Theories of truth for countable languages which conform to classical logic

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Abstract

Every countable language which conforms to classical logic is shown to have an extension which conforms to classical logic, and has a definitional theory of truth. That extension has a semantical theory of truth, if every sentence of the object language is valuated by its meaning either as true or as false. These theories contain both a truth predicate and a non-truth predicate. Theories are equivalent when the sentences of the object language are valuated by their meanings.

1 Introduction

Based on 'Chomsky Definition' (cf. [2]) we assume that a language is a nonempty countable set of sentences with finite length, and formed by a countable set of elements. A theory of syntax is also assumed to provide a language with rules to construct well-formed sentences, formulas etc.

A language is said to conform to classical logic if it has, or if it can be extended to have at least the following properties ('iff' means 'if and only if'):

(i) It contains logical symbols $\neg$ (not), $\vee$ (or), $\wedge$ (and), $\rightarrow$ (if...then), $\leftrightarrow$ (iff), $\forall$ (for all) and $\exists$ (exists), and the following sentences: If A and B are (denote) sentences of the language, so are $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$ and $A \leftrightarrow B$. If $P(x_1, \ldots, x_m)$, $m \geq 1$, is a formula of the language with m free variables, then P is called a predicate with arity m and domain $D_P = D_P^1 \times \cdots \times D_P^m$, where each D_P^i is a subset of a set D of objects, called the domain of discourse, if the following properties hold.

(p1) Every object of D is named by a term. Denote by N_P^i the set of those terms which name the objects of D_P^i, $i = 1, \ldots, m$.

(p2) $P(b_1, \ldots, b_m)$ is a sentence of that language obtained from $P(x_1, \ldots, x_m)$ by substituting for each $i = 1, \ldots, m$ a term b_i of N_P^i for x_i in every free occurrence of x_i in P.

If P is a predicate with arity $m \geq 1$, then the sentences $q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m)$, where each q_i is either $\forall$ or $\exists$, are in the language. $\neg P$ is also a predicate with arity m and domain D_P.

(ii) The sentences of that language are so valuated as true or as false that the following rules of classical logic are valid: If A and B denote sentences of the language, then A is true iff $\neg A$ is false, and A is false iff $\neg A$ is true; $A \vee B$ is true iff A or B is true, and false iff A and B are false; $A \wedge B$ is true iff A and B are true, and false iff A or B is false; $A \rightarrow B$ is true iff A is false or B is true, and false iff A is

true and B is false; $A \leftrightarrow B$ is true iff A and B are both true or both false, and false iff A is true and B is false or A is false and B is true. If P is a predicate with arity m, then the sentence of the form $q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m)$ is true iff the sentence $P(b_1, \ldots, b_m)$ is true for all $b_i \in N_P^i$ when q_i is $\forall$, and for some $b_i \in N_P^i$ when q_i is $\exists$. $q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m)$ is false iff the sentence $P(b_1, \ldots, b_m)$ is false for all $b_i \in N_P^i$ when q_i is $\exists$, and for some $b_i \in N_P^i$ when q_i is $\forall$.

(iii) The language is bivalent, i.e., every sentence of it is either true or false.

Every countable and bivalent first-order language with or without identity conforms to classical logic. A classical example is the language of arithmetic in its standard interpretation.

We say that a language has a theory of truth if truth values are assigned to its sentences, and if it contains a predicate T which satisfies

T-rule: $T(\lceil A \rceil) \leftrightarrow A$ is true for every sentence A of the language.

Term $\lceil A \rceil$ which names the sentence A is defined below. A predicate T which satisfies T-rule is called a truth predicate. A theory of truth is said to be definitional if truth values are defined for sentences, and semantical if truth values of sentences are determined by their meanings.

Main results of this paper are:

Every countable language which conforms to classical logic has an extension which has properties (i)–(iii), and has a definitional theory of truth. That extension has a semantical theory of truth if every sentence of the object language is valuated by its meaning either as true or as false. These theories of truth contain truth and non-truth predicates. Theories are equivalent when the sentences of the object language are valuated by their meanings.

2   Extended languages

Assume that an object language L_0 conforms to classical logic, and is without a truth predicate. L_0 has by definition an extension which has properties (i) – (iii). That extension, denoted by L, is called a basic extension of L_0.

The language L_T is formed by adding to L extra formulas $T(x)$ and $\neg T(x)$, and sentences $T(\mathbf{n})$ and $\neg T(\mathbf{n})$, where $\mathbf{n}$ goes through all numerals which denote numbers $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ ($\mathbf{0}$=0, $\mathbf{1}$=S0, $\mathbf{2}$=SS0,...). Neither valuation nor meaning is yet attached to these sentences. Numerals are added, if necessary, to terms of L_T. Choose a Gödel numbering to sentences of L_T (see Wikipedia). The Gödel number of a sentence denoted by A is denoted by $\#A$, and the numeral of $\#A$ by $\lceil A \rceil$, which names the sentence A.

If P is a predicate of L with arity m, then $P(b_1, \ldots b_m)$ is a sentence of L for each $(b_1, \ldots b_m) \in N_P = N_1^P \times \cdots \times N_P^m$, and $\lceil P(b_1, \ldots b_m) \rceil$ is the numeral of its Gödel number. Thus $T(\lceil P(b_1, \ldots, b_m) \rceil)$ and $\neg T(\lceil P(b_1, \ldots, b_m) \rceil)$ are sentences of L_T for each $(b_1, \ldots, b_m) \in N_P$, so that they are determined by predicates of L_T having the domain D_p of P. Denote these predicates by $T(\lceil P(\dot{x}_1, \ldots, \dot{x}_m) \rceil)$ and $\neg T(\lceil P(\dot{x}_1, \ldots, \dot{x}_m) \rceil)$, and add them to L_T. Notation $P(\dot{x}_1, \ldots, \dot{x}_m)$ stands for the result of formally replacing variables $x_1, \ldots x_m$ of $P(x_1, \ldots, x_m)$ by terms of N_P (cf. [4]).

Add to the language L_T sentences $\forall x T_1(x)$, $\exists x T_1(x)$, $\forall x T_1(\lceil T_2(\dot{x}) \rceil)$ and $\exists x T_1(\lceil T_2(\dot{x}) \rceil)$, where $T_1, T_2 \in \{T, \neg T\}$, and sentences $q_1 x_1 \ldots q_m x_m T(\lceil P(\dot{x}_1, \ldots, \dot{x}_m) \rceil)$ and $q_1 x_1 \ldots q_m x_m \neg T(\lceil P(\dot{x}_1, \ldots, \dot{x}_m) \rceil)$ for each predicate P of L with arity $m \geq 1$ and for each m-tuple $q_1, \ldots, q_m$, where q_i's are $\forall$ or $\exists$.

The so obtained extension of the language L_T is denoted by $\mathcal{L}_0$.

When a language $\mathcal{L}_n$, $n \in \mathbb{N}_0$, is defined, let $\mathcal{L}_{n+1}$ be a language which is formed by adding to $\mathcal{L}_n$ those of the following sentences which are not in $\mathcal{L}_n$: $\neg A$, $A \vee B$, $A \wedge B$, $A \to B$ and $A \leftrightarrow B$, where A and B are sentences of $\mathcal{L}_n$.

The language $\mathcal{L}$ is defined as the union of languages $\mathcal{L}_n$, $n \in \mathbb{N}_0$. Extend the Gödel numbering of the sentences of L_T to those of $\mathcal{L}$, and denote by $\mathcal{D}$ the set of Gödel numbers of the sentences of $\mathcal{L}$.

Our main goal is to extract from $\mathcal{L}$ a sublanguage which under suitable valuations of its sentences has properties (i)–(iii) given in Introduction, and has a theory of truth.

At first we define some subsets of $\mathcal{L}$.

Denote by $\mathcal{P}^m$ the set of those predicates of L which have arity m, $\mathcal{P} = \bigcup_{m=1}^{\infty} \mathcal{P}^m$, and

$$
\begin{cases}
Z_3^m = \{q_1 x_1 \ldots q_m x_m T(\lceil P(\dot{x}_1, \ldots, \dot{x}_m) \rceil)) : P \in \mathcal{P}^m \text{ and } q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m) \text{ is true}\}; \\
Z_4^m = \{q_1 x_1 \ldots q_m x_m \neg T(\lceil P(\dot{x}_1, \ldots, \dot{x}_m) \rceil)) : P \in \mathcal{P}^m \text{ and } q_1 x_1 \ldots q_m x_m \neg P(x_1, \ldots, x_m) \text{ is true}\}.
\end{cases}
\tag{2.1}
$$

Define subsets $Z_1(U)$, $Z_2(U)$, $U \subset \mathcal{D}$, and Z_i, $i = 1 \ldots 4$, of $\mathcal{L}$ by

$$
\begin{cases}
Z_1(U) = \{T(\mathbf{n}) \colon \mathbf{n} = \lceil A \rceil, \text{ where } A \text{ is a sentence of } \mathcal{L} \text{ and } \#A \text{ is in } U\}, \\
Z_2(U) = \{\neg T(\mathbf{n}) \colon \mathbf{n} = \lceil A \rceil, \text{ where } A \text{ is a sentence of } \mathcal{L} \text{ and } \#\lceil \neg A \rceil \text{ is in } U\}, \\
Z_1 = \{\neg \forall x T(x), \exists x T(x), \neg \forall x \neg T(x), \exists x \neg T(x)\}, \\
Z_2 = \{\neg(\forall x T_1(\lceil T_2(\dot{x}) \rceil)), \exists x T_1(\lceil T_2(\dot{x}) \rceil), \ T_1, T_2 \in \{T, \neg T\}, \\
Z_3 = \bigcup_{m=1}^{\infty} Z_3^m, \ Z_4 = \bigcup_{m=1}^{\infty} Z_4^m.
\end{cases}
\tag{2.2}
$$

Subsets $L_n(U)$, $n \in \mathbb{N}_0$, of $\mathcal{L}$ are defined recursively as follows.

$$
L_0(U) = \begin{cases}
Z = \{A : A \text{ is a true sentence of } L\} \text{ if } U = \emptyset \text{ (the empty set)}, \\
Z \cup Z_1(U) \cup Z_2(U) \cup Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \text{ if } \emptyset \subset U \subset \mathcal{D}.
\end{cases}
\tag{2.3}
$$

When a subset $L_n(U)$ of $\mathcal{L}$ is defined for some $n \in \mathbb{N}_0$, and when A and B are sentences of $\mathcal{L}$, denote

$$
\begin{cases}
L_n^0(U) = \{\neg(\neg A) : A \text{ is in } L_n(U)\}, \\
L_n^1(U) = \{A \vee B : A \text{ and } B, \text{ or } A \text{ and } \neg B, \text{ or } \neg A \text{ and } B \text{ are in } L_n(U)\}, \\
L_n^2(U) = \{A \wedge B : A \text{ and } B \text{ are in } L_n(U)\}, \\
L_n^3(U) = \{A \to B : \neg A \text{ and } B, \text{ or } \neg A \text{ and } \neg B, \text{ or } A \text{ and } B \text{ are in } L_n(U)\}, \\
L_n^4(U) = \{A \leftrightarrow B : \ A \text{ and } B, \text{ or } \neg A \text{ and } \neg B \text{ are in } L_n(U)\}, \\
L_n^5(U) = \{\neg(A \vee B) : \neg A \text{ and } \neg B \text{ are in } L_n(U)\}, \\
L_n^6(U) = \{\neg(A \wedge B) : \neg A \text{ and } B, \text{ or } \neg A \text{ and } \neg B, \text{ or } A \text{ and } \neg B \text{ are in } L_n(U)\}, \\
L_n^7(U) = \{\neg(A \to B) : A \text{ and } \neg B \text{ are in } L_n(U)\}, \\
L_n^8(U) = \{\neg(A \leftrightarrow B) : A \text{ and } \neg B, \text{ or } \neg A \text{ and } B \text{ are in } L_n(U)\},
\end{cases}
\tag{2.4}
$$

and define

$$
L_{n+1}(U) = L_n(U) \cup \bigcup_{k=0}^{8} L_n^k(U).
\tag{2.5}
$$

The above constructions imply that $L_n^k(U) \subseteq L_{n+1}^k(U)$ and $L_n(U) \subset L_{n+1}(U) \subset \mathcal{L}$ for all $n \in \mathbb{N}_0$ and $k = 0, \ldots, 8$. Define a subset $L(U)$ of $\mathcal{L}$ by

$$L(U) = \bigcup_{n=0}^{\infty} L_n(U). \tag{2.6}$$

3 Properties of consistent subsets of $\mathcal{D}$

Recall that $\mathcal{D}$ denotes the set of Gödel numbers of the sentences of $\mathcal{L}$. When U is a subset of $\mathcal{D}$, denote by $G(U)$ the set of Gödel numbers of the sentences of $L(U)$ defined by (2.6):

$$G(U) = \{\#A : A \text{ is a sentence of } L(U)\}. \tag{3.1}$$

A subset U of $\mathcal{D}$ is called consistent if both $\#A$ and $\#[\neg A]$ are not in U for any sentence A of $\mathcal{L}$.

Lemma 3.1. *Let U be a consistent subset of $\mathcal{D}$. Then for no sentence A of $\mathcal{L}$ both A and $\neg A$ belong to $L(U)$, and $G(U)$ is consistent.*

Proof. At first we show that there is no sentence A in $\mathcal{L}$ such that both A and $\neg A$ belong to $L_0(U)$. If $U = \emptyset$, then $L_0(U)$ is by (2.3) the set Z of true sentences of L. If A is a sentence of L, then only one of the sentences A and $\neg A$ is true, and hence in $Z = L_0(U)$, since L has properties (i)–(iii).
Assume next that U is nonempty. As a consistent set U is a proper subset of $\mathcal{D}$.
Let $\mathbf{n}$ be a numeral. If $T(\mathbf{n})$ is in $L_0(U)$, it is in $Z_1(U)$, so that, by (2.2), $\mathbf{n} = \lceil A \rceil$, where $\#A$ is in U. Since U is consistent, then $\#[\neg A]$ is not in U. Thus, by (2.2), $\neg T(\mathbf{n})$ is not in $Z_2(U)$, and hence not in $L_0(U)$. This result implies also that $T(\mathbf{n})$ is not in $L_0(U)$ if $\neg T(\mathbf{n})$ is in $L_0(U)$.
(2.2) and (2.3) imply that sentences $\exists x T(x)$, $\neg \forall x T(x)$, $\neg \forall x \neg T(x)$ and $\exists x \neg T(x)$ are in Z_1, and hence in $L_0(U)$, but their negations are not in $L_0(U)$.
By the definitions (2.2) and (2.3) of Z_2 neither both $\exists x T_1(\lceil T_2(\dot{x}) \rceil)$ and $\neg(\exists x T_1(\lceil T_2(\dot{x}) \rceil))$, nor both $\forall x T_1(\lceil T_2(\dot{x}) \rceil)$ and $\neg(\forall x T_1(\lceil T_2(\dot{x}) \rceil))$, are in $L_0(U)$ for any $T_1, T_2 \in \{T, \neg T\}$. By the definitions (2.1), (2.2) and (2.3) the sentences of Z_3 and Z_4, but not their negations, are in $L_0(U)$.
The above proof shows that for no sentence A of $\mathcal{L}$ both A and $\neg A$ belong to $L_0(U)$.
Make the induction hypothesis:

(h0) For no sentence A of $\mathcal{L}$ both A and $\neg A$ belong to $L_n(U)$.

Applying (h0) and (2.4) we obtain the following results.
(h0) and the definition of $L_n^0(U)$ imply that if a sentence A is in $L_n(U)$, then none of the odd-tuple negations of A are in $L_{n+1}(U)$, and if $\neg A$ is in $L_n(U)$, then none of the even-tuple negations of A are in $L_{n+1}(U)$.
If $A \vee B$ is in $L_{n+1}(U)$, it is in $L_n^1(U)$, whence A or B is in $L_n(U)$. If $\neg(A \vee B)$ is in $L_{n+1}(U)$, it is in $L_n^5(U)$, in which case $\neg A$ and $\neg B$ are in $L_n(U)$. Thus $A \vee B$ and $\neg(A \vee B)$ are not both in $L_{n+1}(U)$, for otherwise both A and $\neg A$ or both B and $\neg B$ are in $L_n(U)$, contradicting with (h0).
$A \wedge B$ and $\neg(A \wedge B)$ cannot both be in $L_{n+1}(U)$, for otherwise $A \wedge B$ is in $L_n^2(U)$, i.e., both A and B are in $L_n(U)$, and $\neg(A \wedge B)$ is in $L_n^6(U)$, i.e., at least one of $\neg A$ and $\neg B$ is in $L_n(U)$. Thus both A and $\neg A$ or both B and $\neg B$ are in $L_n(U)$, contradicting with (h0).
If $A \rightarrow B$ is in $L_{n+1}(U)$, it is in $L_n^3(U)$, so that $\neg A$ or B is in $L_n(U)$. If $\neg(A \rightarrow B)$ is in $L_{n+1}(U)$, it is in $L_n^7(U)$, whence both A and $\neg B$ are in $L_n(U)$. Because of these results and (h0) the sentences $A \rightarrow B$ and $\neg(A \rightarrow B)$ are not both in $L_{n+1}(U)$.

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If $A \leftrightarrow B$ is $L_{n+1}(U)$, it is in $L_n^4(U)$, in which case both A and B or both $\neg A$ and $\neg B$ are in $L_n(U)$. If $\neg(A \leftrightarrow B)$ is in $L_{n+1}(U)$, it is in $L_n^8(U)$, whence both A and $\neg B$ or both $\neg A$ and B are in $L_n(U)$. Thus both $A \leftrightarrow B$ and $\neg(A \leftrightarrow B)$ cannot be in $L_{n+1}(U)$, for otherwise both A and $\neg A$ or both B and $\neg B$ are in $L_n(U)$, contradicting with (h0).

The above results and the induction hypothesis (h0) imply that for no sentence A of $\mathcal{L}$ both A and $\neg A$ belong to $L_{n+1}(U) = L_n(U) \cup \bigcup_{k=0}^8 L_n^k(U)$.

Since (h0) is proved when $n = 0$, it is by induction valid for every $n \in \mathbb{N}_0$.

If A and $\neg A$ are in $L(U)$, then A is by (2.6) in $L_{n_1}(U)$ for some $n_1 \in \mathbb{N}_0$, and $\neg A$ is in $L_{n_2}(U)$ for some $n_2 \in \mathbb{N}_0$. Then both A and $\neg A$ are in $L_n(U)$ when $n = \max\{n_1, n_2\}$. This is impossible, because (h0) is proved for every $n \in \mathbb{N}_0$. Thus A and $\neg A$ cannot both be in $L(U)$ for any sentence A of $\mathcal{L}$.

The above result and (3.1) imply that there is no sentence A in $\mathcal{L}$ such that both $\#A$ and $\#[\neg A]$ are in $G(U)$. Thus $G(U)$ is consistent. $\square$

Lemma 3.2. *Assume that U and V are consistent subsets of $\mathcal{D}$, and that $V \subseteq U$. Then $L(V) \subseteq L(U)$ and $G(V) \subseteq G(U)$.*

Proof. As consistent sets V and U are proper subsets of $\mathcal{D}$. At first we show that $L_0(V) \subseteq L_0(U)$. If $V = \emptyset$, then $L_0(V) = Z \subseteq L_0(U)$ by (2.3).

Assume next that V is nonempty. Thus also U is nonempty.

Let A be a sentence of L. Definition (2.3) of $L_0(U)$ implies that A is in $L_0(U)$ and also in $L_0(V)$ iff A is in Z.

Let $\mathbf{n}$ be a numeral. If $T(\mathbf{n})$ is in $L_0(V)$, it is in $Z_1(V)$, so that $\mathbf{n} = \lceil A \rceil$, where $\#A$ is in V. Because $V \subseteq U$, then $\#A$ is also in U, whence $T(\mathbf{n})$ is in $Z_1(U)$, and hence in $L_0(U)$.

If $\neg T(\mathbf{n})$ is in $L_0(V)$, it is in $Z_2(V)$, in which case $\mathbf{n} = \lceil A \rceil$, where $\#[\neg A]$ is in V. Since $V \subseteq U$, then $\#[\neg A]$ is also in U, whence $\neg T(\mathbf{n})$ is in $Z_2(U)$, and hence in $L_0(U)$.

Because U and V are nonempty and proper subsets of $\mathcal{D}$, then Z_1, Z_2, Z_3 and Z_4 are in $L_0(U)$ and in $L_0(V)$ by (2.3).

The above results imply that $L_0(V) \subseteq L_0(U)$. Make the induction hypothesis:

(h1) $L_n(V) \subseteq L_n(U)$

for some $n \in \mathbb{N}_0$. It follows from (2.4) and (h1) that $L_n^k(V) \subseteq L_n^k(U)$ for each $k = 0, \ldots, 8$. Thus

$$L_{n+1}(V) = L_n(V) \cup \bigcup_{k=0}^8 L_n^k(V) \subseteq L_n(U) \cup \bigcup_{k=0}^8 L_n^k(U) = L_{n+1}(U).$$

(h1) is proved when $n = 0$, whence it is by induction valid for every $n \in \mathbb{N}_0$.

If A is in $L(V)$, it is by (2.6) in $L_n(V)$ for some $n \in \mathbb{N}_0$. Thus A is in $L_n(U)$ by (h1), and hence in $L(U)$. Consequently, $L(V) \subseteq L(U)$.

If $\#A$ is in $G(V)$ then A is in $L(V)$ by (3.1). Thus A is in $L(U)$, so that $\#A$ is in $G(U)$ by (3.1). This shows that $G(V) \subseteq G(U)$. $\square$

Denote by $\mathcal{C}$ the family of consistent subsets of $\mathcal{D}$. In the formulation and the proof of Theorem 3.1 transfinite sequences indexed by ordinals are used. A transfinite sequence $(U_\lambda)_{\lambda < \alpha}$ of $\mathcal{C}$ is said to be increasing if $U_\mu \subseteq U_\nu$ whenever $\mu < \nu < \alpha$, and strictly increasing if $U_\mu \subset U_\nu$ whenever $\mu < \nu < \alpha$.

Lemma 3.3. *Assume that $(U_\lambda)_{\lambda < \alpha}$ is a strictly increasing sequence of $\mathcal{C}$. Then*
(a) $(G(U_\lambda))_{\lambda < \alpha}$ is an increasing sequence of $\mathcal{C}$.
(b) The union $\bigcup_{\lambda < \alpha} G(U_\lambda)$ is consistent.

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Proof. Since $U_\mu \subset U_\nu$ when $\mu < \nu < \alpha$, it follows from Lemma 3.2 that $G(U_\mu) \subseteq G(U_\nu)$ when $\mu < \nu < \alpha$, whence the sequence $(G(U_\lambda))_{\lambda<\alpha}$ is increasing. Consistency of the sets $G(U_\lambda)$, $\lambda < \alpha$, follows from Lemma 3.1 because the sets U_λ, $\lambda < \alpha$, are consistent. This proves (a).

To prove that the union $\bigcup_{\lambda<\alpha} G(U_\lambda)$ is consistent, assume on the contrary that there exists such a sentence A in $\mathcal{L}$ that both $\#A$ and $\#[\neg A]$ are in $\bigcup_{\lambda<\alpha} G(U_\lambda)$. Thus there exist μ, $\nu < \alpha$ such that $\#A$ is in $G(U_\mu)$ and $\#[\neg A]$ is in $G(U_\nu)$. Because $G(U_\mu) \subseteq G(U_\nu)$ or $G(U_\nu) \subseteq G(U_\mu)$, then both $\#A$ and $\#[\neg A]$ are in $G(U_\mu)$ or in $G(U_\nu)$. But this is impossible, since both $G(U_\mu)$ and $G(U_\nu)$ are consistent. Thus, the set $\bigcup_{\lambda<\alpha} G(U_\lambda)$ is consistent. $\qquad\square$

Now we are ready to prove the following Theorem.

Theorem 3.1. *Let W denote the set of Gödel numbers of true sentences of L. We say that a transfinite sequence $(U_\lambda)_{\lambda<\alpha}$ of $\mathcal{C}$ is a G-sequence if it has the following properties.*

(G) $(U_\lambda)_{\lambda<\alpha}$ is strictly increasing, $U_0 = W$, and if $0 < \mu < \alpha$, then $U_\mu = \bigcup_{\lambda<\mu} G(U_\lambda)$.

Then the longest G-sequence exists, and it has the last member. This member is the smallest consistent subset U of $\mathcal{D}$ satisfying $U = G(U)$.

Proof. W is consistent, since L has properties (i)-(iii). At first we show that G-sequences are nested:
(1) Assume that $(U_\lambda)_{\lambda<\alpha}$ and $(V_\lambda)_{\lambda<\beta}$ are G-sequences. Then $U_\lambda = V_\lambda$ when $\lambda < \min\{\alpha, \beta\}$.
$U_0 = W = V_0$ by (G). Make the induction hypothesis:
(h) There exists an ordinal ν which satisfies $0 < \nu < \min\{\alpha, \beta\}$ such that $U_\lambda = V_\lambda$ for each $\lambda < \nu$.
It follows from (h) and (G) that $U_\nu = \bigcup_{\lambda<\nu} G(U_\lambda) = \bigcup_{\lambda<\nu} G(V_\lambda) = V_\nu$. Since $U_0 = V_0$, then (h) holds when $\nu = 1$. These results imply (1) by transfinite induction.
Let $(U_\lambda)_{\lambda<\alpha}$ be a G-sequence. Defining $f(0) = \min U_0$, $f(\lambda) = \min(U_\lambda \setminus U_{\lambda-1})$, $0 < \lambda < \alpha$, and $f(\alpha) = \min(\mathcal{D} \setminus \bigcup_{\lambda<\alpha} U_\lambda)$, we obtain a bijection f from $[0, \alpha]$ to a subset of $\mathbb{N}_0$. Thus α is a countable ordinal. Consequently, the set Γ of those ordinals α for which $(U_\lambda)_{\lambda<\alpha}$ is a G-sequence is bounded from above by the smallest uncountable ordinal. Denote by γ the least upper bound of Γ.
To show that γ is a successor, assume on the contrary that γ is a limit ordinal. Given any $\mu < \gamma$, then $\nu = \mu + 1$ and $\alpha = \nu + 1$ are $< \gamma$. $(U_\lambda)_{\lambda<\alpha}$ is a G-sequence, whence $U_\mu = \bigcup_{\lambda<\mu} G(U_\lambda)$, and $U_\mu \subset U_{\mu+1}$.
Thus $(U_\lambda)_{\lambda<\gamma}$ has properties (G) when $\alpha = \gamma$, so that $(U_\lambda)_{\lambda<\gamma}$ is a G-sequence. Denote $U_\gamma = \bigcup_{\lambda<\gamma} G(U_\lambda)$.
U_γ is consistent by Lemma 3.3(b). Because $U_\mu \subset U_\nu = \bigcup_{\lambda<\nu} G(U_\lambda) \subseteq U_\gamma$ for each $\mu < \gamma$, then $(U_\lambda)_{\lambda<\gamma+1}$ is a G-sequence. But this contradicts with the choice of γ.
Thus γ is a successor, say $\gamma = \alpha + 1$. If $\lambda < \alpha$, then $U_\lambda \subset U_\alpha$, so that $G(U_\lambda) \subseteq G(U_\alpha)$. Then $U_\alpha = \bigcup_{\lambda<\alpha} G(U_\lambda) \subseteq \bigcup_{\lambda<\gamma} G(U_\lambda) = G(U_\alpha)$, whence $U_\alpha \subseteq G(U_\alpha)$. Moreover, $U_\alpha = G(U_\alpha)$,
for otherwise $U_\alpha \subset G(U_\alpha) = \bigcup_{\lambda<\gamma} G(U_\lambda) = U_\gamma$, and $(U_\lambda)_{\lambda<\gamma+1}$ would be a G-sequence.
Consequently, $(U_\lambda)_{\lambda<\gamma}$ is the longest G-sequence, U_α is its last member, and $U_\alpha = G(U_\alpha)$.
Let U be a consistent subset of $\mathcal{D}$ satisfying $U = G(U)$. Then $U_0 = W = G(\emptyset) \subseteq G(U) = U$. Make the induction hypothesis:
(h2) There exists an ordinal μ which satisfies $0 < \mu < \gamma$ such that $U_\lambda \subseteq U$ for each $\lambda < \mu$.

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Then $G(U_\lambda) \subseteq G(U)$ for each $\lambda < \mu$, whence $U_\mu = \bigcup_{\lambda<\mu} G(U_\lambda) \subseteq G(U) = U$. Thus, by transfinite induction, $U_\mu \subseteq U$ for each $\mu < \gamma$. In particular, $U_\alpha \subseteq U$. This proves the last assertion of Theorem. $\square$

4 Language $\mathcal{L}_T$ and its properties

Let L_0 be a language which conforms to classical logic and has not a truth predicate. Let $\mathcal{L}, \mathcal{P}$ and $\mathcal{D}$ be as in Section 2, and let $L(U)$ and $G(U)$, $U \subset \mathcal{D}$, be defined by (2.6) and (3.1). Define

$$F(U) = \{A : \neg A \in L(U)\}. \tag{4.1}$$

Recall that a subset U of $\mathcal{D}$ is consistent if there is no sentence A in $\mathcal{L}$ such that both $\#A$ and $\#[\neg A]$ are in U. By Theorem 3.1 the smallest consistent subset of $\mathcal{D}$ which satisfies $U = G(U)$ exists.

Definition 4.1. Let U be the smallest consistent subset of $\mathcal{D}$ which satisfies $U = G(U)$. Denote by $\mathcal{L}_T$ the language formed by the object language L_0, the predicates of $\mathcal{P}$, the sentences of $L(U)$ and $F(U)$, formulas $T(x)$ and $\neg T(x)$, corresponding predicates T and $\neg T$ with their domain D_T and the set N_T of terms defined by

$$D_T = L(U) \cup F(U) \text{ and } N_T = \{\mathbf{n} : \mathbf{n} = \lceil A \rceil, \text{ where } A \text{ is in } D_T\}, \tag{4.2}$$

and predicates $T_1(\lceil T_2(\dot{x})\rceil)$, and $T_1(\lceil P(\dot{x}_1, \ldots, \dot{x}_m)\rceil)$, where $T_1, T_2 \in \cup\{T, \neg T\}$ and $P \in \mathcal{P}$ with arity $m \geq 1$.

A valuation is defined for sentences of $\mathcal{L}_T$ as follows.

(I) A sentence of $\mathcal{L}_T$ is valuated as true iff it is in $L(U)$, and as false iff it is in $F(U)$.

Lemma 4.1. *The language $\mathcal{L}_T$ defined by Definition 4.1 and valuated by (I) is bivalent.*

Proof. The subsets $L(U)$ and $F(U)$ of the sentences of $\mathcal{L}_T$ are disjoint. For otherwise there is a sentence A of $\mathcal{L}_T$ which is in $L(U) \cap F(U)$. Then A is in $L(U)$, and by the definition (4.1) of $F(U)$ also $\neg A$ is in $L(U)$. But this is impossible by Lemma 3.1. Consequently, $L(U) \cap F(U) = \emptyset$.

If A is a sentence of $\mathcal{L}_T$, then it is in $L(U)$ or in $F(U)$. If A is true, it is in $L(U)$, but not in $F(U)$, and hence not false, because $L(U) \cap F(U) = \emptyset$. Similarly, if A is false, it is in $F(U)$, but not in $L(U)$, and hence not true. Consequently, A is either true or false, so that $\mathcal{L}_T$ is bivalent. $\square$

Lemma 4.2. *Let $\mathcal{L}_T$ be defined by Definition 4.1 and valuated by (I). Then a sentence of the basic extension L of L_0 is true (respectively false) in the valuation (I) iff it is true (respectively false) in the valuation of L.*

Proof. Let A denote a sentence of L. A is true in the valuation (I) iff A is in $L(U)$ iff (by the construction of $L(U)$) A is in Z iff A is true in the valuation of L. A is false in the valuation (I) iff A is in $F(U)$ iff (by (4.1)) $\neg A$ is in $L(U)$ iff ($\neg A$ is a sentence of L) $\neg A$ is in Z iff $\neg A$ is true in the valuation of L iff (L has properties (i)–(iii)) A is false in the valuation of L. $\square$

Lemma 4.3. *The language $\mathcal{L}_T$ defined by Definition 4.1 and valuated by (I) has properties (i) and (ii) given in Introduction.*

Proof. Unless otherwise stated, 'true' means true in the valuation (I), and 'false' means false in the valuation (I).

The construction of $L(U)$ and the definition (4.1) of $F(U)$ imply that $\mathcal{L}_T$ has properties (i).

As for properties (ii) we at first derive the following auxiliary rule.

(t0) Double negation: If A is a sentence of $\mathcal{L}_T$, then $\neg(\neg A)$ is true iff A is true.

To prove (t0), assume first that $\neg(\neg A)$ is true. Then it is in $L(U)$, and hence, by (2.6), in $L_n(U)$ for some $n \in \mathbb{N}_0$. If $\neg(\neg A)$ is in $L_0(U)$ then it by (2.3) in Z. Thus $\neg(\neg A)$ is true in the valuation of L. Then (negation rule is valid in L) $\neg A$ is false in the valuation of L, which implies that A is true in the valuation of L. Thus A is by (2.3) in $Z \subset L_0(U) \subset L(U)$, whence A is true.

Assume next that $n \in \mathbb{N}_0$ is the smallest number for which $\neg(\neg A)$ is in $L_{n+1}(U)$. It then follows from (2.4) and (2.5) that $\neg(\neg A)$ is in $L_n^0(U)$, so that A is in $L_n(U)$, and hence in $L(U)$, i.e., A is true. Thus A is true if $\neg(\neg A)$ is true.

Conversely, assume that A is true. Then A is in $L(U)$, so that A is in $L_n(U)$ for some $n \in \mathbb{N}_0$. Thus $\neg(\neg A)$ is in $L_n^0(U)$, and hence in $L_{n+1}(U)$. Consequently, $\neg(\neg A)$ is in $L(U)$, whence $\neg(\neg A)$ is true. This concludes the proof of (t0).

Rule (t0) is applied to prove

(t1) Negation: A is true iff $\neg A$ is false, and A is false iff $\neg A$ is true.

Let A be a sentence of $\mathcal{L}_T$. Then A is true iff (by (t0)) $\neg(\neg A)$ is true iff $\neg(\neg A)$ is in $L(U)$ iff (by (4.1)) $\neg A$ is in $F(U)$ iff $\neg A$ is false.

A is false iff A is in $F(U)$ iff (by (4.1)) $\neg A$ is in $L(U)$ iff $\neg A$ is true. Thus (t1) is satisfied.

Next we prove the following rule.

(t2) Conjunction: $A \wedge B$ is true iff A and B are true. $A \wedge B$ is false iff A or B is false.

Let A and B be sentences of $\mathcal{L}_T$. If A and B are true, i.e., A and B are in $L(U)$, there is by (2.6) an $n \in \mathbb{N}_0$ such that A and B are in $L_n(U)$. Thus $A \wedge B$ is by (2.4) in $L_n^2(U)$, and hence in $L(U)$, so that $A \wedge B$ is true.

Conversely, assume that $A \wedge B$ is true, or equivalently, $A \wedge B$ is in $L(U)$. Then there is by (2.6) an $n \in \mathbb{N}_0$ such that $A \wedge B$ is in $L_n(U)$. If $A \wedge B$ is in $L_0(U)$, it is in Z. Thus $A \wedge B$ is true in the valuation of L. Because L has property (ii), then A and B are true in the valuation of L, and hence also in the valuation (I) by Lemma 4.2.

Assume next that $n \in \mathbb{N}_0$ is the smallest number for which $A \wedge B$ is in $L_{n+1}(U)$. Then $A \wedge B$ is by (2.4) in $L_n^2(U)$, so that A and B are in $L_n(U)$, and hence in $L(U)$, i.e., A and B are true.

The above reasoning proves that $A \wedge B$ is true iff A and B are true. This result and the bivalence of $\mathcal{L}_T$, proved in Lemma 4.1, imply that $A \wedge B$ is false iff A or B is false. Consequently, rule (t2) is valid.

The proofs of the following rules are similar to the above proof of (t2).

(t3) Disjunction: $A \vee B$ is true iff A or B is true. $A \vee B$ false iff A and B are false.

(t4) Conditional: $A \to B$ is true iff A is false or B is true. $A \to B$ is false iff A is true and B is false.

(t5) Biconditional: $A \leftrightarrow B$ is true iff A and B are both true or both false. $A \leftrightarrow B$ is false iff A is true and B is false or A is false and B is true.

Next we show that if $T_1 \in \{T, \neg T\}$ then $\exists x T_1(x)$ and $\forall x T_1(x)$ have the following properties.

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(t6) $\exists x T_1(x)$ is true iff $T_1(\mathbf{n})$ is true for some $\mathbf{n} \in N_T$, and false iff $T_1(\mathbf{n})$ is false for every $\mathbf{n} \in N_T$.

(t7) $\forall x T_1(x)$ is true iff $T_1(\mathbf{n})$ is true for every $\mathbf{n} \in N_T$, and false iff $T_1(\mathbf{n})$ is false for some $\mathbf{n} \in N_T$.

To simplify proofs we derive results which imply that T is a truth predicate and $\neg T$ is a non-truth predicate for $\mathcal{L}_T$.

Let A denote a sentence of $\mathcal{L}_T$. The valuation (I), rule (t1), the definitions of $Z_1(U)$, $Z_2(U)$ and $G(U)$, and the assumption $U = G(U)$ imply that A is true iff A is in $L(U)$ iff $\#A$ is in $G(U) = U$ iff $T(\lceil A \rceil)$ is in $Z_1(U) \subset L(U)$ iff $T(\lceil A \rceil)$ is true iff $\neg T(\lceil A \rceil)$ is false.

A is false iff A is in $F(U)$ iff $\neg A$ is in $L(U)$ iff $\#[\neg A]$ is in $G(U) = U$ iff $\neg T(\lceil A \rceil)$ is in $Z_2(U) \subset L(U)$ iff $\neg T(\lceil A \rceil)$ is true iff $T(\lceil A \rceil)$ is false.

The above results imply that the following results are valid for every sentence $A \in \mathcal{L}_T$.

(T) A is true iff $T(\lceil A \rceil)$ is true iff $\neg T(\lceil A \rceil)$ is false. A is false iff $T(\lceil A \rceil)$ is false iff $\neg T(\lceil A \rceil)$ is true.

Consider the validity of (t6) and (t7) when T_1 is T. Because U is nonempty, then $\exists x T(x)$ is in $L_0(U)$ by (2.2) and (2.3), and hence in $L(U)$ by (2.6). Thus $\exists x T(x)$ is by (I) a true sentence of $\mathcal{L}_T$.

$T(\lceil A \rceil)$ is true iff (by (T)) A is true iff (by (I)) A is in $L(U)$. Thus $T(\mathbf{n})$ is true for some $\mathbf{n} \in N_T$.

The above results imply that $\exists x T(x)$ is true iff $T(\mathbf{n})$ is true for some $\mathbf{n} \in N_T$. In view of this result and the bivalence of $\mathcal{L}_T$, one can infer that $\exists x T(x)$ is false iff $T(\mathbf{n})$ is false for every $\mathbf{n} \in N_T$. This concludes the proof of (p6) when T_1 is T.

$\neg \forall x T(x)$ is in $Z_1 \subset L_0(U)$, and hence in $L(U)$, so that it is true. Thus $\forall x T(x)$ is false by (t1).

$T(\lceil A \rceil)$ is false iff (by (T)) A is false iff (by (I)) A is in $F(U)$. Thus $T(\mathbf{n})$ is false for some $\mathbf{n} \in N_T$.

Consequently, $\forall x T(x)$ is false iff $T(\mathbf{n})$ is false for some $\mathbf{n} \in N_T$. This result and the bivalence of $\mathcal{L}_T$ imply that $\forall x T(x)$ is true iff $T(\mathbf{n})$ is true for every $\mathbf{n} \in N_T$. This proves (t7) when T_1 is T.

To show that (t6) is valid when T_1 is $\neg T$, notice first that $\exists x \neg T(x)$ is in $Z_1 \subset L_0(U)$, and hence in $L(U)$, whence it is true.

$\neg T(\lceil A \rceil)$ is true iff (by (t1)) $T(\lceil A \rceil)$ is false iff (by (T)) A is false iff (by (I)) A is in $F(U)$. Thus $\neg T(\mathbf{n})$ is true for some $\mathbf{n} \in N_T$. Consequently, $\exists x \neg T(x)$ is true iff $\neg T(\mathbf{n})$ is true for some $\mathbf{n} \in N_T$. This result and the bivalence of $\mathcal{L}_T$ imply that $\exists x \neg T(x)$ is false iff $\neg T(\mathbf{n})$ is false for every $\mathbf{n} \in N_T$. This concludes the proof of (t6) when T_1 is $\neg T$.

Next we prove (t7) when T_1 is $\neg T$. $\neg \forall x \neg T(x)$ is in $Z_1 \subset L_0(U)$, and hence in $L(U)$, so that it is true. Thus $\forall x \neg T(x)$ is false by (t1).

$\neg T(\lceil A \rceil)$ is false iff (by (t1)) $T(\lceil A \rceil)$ is true iff (by (T)) A is true iff (by (I)) A is in $L(U)$. Thus $\neg T(\mathbf{n})$ is false for some $\mathbf{n} \in N_T$. From these results it follows that $\forall x \neg T(x)$ is false iff $\neg T(\mathbf{n})$ is false for some $\mathbf{n} \in N_T$. This result and bivalence of $\mathcal{L}_T$ imply that $\forall x \neg T(x)$ is true iff $\neg T(\mathbf{n})$ is true for all $\mathbf{n} \in N_T$. Thus (t7) is valid when T_1 is $\neg T$.

Next we show that the following rules are valid when $T_1, T_2 \in \{T, \neg T\}$.

(tt6) $\exists x T_1(\lceil T_2(\dot{x}) \rceil)$ is true iff $T_1(\lceil T_2(\mathbf{n}) \rceil)$ is true for some $\mathbf{n} \in N_T$.

$\exists x T_1(\lceil T_2(\dot{x}) \rceil)$ is false iff $T_1(\lceil T_2(\mathbf{n}) \rceil)$ is false for every $\mathbf{n} \in N_T$;

(tt7) $\forall x T_1(\lceil T_2(\dot{x}) \rceil)$ is true iff $T_1(\lceil T_2(\mathbf{n}) \rceil)$ is true for every $\mathbf{n} \in N_T$.

$\forall x T_1(\lceil T_2(\dot{x}) \rceil)$ is false iff $T_1(\lceil T_2(\mathbf{n}) \rceil)$ is false for some $\mathbf{n} \in N_T$.

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The sentences $\exists x T_1(\lceil T_2(\dot{x})\rceil)$, where $T_1, T_2 \in \{T, \neg T\}$ are in Z_2, whence they are in $L(U)$ and hence true. Applying (T) one can show that in every case there exists an $\mathbf{n} \in N_T$ so that $T_1(\lceil T_2(\mathbf{n})\rceil)$ is true ($\mathbf{n} = \lceil A\rceil$, where A, depending on the case, is in $L(U)$ or in $F(U)$). These results imply truth part of (tt6) when T_1 and T_2 are in $\{T, \neg T\}$. Falsity part in (tt6) is then valid by bivalence of $\mathcal{L}_T$.

$\forall x T_1(\lceil T_2(\dot{x})\rceil)$ is false because its negation is in Z_2 and hence true. Using (T) it is easy to show that $T_1(\lceil T_2(\mathbf{n})\rceil)$ is false for some $\mathbf{n} \in N_T$ whenever $T_1, T_2 \in \{T, \neg T\}$. This proves the falsity part of (tt7), and also implies the truth part by bivalence of $\mathcal{L}_T$.

Since L has properties (ii), then for every predicate $P \in \mathcal{P}^m$ with arity $m \geq 1$ the following properties hold in the valuation of L, and hence in the valuation (I) by Lemma 4.2.

(p6) The sentence of the form $q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m)$ is true iff $P(b_1, \ldots, b_m)$ is true for all $b_i \in N_P^i$ when q_i is $\forall$, and for some $b_i \in N_P^i$ when q_i is $\exists$. $q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m)$ is false iff $P(b_1, \ldots, b_m)$ is false for all $b_i \in N_P^i$ when q_i is $\exists$, and for some $b_i \in N_P^i$ when q_i is $\forall$.

If P is a predicate of $\mathcal{P}$ with arity $m \geq 1$, and if $q_1, \ldots, q_m$ is any m-tuple of quantifiers $\forall$ and $\exists$, then the sentence $q_1 x_1 \ldots q_m x_m T(\lceil P(\dot{x}_1, \ldots, \dot{x}_m)\rceil)$ is true iff it is in $L_0(U)$ iff it is in Z_1^m iff (by (2.1)) the sentence $q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m)$ is true iff (by (p6)) the sentence $P(b_1, \ldots, b_m)$ is true in L, and hence also in $\mathcal{L}_T$ for all $b_i \in N_P^i$ when q_i is $\forall$, and for some $b_i \in N_P^i$ when q_i is $\exists$ iff (by (T)) the sentence $T(\lceil P(b_1, \ldots, b_m)\rceil)$ is true for all choices of $b_i \in N_P^i$ when q_i is $\forall$, and for some choices of $b_i \in N_P^i$ when q_i is $\exists$.

The above equivalences and the bivalence of $\mathcal{L}_T$ imply the following result.

(tp6) The sentence $q_1 x_1 \ldots q_m x_m T(\lceil P(\dot{x}_1, \ldots, \dot{x}_m)\rceil)$ true iff the sentence $T(\lceil P(b_1, \ldots, b_m)\rceil)$ is true for all choices of $b_i \in N_P^i$ when q_i is $\forall$, and for some choices of $b_i \in N_P^i$ when q_i is $\exists$.

   $q_1 x_1 \ldots q_m x_m T(\lceil P(\dot{x}_1, \ldots, \dot{x}_m)\rceil)$ is false iff the sentence $T(\lceil P(b_1, \ldots, b_m)\rceil)$ is false for all $b_i \in N_P^i$ when q_i is $\exists$, and for some $b_i \in N_P^i$ when q_i is $\forall$.

Similarly, applying (2.1), (T), (p6) with P replaced by $\neg P$ and bivalence of $\mathcal{L}_T$ it one can show that T can be replaced in (tp6) by $\neg T$ when P is in $\mathcal{P}^m$ and $q_1, \ldots, q_m$ is any m-tuple of quantifiers $\forall$ and $\exists$.

The above proof shows that $\mathcal{L}_T$ has properties (ii). □

5   Theories of truth

Now we are ready to present our main results.

Theorem 5.1. *Let L_0 be a countable language which conforms to classical logic and has not a truth predicate. The language $\mathcal{L}_T$ defined in Definition 4.1 and valuated by (I) has properties (i)–(iii), and has a definitional theory of truth (shortly DTT). T is a truth predicate, and $\neg T$ is a non-truth predicate.*

Proof. Properties (i)–(iii) given in Introduction are valid for $\mathcal{L}_T$ by Lemma 4.2 and Lemma 4.3. Thus $\mathcal{L}_T$ conforms to classical logic.

The results (T) derived in the proof of Lemma 4.3 and biconditional rule (t5) imply that the sentence $T(\lceil A\rceil) \leftrightarrow A$ is true and the sentence $\neg T(\lceil A\rceil) \leftrightarrow A$ is false for every sentence A of $\mathcal{L}_T$. T and $\neg T$ are predicates of $\mathcal{L}_T$, and their domain D_T, the set all sentences of $\mathcal{L}_T$, satisfies the condition presented in [3, p. 7] for the domains of truth predicates. Consequently, T is a truth predicate and $\neg T$ is a non-truth predicate. The above results imply that $\mathcal{L}_T$ has a theory of truth. It is definitional, since truth values of sentences are defined by (I). □

Next we show that $\mathcal{L}_T$ has a semantical theory of truth under the following assumptions.

(s1)   The object language L_0 is countable, has not a truth predicate, and every sentence of L_0 is meaningful and is valuated by its meaning either as true or as false.

(s2)   Standard meanings are assigned to logical symbols.

(s3)   The sentence $T(\mathbf{n})$ means: 'the sentence whose Gödel number has $\mathbf{n}$ as its numeral is true'.

At first we prove preliminary Lemmas.

Lemma 5.1. *Under the hypotheses (s1) and (s2) the object language L_0 has an extension L which has properties (i)–(iii) given in Introduction when its sentences are valuated by their meanings. In particular, L_0 conforms to classical logic.*

Proof. The object language L_0 is bivalent by (s1). The basic extension L of L_0 which has properties (i)–(iii) when its sentences are valuated by their meanings is constructed as follows. The first extension L_1 of L_0 is formed by adding those sentences $\neg A$, $A \vee B$, $A \wedge B$, $A \to B$, $A \leftrightarrow B$ and $q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m)$ which are not in L_0 when A and B go through all sentences of L_0, P its predicates and their negations (added if necessary), and $(q_1, \ldots, q_m)$ m-tuples, where q_i's are either $\forall$ or $\exists$. If there exist objects of D without names, add terms to name them. Valuating the sentences of L_1 by their meanings, the assumptions (s1) and (s2) ensure that properties (ii) and (iii) are valid.
When the language L_n, $n \geq 1$ is defined, define the language L_{n+1} by adding those sentences $\neg A$, $A \vee B$, $A \wedge B$, $A \to B$, $A \leftrightarrow B$ which are not in L_n when A and B are sentences of L_n. Valuating the sentences of L_{n+1} by their meanings, the properties (ii) and (iii) are valid by assumptions (s1) and (s2). The union L of languages L_n, $n \in \mathbb{N}_0$ has also properties (ii) and (iii).
If A and B denote sentences of L, there exist n_1 and n_2 in $\mathbb{N}_0$ such that A is in L_{n_1} and B is in L_{n_2}. Denoting $n = \max\{n_1, n_2\}$, then A and B are sentences of L_n. Thus the sentences $\neg A$, $A \vee B$, $A \wedge B$, $A \to B$ and $A \leftrightarrow B$ are in L_{n+1}, and hence in L. If P is a predicate of L_1 with arity $m \geq 1$, then for each m-tuple $(q_1, \ldots, q_m)$, where q_i's are either $\forall$ or $\exists$, the sentence $q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m)$ is in L_1, so that it is in L. Thus L has also properties (i).
Consequently, the basic extension L of L_0 constructed above has all properties (i) – (iii) when its sentences are valuated by their meanings. This implies that L_0 conforms to classical logic. $\qquad\square$

Lemma 5.2. *Make the assumptions (s1)–(s3), and let L be the language constructed in the proof of Lemma 5.1. Let $\mathcal{L}_T$, $\mathcal{D}$ and U be as in Definition 4.1, and let W be the set of Gödel numbers of true sentences of L. Given a consistent subset V of $\mathcal{D}$ which satisfies $W \subseteq V \subseteq U$, assume that every sentence of $\mathcal{L}_T$ whose Gödel number is in V is true and not false by its meaning. Then every sentence of $L(V)$ (defined by (2.6) with U replaced by V) is true and not false by its meaning.*

Proof. Because $V \subseteq U = G(U)$, then every sentence whose Gödel number is in V, is in $\mathcal{L}_T$. At first we prove that every sentence of $L_0(V)$ is true and not false by its meaning.

By Lemma 5.1 L is bivalent. Thus every true sentence of L, i.e., every sentence of Z is true and not false by its meaning.
Let A denote a sentence of $\mathcal{L}_T$. By (s3) the sentence $T(\lceil A \rceil)$ means that 'the sentence whose Gödel number has $\lceil A \rceil$ as its numeral, i.e., the sentence A, is true'. Thus, by its meaning, $T(\lceil A \rceil)$ is true iff A is true and false iff A is false.
Assume that the Gödel number of a sentence A is in V. Since A is by a hypothesis true and not false by its meaning, then $T(\lceil A \rceil)$ is true and not false by its meaning. This implies by (2.2) that the sentences

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of $Z_1(V)$ are true and not false by their meanings. By the standard meaning of negation the sentence $\neg T(\lceil A\rceil)$ is false and not true by its meaning. Replacing A by $T(\lceil A\rceil)$ and $\neg T(\lceil A\rceil)$ it follows from the above results that the sentences $T(\lceil T(\lceil A\rceil)\rceil)$ and $\neg T(\lceil \neg T(\lceil A\rceil)\rceil)$ are true and not false by their meanings, and the sentences $T(\lceil \neg T(\lceil A\rceil)\rceil)$ and $\neg T(\lceil T(\lceil A\rceil)\rceil)$ are false and not true by their meanings. Let A denote such a sentence of $\mathcal{L}_T$, that the Gödel number of the sentence $\neg A$ is in V. $\neg A$ is by a hypothesis true and not false by its meaning, so that A is false and not true by its meaning since V is consistent. Thus the sentence $T(\lceil A\rceil)$ is false and not true by its meaning, and the sentence $\neg T(\lceil A\rceil)$ is true and not false by its meaning. It then follows from (2.2) that the sentences of $Z_2(V)$ are true and not false by their meanings.

Replacing A by $T(\lceil A\rceil)$ and $\neg T(\lceil A\rceil)$ we then obtain that the sentences $T(\lceil T(\lceil A\rceil)\rceil)$ and $\neg T(\lceil \neg T(\lceil A\rceil)\rceil)$ are false and not true by their meanings, and the sentences $\neg T(\lceil T(\lceil A\rceil)\rceil)$ and $T(\lceil \neg T(\lceil A\rceil)\rceil)$ are true and not false by their meanings.

The set N_T of numerals, defined by (4.2), is formed by numerals $\lceil A\rceil$, where A goes through all the sentences of $\mathcal{L}_T$. Thus, by results proved above $T(\mathbf{n})$, $T(\lceil T(\mathbf{n})\rceil)$, $\neg T(\lceil \neg T(\mathbf{n})\rceil)$, $\neg T(\mathbf{n})$, $\neg T(\lceil T(\mathbf{n})\rceil)$ and $T(\lceil \neg T(\mathbf{n})\rceil)$ are for some $\mathbf{n} \in N_T$ true and not false by their meanings and for some $\mathbf{n} \in N_T$ false and not true by their meanings. These results and the standard meanings of quantifiers and negation imply that $\exists x T(x)$, $\exists x T(\lceil T(\dot{x})\rceil)$, $\exists x \neg T(\lceil \neg T(\dot{x})\rceil)$, $\exists x \neg T(x)$, $\exists x \neg T(\lceil T(\dot{x})\rceil)$ and $\exists x T(\lceil \neg T(\dot{x})\rceil)$ are true and not false by their meanings, and their negations are false and not true by their meanings, whereas $\forall x T(x)$, $\forall x T(\lceil T(\dot{x})\rceil)$, $\forall x \neg T(\lceil \neg T(\dot{x})\rceil)$, $\forall x \neg T(x)$, $\forall x \neg T(\lceil T(\dot{x})\rceil)$ and $\forall x T(\lceil \neg T(\dot{x})\rceil)$ are false and not true by their meanings, and their negations are true and not false by their meanings.

By above results and (2.2) the sentences of Z_1 and Z_2 are true and not false by their meanings.

Let P be a predicate in $\mathcal{P}^m$, and let $q_1, \ldots, q_m$ be an m-tuple of quantifiers $\forall$ and $\exists$. Since the sentences $q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m)$ and $P(b_1, \ldots, b_m)$ are in L, they are by their meanings either true and not false, or false and not true.

If $q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m)$ is true and not false by its meaning, then $P(b_1, \ldots, b_m)$ is true and not false by its meaning for all b_i when q_i is $\forall$, and for some b_i when p_i is $\exists$. Thus, by (s3), $T(\lceil P(b_1, \ldots, b_m)\rceil)$ is true and not false by its meaning for all b_i when q_i is $\forall$, and for some b_i when p_i is $\exists$. Consequently, $q_1 x_1 \ldots q_m x_m T(\lceil P(b_1, \ldots, b_m)\rceil)$ is true and not false by its meaning.

By its meaning $\neg P(b_1, \ldots, b_m)$ is true and not false iff $P(b_1, \ldots, b_m)$ is false and not true iff, by (s3), $T(\lceil P(b_1, \ldots, b_m)\rceil)$ is false and not true iff $\neg T(\lceil P(b_1, \ldots, b_m)\rceil)$ is true and not false. Thus the sentence $q_1 x_1 \ldots q_m x_m \neg T(\lceil P(\dot{x}_1, \ldots, \dot{x}_m)\rceil)$ is true and not false by its meaning iff the sentence $q_1 x_1 \ldots q_m x_m \neg P(x_1, \ldots, x_m)$ is true and not false by its meaning.

It follows from the above results and (2.1) that the sentences of Z_1^m and Z_2^m are true and not false by their meanings. Consequently, the sentences of Z_3 and Z_4 are by (2.2) true and not false by their meanings.

The above results and (2.3) imply that every sentence of $L_0(V)$ is true and not false by its meaning. Thus the following property holds when $n = 0$.

(h3)   Every sentence of $L_n(V)$ is true and not false by its meaning.

Make the induction hypothesis: (h3) holds for some $n \in \mathbb{N}_0$.

Given a sentence of $L_n^0(V)$, it is of the form $\neg(\neg A)$, where A is in $L_n(V)$. A is by (h3) true and not false by its meaning. Thus, by standard meaning of negation, its double application implies that the sentence $\neg(\neg A)$, and hence the given sentence, is true and not false by its meaning.

If a sentence is in $L_n^1(V)$, it is of the form $A \vee B$, where A or B is in $L_n(V)$. By (h3) at least one of the sentences A and B is true and not false by its meaning. Thus the sentence $A \vee B$, and hence given sentence, is true and not false by its meaning.

Similarly it can be shown that if (h3) holds, then every sentence of $L_n^k(V)$, where $2 \le k \le 8$, is true and not false by its meaning.

The above results imply that under the induction hypothesis (h3) every sentence of $L_n^k(V)$, where $0 \le k \le 8$, is true and not false by its meaning.

It then follows from the definition (2.5) of $L_{n+1}(V)$ that if (h3) is valid for some $n \in \mathbb{N}_0$, then every sentence of $L_{n+1}(V)$ is true and not false by its meaning.

The first part of this proof shows that (h3) is valid when $n = 0$. Thus, by induction, it is valid for all $n \in \mathbb{N}_0$. This result and (2.6) imply that every sentence of $L(V)$ is true and not false by its meaning. $\quad\square$

Lemma 5.3. *Let U be the smallest consistent subset of $\mathcal{D}$ which satisfies $U = G(U)$. Then under the hypotheses (s1)–(s3) every sentence of $\mathcal{L}_T$ whose Gödel number is in U is true and not false by its meaning.*

Proof. By Theorem 3.1 the smallest consistent subset U of $\mathcal{D}$ which satisfies $U = G(U)$ is the last member of the transfinite sequence $(U_\lambda)_{\lambda < \gamma}$ constructed in the proof of that Theorem. We prove by transfinite induction that the following result holds for all $\lambda < \gamma$.

(H) Every sentence of $\mathcal{L}_T$ whose Gödel number is in U_λ is true and not false by its meaning.

Make the induction hypothesis: There is a μ satisfying $0 < \mu < \gamma$ such that (H) holds for all $\lambda < \mu$.

Let $\lambda < \mu$ be given. Because U_λ is consistent and $W \subseteq U_\lambda \subseteq U$ for every $\lambda < \mu$, it follows from the induction hypothesis and Lemma 5.2 that every sentence of $L(U_\lambda)$ is true and not false by its meaning. This implies by (3.1) that (H) holds when U_λ is replaced $G(U_\lambda)$, for every $\lambda < \mu$. Thus (H) holds when U_λ is replaced by the union of those sets. But this union is U_μ by Theorem 3.1 (G), whence (H) holds when $\lambda = \mu$.

When $\mu = 1$, then $\lambda < \mu$ iff $\lambda = 0$. $U_0 = W$, i.e., the set of Gödel numbers of true sentences of L. Since L, valuated by meanings of its sentences, is bivalent by Lemma 5.1, the sentences of U_0 are true and not false by their meanings. This proves that the induction hypothesis is satisfied when $\mu = 1$.

The above proof implies by transfinite induction properties assumed in (H) for U_λ whenever $\lambda < \gamma$. In particular the last member of $(U_\lambda)_{\lambda < \gamma}$ satisfies (H), which is by Theorem 3.1 the smallest consistent subset U of $\mathcal{D}$ for which $U = G(U)$. This proves the assertion. $\quad\square$

The next result is a consequence of Lemma 5.1, Lemma 5.3 and Theorem 5.1.

Theorem 5.2. *Under the hypotheses (s1)–(s3) the extension $\mathcal{L}_T$ of L_0 defined in Definition 4.1 has a semantical theory of truth (shortly STT), when the valuation (I) is replaced in Theorem 5.1 with the valuation of the sentences of $\mathcal{L}_T$ by their meanings. This valuation is equivalent to valuation (I), and the results of Theorem 5.1 are valid for STT.*

Proof. Let A denote a sentence of $\mathcal{L}_T$. A is by Definition 4.1 in $L(U)$ or in $F(U)$, where U is the smallest consistent subset of $\mathcal{D}$ which satisfies $U = G(U)$. If A is in $L(U)$, its Gödel number is in $G(U) = U$ by (3.1), whence it is by Lemma 5.3 true and not false by its meaning. If A is in $F(U)$, then $\neg A$ is in $L(U)$ by (4.1). Thus $\neg A$ is true and not false by its meaning, so that A is by the standard meaning of negation false and not true by its meaning. Hence the valuation of $\mathcal{L}_T$ by meanings of its sentences is equivalent to valuation (I). In particular, $\mathcal{L}_T$ has properties (i)-(iii). Moreover, the sentence $T(\lceil A \rceil) \leftrightarrow A$ is true by its meaning and the sentence $\neg T(\lceil A \rceil) \leftrightarrow A$ is false by its meaning for every sentence A of $\mathcal{L}_T$. These results imply that T is a truth predicate and $\neg T$ is a non-truth predicate for $\mathcal{L}_T$. Consequently, $\mathcal{L}_T$ has a semantical theory of truth.

$\quad\square$

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6 Compositionality of truth in theories DTT and STT

One of the desiderata introduced in [7] for theories of truth is that truth should be compositional. In this section we present some logical equivalences which theories DTT and STT of truth prove.

Lemma 6.1. *Theories DTT and STT of truth presented in Theorems 5.1 and 5.2 prove the following logical equivalences when A and B are sentences of $\mathcal{L}_T$, and P is in $\mathcal{P}$ or P is T.*

(a0) $T(\ulcorner \ldots \ulcorner T(\ulcorner A \urcorner) \urcorner \ldots \urcorner) \leftrightarrow T(\ulcorner T(\ulcorner A \urcorner) \urcorner) \leftrightarrow T(\ulcorner A \urcorner) \leftrightarrow A.$

(a1) $\neg T(\ulcorner A \urcorner) \leftrightarrow T(\ulcorner \neg A \urcorner) \leftrightarrow \neg A.$

(a2) $T(\ulcorner A \urcorner) \vee T(\ulcorner B \urcorner) \leftrightarrow T(\ulcorner A \vee B \urcorner) \leftrightarrow A \vee B.$

(a3) $T(\ulcorner A \urcorner) \wedge T(\ulcorner B \urcorner) \leftrightarrow T(\ulcorner A \wedge B \urcorner) \leftrightarrow A \wedge B.$

(a4) $(T(\ulcorner A \urcorner) \to T(\ulcorner B \urcorner)) \leftrightarrow T(\ulcorner A \to B \urcorner) \leftrightarrow (A \to B).$

(a5) $(T(\ulcorner A \urcorner) \leftrightarrow T(\ulcorner B \urcorner)) \leftrightarrow T(\ulcorner A \leftrightarrow B \urcorner) \leftrightarrow (A \leftrightarrow B).$

(a6) $\neg T(\ulcorner A \vee B \urcorner) \leftrightarrow \neg(A \vee B) \leftrightarrow \neg A \wedge \neg B \leftrightarrow T(\ulcorner \neg A \urcorner) \wedge T(\ulcorner \neg B \urcorner) \leftrightarrow \neg T(\ulcorner A \urcorner) \wedge \neg T(\ulcorner B \urcorner).$

(a7) $\neg T(\ulcorner A \wedge B \urcorner) \leftrightarrow \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B \leftrightarrow T(\ulcorner \neg A \urcorner) \vee T(\ulcorner \neg B \urcorner) \leftrightarrow \neg T(\ulcorner A \urcorner) \vee \neg T(\ulcorner B \urcorner).$

(a8) $\forall x T(\ulcorner P(\dot{x}) \urcorner) \leftrightarrow T(\ulcorner \forall x P(x) \urcorner) \leftrightarrow \forall x P(x) \leftrightarrow \neg \exists x \neg P(x) \leftrightarrow T(\ulcorner \neg \exists x \neg P(x) \urcorner) \leftrightarrow \neg \exists x \neg T(\ulcorner P(\dot{x}) \urcorner).$

(a9) $\exists x T(\ulcorner P(\dot{x}) \urcorner) \leftrightarrow T(\ulcorner \exists x P(x) \urcorner) \leftrightarrow \exists x P(x) \leftrightarrow \neg \forall x \neg P(x) \leftrightarrow T(\ulcorner \neg \forall x \neg P(x) \urcorner) \leftrightarrow \neg \forall x \neg T(\ulcorner P(\dot{x}) \urcorner).$

(a10) $\neg T(\ulcorner \forall x P(x) \urcorner) \leftrightarrow T(\ulcorner \neg \forall x (P(x)) \urcorner) \leftrightarrow \neg \forall x P(x) \leftrightarrow \exists x \neg P(x) \leftrightarrow T(\ulcorner \exists x \neg P(x) \urcorner).$

(a11) $\neg T(\ulcorner \exists x P(x) \urcorner) \leftrightarrow T(\ulcorner \neg \exists x P(x) \urcorner) \leftrightarrow \neg \exists x P(x) \leftrightarrow \forall x \neg P(x) \leftrightarrow T(\ulcorner \forall x \neg P(x) \urcorner).$

Proof. T-rule implies equivalences of (a0).

The first equivalences in (a1)–(a5) are easy consequences of rules (t1)–(t5) and T-rule (cf. [5, Lemma 4.1]). Their second equivalences are consequences of T-rule.

The first and last equivalences of (a6) and (a7) are consequences of (a1). Their second equivalences are De Morgan's laws of classical logic (cf. [1]). Third equivalences of (a6) and (a7) follow from T-rule. The first equivalences of (a8) and (a9) are easy consequences of rules (tp6) and (tp7) and T-rule (cf. [5, Lemma 4.2]). T-rule implies their second equivalences. The third equivalences are De Morgan's laws for quantifiers (cf. [1]). The fourth ones follow from T-rule. De Morgan's laws with $P(x)$ replaced by $T(\ulcorner P(\dot{x}) \urcorner)$ imply equivalence of the last and the first sentences. (a10) and (a11) are negations to some equivalences of (a8) and (a9). $\square$

Let L_0 be a countable and bivalent first-order language with or without identity. It conforms to classical logic. If P and Q are predicates of L_0 with arity 1 and domain D, then $\neg P$, $P \vee Q$, $P \wedge Q$, $P \to Q$ and $P \leftrightarrow Q$ are predicates of L_0 with domain D. Replacing P and/or Q by some of them we obtain new predicates with domain D, and so on. Thus P in (a8) and (a9) can be replaced by anyone of these predicates. Their universal and existential quantifications are sentences of L. They are also sentences of $\mathcal{L}_T$. Anyone of them can be the sentence A and/or the sentence B in results (a1)–(a7) derived above. Moreover, P can be replaced by anyone of those predicates in (a8)–(a11). Take a few examples.

$\forall x T(\ulcorner P(\dot{x}) \to Q(\dot{x}) \urcorner) \leftrightarrow T(\ulcorner \forall x (P(x) \to Q(x)) \urcorner) \leftrightarrow \forall x (P(x) \to Q(x)).$
$\exists x T(\ulcorner P(\dot{x}) \wedge Q(\dot{x}) \urcorner) \leftrightarrow T(\ulcorner \exists x (P(x) \wedge Q(x)) \urcorner) \leftrightarrow \exists x (P(x) \wedge Q(x)).$
$\forall x T(\ulcorner P(\dot{x}) \to \neg Q(\dot{x}) \urcorner) \leftrightarrow T(\ulcorner \forall x (P(x) \to \neg Q(x)) \urcorner) \leftrightarrow \forall x (P(x) \to \neg Q(x)).$
$\exists x T(\ulcorner P(\dot{x}) \wedge \neg Q(\dot{x}) \urcorner) \leftrightarrow T(\ulcorner \exists x (P(x) \wedge \neg Q(x)) \urcorner) \leftrightarrow \exists x (P(x) \wedge \neg Q(x)).$

These equivalences correspond to the four Aristotelian forms: 'All P's are Q's', 'some P's are Q's', 'no P's are Q's' and 'some P's are not Q's' (cf. [1]).

Let P be a predicate of L_0 with arity $m > 1$, and let $q_1, \ldots q_m$ be any of the 2^m different m-tuples which can be formed from quantifiers $\forall$ and $\exists$. Theories DTT and STT presented in Theorems 5.1 and 5.2 prove the following logical equivalences.

$T(\lceil q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m) \rceil) \leftrightarrow q_1 x_1 \ldots q_m x_m P(x_1, \ldots, x_m) \leftrightarrow q_1 x_1 \ldots q_m x_m T(\lceil P((\dot{x}_1, \ldots, \dot{x}_m)) \rceil)$,

$T(\lceil q_1 x_1 \ldots q_m x_m \neg P(x_1, \ldots, x_m) \rceil) \leftrightarrow q_1 x_1 \ldots q_m x_m \neg P(x_1, \ldots, x_m) \leftrightarrow q_1 x_1 \ldots q_m x_m \neg T(\lceil P(\dot{x}_1, \ldots, \dot{x}_m) \rceil)$.

T-rule implies the first equivalences, and the second equivalences are consequences of (2.1) and bivalence of $\mathcal{L}_T$. An application of T-rule proves the universal T-schema:

(UT) $\forall x_1 \ldots \forall x_m \big(T(\lceil P((\dot{x}_1, \ldots, \dot{x}_m)) \rceil) \leftrightarrow P(x_1, \ldots, x_m) \big)$.

Example 6.1. Assume that L_0 is the language of arithmetic with its standard interpretation. Let $R(x, y)$ be formula $x = 2y$, and let R be the corresponding predicate with domain $D_R = \mathbb{N}_0 \times \mathbb{N}_0$. Then the truth theories DTT and STT of the extension $\mathcal{L}_T$ of L_0 prove the universal T-schema

(UTR) $\forall x \forall y \big(T(\lceil R((\dot{x}, \dot{y})) \rceil) \leftrightarrow R(x, y) \big)$,

and the logical equivalences

(q1) $q_1 x q_2 y T(\lceil R(\dot{x}, \dot{y}) \rceil) \leftrightarrow T(\lceil q_1 x q_2 y R(x, y) \rceil) \leftrightarrow q_1 x q_2 y R(x, y)$,

(q2) $q_1 x q_2 y \neg T(\lceil R(\dot{x}, \dot{y}) \rceil) \leftrightarrow \neg T(\lceil q_1 x q_2 y R(x, y) \rceil) \leftrightarrow q_1 x q_2 y \neg R(x, y)$.

The sentences in (q1) are true iff $q_1 q_2$ is $\forall\exists$ or $\exists\exists$, and false iff $q_1 q_2$ is $\forall\forall$ or $\exists\forall$. In (q2) the sentences are true iff $q_1 q_2$ is $\forall\forall$ or $\forall\exists$, and false iff $q_1 q_2$ is $\exists\forall$ or $\exists\exists$.

The next example is a modification of an example presented in [10, p. 704].

Example 6.2. Let L_0, $\mathcal{L}_T$, T and $R(x, y)$ be as in Example 6.1. Denote by T' the predicate with domain $\mathbb{N}_0$ corresponding to formula $\exists y R(x, y)$. Let the sublanguage L_0' of L_0 be formed by the syntax of L_0, predicate T' and sentences $\exists y R(\mathbf{n}, y)$, where $\mathbf{n}$ goes through all numerals, the names of natural numbers n. Choose n to be the code number of the sentence $\exists y R(\mathbf{n}, y)$ in L_0' for each numeral $\mathbf{n}$. If A is a sentence of L_0', i.e. $\exists y R(\mathbf{n}, y)$ for some numeral $\mathbf{n}$, then n is the code number of A, so that $\lceil A \rceil$ is $\mathbf{n}$. Since $T'(\mathbf{n})$ denotes the sentence $\exists y R(\mathbf{n}, y)$ for each numeral $\mathbf{n}$, then $T'(\lceil A \rceil)$ or equivalently $T'(\mathbf{n})$, is true (respectively false) in L_0' iff $\exists y R(\mathbf{n}, y)$, or equivalently A, is true, (respectively false) in L_0' iff (by definition of $R(x, y)$) n is even (respectively odd). Thus T' is a truth predicate of L_0'. The restriction of T to L_0' is also a truth predicate of L_0'. It is not equal to T' because the coding of L_0' is not the restriction of the Gödel numbering of $\mathcal{L}_T$, whereas L_0 can have that syntax.

7   Remarks

Theories DTT and STT conform to the seven desiderata presented in [7] for theories of truth (cf. [5, Theorem 4.2]). This challenges the current view (cf, e.g., [3, 7, 9]). Theories DTT and STT are also free from paradoxes. The fact that the languages $\mathcal{L}_T$ which have these theories of truth contain their own truth predicates seems to be in contrast to some Tarski's limiting theorems for theories of truth described, e.g., in [10]. Conformity of theories DTT and STT to the principles of classical logic can be vitiated by adding first-order syntax to $\mathcal{L}_T$. The object language L_0 is allowed to have that syntax.

Theorems 5.1 and 5.2 imply that theories DTT and STT of truth together contain the theory DSTT of truth introduced in [5, Theorem 4.1] for languages $\mathcal{L}^0$ when those sentences of the form $A \vee B$, $A \to B$ and $\neg(A \wedge B)$ whose one component has not a true value are deleted from $\mathcal{L}^0$. Otherwise the languages $\mathcal{L}_T$ for which theories DTT and STT of truth are introduced extend languages $\mathcal{L}^0$. The amount of predicates and compositional sentences are multiplied by means of the added non-truth predicate $\neg T$, and predicates of the object language L_0 which have several free variables.

The family of those languages which conform to classical logic is considerably larger than the families of those object languages considered in [5]. For instance, the object language L_0 can be any language whose every sentence is valuated by its meaning either as true or as false. Every language L_0 which has properties (i) – (iii), e.g., every countable and bivalent first-order language with or without identity, conforms to classical logic. If L_0 is any countable language conforming to classical logic, and if L_0' is any sublanguage of L_0 formed by the syntax of L_0, any nonempty subset of its sentences and any subset of its predicates, then L_0' conforms to classical logic. For instance, the language L_0' in Example 6.2 conforms to classical logic.

The set U used in the definition of $\mathcal{L}_T$ is the smallest consistent set for which $U = G(U)$, where $G(U)$ is the set of Gödel numbers of sentences of $L(U)$. Thus U is the minimal fixed point of the mapping $G : \mathcal{C} \to \mathcal{C}$, where $\mathcal{C}$ is the set of consistent sets of Gödel numbers of sentences of $\mathcal{L}$. Moreover $\mathcal{L}_T$ conforms to classical logic. Thus the sentences of $\mathcal{L}_T$ are grounded in the sense defined by Kripke in [6, p. 18]. The language $\mathcal{L}_\sigma$ determined by the minimal fixed point in Kripke's construction contains also sentences which don't have truth values. For instance, the sentence $A \leftrightarrow T(\lceil A \rceil)$ has not a truth value for every sentence A of $\mathcal{L}_\sigma$. Thus a three-valued logic is needed in [6], as well as in [3] and in [4]. The only logic used here is classical.

The equivalence of truth values of sentences in theories DTT and STT show that the notion of 'grounded truth' defined by valuation (I) conforms to the 'ordinary' notion of truth.

In the metalanguage used in the above presentation some concepts dealing with predicates and their domains are revised from those used in [5] so that they agree better with the corresponding concepts in informal languages of first-order logic (cf. [1]). The circular reasoning used in [5] to show that $G(U)$ is consistent if U is consistent is corrected in the proof of Lemma 3.1.

Mathematics, especially ZF set theory, plays a crucial role in this paper. Metaphysical necessity of pure mathematical truths is considered in [8].

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