A theory of truth for a class of mathematical languages and an application

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November 10, 2014

Abstract

In this paper a class of so called mathematically acceptable (shortly MA) languages is introduced. First-order formal languages containing natural numbers and numerals belong to that class. MA languages which are contained in a given fully interpreted MA language augmented by a monadic predicate are constructed. A mathematical theory of truth (shortly MTT) is formulated for some of these languages. MTT makes them fully interpreted MA languages which posses their own truth predicates, yielding consequences to philosophy of mathematics. MTT is shown to conform well with the eight norms presented for theories of truth in the paper 'What Theories of Truth Should be Like (but Cannot be)', by Hannes Leitgeb. MTT is also free from infinite regress, providing a proper framework to study the regress problem.

MSC: 00A30, 03A05, 03B10, 03C62, 03E04, 03F50, 06A07, 47H04, 47H10

Keywords: language, theory, truth, logic, first-order, set theory, Zermelo-Fraenkel, model, fixed point, regress problem.

1 Introduction

In this paper a theory of truth is formulated for languages which are formal enough for mathematical reasoning. The regress problem is studied within the framework of that theory. By Chomsky's definition (cf. [2]) a "language is a set (finite or infinite) of sentences of finite length, and constructed out of finite sets of symbols". Allowing countable sets of sentences and symbols we call such a language L *mathematically acceptable* (shortly MA), if the syntax and the lexicon of L are those of first-order predicate logic (cf., e.g., [17, Definitions II.5.1 and II.5.2]), added by natural numbers and their names; numerals, and if L is closed with respect to logical connectives and quantifiers. L is called fully interpreted, if every sentence of L is interpreted either as true or as false, and if standard truth tables are valid. Any countable first-order formal language, equipped with a consistent theory interpreted by a countable model, and containing natural numbers and numerals, is a fully interpreted MA language. A classical example is the language of arithmetic with its standard model and interpretation.
Basic ingredients in our approach are: 1. A fully interpreted MA language L (base language).
2. A monadic predicate T having the set of numerals as its domain (truth predicate).
3. The MA language $\mathcal{L}$ obtained as a closure with respect to logical connectives of the set formed by sentences of L, elements of the range of T and quantifications $\forall x T(x)$ and $\exists x T(x)$.
4. The set D of Gödel numbers of sentences of $\mathcal{L}$ in its fixed Gödel numbering.

The paper is organized as follows. In Section 2 we construct to each subset U of D new subsets $G(U)$ and $F(U)$ of D, and denote by $\mathcal{L}_U$ the language of those sentences which have Gödel numbers in $G(U) \cup F(U)$. The construction implies that $\mathcal{L}_U$ is an MA language. U is called consistent if for no sentence A of $\mathcal{L}$ the Gödel numbers of both A and $\neg A$ are in U.

In Section 3 we define a mathematical theory of truth (shortly MTT) for languages $\mathcal{L}_U$, where U is consistent and is a fixed point of G, i.e., $U = G(U)$. A sentence A of $\mathcal{L}_U$ is interpreted as true if its Gödel number $\#A$ is in $G(U)$, and as false if $\#A$ is in $F(U)$. This makes $\mathcal{L}_U$ a fully interpreted MA language. Thus *$\mathcal{L}_U$ has the same formal and semantical properties as assumed for L*. We call T a truth predicate for $\mathcal{L}_U$, and show that T-biconditionality: $A \leftrightarrow T(\lceil A \rceil)$, where $\lceil A \rceil$ is the numeral of the Gödel number of A, is true for all sentences A of $\mathcal{L}_U$. The truth in L and the truth in $\mathcal{L}_U$ are connected so that each sentence A of L, either
(a) A is true in the interpretation of L, iff A is true in $\mathcal{L}_U$, iff $T(\lceil A \rceil)$ is true in $\mathcal{L}_U$, or
(b) $\neg A$ is true in the interpretation of L, iff $\neg A$ is true in $\mathcal{L}_U$, iff $\neg T(\lceil A \rceil)$ is true in $\mathcal{L}_U$.

In Section 4 we study consequences of the presented theory to mathematical philosophy. MTT is immune to 'Tarski's Commandment' (cf. [19]), to Tarski's Undefinability Theorem (cf. [25]), to 'Tarskian hierarchies' (cf. [8]), and to 'Liar paradox' (cf. [12]). MTT is also shown to conform well with the norms presented in [18] for theories of truth.
Section 5 is devoted to the study of the regress problem within the framework of MTT. In the study we question the following conclusion stated in [23]: "it is logically impossible for there to be an infinite parade of justifications". Notwithstanding this conclusion we present an example of an infinite parade (regress) of justifications that satisfies the conditions imposed on them in [22]. Example is inconsistent with the above quoted conclusion that is used in [22, 23] as a basic argument to refute Principles of Sufficient Reasons.

Main tools used in proofs are Zermelo-Fraenkel (ZF) set theory and classical logic.

2 Construction of languages

Let basic ingredients L, T, $\mathcal{L}$ and D be as in the Introduction. We shall construct a family of sublanguages for the language $\mathcal{L}$. As for the used terminology, cf. e.g., [17]. Given a subset U of D, define subsets $G(U)$ and $F(U)$ of D by following rules (iff abbreviates if and only if):

(r1) If A is a sentence of L, then the Gödel number $\#A$ of A is in $G(U)$ iff A is true in the interpretation of L, and in $F(U)$ iff A is false in the interpretation of L.

(r2) If A is a sentence of $\mathcal{L}$, then $\#T(\lceil A \rceil)$ is in $G(U)$ iff $\#A$ is in U, and in $F(U)$ iff the Gödel number $\#[\neg A]$ of the negation $\neg A$ of A is in U.

Sentences determined by rules (r1) and (r2), i.e., all sentences A of L and those sentences $T(\lceil A \rceil)$ of $\mathcal{L}$ for which $\#A$ or $\#[\neg A]$ is in U, are called *basic sentences*.
Next rules deal with logical connectives. Let A and B be sentences of $\mathcal{L}$.

(r3) Negation rule: $\#[\neg A]$ is in $G(U)$ iff $\#A$ is in $F(U)$, and in $F(U)$ iff $\#A$ is in $G(U)$.

(r4) Disjunction rule: $\#[A \vee B]$ is in $G(U)$ iff $\#A$ or $\#B$ is in $G(U)$, and in $F(U)$ iff $\#A$ and $\#B$ are in $F(U)$.

(r5) Conjunction rule: $\#[A \wedge B]$ is in $G(U)$ iff $\#[\neg A \vee \neg B]$ is in $F(U)$ iff (by (r3) and (r4)) both $\#A$ and $\#B$ are in $G(U)$. Similarly, $\#[A \wedge B]$ is in $F(U)$ iff $\#[\neg A \vee \neg B]$ is in $G(U)$ iff $\#A$ or $\#B$ is in $F(U)$.

(r6) Implication rule: $\#[A \rightarrow B]$ is in $G(U)$ iff $\#[\neg A \vee B]$ is in $G(U)$ iff (by (r3) and (r4)) $\#A$ is in $F(U)$ or $\#B$ is in $G(U)$. $\#[A \rightarrow B]$ is in $F(U)$ iff $\#[\neg A \vee B]$ is in $F(U)$ iff $\#A$ is in $G(U)$ and $\#B$ is in $F(U)$.

(r7) Biconditionality rule: $\#[A \leftrightarrow B]$ is in $G(U)$ iff $\#A$ and $\#B$ are both in $G(U)$ or both in $F(U)$, and in $F(U)$ iff $\#A$ is in $G(U)$ and $\#B$ is in $F(U)$ or $\#A$ is in $F(U)$ and $\#B$ is in $G(U)$.

If $A(x)$ is a formula in L, then quantifications $\exists x A(x)$ and $\forall x A(x)$ are sentences of L. Hence rule (r1) is applicable for them. So it suffices to set rules for $\exists x T(x)$ and $\forall x T(x)$. Assume that the set X of numerals of Gödel numbers of sentences of $\mathcal{L}$ is the intended domain of discourse of T. We set the following rules:

(r8) $\#[\exists x T(x)]$ is in $G(U)$ iff $\#T(\mathbf{n})$ is in $G(U)$ for some $\mathbf{n} \in X$, and $\#[\exists x T(x)]$ is in $F(U)$ iff $\#T(\mathbf{n})$ is in $F(U)$ for every $\mathbf{n} \in X$.

(r9) $\#[\forall x T(x)]$ is in $G(U)$ iff $\#T(\mathbf{n})$ is in $G(U)$ for every $\mathbf{n} \in X$, and $\#[\forall x T(x)]$ is in $F(U)$ iff $\#T(\mathbf{n})$ is in $F(U)$ at least for one $\mathbf{n} \in X$.

Rules (r1)–(r9) and induction on the complexity of formulas determine uniquely subsets $G(U)$ and $F(U)$ of D whenever U is a subset of D. Denote by $\mathcal{L}_U$ the language formed by those sentences A of $\mathcal{L}$ for which $\#A$ is in $G(U)$ or in $F(U)$. $\mathcal{L}_U$ contains by rule (r1) all sentences

of the base language L, and is by construction closed with respect to logical connectives and quantifiers. In particular, $\mathcal{L}_U$ is an MA language.
We say that a subset U of D is *consistent* iff both $\#A$ and $\#[\neg A]$ are not in U for any sentence A of $\mathcal{L}$. For instance, the empty set $\emptyset$ is consistent.
The following two lemmas have counterparts in [10].

Lemma 2.1. *Let U be a consistent subset of D. Then $G(U) \cap F(U) = \emptyset$.*

Proof. Let A be a sentence of L. Because L is fully interpreted, then either A is true or false in the interpretation of L. Thus, by rule (r1), $\#A$ is either in $G(U)$ or in $F(U)$.
If A is a sentence of $\mathcal{L}$, then by rule (r2), $\#T(\lceil A \rceil)$ is in $G(U)$ iff $\#A$ is in U, and in $F(U)$ iff $\#[\neg A]$ is in U. Thus $\#T(\lceil A \rceil)$ cannot be both in $G(U)$ and in $F(U)$ because U is consistent.
Make an induction hypothesis:

(h0) A and B are such sentences of $\mathcal{L}$ that neither $\#A$ nor $\#B$ is in $G(U) \cap F(U)$.

As shown above, (h0) holds if A and B are basic sentences.
If $\#[\neg A]$ is in $G(U) \cap F(U)$, then $\#A$ is in $F(U) \cap G(U)$. Hence, if (h0) holds, then $\#[\neg A]$ is not in $G(U) \cap F(U)$.
If $\#[A \vee B]$ is in $G(U) \cap F(U)$, then $\#A$ or $\#B$ is in $G(U)$, and both $\#A$ and $\#B$ are in $F(U)$ by (r4), so that $\#A$ or $\#B$ is in $G(U) \cap F(U)$. Hence, if (h0) holds, then $\#[A \vee B]$ is not in $G(U) \cap F(U)$.
$\#[A \wedge B]$ cannot be in $G(U) \cap F(U)$, for otherwise both $\#A$ and $\#B$ are in $G(U)$, and at least one of $\#A$ and $\#B$ is in $F(U)$, contradicting with (h0).
If $\#[\neg A]$ is in $G(U) \cap F(U)$, then $\#A$ is in $F(U) \cap G(U)$, and (h0) is not valid. Thus, under the hypothesis (h0) neither $\#[\neg A]$ nor $\#B$ is in $G(U) \cap F(U)$. This result and the above result for disjunction imply that $\#[\neg A \vee B]$, or equivalently, $\#[A \rightarrow B]$, is not in $G(U) \cap F(U)$.
Similarly, $\#[A \leftrightarrow B]$ is not in $G(U) \cap F(U)$, for otherwise, $\#A$ or $\#B$ would be in $G(U) \cap F(U)$ by rule (r7), contradicting with (h0).
It remains to show that $\#[\exists x T(x)]$ and $\#[\forall x T(x)]$ are not in $G(U) \cap F(U)$.
If U is empty, then $T(\mathbf{n})$ is by rule (r2) neither in $G(U)$ nor in $F(U)$ for any $\mathbf{n} \in X$. Thus $\#[\exists x T(x)]$ is by rule (r8) neither in $G(U)$ nor in $F(U)$, and hence not in $G(U) \cap F(U)$.
If U is nonempty, then $\#A$ is in U for some A in $\mathcal{L}$. Since U is consistent, then $\#[\neg A]$ is not in U, whence $\#T(\lceil A \rceil)$ is by rule (r2) not in $F(U)$. Thus $\#[\exists x T(x)]$ is by rule (r8) not in $F(U)$, and hence not in $G(U) \cap F(U)$.
Because U is consistent, it is a proper subset of D. Thus there is $n \in D$ such that $n \notin U$. But $n = \#A$ for some sentence A of $\mathcal{L}$, whence $\#T(\mathbf{n}) = \#T(\lceil A \rceil)$ is not in $G(U)$ by rule (r2). Consequently, $\#[\forall x T(x)]$ is by rule (r9) not in $G(U)$, and hence not in $G(U) \cap F(U)$.
The above results and induction on the complexity of formulas imply that $\#A$ is not in $G(U) \cap F(U)$ for any sentence A of $\mathcal{L}$. $\square$

Lemma 2.2. *If U is a consistent subset of D, then both $G(U)$ and $F(U)$ are consistent.*

Proof. If $G(U)$ is not consistent, then there is such a sentence A of $\mathcal{L}$, that $\#A$ and $\#[\neg A]$ are in $G(U)$. Because $\#[\neg A]$ is in $G(U)$, then $\#A$ is also in $F(U)$ by rule (r3), and hence in $G(U) \cap F(U)$. But then, by Lemma 2.1, U is not consistent. Consequently, if U is consistent, then $G(U)$ is consistent. The proof that $F(U)$ is consistent if U is, is similar. $\square$

3 A mathematical theory of truth

Recall that D denotes the set of Gödel numbers of sentences of the language $\mathcal{L}$. Given a subset U of D, let $G(U)$ and $F(U)$ be the subsets of D constructed in Section 2. In the next definition we formulate our mathematical theory of truth (shortly MTT).

Definition 3.1. *Assume that U is a consistent subset of D, and that $U = G(U)$. Denote by $\mathcal{L}_U$ the language containing those sentences A of $\mathcal{L}$ for which $\#A$ is in $G(U)$ or in $F(U)$. A sentence A of $\mathcal{L}_U$ is interpreted as true iff $\#A$ is in $G(U)$, and as false iff $\#A$ is in $F(U)$. T is called a truth predicate for $\mathcal{L}_U$.*

The existence of consistent fixed points of G, i.e., those consistent subsets U of D satisfying $U = G(U)$, including the smallest one, can be proved, by applying ZF set theory, as in [11, Section 4], when 'true in M' is replaced by 'true in the interpretation of L'.

In view of Definition 3.1, '$\#A$ is in $G(U)$' can be replaced by 'A is true' and '$\#A$ is in $F(U)$' by 'A is false' in (r3)–(r9). Thus standard truth tables of classical two-valued logic hold for logical connectives of sentences of $\mathcal{L}_U$. This makes $\mathcal{L}_U$ *a fully interpreted MA language that posses those formal and semantical properties which are assumed for the base language L.*

The following result justifies to call T as a truth predicate of $\mathcal{L}_U$.

Lemma 3.1. *If U is a consistent subset of D, and if $U = G(U)$, then T-biconditionality: $A \leftrightarrow T(\lceil A \rceil)$ is true for every sentence A of $\mathcal{L}_U$.*

Proof. Assume that $U \subset D$ is consistent, and that $U = G(U)$. Let A be a sentence of $\mathcal{L}_U$. Applying rules (r2) and (r3), and the assumption $U = G(U)$, we obtain
– $\#A$ is in $G(U)$ iff $\#A$ is in U iff $\#T(\lceil A \rceil)$ is in $G(U)$;
– $\#A$ is in $F(U)$ iff $\#[\neg A]$ is in $G(U)$ iff $\#[\neg A]$ is in U iff $\#T(\lceil A \rceil)$ is in $F(U)$.
Consequently, $\#A$ and $\#T(\lceil A \rceil)$ are both either in $G(U)$ or in $F(U)$. Thus $\#[A \leftrightarrow T(\lceil A \rceil)]$ is by rule (r7) in $G(U)$, so that $A \leftrightarrow T(\lceil A \rceil)$ is true by Definition 3.1. This holds for every sentence A of $\mathcal{L}_U$. □

Our main result on the connection between the valuations determined by the interpretation of L and that of $\mathcal{L}_U$ defined in Definition 3.1 reads as follows:

Theorem 3.1. *Let U be a consistent fixed point of G. If A is a sentence of L, then either*
(a) A is true in the interpretation of L, iff A is true, iff $T(\lceil A \rceil)$ is true, or
(b) A is false in the interpretation of L, iff A is false, iff $T(\lceil A \rceil)$ is false.

Proof. Assume that A is a sentence of L. Because L is completely interpreted, then A is either true or false in the interpretation of L.
– A is true in the interpretation of L iff $\#A$ is in $G(U)$, by rule (r1), iff $\#A$ is in U, because $U = G(U)$, iff $\#T(\lceil A \rceil)$ is in $G(U)$ by rule (r2), iff $T(\lceil A \rceil)$ is true, by Definition 3.1.
– A is false in the interpretation of L iff $\neg A$ is true in the interpretation of L iff $\#[\neg A]$ is in $G(U)$, by rule (r1), iff $\#[\neg A]$ is in U, because $U = G(U)$, iff $\#\lceil T(A) \rceil$ is in $F(U)$, by rule (r2), iff $T(\lceil A \rceil)$ is false, by Definition 3.1.
Consequently, a sentence A of L is true in the interpretation of L iff $T(\lceil A \rceil)$ is true, and false in the interpretation of L iff $T(\lceil A \rceil)$ is false. These results and the result of Lemma 3.1 imply the conclusions (a) and (b). □

4 Consequences to philosophy of mathematics

Let $S = (L, \Sigma)$ be a mathematical system, where L is a first-order formal language, and Σ is a set of axioms. Assume that Σ is consistent, and is either an extension of Robinson arithmetic Q (e.g., Q itself or Peano arithmetic, L being the language of arithmetic), or Q can be interpreted in Σ (e.g., Σ axiomatizes ZF set theory, and L is the language of set theory). By Löwenheim-Skolem theorem that theory has a countable model M. Interpret a sentence A of L as true in L if $M \models A$, and false in L if $M \models \neg A$, in the sense defined in [17, II.2.7]. By [17, Lemma II.2.8.22] this interpretation makes L fully interpreted, and L is an MA language. Tarski's Undefinability Theorem (cf. [25]) implies that L cannot contain it's truth predicate, yielding 'Tarski's Commandment' (cf. [19]). Let $\mathcal{L}$ be a formal language obtained by augmenting L with a monadic predicate T. T cannot be a truth predicate of $\mathcal{L}$, for otherwise one could construct a Liar sentence, which implies the 'Liar paradox' (cf. [12]). Thus $\mathcal{L}$ does not contain its truth predicate, either, yielding 'Tarskian hierarchies' (cf. [8]). Surprisingly, a Liar sentence is accepted in many axiomatic theories of truth (cf., e.g, [5])!

Theory MTT provides an alternative. Given a completely interpreted MA language L, let $\mathcal{L}_U$, where U is a consistent fixed point of G, be an extension of L constructed in Section 2. The interpretation given for $\mathcal{L}_U$ in Definition 3.1 makes it fully interpreted. Moreover, $\mathcal{L}_U$ contains by Definition 3.1 a truth predicate T defined within $\mathcal{L}_U$ itself. It follows from Lemma 3.1 that there is no Liar sentence in $\mathcal{L}_U$. As an MA language $\mathcal{L}_U$ *is closed with respect to logical connectives and quantifiers, and hence formal enough for mathematical reasoning.* In particular, the language L of the above system S is extended in $S_U = (\mathcal{L}_U, \text{MTT})$ to a fully interpreted MA language $\mathcal{L}_U$ that contains its truth predicate and is free from paradoxes.

If $S = (L, \Sigma)$ is as above, L contains by Gödel's First Incompleteness Theorem a true arithmetical sentence, say B, that is not provable from the axioms of Σ (cf. [24]). By Theorem 3.1 both B and $T(B)$ are true in the interpretation of $\mathcal{L}_U$. This provides some support to the following opinions on mathematical truth presented in [21, Chapter 4]: "The notion of mathematical truth goes beyond the whole concept of formalism. There is something absolute and 'God-given' about mathematical truth. Real mathematical truth goes beyond mere man-made constructions." Because of consistency assumption for Σ these opinions are questioned (cf. [24]). But inability of human mind to verify that Σ is consistent rather *supports* than questions these opinions. Mathematics rests on the belief that its theories are consistent.

MTT has properties that conform quite well with the eight norms formulated in [18] for theories of truth. Truth is expressed by a predicate T. The syntax of the MA language $\mathcal{L}_U$ is that of first-order logic with equality added by natural numbers as constants and numerals as terms. It is closed under logical connectives and quantifiers. A theory of truth is added to the base language L. If the interpretation of L is determined by a consistent mathematical theory (Peano arithmetic, ZF set theory, e.t.c.), then MTT proves the theory in question true, by Theorem 3.1. Truth predicate is not subject to any restrictions within fixed point languages $\mathcal{L}_U$. T-biconditionals are derivable unrestrictedly within fixed point languages $\mathcal{L}_U$, by the proof of Lemma 3.1. Truth is compositional, by Definition 3.1 and rules (r3)–(r9). The theory allows for standard interpretations if the interpretation of L is standard. In particular, *the outer logic and the inner logic coincide, and they are classical.*

5 On the Regress Problem

First of ten theses presented in [1, p. 6] is: "The Regress Problem is a real problem for epistemology." We are going to study the regress problem in the framework of MTT. We adjust first our terminology to that used in [22] in the study of the regress problem. Given a fully interpreted MA language L, let a fully interpreted MA language $\mathcal{L}_U$ that contains L be determined by Definition 3.1, U being the smallest fixed point of G. By statements we mean the sentences of $\mathcal{L}_U$, which are valued by Definition 3.1. A statement A is said to entail B, if it is not possible that A is true and B is false simultaneously. For instance, if $A \to B$ is true, then A entails B. We say that a statement A justifies a statement B if A confirms the truth of B. For instance, if $A \leftrightarrow \neg B$ is true, then A justifies B iff A is false. If $A \to B$ is true, then A justifies B iff A is true (Modus Ponens). A is called contingent if the truth value of A is unknown.

Consider an infinite regress

$$\dots F_i, \dots, F_1, F_0 \tag{5.1}$$

of statements F_i, $i \geq 0$, where the statement F_0 is contingent. We shall impose the following conditions on statements F_i, $i > 0$ (cf. [22]):

(i) F_i entails F_{i-1};

(ii) $F_0 \vee \cdots \vee F_{i-1}$ does not entail F_i;

(iii) $F_0 \vee \cdots \vee F_{i-1}$ does not justify F_i.

Regress (5.1) is called justification-saturated if the following condition holds:

(iv) ... what justifies F_{i-1} is $F_i, \dots$, what justifies F_1 is F_2, what justifies F_0 is F_1.

Lemma 5.1. *Assume that in regress (5.1) the statement F_0 is contingent, and that the statements F_i, $i > 0$, satisfy conditions (i)–(iii).*
(a) If F_1 is false, then F_i is false for each $i > 0$. F_0 is justified iff $F_0 \leftrightarrow \neg F_1$ is true.
(b) If F_n is true for some $n > 0$, then F_i is true when $0 \leq i \leq n$.
(c) The regress (5.1) is justification saturated iff F_i is true for all $i > 0$, in which case F_0 is justified.

Proof. (a) Assume that F_1 is false. If F_i would be true for some $i > 1$, there would be the smallest such an i. Then F_{i-1} would be true by (i). Replacing i by $i - 1$, and so on, this reasoning would imply after $i - 1$ steps that F_1 is true; a contradiction. Thus all statements F_i, $i > 0$, are false. Because F_1 is false, it confirms the truth of F_0 iff F_0 and $\neg F_1$ have same truth values iff $F_0 \leftrightarrow \neg F_1$ is true.
(b) Assume that F_n is true for some $n > 0$. Since F_i entails F_{i-1}, $i = n, n-1, \dots, 1$, then F_i is true for every $i = n-1, \dots, 0$.
(c) If F_n is false for some $n > 0$, then F_{n+1} is false by property (i), and it does not justify F_n, so that condition (iv) is not valid. On the other hand, condition (i) ensures that condition (iv) is valid if F_i is true for all $i > 0$. In this case F_1 justifies F_0, i.e., F_0 is true. □

Example 5.1. Let L be the first-order language $L = \{\in\}$ of set theory, and M the minimal model of ZF set theory constructed in [3]. M is countable and contains the set ω of natural numbers and their set $S(\omega) = \omega \cup \{\omega\}$ (cf. [3, 14]). We assume that numerals are defined in L, e.g., as in [6]. The interpretation induced by M makes L fully interpreted (cf. Section 4). L is also an MA language. Choose L as the base language of theory MTT. Equip $S(\omega)$ with the natural ordering $<$ of natural numbers plus $n < \omega$ for every natural number n. If Z denotes a nonempty subset of $S(\omega)$, it is easy to verify that the infinite regress (5.1) of statements

$$F_i: \quad i < \beta, \text{ for every } \beta \in Z, \quad i = 0, 1, \ldots, \tag{5.2}$$

satisfy conditions (i)–(iii), and that F_0 is contingent. Moreover, condition (iv) is valid by Lemma 5.1 if and only if F_i is true for all $i > 0$. This holds if and only if $Z = \{\omega\}$.

Comments. Example 5.1 is inconsistent with the conclusion of [23] cited in the Introduction. In this example the property that regress (5.1) is justification-saturated both implies and is implied by truth of a 'foundational' statement $F_b : Z = \{\omega\}$. Thus it does not support the form of infinitism presented in [15]: "infinitism holds that there are no ultimate, foundational reasons". On the other hand, it supports "impure" infinitism and the form of foundationalism presented in [1, 26].

Example 5.1 implies that the proofs in [22, 23] to the assertion that "any version of Principle of Sufficient Reason is false" are based on the questionable premise that infinite regresses of justifications don't exist. In fact, this example gives some support to Principles of Sufficient Reason, as well as to many other arguments whose validity is questioned in [22, 23]. For instance, in the 'universe' $S(\omega)$ of Example 5.1,

- $\{\omega\}$ provides a *sufficient reason* for F_0;
- $\{\omega\}$ affords an *ultimate and foundational reason* that justifies F_0;
- $\{\omega\}$ is the *final explainer* of F_0;
- $\{\omega\}$ gives the *first cause* that makes regress (5.1),(5.2) justification-saturated;
- ω *explains the existence of the 'universe'* $\mathbb{N}$ of natural numbers ($\mathbb{N} = \omega$ by [13]);
- ω and $\{\omega\}$ *explain the existence of the 'universe'* $S(\omega)$ ($S(\omega) = \omega \cup \{\omega\}$ by [13]);
- ω is *something beyond natural numbers*;
- ω is *infinite and greatest* in the 'universe' $S(\omega)$;
- ω is *'self-justified'* (The Axiom of Infinity).

Belief that ω exists is a matter of faith. In Example 5.1 we have assumed it because the model M of ZF set theory contains the set $\omega \cup \{\omega\}$. Notice that this set does not belong to the standard model of arithmetic. Thus MTT, where the base language L is the language of arithmetic, is not a sufficient framework for Example 5.1.

6 Remarks

The main purpose of this paper is to present fully interpreted languages which contain their own truth predicates, are free from paradoxes, and are formal enough for mathematical reasoning.

Another purpose of the presented theory of truth is to establish a proper framework to study the regress problem. Tarski's theory of truth (cf. [25]) does not offer it because that theory itself is not free from infinite regress. According to [20, p.189]: "the most important problem with a Tarskian truth predicate is its demand for a hierarchy of languages. ... within that hierarchy of languages, we cannot seem to have any valid method of ending the regression to introduce the "basic" metalanguage." Pure infinite regress is even refused in ([4, p.13]).

Kripke's theory of truth is also a problematic framework because of three-valued inner logic. As stated in [18, p.283]: "Classical first-order logic is certainly the default choice for any selection among logical systems. It is presupposed by standard mathematics, by (at least) huge parts of science, and by much of philosophical reasoning." Moreover, T-biconditionality rule does not hold in Kripke's theory of truth because of paradoxical sentences.

Paradoxes led Zermelo to axiomatize set theory. To avoid paradoxes Tarski "excluded all Liar-like sentences from being well-formed", as noticed in [18]. As a contemporary mathematician I consider as unacceptable that a mathematical theory of truth contains a Liar sentence. Construction of languages $\mathcal{L}_U$ limits the set of those sentences of $\mathcal{L}$ which contain T in such a way that MTT makes them free from paradoxes. The smallest of those languages for which MTT is formulated is $\mathcal{L}_U$, where U is the smallest consistent fixed point of G. It relates to that of the grounded sentences defined in [10, 16]. See also [7], where considerations are restricted to signed statements.

In spite of limitations the *languages $\mathcal{L}_U$ have the same formal and semantical properties as the base language L*. Being MA languages, L and $\mathcal{L}_U$ don't need to posses all the formal properties of first-order formal languages. In particular, the class of fully interpreted MA languages extends the class of languages for which theories of truth are usually formulated.

Acknowledgments: The author is indebted to Ph.d. Markus Pantsar for valuable discussions on the subject. The present work is influenced by [10].

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