorts. Suppose, for example, that Axiom V is in place and that predicates true or false of value-ranges are substitutable into Hume’s Principle (i.e., that value-ranges are ‘basic objects’). Consider the sequence of value-ranges: $\dot{\varepsilon}.\epsilon \neq \epsilon$, $\dot{\alpha}.\alpha = \dot{\varepsilon}.\epsilon \neq \epsilon$, etc., i.e., the sequence beginning with the empty value-range which is determined by taking, at each step, the singleton of that at the previous step. It is not difficult to show that, for each natural number *n*, there is a concept true of exactly the first *n* members of this series; since the series is unending, there will be a concept Fξ under which fall those value-ranges and the next one in the series.[31] It will then be an easy matter to show that the number of Fs is the successor of *n*.

As a technical observation, this is plainly correct: If Frege had been willing to make such an appeal to Axiom V, or if he had been willing to appeal to an axiom of infinity, he could have done without the claim that numbers are of the same Sort as 'basic objects'. But it is precisely appeal to a special axiom of infinity that Frege is trying to avoid, since part of his purpose is to explain the genesis of our knowledge that *there are* infinitely many objects. Of course, Hume's Principle, as it is usually formulated, *is* an axiom of infinity; but that fact is uninteresting, since it is completely obvious that any principles from which the truths of arithmetic can be derived will imply the existence of infinitely many objects. To object to Frege's use of Hume's Principle simply on the ground that it is an axiom of infinity is to object that his premises imply his conclusion. What is interesting about Hume's Principle is not so much that it implies the existence of infinitely many objects, but that the infinitely many objects whose existence it implies are, or at least are intended to be, *precisely the natural numbers*.[32] Even if one does not go on to claim that Hume's Principle is a conceptual truth, or an explanatory principle, or any other such thing, it is still one thing to rest one's development of arithmetic upon a principle implying that there are infinitely many cardinal numbers, and something else entirely to presuppose that there are infinitely many objects of some other, possibly even non-mathematical, kind (as do both Dedekind and Russell). This is why I said, above, that Frege wants to avoid appeal to a *special* axiom of infinity.

What then of proving the infinity of the series of natural numbers by means of a direct appeal to Axiom V? The answer is similar, but more complicated. Note first that Frege would certainly have refused to make any direct appeal to his explicit definition of numbers in giving his proofs. Though Hume's Principle is derived from the explicit definition, by means of Axiom V, it is essential, if he is to draw certain of the philosophical conclusions he bases upon his formal project, that Frege make no further appeal to the explicit definition. The reason is easy enough to state. Any explicit definition of the natural numbers as value-ranges will have some arbitrary features. There is no reason Frege had to define the number of Fs as the class of all concepts equinumerous with Fξ; he could just as well have defined it as the class of all concepts smaller than Fξ, or as any of indefinitely many other classes. Now, Frege wishes to claim that the truths of arithmetic, being consequences of this definition, are analytic of the concept of number. But how can that be, if the definition has admittedly arbitrary features? The answer is that Frege need not claim that *all* consequences of the definition are analytic of the concept of number, but only that those which do not depend upon any of its arbitrary features are. But what is non-arbitrary about the definition is precisely that it yields Hume's Principle: So a theorem depends upon no arbitrary features of the definition just in case it is provable from Hume's Principle, without any further appeal to the explicit definition.[33]

In itself, this does not show that Frege would have refused to appeal directly to Axiom V in his proof of the infinity of the number-series. That he would is suggested by the fact that he did not do so, for it would be at least a little bit surprising if he had avoided making essential appeal to Axiom V in his proofs of the axioms of arithmetic for no very good reason. My own view is that Frege thought that arithmetic rests, in a sense, upon Hume's Principle alone, that nothing besides logic and this very general principle concerning the conditions under which numbers are to be ascribed to concepts is required for the proof of the axioms of arithmetic. That proof, however, does not establish logicism, for the question of the epistemological status of Hume's Principle just begs to be asked. The role of Axiom V, I think, is *simply* to underwrite the logical status of Hume's Principle and not in any way to undermine its centrality. But all of that is really just so much speculation at this point. I shall have to leave matters there for the present and simply claim that, for some such reasons as these, Frege would politely have declined the appeal to Axiom V offered him earlier. If so, then he would have had reason to regard the claim that numbers are of the same Sort as basic objects as essential to the sort of proof of the infinity of the number-series required by his philosophical purposes.

The argument given so far shows merely that the proofs Frege gave of the axioms of arithmetic require the assumption that numbers are of the same Sort as basic objects: It does not show that those axioms can not be proven, or that Frege would have had reason to suppose they can not be proven, without that assumption. Consider, then, a formulation of Hume's Principle in a two-sorted language. There are 'basic' individual variables, 'x', 'y', and the like, and 'numeric' individual variables, '**x**', '**y**', and the like. There are also basic and numeric predicate variables, ‘F’, ‘G’, and ‘**F**’, ‘**G**’, respectively. And there are relation variables of various kinds, the logical types of which may be indicated by subscripts: ‘R_{bn}’ is a relation variable whose first argument must be basic and whose second must be numeric; ‘R_{bb}’, one both of whose arguments must be basic; and so on.[34] As for identity, there is not one identity-sign in this language, but two, which we might write as ‘$=_b$’ and ‘$=_n$’; no ‘mixed’ identity-statement will be well-formed. A version of Hume’s Principle may then be formulated as follows:

$$\mathbf{N}x{:}Fx =_n \mathbf{N}x{:}Gx \text{ iff } \exists R_{bb}\{\forall x\forall y\forall z\forall w(R_{bb}xy \,\&\, R_{bb}zw \rightarrow x=_b z \equiv y=_b w) \,\& \\ \forall x[Fx \rightarrow \exists y(Gy \,\&\, R_{bb}xy)] \,\&\, \forall y[Gy \rightarrow \exists x(Fx \,\&\, R_{bb}xy)]\}$$

That the ‘N’ is boldface indicates that terms governed by Hume’s Principle are numeric. Note that, in this language, there is no well-formed term such as “$\mathbf{N}x{:}(x =_? \mathbf{N}y{:}y\neq y)$”, the terms appearing on either side of ‘=’ being of different Sorts.

What will one be able to prove in a theory whose sole non-logical axiom is this version of Hume’s Principle? The answer is somewhat surprising. Frege’s definition of predecession can be formulated as follows:

$$\mathbf{P}(\mathbf{m},\mathbf{n}) \text{ iff } \exists F\exists x[\mathbf{n} =_n \mathbf{N}x{:}Fx \,\&\, Fy \,\&\, \mathbf{m} =_n \mathbf{N}x{:}(Fx \,\&\, x\neq_b y)]$$

His definitions of “0” and the predicate “ξ is a natural number” can also be stated without difficulty:[35]

$$\mathbf{0} =_n\text{df } \mathbf{N}x{:}x\neq x$$
$$\mathbb{N}\mathbf{n} \equiv\text{df } \forall\mathbf{F}[\mathbf{F0} \,\&\, \forall\mathbf{x}(\mathbf{Fx} \,\&\, \mathbf{P}(\mathbf{x},\mathbf{y}) \rightarrow \mathbf{Fy}) \rightarrow \mathbf{Fn}]$$

Now, where ‘$\mathbf{P}^*(\xi,\eta)$’ denotes the strong ancestral of ‘$\mathbf{P}(\xi,\eta)$’, the axioms of arithmetic, as Frege formulates them, may be stated as follows:[36]

1. $\forall\mathbf{x}\forall\mathbf{y}\forall\mathbf{z}[\mathbf{P}(\mathbf{x},\mathbf{y}) \,\&\, \mathbf{P}(\mathbf{x},\mathbf{z}) \rightarrow \mathbf{y} =_n \mathbf{z})$
2. $\neg\exists\mathbf{x}.[\mathbb{N}\mathbf{x} \,\&\, \mathbf{P}^*(\mathbf{x},\mathbf{x})]$
3. $\forall\mathbf{x}[\mathbb{N}\mathbf{x} \rightarrow \exists\mathbf{y}.\mathbf{P}(\mathbf{x},\mathbf{y})]$
4. $\mathbb{N}\mathbf{n} \equiv \forall\mathbf{F}[\mathbf{F0} \,\&\, \forall\mathbf{x}(\mathbf{Fx} \,\&\, \mathbf{P}(\mathbf{x},\mathbf{y}) \rightarrow \mathbf{Fy}) \rightarrow \mathbf{Fn}]$

Frege’s own proofs of axioms (1) and (2) may simply be carried over into this new theory; axiom (4) follows immediately from the definition of ‘$\mathbb{N}\xi$’. That one can not prove axiom (3), however, may be easily shown: Simply consider the model in which the basic domain contains only Caesar, in which the numerical domain contains only 0 and 1, and in which terms of the form “**N**x:Fx” are interpreted in the obvious fashion. It follows that all of Frege’s axioms for arithmetic, other than the third, may be proven in this theory.

The foregoing was known to Frege. He should be understood as arguing, when he first raises the Caesar objection in *Die Grundlagen*, precisely this: That, without the assumption that numbers are of the same Sort as basic objects, the logicist program is impossible. Recall that the inductive definition of cardinal numbers which Frege considers in *Gl* §55 may be formalized as follows:[37]

$$\exists_0 x.Fx \quad \equiv_{df} \neg\exists x.Fx$$
$$\exists_{n+1} x.Fx \quad \equiv_{df} \exists y[Fy \,\&\, \exists_n x(Fx \,\&\, x \neq y)]$$

In *Gl* §56, he writes that "these definitions suggest themselves so spontaneously in the light of our previous results that we shall have to go into the reasons why they cannot be reckoned satisfactory". He makes three objections. The first is that "we can never...decide whether any concept has the number Julius Caesar belonging to it, or whether that same familiar conqueror of Gaul is a number or not". The second is that "we cannot by the aid of our suggested definitions prove that, if the number *a* belongs to the concept F and the number *b* belongs to this same concept, then necessarily *a*=*b*". The third is that "[i]t is only an illusion that we have defined 0 and 1; in reality we have only fixed the senses of the phrases 'the number 0 belongs to' and 'the number 1 belongs to'...". Our task here is to understand the nature of these objections.

Note first that Frege's concern, in *Gl* §§55-6, is not just with the question whether the inductive definition suffices to explain ascriptions of number to concepts. Were that Frege's chief concern, one would have expected him to point out that, if the numbers are defined in accord with the inductive definition, one can not prove that each concept has *at least* one number.[38] What are defined are the second-level predicates: $\exists_0 x.\Phi x$, $\exists_1 x.\Phi x$, $\exists_{1+1} x.\Phi x$, and so on. The inductive definition does not provide an answer for every question of the form "How many Fs are there?" and, in particular, does not do so if there are infinitely many Fs. But Frege does not even raise this objection: He is here concerned only with finite numbers.

At this point in *Die Grundlagen*, Frege is beginning his discussion of the concept of number. As noted, he says that the inductive definitions "suggest themselves...spontaneously in the light of our previous results..." (*Gl* §56). The results in question are, essentially, two: First, "that the individual numbers are best derived...from the number one [better, zero] together with increase by one" (*Gl* §18); and, secondly, "that the content of a statement of number is an assertion about a concept" (*Gl* §46). Now, the 'inductive' character of the definitions certainly embodies the first of these results. What embodies the second is the fact that, as the numbers are here defined, they are second-level concepts, properties of concepts. That this view is Frege's target is confirmed by his pointing out, at the beginning of *Gl* §57, that, contrary to what one might have thought, it does not follows from the fact that a statement of number ascribes a property to a concept that "a number [is] a *property* of a concept". It is by disposing of this view that Frege means to motivate his alternate view, that numbers are *objects*.[39] His central point, I think, is that, if the numbers are defined as second-level concepts, as numerically definite quantifiers, the basic laws of arithmetic will not be provable, not at least without some further assumptions.

There is no difficulty with the definitions of the fundamental notions. 'Zero' may be defined as '$\exists_0$x.Φx'; and the 'inductive' part of the definition may be recast as a definition of the relation of predecession:[40]

$$\text{Pred}_\Phi[Qx.\Phi x, Rx.\Phi x] \equiv df\ \forall F[Rx.Fx \equiv \exists y(Fy \ \&\ Qx.(Fx \ \&\ y \neq x))]$$

That is to say, the second-level concept Rx.Φx will be the successor of the second-level concept Qx.Φx just in case: A concept Fξ falls under Rx.Φx if, and only if, there is an object, y, which is F, such that the concept Fξ & y≠ξ falls under Qx.Φx. Finally, then, the concept of a natural number may be defined by means of a fourth-order analogue of Frege's definition of the ancestral. An easily proven Lemma, to be needed shortly, is that, if Qx.Φx is a 'natural number', then the concepts falling under Qx.Φx are equinumerous with one another, that is, Qx.Φx is numerically definite.

Given these definitions, versions of Frege's axioms of arithmetic will be formulable. The first axiom, stating that the relation of predecession is functional, is a trivial consequence of the extensionality of (second-level) concepts: For if both Rx.Φx and Tx.Φx succeed Qx.Φx, then Rx.Fx ≡ ∃y(Fy & Qx.(Fx & y≠x)) ≡ Tx.Fx, so ∀F(Rx.Fx ≡ Tx.Fx).[41] The fourth axiom, induction, is again a trivial consequence of the definition of natural number. In *this* case, moreover, it is immediate that every number has a successor. For Qx.Φx will have a successor just in case some second-level concept Rx.Φx satisfies

$$\forall F[Rx.Fx \equiv \exists y(Fy \ \&\ Qx.(Fx \ \&\ y \neq x))]$$

But the sentence stating that there is such a concept is just an instance of (third-order) comprehension. The only interesting case, really, is that in which there are exactly *n* objects in the domain, for some natural number *n*. In this case, the successor of $\exists_n$x.Φx will be the *empty* second-level concept, ∅x.Φx, which is false of *every* concept: For, if there are exactly *n* objects, then there are *no* concepts $F\xi$ for which there is some object y such that $F\xi$ & $y \neq \xi$ falls under $\exists_n x.\Phi x$. Similarly, $\varnothing x.\Phi x$ succeeds itself.

Thus is the third of Frege's axioms for arithmetic proved. But the second axiom, which states that no natural number ancestrally follows itself, is not provable. For, as just noted, if the domain is finite, $\varnothing x.\Phi x$ will be a 'natural number' which *immediately*, and so ancestrally, succeeds itself. Where, then, would an attempt to mimic Frege's proof of this axiom fail? As an examination of the proof will show, the only arithmetical facts upon which it depends are that zero has no predecessor and that different numbers have different successors.[42] But the latter is not provable. For suppose that the domain contains exactly Caesar. Then the concept $\xi \neq \xi$ will be the only concept falling under $\exists_0 x.\Phi x$; the concept $\xi = \xi$, the only concept falling under $\exists_1 x.\Phi x$; and there will be *no* concept falling under $\exists_{1+1} x.\Phi x$, nor any falling under $\exists_{1+1+1} x.\Phi x$; and so on. $\exists_{1+1} x.\Phi x$ will therefore be the same second-level concept as $\exists_{1+1+1} x.\Phi x$, namely $\varnothing x.\Phi x$; hence, the successor of $\exists_1 x.\Phi x$ will be the same as that of $\exists_{1+1} x.\Phi x$.

What is needed if we are to prove that $\exists_{1+1} x.\Phi x$ is different from $\exists_{1+1+1} x.\Phi x$ is a proof that there are at least two objects. What is needed to prove that $\exists_n x.\Phi x$ is different from $\exists_{n+1} x.\Phi x$ is a proof that there are at least *n* objects. What is needed if we are to prove that all such quantifiers are distinct is a proof that, for each *n*, there are at least *n* objects. What Russell and Whitehead do in *Principia Mathematica* is simply to assume this claim: It is their axiom of infinity. Frege's objection to this would have been that to adopt an axiom of infinity is to abandon the epistemological project of accounting for our knowledge of the basic laws of arithmetic, for our knowledge of the infinity of the series of natural numbers, in particular. It would be more honest simply to assume that different numbers have different successors (a proposition to which the axiom of infinity is equivalent in *Principia*).[43]

Frege's own proof that there are infinitely many numbers will not even be formulable, for one can not, given the inductive definitions alone, so much as make sense of such predicates as "$\xi=0 \vee \xi=1$", which is what is required if "$\exists_2 x.(x=0 \vee x=1)$" is to be proven. We unable to make sense of such predicates because, as Frege points out in his third objection, the inductive definitions do not so much as define the expressions '0' and '1', but only the (orthographically) complex expressions '$\exists_0 x.\Phi x$' and '$\exists_1 x.\Phi x$'. Even if we waive this objection, we have the first objection, the Caesar objection, to face. We can not simply assume that the numerals, so defined, are of the same Sort as the variables bound by these quantifiers. And, as we saw earlier, without *that* assumption, the proof of the third axiom will fail.

Frege's second objection was that "we cannot by the aid of our suggested definitions prove that, if the number *a* belongs to the concept F and the number *b* belongs to this same concept, then necessarily *a*=*b*". What exactly does Frege mean that we can not prove? One idea would be that he is anticipating the third objection, that he really means that we have not even given a *sense* to sentences of the form "the number *a* is the same as the number *b*". But, even so, one can formulate something *like* a notion of identity for the second-level concepts introuduced by the inductive definition, in terms of their being co-extensive. And it is not difficult to show that no concept can fall under more than one second-level concept of the relevant kind, one numerically definite quantifier.[44] So either Frege is blundering badly, whence we must agree with Dummett that "§56 [is] the weakest in the whole of *Grundlagen*",[45] or this interpretation of the second objection is unsatisfactory.

What is at issue in the second objection is whether we can show that every concept has *at most* one number. Frege mentions this question only one other time in his (extant) writings, when introducing his second axiom in *Grundgesetze*. In *Gg* I §108, he argues that it follows from the second axiom that the fact that the results of counting are well-defined—that is to say, that any counting of the objects falling under a given (finite) concept must yield the same number. It is thus clear that Frege at least connected the second axiom with the question whether every concept has at most one number. Perhaps the second objection is, then, but a misleading way of putting the claim that the second axiom can not be proven from the inductive definitions. Frege misspeaks in so far as the problem is not that a single concept might fall under two of the quantifiers defined by the inductive definition: The problem concerns, not the *non-empty* quantifiers so defined, but rather any *empty* ones. Such sloppiness would be excusable, for it is not at all easy to express the point precisely, not at least without greater technicality than would have been appropriate to *Gl* §56. One could try putting the point by saying that one can not prove that all the quantifiers occuring in the series $\exists_0 x.\Phi x$, $\exists_1 x.\Phi x$, $\exists_{1+1} x.\Phi x$, etc., are distinct. But this too is misleading, because the concepts belonging to this series are, of course, distinct from one another—unless one means to allow that a given concept might 'belong' to the series more than once. The only precise way to state this claim with its intended meaning is to say that what is wanted is a proof that no concept ever occurs twice in this series. But this will remain obscure until the notion of ‘series’ being employed is explained—which would require the introduction of the ancestral, which is in fact delayed until *Gl* §79 and would have been quite out of place in the context of *Gl* §56.

4. Avoiding the Caesar Objection

Frege thus thought, for good reason, that the sort of proof of the axioms of arithmetic he needed required numbers to be treated as being of the same Sort as basic objects. It is for this reason, and not because of some dubious metaphysical thesis that all objects simply *must* be of a single Sort, that the Caesar objection arises: The Caesar objection is a reflection of a deep formal difficulty, one which Russell and Whitehead would rediscover some twenty-five years later. What I want to argue now, however, is that the foregoing considerations do not really show that Frege’s project requires numbers to be of the same Sort as basic objects, that Frege has overlooked a way of carrying out his project in the context of the many-sorted theory mentioned above.

Our knowledge of arithmetic begins with our understanding of attributions of number to concepts true or false of basic objects. At this early stage, our use of names of numbers is formalizable by means of the higher-order theory discussed in the last section, numerals being understood merely as orthographic parts of numerically definite quantifiers. What further is required if one is to achieve an understanding of names of numbers sufficient to ground our knowledge of the infinity of the number-series? A very natural suggestion, compatible with Frege’s general outlook, is that one must come to know that numbers are objects. But clearly that is not sufficient: If one treats numbers as objects of a Sort different from that of basic objects—and so in a way formalizable by means of the two-sorted theory discussed in the last section—then one will have no way of proving that there are infinitely many numbers. So, it would seem, if anything like Frege’s explanation of the genesis of our knowledge of the axioms of arithmetic is to succeed, we must confront the question *with what right* we suppose that numbers are of the same Sort as basic objects.[46] One way to press this point is to note that no sense has been given to such statements as “Caesar is the number zero”: For that such statements *do* make sense is precisely the content of the claim that numbers are of the same Sort as basic objects, that numbers and basic objects are members of a single domain of quantification. Now, I am not going to discuss the question whether a general solution to the Caesar problem might be possible. Even if one is, it would surely be better if Frege's position did not have to rest upon it.

Consider a speaker who has mastered only attributions of number to concepts true or false of basic objects. What distinguishes our mature understanding of names of numbers from hers is, most fundamentally, that we know that numbers can themselves be counted. What this comes to is that we understand terms of the form "the number of numbers satisfying such-and-such a condition". How is our understanding of such terms to be explained? Plainly, the two-sorted form of Hume's Principle discussed in the last section utterly fails to explain terms of this form; equally plainly, the familiar single-sorted version of Hume's Principle does explain such terms. If such terms can be explained only in terms of the single-sorted version of Hume's Principle, then Frege's project really does require numbers to be of the same Sort as basic objects. What I am going to argue, however, is that there is another way of explaining these terms, one equally in the spirit of Frege's project, which does not lead to a problematic instance of the Caesar objection.

One tempting suggestion is that an analogue of Hume's Principle may be introduced, one which specifically governs terms of the form "the number of numbers which are F". One way of taking this proposal may be explained by formalizing it within a many-sorted, higher-order language. There are to be variables of various types, the types indexed by numerals: Thus, variables of type 0 are written 'x_0', 'y_0', etc.; variables of type 1, 'x_1', 'y_1', etc.; and so forth. Similarly, there will be predicate variables and relation variables of various types, the types of whose arguments are indicated by subscripts: written 'F_0', 'G_0', 'F_1', etc.; '$R_{0,0}$', '$R_{0,1}$', '$R_{5,2,3}$', and so forth. The language also contains countably many term-forming operators "$N_{n+1}v_n{:}\Phi_n v_n$", which form terms of type n+1 from formulae containing a free variable of type n. Hume's Principle, in its original form, may then be formalized as:[47]

$$
\begin{aligned}
&N_1x_0{:}F_0x_0 = N_1x_0{:}G_0x_0 \text{ iff}\\
&\qquad \exists R_{0,0}\{\forall x_0\forall y_0\forall z_0\forall w_0(R_{0,0}x_0y_0 \,\&\, R_{0,0}z_0w_0 \rightarrow x_0{=}z_0 \equiv y_0{=}w_0) \,\&\\
&\qquad \forall x_0[F_0x_0 \rightarrow \exists y_0(G_0y_0 \,\&\, R_{0,0}x_0y_0)] \,\&\, \forall x_0[G_0x_0 \rightarrow \exists y_0(F_0y_0 \,\&\, R_{0,0}y_0x_0)]\}
\end{aligned}
$$

The suggested analogue of Hume's Principle, required for 'counting numbers', may then be formalized as:

$$
\begin{aligned}
&N_2x_1{:}F_1x_1 = N_2x_1{:}G_1x_1 \text{ iff}\\
&\qquad \exists R_{1,1}\{\forall x_1\forall y_1\forall z_1\forall w_1(R_{1,1}x_1y_1 \,\&\, R_{1,1}z_1w_1 \rightarrow x_1{=}z_1 \equiv y_1{=}w_1) \,\&\\
&\qquad \forall x_1[F_1x_1 \rightarrow \exists y_1(G_1y_1 \,\&\, R_{1,1}x_1y_1)] \,\&\, \forall x_1[G_1x_1 \rightarrow \exists y_1(F_1y_1 \,\&\, R_{1,1}y_1x_1)]\}
\end{aligned}
$$

Thus, the number of *type 1* numbers satisfying a given condition is a *type 2* number. And, more generally, the number of *type n* numbers satisfying a given condition will be a *type n+1* number:

$$N_{n+1}x_n{:}F_nx_n = N_{n+1}x_n{:}G_nx_n \text{ iff}$$
$$\exists R_{n,n}\{\forall x_n\forall y_n\forall z_n\forall w_n(R_{n,n}x_ny_n \;\&\; R_{n,n}z_nw_n \rightarrow x_n{=}z_n \equiv y_n{=}w_n) \;\&$$
$$\forall x_n[F_nx_n \rightarrow \exists y_n(G_ny_n \;\&\; R_{n,n}x_ny_n)] \;\&\; \forall x_n[G_nx_n \rightarrow \exists y_n(F_ny_n \;\&\; R_{n,n}y_nx_n)]\}$$

Similarities to the theory of types are, of course, non-accidental. In this theory, as in the two-sorted theory discussed earlier, it will be possible, at each type i>0, to formulate versions of the Frege's axioms for arithmetic and to prove all of them except that every number has a successor, for the same reason discussed above in connection with the two-sorted theory.

But is this treatment really compatible with Frege's observation that numbers can themselves be counted? Does this theory really capture our mature understanding of names of numbers? I think not. Consider the question what, on this view, is to be said about the use of numerals in ordinary language (or, for that matter, in informal mathematics). Presumably, when speakers make remarks such as "The number of numbers greater than 5 but less than 15 is 9",[48] this is to be taken as *systematically ambiguous*. The speaker should be understood not as making some specific claim of the form "The number$_{n+1}$ of numbers$_n$ greater than 5$_n$ but less than 15$_n$ is 9$_{n+1}$", but rather as making an *ambiguous* claim: Roughly speaking, she claims that, for all permissible assignments of types, the resulting sentence is true.[49]

Whatever the prospects of this sort of manoeuver elsewhere, it will not work here. When one asks oneself, say, how many numbers there are which are less than 5, one supposes that the answer will be a *number*, an object which (logically speaking) *could be*, though it is not, one of the numbers less than 5; the type-theoretic treatment has it, however, that the answer is a 'number' of a different type, a different Sort. To put the point differently, one supposes that the statement "The number of numbers less than 5 is 5" is at least *well-formed*. The type-theoretic view can, of course, concede this point and suggest that what is really meant is something like: "The number$_{n+1}$ of numbers$_n$ less than$_{n,n}$ 5$_n$ is$_{n+1}$ 5$_{n+1}$". But even if this re-interpretation of the claim should be deemed acceptable, the trick does not work more generally. Consider the statement "There is a number which is the number of numbers less than it", formally: $\exists x.[x = Ny{:}y<x]$. *This* claim admits of no acceptable assignment of types, for "$\exists x_n[x_n =_? N_{n+1}y_n{:}y_n<_{n,n}x_n]$" is not well-formed, since terms of different types appear on the two sides of the identity-sign. Or again, consider the (Boolosese) claim "There are some numbers the number of which is one of them”, formally: $\exists F.F(Nx{:}Fx)$. This claim, again, is not only well-formed but true. But there is no permissible assignment of types.

What do these examples show? Not, admittedly, that there is *no* type-theoretic way to handle such statements. I am not about to make such a prediction, given the malleability of formal methods and the general cleverness of logicians.[50] What the examples show is that, though we have no strong intuitions regarding whether numbers and people are of a single Sort, we *do* consider all *numbers* to be of a single Sort. That is to say, we regard terms of the form “the number of Roman emperors” and “the number of numbers less than 5” as being of the same Sort. Of course, if we so regard them, we must be prepared to face an instance of the Caesar objection. But *this* instance of the Caesar objection is innocuous. For consider an identity-statement linking terms of these two kinds:

The number of Roman emperors is the same as the number of numbers less than 5

We discussed this statement earlier, and whether numbers are of the same Sort as people is irrelevant to its truth-conditions: The sentence will be true just in case there is a one-one correlation between the Roman emperors and the numbers less than 5. More generally, a statement of the form “The number of basic objects which are F is the same as the number of numbers which are **G**” will be true just in case there is a one-one correlation between the Fs and the **G**s. This *specific* instance of the Caesar objection thus has a simple, obvious, and obviously correct answer.

Now, should we conclude from this that numbers must be of the same Sort as basic objects? Such an inference would be completely fallacious. The fact that “the number of basic objects which are F” is of the same Sort as “the number of numbers which are **F**” says nothing whatsoever about the Sorts of the predicates in question. That functional expressions have *values* of the same Sort does not imply that they have *arguments* of the same sort. What we need, then, is to formulate a theory which distinguishes the Sorts of numbers and basic objects, but which treats terms of the form “the number of Fs” and “the number of **F**s” as being of the same sort.

We revert to the two-sorted language employed in the last section. A version of Hume’s Principle for concepts true or false of basic objects may be formulated as follows:

$$(HP_{bb})\quad \mathbf{N}x{:}Fx = \mathbf{N}x{:}Gx \text{ iff } \exists R_{bb}\{\forall x\forall y\forall z\forall w(R_{bb}xy \,\&\, R_{bb}zw \rightarrow x{=}z \equiv y{=}w) \,\&\, \forall x[Fx \rightarrow \exists y(Gy \,\&\, R_{bb}xy) \,\&\, \forall y(Gy \rightarrow \exists x(Fx \,\&\, R_{bb}xy)]\}$$

A version of Hume’s Principle which governs concepts true or false of numbers themselves may then be formulated as:

$$(HP_{nn})\quad \mathbf{Nx}:\mathbf{Fx} = \mathbf{Nx}:\mathbf{Gx} \text{ iff } \exists R_{nn}\{\forall\mathbf{x}\forall\mathbf{y}\forall\mathbf{z}\forall\mathbf{w}(R_{nn}\mathbf{xy} \,\&\, R_{nn}\mathbf{zw} \rightarrow \mathbf{x}=\mathbf{z} \equiv \mathbf{y}=\mathbf{w}) \,\&\, \forall\mathbf{x}[\mathbf{Fx} \rightarrow \exists\mathbf{y}(\mathbf{Gy} \,\&\, R_{nn}\mathbf{xy})] \,\&\, \forall\mathbf{y}[\mathbf{Gy} \rightarrow \exists\mathbf{x}(\mathbf{Fx} \,\&\, R_{nn}\mathbf{xy})]\}$$

Note that terms of the form “**Nx**:**Fx**” are to be of the same Sort as those of the form “**N**x:Fx”. The version of Hume’s Principle which governs *mixed* identities is then:[51]

$$(HP_{bn})\quad \mathbf{N}x{:}Fx = \mathbf{Nx}:\mathbf{Gx} \text{ iff } \exists R_{bn}\{\forall x\forall\mathbf{y}\forall z\forall\mathbf{w}(R_{bn}x\mathbf{y} \,\&\, R_{bn}z\mathbf{w} \rightarrow x=z \equiv \mathbf{y}=\mathbf{w}) \,\&\, \forall x[Fx \rightarrow \exists\mathbf{y}(\mathbf{Gy} \,\&\, R_{bn}x\mathbf{y}) \,\&\, \forall\mathbf{y}(\mathbf{Gy} \rightarrow \exists x(Fx \,\&\, R_{bn}x\mathbf{y})]\}$$

There is no formal obstacle to formulating such a principle, even if basic objects and numbers are assumed to be of different Sorts.

Call the two-sorted, second-order theory whose (non-logical) axioms are (HP_{bb}), (HP_{nn}), and (HP_{bn}), *Two-sorted Fregean Arithmetic* (2FA). Axioms for arithmetic are provable in 2FA. What is required for Frege’s proof of the infinity of the number-series is that we should be able to prove that the number of numbers less than or equal to *n* is the successor of *n* (if *n* is a natural number). As was seen above, if this is to be proven, all that is required is that “the number of numbers less than or equal to *n*” be of the same Sort as “*n*” itself. But, in 2FA, this is so. More formally, suppose we simply *drop* all reference to basic objects from 2FA. What remains is a *single-sorted* theory whose sole (non-logical) axiom is:

$$\mathbf{Nx}:\mathbf{Fx} = \mathbf{Nx}:\mathbf{Gx} \text{ iff } \exists R\{\forall\mathbf{x}\forall\mathbf{y}\forall\mathbf{z}\forall\mathbf{w}(R\mathbf{xy} \,\&\, R\mathbf{zw} \rightarrow \mathbf{x}=\mathbf{z} \equiv \mathbf{y}=\mathbf{w}) \,\&\, \forall\mathbf{x}[\mathbf{Fx} \rightarrow \exists\mathbf{y}(\mathbf{Gy} \,\&\, R\mathbf{xy}) \,\&\, \forall\mathbf{y}(\mathbf{Gy} \rightarrow \exists\mathbf{x}(\mathbf{Fx} \,\&\, R\mathbf{xy})]\}$$

But this axiom, *modulo* the boldface, is just Hume’s Principle in its simple form: So 2FA is an extension of single-sorted Fregean Arithmetic and so proves whatever it does.[52]

To see how insignificant it ultimately is that numbers and basic objects belong to different Sorts, consider the fact that not just *objects* but also *concepts and functions* can be counted.[53] Not only is it a sensible question how many (Fregean) concepts there are which are true of only Caesar, the answer is obvious: One. Nor does the fact that one is counting *concepts* rather than objects imply that the answer is not a *number* of the usual kind. Nor is there any difficulty in saying under what circumstances the number of concepts $F\xi$ falling under some second-level concept $\Phi_x\varphi x$ is the same as the number of objects x falling under some first-level concept $P\xi$: The numbers will be the same just in case there is a one-one correlation between the concepts and the objects. Where 'Σ_z' ranges over relations between concepts and objects, 'Φ_z', over second-level concepts, this thought can be formalized as follows:

$$\text{NF:}\Phi_x Fx = Nx{:}Px \text{ iff } \exists\Sigma\{\forall x\forall F\forall y\forall G[\Sigma_z(x,Fz) \;\&\; \Sigma_z(y,Gz) \rightarrow x=y \equiv \forall x(Fx \equiv Gx)] \;\&\; \\ \forall x[Px \rightarrow \exists F(\Phi_z Fz \;\&\; \Sigma_z(x,Fz))] \;\&\; \forall F[\Phi_z Fz \rightarrow \exists x(Px \;\&\; \Sigma_z(xFz))]\}$$

There is no formal obstacle *even* to the formulation of a version of Hume's Principle governing mixed identity-statements of *this* kind. Indeed, one can formulate versions of Hume's Principle to govern terms of the form "the number of entities satisfying such and such a condition", in both pure and mixed identity-statements, no matter what sorts of entities might be in question.[54] Hume's Principle may now be thought of as *schematic*, as specifying the truth-conditions of pure and mixed identity-statements containing terms ascribing number to concepts (whose arguments may be) of any level or Sort.

5. Closing

The fundamental epistemological question of the philosophy of arithmetic is of the basis of our knowledge of the infinity of the series of natural numbers. On the position developed here, the acquisition of such knowledge occurs in three stages: First, one must come to understand ascriptions of number to concepts true or false of basic objects, of the various objects to which one is capable of referring before acquiring a capacity to refer to numbers; Secondly, one must come to understand ascriptions of number to concepts true or false of numbers themselves; and, Thirdly, one must come to understand the conditions under which a number ascribed to a concept true or false of basic objects will be the same as one ascribed to a concept true or false of numbers. (These three stages correspond to the three axioms of 2FA: The first and second correspond to the axioms governing the 'pure' identity-statements involving terms ascribing number to concepts true or false of basic objects and numbers, respectively; the third, to the axiom governing mixed identity-statements.) At this point, one will know that terms of the form "the number of numbers which are **F**" and "the number of basic objects which are F" are of the same logical Sort and so will have acquired knowledge from which the infinity of the series of natural numbers may be inferred. I find this philosophical 'reconstruction' of the genesis of our knowledge of arithmetic extremely compelling.

This position is subtly but importantly different from that mentioned at the beginning of §4. On that view, there is, in place of the second and third steps, a single step at which the speaker comes to understand ‘that numbers are objects’, that numbers are objects of the same Sort as basic objects. If one formulates what a speaker must come to know in this way, the Caesar objection looms; the question with what right we suppose ourselves to know that numbers are of the same Sort as basic objects may well be unanswerable. However, though one *could* acquire ‘knowledge’ from which the axioms of arithmetic are derivable by coming to ‘know’ that numbers are of the same Sort as basic objects, such ‘knowledge’ is not necessary: What one needs to know is that numbers ascribed to concepts true or false of numbers are of the same Sort as numbers ascribed to concepts true or false of basic objects. It is quite fallacious to infer from this that numbers must be of the same Sort as basic objects—and it is, I think, just such a fallacious inference that explains Frege’s obsession with the Caesar problem.

Moreover, we are now in a position to see that *not even the claim that numbers are objects is required for Frege’s proofs of the axioms of arithmetic*. What is required is that expressions of the form “the number of numbers less than 5” should be of the same logical Sort as those of the form “the number of Roman emperors”. It matters not one bit whether this Sort is that of basic objects, of objects of some other kind, or of second-level concepts.[55] What are required are axioms similar to those of 2FA. Write ‘Eq’ for ‘...is equinumerous with...’. The first axiom is then:

$(HP_{1,1})$ $\forall H\{N_{xy}(Fx)(Hy) \equiv N_{xy}(Gx)(Hy)\}$ iff $Eq_x(Fx,Gx)$

What this says is that, for each concept $F\xi$, the second-level concept $N_{xy}(Fx)(\varphi y)$ is (co-extensive with) the concept $N_{xy}(Gx)(\varphi y)$ if, and only if, $F\xi$ and $G\xi$ are equinumerous. What we now require is an analogue of this for *third-level* concepts, i.e., for concepts true or false of the ‘numbers’, which are the second-level concepts $N_{xy}(Fx)(\varphi y)$. Let ‘Ψ’ and ‘Φ’ be variables for third-level concepts; ‘χ’, for second-level. Then the needed axiom is:

$(HP_{3,3})$ $\forall H\{N_{\chi,x}[\Psi(\chi)](Hx) \equiv N_{\chi,x}[\Phi(\chi)](Hx)\}$ iff $Eq_\chi[\Psi(\chi),\Phi(\chi)]$

What this says is that, for each third-level concept Ψ, the second-level concept $N_{\chi,x}[\Psi(\chi)](\varphi y)$ is (co-extensive with) the concept $N_{\chi,x}[\Phi(\chi)](\varphi y)$ if, and only if, $\Psi(\chi)$ and $\Phi(\chi)$ are equinumerous. The third axiom is that governing mixed identities:

$(HP_{1,3})$ $\forall H\{N_{\chi,x}[\Psi(\chi)](Hx) \equiv N_{xy}(Gx)(Hy)\}$ iff $Eq_{\chi,x}[\Psi(\chi),Gx]$

It is easy to see that the resulting system is consistent and proves axioms for arithmetic.[56] Frege’s proofs can simply be mimicked, using $(HP_{3,3})$.

It does not follow, of course, that there are no objections to a form of logicism based upon the derivability of axioms for arithmetic in 2FA. In particular, I have not addressed the ‘bad company’ objections to the claim that Hume’s Principle’s has a favored epistemological status—objections which derive from the observation that formally similar principles are inconsistent, are (though consistent) inconsistent with Hume’s Principle, or are otherwise naughty.[57] But let us not forget, in our haste to evaluate such a version of logicism, that the attractions of the genetic story told at the beginning of this section do not depend upon the claim that the various instances of Hume’s Principle are logical truths, analytic truths, or any such thing. Frege’s most fundamental thought—that our knowledge of the truths of arithmetic derives, in some sense, from our knowledge of Hume’s Principle—could well be true, even if it does not have the epistemological implications he had hoped it would.[58]

Notes

1. See Gottlob Frege, *The Foundations of Arithmetic*, tr. by J.L. Austin (Evanston IL, 1980). Hume's Principle is introduced in §63; the proofs are sketched in §§74-83. Further references are in the text, marked by '*Gl*' and the section number. Of course, the mathematical importance of equinumerosity was first fully understood by Cantor.

2. This was first noted, in recent times, in Charles Parsons, "Frege's Theory of Number", in *Mathematics in Philosophy* (Ithaca NY, 1983), 150-75, reprinted in W. Demopoulos, ed., *Frege's Philosophy of Mathematics* (Cambridge MA, 1995), 182-210. The potential interest of the result was, however, only emphasized in Crispin Wright, *Frege's Conception of Numbers as Objects* (Aberdeen, 1983), the fourth chapter of which contains a proof.

3. The consistency of the theory was independently noted by John Burgess, Allen Hazen, and Harold Hodes, in their respective reviews of *Frege's Conception*. The result mentioned, however, was first proved in George Boolos, "The Consistency of Frege's *Foundations of Arithmetic*", in J. Thomson, ed., *On Being and Saying: Essays for Richard Cartwright* (Cambridge MA, 1987), 1-20, reprinted in Demopoulos, ed., 211-33. For an improved proof, see the first appendix to George Boolos and Richard G. Heck, Jnr., "*Die Grundlagen der Arithmetik* §§82-3", *forthcoming* in M. Schirn, ed., *Philosophy of Mathematics Today*.

4. Gottlob Frege, *Grundgesetze der Arithmetic* (Hildesheim, 1966), Part II, sections Alpha-Iota. Further references are in the text, marked by '*Gg*' and a volume and section number. Frege does frequently make *inessential* uses of Axiom V, but these are for convenience. For discussion, see my "The Development of Arithmetic in Frege's *Grundgesetze der Arithmetik*", *Journal of Symbolic Logic* 58 (1993), 579-601; reprinted, with a postscript, in Demopoulos, ed., 257-94.

5. For discussion, see my "Frege's Principle", in J. Hintikka, ed., *From Dedekind to Gödel* (Boston, 1995), 119-42, and the references to Michael Dummett's work to be found there. We shall touch upon the matter below, too.

6. Gottlob Frege, *Philosophical and Mathematical Correspondence*, ed. by G. Gabriel, *et al*., tr. by H. Kaal (Chicago, 1980), 141. This is letter xxxvi/7.

7. Note that the objection is not so much that the contextual definition fails to decide the truth-value of this sentence, but that it fails to give any sense to it at all.

8. One might want to object that there is no reason to suppose that our recognition that Caesar is not a number must be explained in terms of the resources employed in an explanation of our apprehension of numbers as objects. But it is hard to see how one's recognition that Caesar is not a number could, so to speak, flow from anything but one's understanding of numerical terms. In any event, Frege does not discuss this assumption; for present purposes, let us just record it.

9. This is the point of Frege's mentioning of sentences of the form "*q* is the direction of *a*". I take it that a nice, rhetorical way to make this point is to emphasize that the definition gives us no purchase on the question whether this open sentence is true or false of England. To the best of my knowledge, this point, like many others relevant to the present topic, was first made in Charles Parsons's "Frege's Theory of Number".

10. For an approach to this problem, see my "The Existence (and Non-existence) of Abstract Objects", *forthcoming*.

11. This is what Wright attempts to do in §xiv of *Frege's Conception*. The use of the term "sort" is intentional: Adopting the view discussed below, that numbers are simply a different *Sort* from people, will not relieve one of the obligation to explain the origins of speakers' knowledge of this fact.

12. Gottlob Frege, “Function and Concept”, tr. by P. Geach, in *Collected Papers*, ed. by B. McGuinness (Oxford, 1984), 137-56, at original page 20. I have argued elsewhere that this requirement itself results from Frege’s desire to provide his formal theory with a classical semantics. See my “Frege and Semantics”, *forthcoming* in T. Ricketts, ed., *The Cambridge Companion to Frege*.

13. Peter Schroeder-Heister has shown that the permutation argument can be so reconstructed that it does indeed show, with respect to the first-order fragment of Frege’s theory, what is here claimed. See his “A Model-theoretic Reconstruction of Frege’s Permutation Argument”, *Notre Dame Journal of Formal Logic* 28 (1987), 69-79. The argument can be modified to establish a similar claim about the predicative second-order fragments of the theory.

Note that both the first-order and predicative second-order fragments of Frege’s theory are consistent. For the former result, see Terrence Parsons, “On the Consistency of the First-Order Portion of Frege’s Logical System”, *Notre Dame Journal of Formal Logic* 28 (1987), 161-8. For the latter, my “The Consistency of Predicative Fragments of Frege’s *Grundgesetze der Arithmetik*”, unpublished.

14. One of the interesting facts about the stipulation is that it is not embodied in the axioms of the formal theory. The reason, one might conjecture, is that the problem under discussion in *Gg* I §10 is a *semantical* problem; that Frege appeals to his stipulation only during *Gg* I §31 also suggests this, for *Gg* I §31 is a (flawed) attempt to prove the consistency of Frege’s theory. But that is another paper, namely, my “*Grundgesetze der Arithmetik* I §31”, *forthcoming*.

15. Tyler Burge has argued that there are good, philosophical reasons both for Frege to identify truth-values as objects and for him to identify them with the value-ranges with which he does. See his “Frege on Truth”, in J. Hintikka and L. Haaparanta, eds., *Frege Synthesized* (Dordrecht, 1986), 97-154. His arguments merit a reply, but I can not discuss them here. Let me say, though, that *if* we assume Frege has reason to identify truth-values with value-ranges, Burge gives as good an account as one can of why Frege chooses the value-ranges he does.

Thomas Ricketts has suggested to me that, as I am reading it, the point of the ‘argument’ is just that, since sentences are saturated, allowing them to stand where names stand is legitimate in so far as doing so will not, at least, leave unfilled argument places. This seems right.

16. Michael Dummett, *Frege: Philosophy of Language*, 2nd ed. (London, 1981), 183-5.

17. Until, of course, one realized that (Vc) is inconsistent and (Vf) implies that there is exactly one object. We shall, however, abstract from these problems with (Vc) and (Vf) here: One could transpose much of the present discussion to the context of predicative second-order logic, where (Vc) and (Vf) raise no such problems.

18. See my “Development”, 581-3, for discussion of this point.

19. For Frege, ordered pairs are defined in terms of value-ranges, so value-ranges are not sets of ordered pairs. But if we pretend that they are, the points made in the text can be put as follows. The value-range of the function, ἐ.∈+2, is the set of those ordered pairs {<∈,∈+2>}. The value-range of ἐ.∈+*n* will be the set of ordered pairs {<∈,∈+*n*>}. And the value-range of the function ἐ.∈+ξ is the *double* value-range ἀ[ἐ.∈+α], the set of ordered pairs {<α,ἐ.∈+α>}, or {<α,{<∈,∈+α>}}.

20. Frege is thus treating, for this purpose, every function, of however many arguments, as a one-place function, with values possibly themselves being functions. So, for example, a two-place function is here treated as being a one-place function whose value is a one-place function. This treatment is, I am told, standard in combinatory logic and is credited to Moses Schönfinkel, for which see his “On the Building Blocks of Mathematical Logic”, in J. van Heijenoort, *A Source Book in Mathematical Logic, 1879-1931* (Cambridge MA, 1967), 255-66. In his introductory note (which is on pp. 255-7), Quine notes that Frege “anticipated” Schönfinkel’s construction in his discussion of double value-ranges. Thanks to George Boolos and Jason Stanley for this information.

21. Thanks to Michael Glanzberg for asking a question which made what follows clear to me.

22. In set-theory, the characteristic function associated with a set S is the function $\varphi_S(\xi)$ whose value, for argument x, is 1 if $x \in S$; 0, otherwise.

23. Frege has a description operator in the formal system of *Grundgesetze*, but it is applied to value-range terms, not to predicates. What is needed here is description operator of the more usual sort. The axiom "$a = \iota x.Fx$ iff Fa & $\forall x \forall y(Fx$ & $Fy \rightarrow x=y)$" will serve as an analogue of Frege's Axiom VI. A pure analogue of Axiom VI would be "$a = \iota x.x=a$", but this will not imply the above without an axiom asserting the extensionality of concepts. Such an axiom is not present in Frege's system (and is not usually present in axiomatic second-order logic), though his semantics for second-order logic does justify it.

24. See Gottlob Frege, *Begriffsschrift: A Formula Language, Modeled Upon That of Arithmetic, for Pure Thought*, in J. van Heijenoort, ed., 1-82, §3.

25. Moreover, we can see now why Frege thought himself free to *stipulate* which value-ranges the truth-values are: For the selection of objects in terms of which to define the characteristic functions is arbitrary, constrained only by the resources available in the language. One might wonder, though, how Frege might have proven that *there are* two objects in the domain. Treating the truth-values as objects does resolve this problem (as well as the more general problem of the non-emptiness of the domain).

26. To emphasize that I am using the term, on Frege's behalf, in this sense, I shall speak of expressions as being of the same logical Sort, rather than of the same logical type.

27. For simplicity, I assume that these are of a single Sort. Just how unimportant this assumption is will be clear by the end of the paper.

28. It is exactly at this point that the second problem raised by the Caesar objection, discussed in section 1, becomes pressing.

29. See *Gl* §14: "The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs..everything thinkable". See also "Formal Theories of Arithmetic", in the *Collected Papers*, 112-21, at original page 94: "...[W]e can count just about everything that can be an object of thought: ...even numbers can in their turn be counted".

30. It is more usual to make this argument by concentrating on the Sort of object over which the *bound variables* appearing in Hume's Principle range. The argument given here is really equivalent to that one, though putting it in the terms in which it is put here sheds some additional light on it and makes for a change of pace.

31. This is done, of course, by inductively defining a map from the numbers into this series. The techniques for doing so were known to Frege. See my "Definition by Induction in Frege's *Grundgesetze der Arithmetik*", in W. Demopoulos, ed., *Frege's Philosophy of Mathematics* (Cambridge MA, 1995), 295-333, and "The Finite and the Infinite in Frege's *Grundgesetze der Arithmetik*", *forthcoming* in M. Schirn, ed., *Philosophy of Mathematics Today*.

It is worth mentioning that this same construction can be carried out given various consistent axioms weaker than Axiom V. For example, the same construction can be carried out given George's Boolos's Axiom New V, and an analogous construction can be carried out given the ordered pair axiom and another axiom asserting that there are at least two objects.

32. Another way to put this point is as follows: Any (reasonably typical, non-modal, etc.) development of arithmetic will imply the existence of the natural numbers, but Frege's has the virtue of implying the existence of nothing else.—Actually, Hume's Principle does imply the existence of the first transfinite cardinal, though that is not really relevant in the present context. Frege is looking for a general theory of

cardinal number.

33. I owe W.W. Tait special thanks for discussions connected with the preceding few paragraphs.

34. Here and below, the comprehension axioms for the theory are to be of the usual, impredicative sort. That is to say, where 'F' and '**F**' are not free in 'A(x)' and 'A(**x**)', respectively, all formulae of the forms

$\exists F \forall x[Fx \equiv A(x)]$

$\exists \mathbf{F} \forall \mathbf{x}[\mathbf{Fx} \equiv A(\mathbf{x})]$

and so forth are to be comprehension axioms.

35. I ignore here some of the complexity inherent if Frege's actual definition of "ξ is a natural number". See "Development of Arithmetic", 590-1, for discussion of this.

36. The strong ancestral of a relation Rξη is defined as follows:

$R^*(a,b) \equiv df\ \forall F[\forall x(Rax \rightarrow Fx)\ \&\ \forall x \forall y(Fx\ \&\ Rxy \rightarrow Fy) \rightarrow Fb]$

Intuitively, a bears the strong ancestral of Rξη to b if there is a non-empty, but finite, sequence of R-steps connecting a to b. Thus, an object does not, in general, bear the strong ancestral of a relation to itself, but only to its 'proper' ancestors.

The second axiom requires that there be no 'loops' in the sequence of natural numbers. For a discussion of Frege's axioms for arithmetic, see "Definition by Induction". It is perhaps worth noting, too, that the second axiom is specially emphasized in Frege's discussion in *Gl* §83.

37. The second definition should, strictly speaking, be understood as a schema. This is the point of Frege's remark that we can so explain only expressions of the form "the number 1+1+...+1 belongs to the concept F" (*Gl* §56). An analogue of this point was made in §vi of Wright's *Frege's Conception of Numbers as Objects*. My work on *Gl* §§55-6 owes heavily to Wright's discussion, as well as to that in Ch. 9 of Michael Dummett's *Frege: Philosophy of Mathematics* (Cambridge MA, 1992).

38. Thanks to Ori Simchen for emphasizing this point to me.

39. This is how Michael Dummett reads these sections in *Frege: Philosophy of Mathematics*. My debt to his discussion there is enormous.

40. Note that 'Φ', on the left-had side, is just a bound (second-order) variable. That the definition is so easily recast perhaps further supports my claim that the inductive definitions are intended as a 'first attempt' at proving the axioms of arithmetic.

41. One might have wanted to suggest that it is *this* Frege thinks can not be proved: The proof does require an axiom stating that second-level concepts are extensional, and one might have supposed that Frege did not come to that view until later. But see the footnote to *Gl* §68.

42. See "Development of Arithmetic", 593-5.

43. See Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, 2nd. ed. (Cambridge, 1925). The axiom of infinity is introduced at *120.03 and is discussed on the following two pages, vol. II, 203-4. The equivalence mentioned is proved at *125.14. Interestingly enough, the axiom of infinity is also equivalent to the claim that ∅x.Φx is not a natural number: see *125.13. For a more sympathetic discussion of the role of the axiom of infinity in the theory of types, see George Boolos, "The Advantages of Honest Toil over Theft", in Alexander George, ed., *Mathematics and Mind* (Oxford, 1994), 27-44.

44. A numerically definite quantifier is one all the concepts falling under which are equinumerous with one another: I.e., one equivalent, for some Fξ, to: Φξ is equinumerous with Fξ. I shall abuse use and mention and use 'quantifier' to speak both of the symbols in question and of the second-level concepts they denote.

45. See Dummett, *Frege: Philosophy of Mathematics*, 102-8. The quotation is taken from p. 105.

46. The importance of this question was first suggested to me by Charles Parsons's paper "Intuition and Number", in George, ed., 141-57, see esp. pp. 150-1.

47. I apologize for the low level of readability of this formula, due to the presence of so many subscripts. It is crucial, though, that it be clear that everything being done here can be done without violating any type-restrictions, and the only way to establish this point is to be as rigorous as possible.

48. It is frequently noted in such contexts that number theory does concern itself with e.g. the number of prime numbers less than a given number. This is, indeed, the example Dummett uses in his discussion of this matter.

49. On the notion of systematic ambiguity, see Charles Parsons, "Sets and Classes", reprinted in his *Mathematics in Philosophy*, 209-20. If one were to insist that such statements involve definite assignments of types, that would only make matters worse.

50. It should be said, however, that there is reason to suppose that no such treatment will be forthcoming. The reason is simple: If there is some way of construing such claims so that they are well-formed and have the correct truth-conditions, then "If *n* is finite, the number of numbers less than or equal to *n* is the successor of *n*" will have to come out true. But then Frege's project goes forward. Compare my "Critical Notice of Michael Dummett, *Frege: Philosophy of Mathematics*", *Philosophical Quarterly* 43 (1993), 223-33, at 231.

51. Identity-statements of the form "**Nx**:**Fx** = **N**x:Fx" may either be governed by another form of Hume's Principle, whose formulation should be obvious at this point, or simply be stipulated to be equivalent to those of the form "**N**x:Fx = **Nx**:**Fx**".

52. Frege's actual proofs of the axioms of arithmetic, that is to say, can simply be carried out by boldfacing all his object- and concept- variables and adding the subscript 'nn' to all relation-variables.

53. Frege makes this claim in at least two different places. First, in his letter to Marty of 29 August 1882: "Everything is enumerable, ...even concepts..." (*Correspondence* 100, letter xxx/1). Secondly, in "Formal Theories of Arithmetic" (1885), op. 94: "...[W]e can count just about everything that can be an object of thought: ...concepts as well as objects...". Frege's insistence that "everything thinkable" can be counted, at *Gl* §14, must presumably include concepts as well as objects. (Thanks to Jamie Tappenden for these references.)

54. The higher-order theory containing as (non-logical) axioms all the infinitely many such Principles is provably (formally) consistent. For a sketch of the proof, see my "Critical Notice", 231. (The proof as formulated there is, unfortunately, a bit sloppier than I would now have it be.)

55. This point is made in my "Critical Notice", 230, but I have only recently understood its force. What this means is that something like the program David Bostock pursues in his *Logic and Arithmetic: Natural Numbers* (Oxford, 1974) can be executed. Bostock's idea was to treat numbers as second-level concepts, but Dana Scott showed the system in which he worked to be inconsistent.

56. It is crucial that these axioms say nothing about what *kinds* of second-level concepts the numbers are. For all the axioms say, $N_{xy}(x \neq x)(\varphi y)$ might be the second-level concept: φ(Caesar). One could try to proceed differently and simply *define* '$N_{xy}(Gx)(\varphi y)$' as '$Eq_x(Gx,\varphi x)$', whence ($HP_{1,1}$) will be provable. Indeed, one should probably think of Frege's discussion in *Gl* §§55-6 as being directed toward a theory whose sole axiom (really, definition) is

$$N_{xy}(Gx)(Fy) \equiv Eq_x(Gx,Fx)$$

But, though one can also define '$N_{\chi,x}[\Psi(\chi)](\varphi x)$' as '$Eq_{\chi,x}[\Psi(\chi),\varphi x]$', one will not them be able to prove ($HP_{3,3}$); the numbers of non-equinumerous concepts will end up being the same, namely, the empty second-level concept.

The difficulty with a theory based upon ($HP_{3,3}$) is, to my mind, that it will imply the existence of infinitely many *objects*. The numbers here are not objects, so the theory is implying that there are infinitely many objects of what kind, exactly? Compare here the discussion of the "deep misunderstanding" in §3.

57. See, for example, George Boolos, "The Standard of Equality of Numbers", in G. Boolos, ed., *Meaning and Method* (Cambridge, 1990), 261-77, reprinted in Demopoulos, ed., 234-54; Michael Dummett, *Frege: Philosophy of Mathematics*, 187-89; and my "On the Consistency of Second-order Contextual Definitions", *Noûs* 26 (1992), 491-4. For further discussion, see the papers by Wright and Boolos in this volume.

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