ADMISSIBILITY IN A LOGICAL FRAMEWORK

JEFFREY HELZNER

Abstract. Theories of rational choice are often based on the assumption that the agent's choices are reducible to a preference ordering on the set of alternatives. This assumption is reflected in work from a variety of disciplines including economics [19], psychology [15], and statistics [32], as well as in philosophy where the "logic of preference" is an established topic of study [12]. However, a persistent minority of theorists have abandoned this fundamental assumption for a variety of reasons. The accounts that have emerged from this minority can be understood in terms of *admissibility*, a concept that does not in general reduce to preference. The purpose of this paper is to present a logic-based approach to admissibility.

1. Introduction

Many, perhaps most, accounts of rationality are based on the concept of *preference*. The various analyses of expectation, from the expected value theory that was the target of Bernoulli's St. Petersburg example to the subjective expected utility theory of Savage [32], assume a preference relation in the form of an ordering on the set of alternatives. In many ways this basic assumption has been the most resilient in the expected utility tradition, surviving in descriptive accounts such as Kahneman and Tversky's prospect theory [15] and normative theories such as those offered by Ellsberg [7, 8] and by Gardenfors and Sahlin [10]. These accounts retain an ordering on the set of alternatives while abandoning other aspects of the expected utility tradition in order to accommodate well-known challenges, e.g., the examples of Allais [3] and of Ellsberg [7, 8]. In contrast, some theorists have relaxed the requirement that choice is reducible to an ordering on the set of alternatives. The reasons for relaxing the indicated ordering requirement range from proposals for accommodating the aforementioned challenges of Allais and Ellsberg [24], compromises in group decision making [35], models of bounded rationality [31], and general arguments within the standard framework of choice functions [38]. We will return to Sen's arguments from [38] in a later section. For now, we recall some motivating work on indeterminacy, a topic that provides what is perhaps the most familiar motivation for abandoning the standard reduction to preference.

Credal indeterminacy, or uncertainty, is the most well-known form of uncertainty that is relevant to the ordering assumption of rational choice theory. While probability functions are the most familiar models of uncertainty, there is significant literature that documents objections to this use of probability functions. Early objections of this sort were expressed by Knight [18] in his discussions of *unmeasurable uncertainty* and Keynes [17] in his discussions concerning *weight of evidence*.

More recent, but influential, critiques relating to the use of probability functions as models of uncertainty include those by Kyburg in [20] and by Levi in [21]. [1] [2]

The limitations that have been identified in connection with the use of probability functions as models of uncertainty *simpliciter* are even more apparent in the context of decision making under uncertainty, i.e., cases of decision making in which the decision maker does not have access to an objective probability distribution over the relevant state space. According to *subjective expected utility* theory, which is still the received view concerning decision making under uncertainty, the rational agent who is confronted with such a situation has a credal state that can be represented by a probability function over the relevant state space and values that can be represented by a cardinal utility function over the relevant outcome space and, moreover, that such an agent is obligated to select an alternative that maximizes expectation with respect to the indicated probability distribution and cardinal utility function [32, 25].

Classic examples, such as those offered by Ellsberg in [7, 8], have motivated some people to reject subjective expected utility theory as the proper account of rational decision making under uncertainty and, in light of this rejection, to search for other normative accounts. Prominent among such accounts is the decision rule defended by Ellsberg in [7], the decision rule defended by Gardefors and Sahlin in [10], and the decision rule defended by Levi in [21]. All three of these rules use sets of probabilities to represent uncertainty. However, there are significant differences between these rules, and some reasons for preferring the rule defended by Levi to the other two are given in [24, 34, 33]. Such reasons aside, the decision rule defended by Levi might be viewed as coming with its own conceptual price in that it is not reducible to preference.

Given that other sources of indeterminacy – e.g., value conflict [23] and indeterminacy in the weighting of attributes [13] – lead to analogous challenges, and given the rather different motivations that are presented in [39], it seems reasonable to investigate departures from ordering in a more general setting. There have been several studies of this sort [2, 1, 26, 27, 28]. Set-valued choice functions provide a common foundation for these studies. If X is a set, then we write $\mathcal{P}(X)$ for the powerset of X and $\mathcal{P}_\omega(X)$ for the set of all finite subsets of X. $C : \mathcal{P}_\omega(X) \to \mathcal{P}_\omega(X)$ is a *choice function* on X just in case C is a function that satisfies $C(Y) \subseteq Y$ for all $Y \in \mathcal{P}_\omega(X)$. In addition, it is usually assumed that $C(Y)$ is nonempty whenever Y is nonempty. Empty domains have not attracted much attention in the standard literature on choice functions. However, the issue does have some significance within the context of the current investigation. We return to this point in Section 5.

[1] There are, of course, numerous psychological studies that are said to show that probability functions are not descriptively adequate with respect to judgments of uncertainty, but this work, some of which is extremely interesting, is not as immediately relevant in the present context as our concerns are restricted mainly to the judgments of rational agents.

[2] In the critiques cited above, both Kyburg and Levi identify the inability of probability functions to accommodate indeterminacy as being central to their shortcomings as models of uncertainty. There is more than one way to generalize the concept of a probability function in order to provide for indeterminacy. Kyburg is among those who favor "interval-valued" probabilities, i.e., certain functions that assign an interval of values in from $[0, 1]$, as opposed to a mere point in $[0, 1]$, to each event (or proposition). Levi is among those who favor using sets of probability functions to represent the rational agent's credal state. It should be noted that sets of probability functions provide a framework that is more general than the interval-valued probabilities approach [11].

Set-valued choice functions support a general notion of *admissibility*: $C(Y)$ is interpreted as the set of alternatives that the agent would judge as admissible if it were presented with menu Y. Note that we are taking admissibility as a primitive notion, much as preference is taken in many traditional studies of rational choice. It might be suggested that preference benefits from having an operational analysis, that the concept can be unpacked in terms of behavior. However, it should be cautioned that (1) any such account must rely on some non-trivial assumptions that relate preferences to observed behavior and (2) any such account will need to distinguish 'picking' from 'choosing', to borrow the terminology of [40]. Roughly, rationality determines a set of admissible alternatives among the available alternatives. Since the remaining set of admissible alternatives may contain more than one element, [3] it is not clear how to make an appropriate, behavioral distinction between choosing a from $\{a, b, c\}$ when a is uniquely admissible and choosing a from $\{a, b, c\}$ when a is admissible but not uniquely so. Although it is not central to our main concerns, as we are content to take admissibility as a primitive concept, it is worth noting that ideas concerning an operational analysis of admissibility, or at least approaches to eliciting admissibility judgments, have been considered [31, 36]. Perhaps the most immediate advantage to taking admissibility rather than preference as the fundamental notion is that it provides a framework in which accounts that relax ordering can be investigated in a neutral setting. Cases in which admissibility is reducible to preference can be characterized in terms of appropriate conditions on choice functions, e.g the following conditions from Sen [37, 19]:

- **α:** For all $Y, Z \in \mathcal{P}_\omega(X)$, if $x \in Y \subseteq Z$ and $x \in C(Z)$, then $x \in C(Y)$.
- **β:** For all $Y, Z \in \mathcal{P}_\omega(X)$, if $x, y \in C(Y)$, $Y \subseteq Z$, and $y \in C(Z)$, then $x \in C(Z)$.

These conditions are well-known to be jointly necessary and sufficient for reducing admissibility to an ordering.

Despite the number of formal studies examining choice in the absence of a complete preference ranking, we are not aware of any attempts to provide a logical analysis of admissibility (i.e., attempts to provide truth conditions for statements of the form 'x is admissible from the menu consisting of y_1,...,y_n'). The main purpose of the present work is to offer such an analysis. The basic idea of this account is as follows: Assume a propositional language L that has been augmented by the introduction of a unary operator $\Box$ and a variable-arity operator A. The intended reading of $\Box$ is that $\Box\phi$ asserts that ϕ is necessary. The intended reading of A is that $A(\phi \mid \psi_1, ..., \psi_n)$ asserts that ϕ is admissible in the menu consisting of $\psi_1, ..., \psi_n$. We assume that alternatives can be represented as L-formulas. Our objective is to present a semantics for L. In particular, we seek to provide truth conditions for formulas such as $A(\phi \mid \psi_1, ..., \psi_n)$. Roughly, the idea is that $A(\phi \mid \psi_1, ..., \psi_n)$ is true at state w just in case the alternative of moving to an otherwise unspecified state at which ϕ holds is admissible among the menu that, for $1 \leq i \leq n$, includes the alternative of moving to an otherwise unspecified state at which ψ_i holds; here the reference to admissibility is unpacked in terms of an appropriate set-valued choice function at w.

The rest of this paper is organized as follows: Section 2 and Section 3 present the relevant syntax and semantics, respectively. Formal systems are developed in

[3]This is certainly possible under standard theories like expected utility maximization.

Section 4, Section 5, and Section 6. Completeness results are presented in Section 7 and Section 8, respectively. Proofs of all results are in the appendices. In Section 9 we consider some directions for future work, including the possibility of addressing challenges of the sort raised by Sen in [38].

2. Syntax

Let Ω be a countable set of atoms. The *language* L (over Ω) is defined by the following inductive clauses:

Atoms: $\Omega \subseteq L$
Negation: If $\phi \in L$, then $\neg\phi \in L$
Conjunction: If $\phi, \psi \in L$, then $(\phi \wedge \psi) \in L$
Admissibility: If $\phi, \psi_1, ..., \psi_n \in L$, then $A(\phi \mid \psi_1, ..., \psi_n) \in L$
Necessity: If $\phi \in L$, then $\Box\phi \in L$

Only the fourth clause is nonstandard. The intended reading of $A(\phi \mid \psi_1, ..., \psi_n)$ is that ϕ is admissible in the menu consisting of $\psi_1, ..., \psi_n$. We return to these matters in more detail in the next section. As far as additional syntactic issues, the usual conventions regarding the omitting of parentheses will be observed and the other standard connectives will be taken as abbreviations in the usual manner, e.g., $\phi \vee \psi =_{df} \neg(\phi \wedge \psi)$, and we will take '$\Diamond$', the dual of '$\Box$', as an abbreviation for '$\neg\Box\neg$'. Also, we will adopt the familiar prefix notation for iterated conjunctions and disjunctions, i.e.,

$$\bigwedge_{i=1}^{n} \phi_i =_{df} \phi_1 \wedge ... \wedge \phi_n$$

and

$$\bigvee_{i=1}^{n} \phi_i =_{df} \phi_1 \vee ... \vee \phi_n$$

where n is a positive integer greater than 1. We extend this notation to the case where $n = 1$ by $\bigwedge_{i=1}^{n} \phi_i =_{df} \phi_1 =_{df} \bigvee_{i=1}^{n} \phi_i$.

3. Semantics

The main objective in this section is to present a semantics for languages of the sort that are described in Section 2. In particular, we present a semantics for formulas such as $A(\phi \mid \psi_1, ..., \psi_n)$. The basic idea noted above can be refined so that $A(\phi \mid \psi_1, ..., \psi_n)$ is true just in case the alternative of moving to an otherwise unspecified *and possible* state at which ϕ holds is admissible among the menu that, for $1 \leq i \leq n$, includes the alternative of moving to an otherwise unspecified and possible state at which ψ_i holds. That fact that the relevant notion of possibility may vary from state to state provides a way for us to distinguish the description of an alternative from its local denotation. [4]

[4]This is crucial to addressing the "epistemic value of the menu" problem in the manner that is discussed in Section 9.

3.1. **Frames.** A *frame* is a tuple $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}\rangle$ that satisfies the following requirement:

F1: W is a nonempty set
F2: R is a binary relation on W.
F3: $\mathcal{X}$ is a nonempty subset of $\mathcal{P}(W)$
F4: C_w is a choice function on $\mathcal{P}_\omega(\mathcal{X}_w)$ for all $w \in W$, where

$$\mathcal{X}_w = \{Y \mid Y \neq \emptyset \text{ and } Y = S_w \cap Z \text{ for some } Z \in \mathcal{X}\}$$

and where, for all $w \in W$, $S_w = \{x \mid (w, x) \in R\}$.
F5: $\mathcal{X}$ is closed under the following operations:
- $U \mapsto W - U$
- $(U, V) \mapsto U \cap V$
- $U \mapsto \{w \mid S_w \subseteq U\}$
- $(U, V_1, ..., V_n) \mapsto \{w \mid (S_w \cap U) \in C_w(\{S_w \cap V_1, ..., S_w \cap V_n\})\}$ [5]

W is a set of (logically) possible states. R encodes an agent-relative notion of possibility at w. In keeping with the informal sketch of the semantics that was given earlier, the notion of possibility encoded by R is not the usual notion of epistemic possibility. Whereas the traditional notion of epistemic possibility concerns the agent's beliefs about the current state while at w, the present notion concerns the agent's beliefs about possible successor states while at w. The two notions are not mutually exclusive. However, for simplicity of exposition, and since the traditional epistemic notion is well-known, we will restrict our use of agent-relative notions of possibility to the aforementioned one involving successor states. Continuing on with the components of a frame, $\mathcal{X}$ is a set of propositions and each element of $\mathcal{X}$ determines an alternative at w by intersecting with the set of possibilities at w. $\mathcal{X}_w$ is the resulting set of alternatives at w, and C_w is a choice function on that set of alternatives. The closure conditions in **F5** ensure that every formula is interpreted as a proposition in $\mathcal{X}$ (cf. Section 3.2).

3.2. **Interpretations.** An *interpretation* of L is a frame $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}\rangle$ along with a function π from Ω to $\mathcal{X}$. Given such an interpretation, π is extended to a function π^* on L according to the following inductive clauses:

Atoms: $\pi^*(\phi) = \pi(\phi)$
Negation: $\pi^*(\neg\phi) = W - \pi^*(\phi)$
Conjunction: $\pi^*(\phi \wedge \psi) = \pi^*(\phi) \cap \pi^*(\psi)$
Necessity: $\pi^*(\Box\phi) = \{w \mid S_w \subseteq \pi^*(\phi)\}$
Admissibility: $\pi^*(A(\phi \mid \psi_1, ..., \psi_n)) =$

$$\{w \mid (S_w \cap \pi^*(\phi)) \in C_w(\{S_w \cap \pi^*(\psi_1), ..., S_w \cap \pi^*(\psi_n)\})\}$$

Given an interpretation $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi\rangle$, we write $(\mathcal{I}, w) \models \phi$ just in case $w \in \pi^*(\phi)$ and write $\mathcal{I} \models \phi$ just in case $(\mathcal{I}, w) \models \phi$ for all $w \in W$.

[5]Note that $(U, V_1, ..., V_n) \subseteq \{w \mid S_w \cap U \neq \emptyset\} \cap \{w \mid S_w \cap V_i \neq \emptyset \text{ for all } i \in \{1, ..., n\}\}$, since $\mathcal{X}_w$ is the domain of C_w

4. Basic axiom schemes

$$\mathbf{K}: \ (\Box\phi \wedge \Box(\phi \rightarrow \psi)) \rightarrow \Box\psi$$

$$\mathbf{C1}: \ A(\psi_1 \mid \psi_2, ..., \psi_n) \rightarrow \bigwedge_{i=1}^{n} \Diamond\psi_i$$

$$\mathbf{C2}: \ \bigwedge_{i=1}^{n} \Diamond\psi_i \rightarrow \bigvee_{i=1}^{n} A(\psi_i \mid \psi_1, ..., \psi_n)$$

$$\mathbf{C3}: \ A(\phi \mid \psi_1, ..., \psi_n) \rightarrow \bigvee_{i=1}^{n} \Box(\phi \leftrightarrow \psi_i)$$

$$\mathbf{C4}: \ (\Box(\phi \leftrightarrow \phi') \wedge \bigwedge_{i=1}^{n} \Box(\psi_i \leftrightarrow \psi_i') \wedge A(\phi \mid \psi_1, ..., \psi_n)) \rightarrow A(\phi' \mid \psi_1', ..., \psi_n')$$

K is the familiar "distribution" axiom of relational semantics. **C1** requires that admissibility judgments are made with respect to alternatives that are possible. **C2** requires that a menu of possible alternatives has at least one admissible alternative. **C3** requires that every admissible alternative is an available alternative. **C4** requires that admissibility judgments are extensional.

Proposition 4.1. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi \rangle$ be an interpretation of L. If θ is an instance of **K, C1, C2, C3** or **C4**, then $\mathcal{I} \models \theta$.*

5. Additional axiom schemes

5.1. Let p be an atom of L, i.e., $p \in \Omega$. We introduce the following 0-ary connective as an abbreviation: $\top =_{df} p \vee \neg p$. The following axiom is mentioned in [5].

$$\mathbf{P}: \ \Diamond\top$$

Recall from Section 1 that we do not rule out having an empty set of alternatives. If we were to do so, then this would amount to requiring that S_w is nonempty, which would amount to requiring that there is something that is possible at w in the relevant sense of possibility considered above; we see no reason to rule out situations that, at least from the agent's perspective, appear to be the "end of the line". **P** guarantees that S_w is nonempty, and, moreover, it is satisfied at w if S_w is nonempty.

Proposition 5.1. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi \rangle$ be an interpretation of L. If $w \in W$, then $w \models \Diamond\top$ iff $S_w \neq \emptyset$.*

5.2. The following two axiom schemes are intended as counterparts to Sen's α and β conditions as presented in Section 1:

$$\mathbf{C}_\alpha: \ (A(\phi \mid \psi_1, ..., \psi_m, \theta_1, ..., \theta_n) \wedge \bigvee_{i=1}^{m} \Box(\phi \leftrightarrow \psi_i)) \rightarrow A(\phi \mid \psi_1, ..., \psi_m)$$

$$\mathbf{C}_\beta: \ (A(\phi_1 \mid \psi_1, ..., \psi_m) \wedge A(\phi_2 \mid \psi_1, ..., \psi_m) \wedge A(\phi_1 \mid \psi_1, ..., \psi_m, \theta_1, ..., \theta_n)) \rightarrow A(\phi_2 \mid \psi_1, ..., \psi_m, \theta_1, ..., \theta_n))$$

The following two propositions state that these axioms schemes are satisfied when the C_w satisfies the corresponding condition on choice functions:

Proposition 5.2. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi \rangle$ be an interpretation of L. If $w \in W$ and ϕ is an instance of $\boldsymbol{C}_\alpha$ and C_w satisfies α, then $w \models \phi$.*

Proposition 5.3. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi \rangle$ be an interpretation of L. If $w \in W$ and ϕ is an instance of $\boldsymbol{C}_\beta$ and C_w satisfies β, then $w \models \phi$.*

6. Systems

A *system* consists of a collection of *axioms* and a set of *inference rules*. $\mathcal{S}$ is a system over L just in case (1) every axiom of $\mathcal{S}$ is in L and (2) L is closed under every inference rule of $\mathcal{S}$. In what follows, every system that is considered is assumed to be a system over L. If $\mathcal{S}$ is a system, then a *derivation* in S is a sequence of formulas $\phi_1, ..., \phi_n$ such that each element of the sequence is either an axiom of $\mathcal{S}$ or follows from earlier elements of the sequence by an application of one of the inference rules for $\mathcal{S}$. ϕ is *derivable* in $\mathcal{S}$ iff there is a derivation in $\mathcal{S}$ that ends with ϕ. We will write $\mathcal{S} \vdash \phi$ just in case ϕ is derivable in $\mathcal{S}$, and we will write $\mathcal{S} \nvdash \phi$ if ϕ is not derivable in $\mathcal{S}$. Finally, if $\mathcal{S}$ and $\mathcal{S}'$ are systems, then $\mathcal{S}$ *extends* $\mathcal{S}'$ just in case every axiom of $\mathcal{S}'$ is an axiom of $\mathcal{S}$ and every inference rule of $\mathcal{S}'$ is an inference rule of $\mathcal{S}$.

Let $\mathcal{L}$ be the system that has all tautologies in P_L (see Appendix B) as axioms and has the following inference rule:

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi}\ \text{MP}$$

Let $\mathcal{C}$ be the system that extends $\mathcal{L}$ by adding all instances of **K**, **C1**, **C2**, **C3**, and **C4** as axioms and adds the following inference rule:

$$\frac{\phi}{\Box\phi}\ \text{Gen}$$

Let $\mathcal{C}_\alpha$ be the system that extends $\mathcal{C}$ by adding all instances $\mathbf{C}_\alpha$. Let $\mathcal{C}_\beta$ be the system that extends $\mathcal{C}$ by adding all instances $\mathbf{C}_\beta$. Let $\mathcal{C}_{\alpha,\beta}$ be the system that extends $\mathcal{C}$ by adding all instances $\mathbf{C}_\alpha$ and all instances of $\mathbf{C}_\beta$. Finally, if $\mathcal{S}$ is a system, then let $\mathcal{S}^+$ be the system that extends $\mathcal{S}$ by adding **P**.

Proposition 6.1 (Soundness)**.** *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi \rangle$ be an interpretation of L. The following soundness results hold: (1) System $\mathcal{C}$ is sound with respect to $\mathcal{I}$. (2) System $\mathcal{C}^+$ is sound with respect to $\mathcal{I}$ if $S_w \neq \emptyset$ for all $w \in W$. (3) System $\mathcal{C}_\alpha$ is sound with respect to $\mathcal{I}$ if C_w satisfies α for all $w \in W$. (4) System $\mathcal{C}_\alpha^+$ is sound with respect to $\mathcal{I}$ if C_w satisfies α and $S_w \neq \emptyset$ for all $w \in W$. (5) System $\mathcal{C}_\beta$ is sound with respect to $\mathcal{I}$ if C_w satisfies β for all $w \in W$. (6) System $\mathcal{C}_\beta^+$ is sound with respect to $\mathcal{I}$ if C_w satisfies β and $S_w \neq \emptyset$ for all $w \in W$. (7) System $\mathcal{C}_{\alpha,\beta}$ is sound with respect to $\mathcal{I}$ if C_w satisfies α and β for all $w \in W$. (8) System $\mathcal{C}_{\alpha,\beta}^+$ is* *sound with respect to $\mathcal{I}$ if C_w satisfies α and β for all $w \in W$ and $S_w \neq \emptyset$ for all $w \in W$.*

7. The canonical frame

If $\mathcal{S}$ extends $\mathcal{C}$ and $\phi \in L$, then let $[\phi]_{\mathcal{S}}$ denote the set of $\Gamma \subseteq L$ such that Γ is maximally consistent with respect to $\mathcal{S}$ and $\phi \in \Gamma$. Where there is no risk of confusion we will suppress the subscript and write $[\phi]$ rather than $[\phi]_{\mathcal{S}}$. Let $\mathcal{S}$ be a system that extends $\mathcal{C}$. The *canonical frame* for $\mathcal{S}$, which we denote as $\mathcal{I}_{\mathcal{S}}$, is defined as follows:

- W is the set of all $\Gamma \subseteq L$ such that Γ is maximally consistent with respect to $\mathcal{S}$
- $(\Gamma, \Gamma') \in R$ just in case $\{\phi \mid \Box\phi \in \Gamma\} \subseteq \Gamma'$
- $Y \in \mathcal{X}$ iff there is a ϕ such that $Y = [\phi]$
- $U \in C_\Gamma(\{V_1, ..., V_n\})$ iff the following conditions are satisfied:
  (1) $U, V_1, ..., V_n \in \mathcal{X}_\Gamma$,
  (2) $U = V_i$ for some $i \in \{1, ..., n\}$,
  (3) if $\phi, \psi_1, ..., \psi_n$ are L formulas such that $U = S_\Gamma \cap [\phi]$ and $V_i = S_\Gamma \cap [\psi_i]$ for all $i \in \{1, ..., n\}$, then $A(\phi \mid \psi_1, ..., \psi_n) \in \Gamma$.

8. Completeness

Let $\mathcal{S}$ be a system that extends $\mathcal{C}$. Let $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}\rangle$ be the canonical frame for system $\mathcal{S}$. Define $\pi : \Omega \to \mathcal{X}$ by $\pi(p) = \{\Gamma \in W \mid p \in \Gamma\}$. We will refer to $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi\rangle$ as the *canonical interpretation* for $\mathcal{S}$.

Proposition 8.1. *Let $\mathcal{I}_{\mathcal{S}}$ be the canonical interpretation for $\mathcal{S}$. If $\mathcal{I}_{\mathcal{S}} \models \phi$, then $\mathcal{S} \vdash \phi$.*

Corollary 8.2 (Basic completeness)**.** *If $\mathbf{M}$ is a class of interpretations that includes $\mathcal{I}_{\mathcal{S}}$ and $\mathcal{I} \models \phi$ for all $\mathcal{I} \in \mathbf{M}$, then $\mathcal{S} \vdash \phi$.*

Proposition 5.1 (in conjunction with Proposition H.1) states that if $\mathcal{S}$ includes $\mathbf{P}$, then $S_w \neq \emptyset$ for all w in $\mathcal{I}_{\mathcal{S}}$. Similarly, the following propositions state that if $\mathcal{S}$ extends $\mathcal{C}_\alpha$ (resp. $\mathcal{C}_\beta$), then C_w satisfies α (resp. β) for all w in $\mathcal{I}_{\mathcal{S}}$.

Proposition 8.3. *Let $\mathcal{S}$ be a system that extends $\mathcal{C}_\alpha$. Let $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}\rangle$ be the canonical frame for $\mathcal{S}$. C_w satisfies α for all $w \in W$.*

Proposition 8.4. *Let $\mathcal{S}$ be a system that extends $\mathcal{C}_\beta$. Let $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}\rangle$ be the canonical frame for $\mathcal{S}$. C_w satisfies β for all $w \in W$.*

In light of these observations we have the following completeness results.

Proposition 8.5 (Completeness)**.** *(1) System $\mathcal{C}$ is complete with respect to the class of all interpretations. (2) System $\mathcal{C}^+$ is complete with respect to the class of all interpretations such that $S_w \neq \emptyset$ for all $w \in W$. (3) System $\mathcal{C}_\alpha$ is complete with respect to the class of all interpretations such that C_w satisfies α for all $w \in W$. (4) System $\mathcal{C}_\alpha^+$ is complete with respect to the class of all interpretations such that C_w satisfies α and $S_w \neq \emptyset$ for all $w \in W$. (5) System $\mathcal{C}_\beta$ is complete with respect to the class of all interpretations such that C_w satisfies β for all $w \in W$. (6) System $\mathcal{C}_\beta^+$ is complete with respect to the class of all interpretations such that C_w satisfies β*

and $S_w \neq \emptyset$ for all $w \in W$. (7) System $\mathcal{C}_{\alpha,\beta}$ is complete with respect to the class of all interpretations such that C_w satisfies α and β for all $w \in W$. (8) System $\mathcal{C}^+_{\alpha,\beta}$ is complete with respect to the class of all interpretations such that C_w satisfies α and β for all $w \in W$ and $S_w \neq \emptyset$ for all $w \in W$.

9. Discussion

Conditions such as α and β are instances of what Amartya Sen calls *axioms of internal consistency of choice* [38]. As Sen explains, such conditions are "'internal' to the choice function in the sense that they require correspondence between different parts of a choice function, without invoking anything *outside* choice (such as motivations, objectives, and substantive principles)."(495) While acknowledging the widespread use of internal consistency conditions in a variety of fields, Sen, in his well-known Presidential Address of the Econometric Society in 1984, argues that internal consistency is "essentially confused". [6] [7] At the heart of Sen's critique of internal consistency are three types of examples, the second of which, according to Sen, concerns the "epistemic value of the menu". This sort of example has received attention in theoretical economics [4, 16] as well as in formal epistemology [30, 6]. The following is an instance of such an example from [38]:

> What is offered for choice can give us information about the underlying situation, and can thus influence our preference over the alternatives, as we see them. For example, the chooser may learn something about the person offering the choice on the basis of what he or she is offering. To illustrate, given the choice between having tea at a distant acquaintance's home (x), and not going there (y), a person who chooses to have tea (x), may nevertheless choose to go away (y), if offered – by that acquaintance – a choice over having tea (x), going away (y), and having some cocaine (z). The menu offered may provide information about the situation-in this case say something about the distant acquaintance, and this can quite reasonably affect the ranking of the alternatives x and y, and yield the pair of choices [that violate α].(502)

The alleged violation of α in the given example is clear: the agent judges x to be uniquely admissible from the menu $\{x, y\}$ but judges y to be uniquely admissible from the menu $\{x, y, z\}$. If we want to count these judgments as rational, then it seems we have to abandon α as a basic principle of rationality – here we are putting aside other arguments for abandoning α, in particular the arguments from indeterminacy that were discussed earlier. One response, which Sen anticipates, suggests that the x from $\{x, y\}$ is not the one that is offered in $\{x, y, z\}$:

> It is, of course, true that the chooser has different information even about x (i.e., having tea with the acquaintance) when the acquaintance gives him the choice of having cocaine with him, and it can certainly be argued that in the "intentional" (as opposed to "extensional") sense the alternative x is no longer the same. But an "intentional" definition of alternatives would be, in general,

[6] Sen mentions that "Internal consistency of choice has been a central concept in demand theory, social choice theory, decision theory, behavioral economics, and related fields."(495)

[7] Sen's Presidential Address was published as [38].

> quite hopeless in invoking inter-menu consistency, especially when (as in this case) the intentional characterization changes precisely with the alternatives available for choice (i.e. with the menus offered).(502)

Sen's point is easy enough to appreciate. However, if inter-menu consistency conditions are translated into the object language that was presented in Section 2, then the framework that has been presented seems to provide a way to preserve the rationality of the choices in Sen's example without trivializing inter-menu consistency. Let us consider Sen's example more closely. It is suggested that the example demonstrates a rational violation of α. Recall that the judgments of admissibility which are to be constrained by α are purely synchronic. Of course such judgments may change over time as a result of changes to antecedent states such as the agent's relevant beliefs or desires – e.g., consider how a subjective expected utility maximizer's judgments of admissibility change as a result of changes in that agent's subjective probability or cardinal utility functions – but the judgments encoded by a set-valued choice functions are taken to be suppositional.

In Sen's example it is implicit that the agent regards the alternative of having tea at the distant acquaintance's home as uniquely admissible from the menu that consists of (x) the alternative of having tea at the distant acquaintance's house and (y) the alternative of being away from the acquaintance. Let ϕ assert that the agent has tea at the distant acquaintance's home. Let ψ assert that the agent is away from the acquaintance. Letting w denote the current state, and working within the proposed framework, the example suggests that $w \models \neg A(\psi \mid \phi, \psi)$. Similarly, Sen's remarks concerning the three-way choice suggest that $w \models A(\psi \mid \phi, \psi, \theta)$, where θ asserts that the agent has cocaine with the the acquaintance. According to the relevant semantic clause in Section 3 there must be a $w' \in W$ such that wRw' and $w' \models \theta$. That is, since θ is a genuine alternative for the agent at w, it is required that at w the agent regards some θ state as a possible successor. The agent's recognition of such possibilities when considering the two-way choice would seem to undermine the plausibility of the rational agent judging ϕ as admissible in the pair and, as a result, renders doubtful claim that $w \models \neg A(\psi \mid \phi, \psi)$. In general, allowing for the "epistemic value of the menu" requires a framework that allows for changes of epistemic state, something that exceeds the standard interpretation of set-valued choice functions. The proposed framework, which uses an indexed set of choice functions in the semantic account, does allow for such changes. Let w_0 be a state in which the agent does not regard any θ states as possible successors. Let w_1 be a state in which the agent does regard some θ states as possible successors. According to the semantics, w_0 will not satisfy $A(\psi \mid \phi, \psi, \theta)$, since θ will not even pick out an alternative at w_0. Moreover, interpreting the example to imply that $w_0 \models \neg A(\psi \mid \psi, \phi)$ and $w_1 \models A(\psi \mid \phi, \psi, \theta)$ is obviously compatible with C_{w_0} and C_{w_1} both satisfying α. Such a story is compatible with a frame that satisfies $\mathbf{C}_\alpha$, which, in light of the completeness results presented presented above, is a non-trivial condition on frames.

Finally, another direction for future work, apart from further investigation into the examples raised by Sen, concerns the interaction between the admissibility and traditional epistemic modalities. As noted in Section 3.1, the accessibility relation R used in the semantic frames above is *not* given the standard epistemic interpretation. While we have avoided the introduction of a second accessibility relation in the present work in order to focus on the semantics of admissibility, we do plan to introduce a belief operator in future work. Combining the present semantics of admissibility with a belief operator would seem to offer a novel way of pursuing the program of taking belief change to be a species of rational choice [22, 29]. With the present framework augmented by the addition of a belief operator B, the admissibility of $B\phi_i$ in menu $\{B\phi_1, ..., B\phi_n\}$ would be represented by the truth of $A(B\phi_i \mid B\phi_1, ..., B\phi_n)$. This approach to a rational-choice gloss on belief change seems natural enough in the present framework and worthy of further investigation. Rott has remarked that Sen's examples concerning the epistemic value of the menu are even more problematic within the context of belief change [30]. It is reasonable to suspect that something like the above analysis of Sen's example might extend to the context of belief revision through the suggested representation of theoretical choice problems.

Technical Appendices

Appendix A. Results from Section 3.2

Proposition A.1. *$\pi^*(\phi) \in \mathcal{X}$ for all $\phi \in L$.*

Proof. Straightforward induction on L. □

Appendix B. Inclusion of propositional logic

Let Ω_L be the set consisting of Ω and all formulas in L of the form $A(\phi \mid \psi_1, ..., \psi_n)$ and all formulas in L of the form $\Box\phi$. Let P_L be the propositional language over Ω_L. That is, P_L is defined inductively by the following clauses:

Atoms: $\Omega_L \subseteq P_L$
Negation: If $\phi \in P_L$, then $\neg\phi \in P_L$
Conjunction: If $\phi, \psi \in L$, then $(\phi \wedge \psi) \in P_L$

Proposition B.1. $P_L \subseteq L$

Proof. Trivial proof by induction on the structure of P_L. □

Proposition B.2. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w\in W}, \pi\rangle$ be an interpretation of L. Define the function $v_w : \Omega_L \to \{T, F\}$ by $v_w(\phi) = T$ iff $(\mathcal{I}, w) \models \phi$. Let $v^*_w : P_L \to \{T, F\}$ be the canonical extension of v_w to all of P_L. For all $\phi \in P_L$, $v^*_w(\phi) = T$ iff $(\mathcal{I}, w) \models \phi$.*

Proof. Trivial proof by induction on the structure of P_L. □

Corollary B.3. *If θ is a tautology in P_L and $\mathcal{I} = \langle W, \mathcal{X}, C, \pi\rangle$ is an interpretation of L, then $\mathcal{I} \models \theta$.*

Proof. Suppose that θ is a tautology in P_L. Hence, $v^*_w(\theta) = T$ for all $w \in W$. Thus, by Proposition B.2, it follows that $(\mathcal{I}, w) \models \theta$ for all $w \in W$. Hence, $\mathcal{I} \models \theta$. □

Appendix C. Results from Section 4

Proof of Proposition 4.1. The result follows from the lemmas in this section. □

Lemma C.1. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi\rangle$ be an interpretation of L. If θ is an instance of **K**, then $\mathcal{I} \models \theta$.*

Proof. Standard. Suppose that $w \models \Box\phi \wedge \Box(\phi \rightarrow \psi)$. Hence, $w \models \Box\phi$ and $w \models \Box(\phi \rightarrow \psi)$. Suppose that $v \in S_w$. It follows that $v \models \phi \rightarrow \psi$. Thus, if $v \models \phi$, then $v \models \psi$. Since $v \in S_w$, it follows that $v \models \phi$. Hence, $v \models \psi$. Thus, $v \models \psi$ for all $v \in S_w$, and we have shown that $w \models \Box\psi$. □

Lemma C.2. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi\rangle$ be an interpretation of L. If $w \in W$ and $\phi \in L$, then $S_w \cap \pi^*(\phi) \neq \emptyset$ iff $w \in \pi^*(\Diamond\phi)$.*

Proof. The result follows easily from the fact that $\Diamond\phi =_{df} \neg\Box\neg\phi$. □

Lemma C.3. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi\rangle$ be an interpretation of L. If θ is an instance of **C1**, then $\mathcal{I} \models \theta$.*

Proof. Suppose that $w \models A(\psi_1 \mid \psi_2, ..., \psi_n)$. From the F4 condition on frames and the admissibility clause for interpretations it follows that $S_w \cap \pi^*(\psi_i) \neq \emptyset$ for all $i \in \{1, ..., n\}$. Thus, by Lemma C.2, $w \models \Diamond\psi_i$ for all $i \in \{1, ..., n\}$, and it follows that $w \models \bigwedge_{i=1}^n \Diamond\psi_i$. □

Lemma C.4. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi\rangle$ be an interpretation of L. If θ is an instance of **C2**, then $\mathcal{I} \models \theta$.*

Proof. Suppose that $w \models \bigwedge_{i=1}^n \Diamond\psi_i$. Hence, $w \models \Diamond\psi_i$ for all $i \in \{1, ..., n\}$. From Lemma C.2, it follows that $S_w \cap \pi^*(\psi_i) \neq \emptyset$ for all $i \in \{1, ..., n\}$. Hence, by Proposition A.1, $S_w \cap \pi^*(\psi_i) \in \mathcal{X}_w$ for all $i \in \{1, ..., n\}$. Since C_w is a choice function, there is a $j \in \{1, ..., n\}$ such that $S_w \cap \pi^*(\psi_j)$ is an element of $C_w(\{S_w \cap \pi^*(\psi_1), ..., S_w \cap \pi^*(\psi_n)\})$. Hence, there is a $j \in \{1, ..., n\}$ such that $w \models A(\psi_j \mid \psi_1, ..., \psi_n)$. It follows that $w \models \bigvee_{i=1}^n A(\psi_i \mid \psi_1, ..., \psi_n)$. □

Lemma C.5. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi\rangle$ be an interpretation of L. If $w \in W$ and $\phi, \psi \in L$, then $w \in \pi^*(\Box(\phi \leftrightarrow \psi))$ iff $S_w \cap \pi^*(\phi) = S_w \cap \pi^*(\psi)$.*

Proof. The result follows easily from the fact that $\phi \leftrightarrow \psi =_{df} (\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$. □

Lemma C.6. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi\rangle$ be an interpretation of L. If θ is an instance of **C3**, then $\mathcal{I} \models \theta$.*

Proof. Suppose that $w \models A(\phi \mid \psi_1, ..., \psi_n)$. It follows that $S_w \cap \pi^*(\phi) \in C_w(S_w \cap \psi_1, ..., S_w \cap \psi_n)$. Thus, $S_w \cap \pi^*(\phi) = S_w \cap \pi^*(\psi_j)$ for some $j \in \{1, ..., n\}$. Hence, from Lemma C.5, $w \models \Box(\phi \leftrightarrow \psi_j)$ for some $j \in \{1, ..., n\}$. It follows that $w \models \bigvee_{i=1}^n \Box(\phi \leftrightarrow \psi_i)$. □

Lemma C.7. *Let $\mathcal{I} = \langle W, R, \mathcal{X}, \{C_w\}_{w \in W}, \pi\rangle$ be an interpretation of L. If θ is an instance of **C4**, then $\mathcal{I} \models \theta$.*

Proof. Suppose that $w \models \Box(\phi \leftrightarrow \phi') \wedge \bigwedge_{i=1}^n \Box(\psi_i \leftrightarrow \psi_i') \wedge A(\phi \mid \psi_1, ..., \psi_n)$. Hence, $w \models \Box(\phi \leftrightarrow \phi')$, $w \models \bigwedge_{i=1}^n \Box(\psi_i \leftrightarrow \psi_i')$, and $w \models A(\phi \mid \psi_1, ..., \psi_n)$. From the last of these it follows that $S_w \cap \pi^*(\phi) \neq \emptyset$, $S_w \cap \pi^*(\psi_i) \neq \emptyset$ for all $i \in \{1, ..., n\}$, and $S_w \cap \pi^*(\phi)$ is in $C_w(S_w \cap \pi^*(\psi_1), ..., S_w \cap \pi^*(\psi_n))$. Since $w \models \Box(\phi \leftrightarrow \phi')$, it follows from Lemma C.5 that $S_w \cap \pi^*(\phi) = S_w \cap \pi^*(\phi')$. Likewise, since $w \models \Box(\psi_i \leftrightarrow \psi_i')$, it follows from Lemma C.5 that $S_w \cap \pi^*(\psi_i) = S_w \cap \pi^*(\psi_i')$ for all $i \in \{1, ..., n\}$. Thus, $S_w \cap \pi^*(\phi') \neq \emptyset$, $S_w \cap \pi^*(\psi_i') \neq \emptyset$ for all $i \in \{1, ..., n\}$, and $S_w \cap \pi^*(\phi')$ is in $C_w(S_w \cap \pi^*(\psi_1'), ..., S_w \cap \pi^*(\psi_n'))$. It follows that $w \models A(\phi' \mid \psi_1', ..., \psi_n')$. □

Appendix D. Results from Section 5

Proof of Proposition 5.1. Trivial □

Proof of Proposition 5.2. Assume that $w \in W$ and ϕ is an instance of $\mathbf{C}_\alpha$ and C_w satisfies α. Suppose that $w \models A(\phi \mid \psi_1, ..., \psi_m, \theta_1, ..., \theta_n) \wedge \bigvee_{i=1}^m \Box(\phi \leftrightarrow \psi_i)$. Thus, (1) $w \models A(\phi \mid \psi_1, ..., \psi_m, \theta_1, ..., \theta_n)$ and (2) $w \models \bigvee_{i=1}^m \Box(\phi \leftrightarrow \psi_i)$. From (1) it follows that $S_w \cap \pi^*(\phi)$ is in $C_w(\{S_w \cap \pi^*(\psi_1), ..., S_w \cap \pi^*(\psi_m), S_w \cap \pi^*(\theta_1), ..., S_w \cap \pi^*(\theta_n)\})$. From (2) it follows that $w \models \Box(\phi \leftrightarrow \psi_i)$ for some $i \in \{1, ..., m\}$. Thus, from Lemma C.5, it follows that $S_w \cap \pi^*(\phi) = S_w \cap \pi^*(\psi_i)$ for some $i \in \{1, ..., m\}$. Thus, since C_w satisfies α, it follows that $S_w \cap \pi^*(\phi)$ is in $C_w(\{S_w \cap \pi^*(\psi_1), ..., S_w \cap \pi^*(\psi_m)\})$. Finally, from (1), $S_w \cap \pi^*(\phi) \neq \emptyset$ and $S_w \cap \pi^*(\psi_i) \neq \emptyset$ for all $i \in \{1, ..., m\}$. Hence, $w \models A(\phi \mid \psi_1, ..., \psi_m)$. □

Proof of 5.3. Assume that $w \in W$ and ϕ is an instance of $\mathbf{C}_\beta$ and C_w satisfies β. Suppose that $w \models A(\phi_1 \mid \psi_1, ..., \psi_m) \wedge A(\phi_2 \mid \psi_1, ..., \psi_m) \wedge A(\phi_1 \mid \psi_1, ..., \psi_m, \theta_1, ..., \theta_n)$. Thus, (1) $w \models A(\phi_1 \mid \psi_1, ..., \psi_m)$, (2) $w \models A(\phi_2 \mid \psi_1, ..., \psi_m)$, and (3) $w \models A(\phi_1 \mid \psi_1, ..., \psi_m, \theta_1, ..., \theta_n)$. From (1) it follows that $S_w \cap \pi^*(\phi_1)$ is in $C_w(\{S_w \cap \pi^*(\psi_1), ..., S_w \cap \pi^*(\psi_n)\})$. From (2) it follows that $S_w \cap \pi^*(\phi_2)$ is in $C_w(\{S_w \cap \pi^*(\psi_1), ..., S_w \cap \pi^*(\psi_n)\})$. From (3) it follows that $S_w \cap \pi^*(\phi_1)$ is in $C_w(\{S_w \cap \pi^*(\psi_1), ..., S_w \cap \pi^*(\psi_m), S_w \cap \pi^*(\theta_1), ..., S_w \cap \pi^*(\theta_n)\})$. Hence, since C_w satisfies β, it follows that $S_w \cap \pi^*(\phi_2)$ is in $C_w(\{S_w \cap \pi^*(\psi_1), ..., S_w \cap \pi^*(\psi_m), S_w \cap \pi^*(\theta_1), ..., S_w \cap \pi^*(\theta_n)\})$. Finally, from (2), $S_w \cap \pi^*(\phi_2) \neq \emptyset$ and $\pi^*(\psi_i) \neq \emptyset$ for all $i \in \{1, ..., m\}$, and, from (3), $S_w \cap \pi^*(\theta_i) \neq \emptyset$ for all $i \in \{1, ..., n\}$. Hence, $w \models A(\phi_2 \mid \psi_1, ..., \psi_m, \theta_1, ..., \theta_n)$. □

Appendix E. Results from Section 6

Proof of Proposition 6.1. Straightforward inductive proof using the soundness of the two inference rules and the soundness of the various axioms, as established in Propositions 4.1, 5.1, 5.2, and 5.3. □

Appendix F. Maximally consistent sets

If $\mathcal{S}$ is a system and $\phi \in L$, then ϕ is *consistent* (with respect to $\mathcal{S}$) just in case $\mathcal{S} \nvdash \neg\phi$. A finite set $\{\phi_1, ..., \phi_n\} \subseteq L$ is consistent iff $\bigwedge_{i=1}^n \phi_i$ is consistent. An infinite set $\Gamma \subseteq L$ is consistent just in case every finite subset of Γ is consistent. We will say that a formula (or set of formulas) is *inconsistent* iff it is not consistent. A set $\Gamma \subseteq L$ is *maximally consistent* just in case (1) it is consistent and (2) $\Gamma \cup \{\phi\}$ is inconsistent for all $\phi \in L - \Gamma$. The following two lemmas are standard results that apply to all systems in which classical, propositional reasoning can be conducted. Proofs are readily available, e.g. Lemma 1.4.3 in [14] or Lemma 3.1.2 in [9].

Lemma F.1. *If $\mathcal{S}$ is a system that extends $\mathcal{L}$ and Γ is consistent with respect to $\mathcal{S}$, then there is a set Γ' such that $\Gamma \subseteq \Gamma'$ and Γ' is maximally consistent.*

Lemma F.2. *If $\mathcal{S}$ is a system that extends $\mathcal{L}$ and Γ is maximally consistent with respect to $\mathcal{S}$, then the following hold for all $\phi, \psi \in L$:*

- *$\phi \in \Gamma$ or $\neg\phi \in \Gamma$;*

- *$\phi \wedge \psi \in \Gamma$ iff $\phi \in \Gamma$ and $\psi \in \Gamma$;*
- *if $\phi \in \Gamma$ and $\phi \to \psi \in \Gamma$, then $\psi \in \Gamma$;*
- *if $\mathcal{S} \vdash \phi$, then $\phi \in \Gamma$.*

Corollary F.3. *If $\mathcal{S}$ is a system that extends $\mathcal{L}$ and Γ is maximally consistent with respect to $\mathcal{S}$, then $\bigvee_{i=1}^{n} \theta_i \in \Gamma$ just in case $\theta_i \in \Gamma$ for some $i \in \{1, ..., n\}$.*

Appendix G. Results from Section 7

Note that it is not immediate that $\mathcal{I}_{\mathcal{S}}$ satisfies the frame conditions. In particular, it is not clear that **F4** and **F5** are satisfied. We now turn our attention to this issue and show that $\mathcal{I}_{\mathcal{S}}$ does indeed satisfy the frame conditions.

Lemma G.1. *Let $\mathcal{S}$ be a system that extends $\mathcal{C}$. Let $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}\rangle$ be the canonical frame for $\mathcal{S}$. If $S_\Gamma \subseteq [\phi]$, then $\Box\phi \in \Gamma$.*

Proof. P.21 in van der Hoek [14]. Note that this uses **Gen** and the **K** schema. □

Lemma G.2. *Let $\mathcal{S}$ be a system that extends $\mathcal{C}$. Let $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}\rangle$ be the canonical frame for $\mathcal{S}$. For all $\Gamma \in W$, $S_\Gamma \cap [\phi] \neq \emptyset$ iff $\Diamond\phi \in \Gamma$.*

Proof. Suppose that $S_\Gamma \cap [\phi] \neq \emptyset$. Hence, there is a $\Gamma' \in S_\Gamma$ such that $\phi \in \Gamma'$. If $\Diamond\phi \notin \Gamma$, then it follows from Lemma F.2 that $\Box\neg\phi \in \Gamma$. Hence, $\neg\phi \in \Gamma'$, contradicting the assumption that Γ' is consistent. Suppose that $\Diamond\phi \in \Gamma$ but that there is no $\Gamma' \in S_\Gamma$ such that $\phi \in \Gamma'$. It follows that $\neg\phi \in \Gamma'$ for all $\Gamma' \in S_\Gamma$. Hence, by Lemma G.1, $\Box\neg\phi \in \Gamma$, contradicting the assumption Γ is consistent. □

Lemma G.3. *Let $\mathcal{S}$ be a system that extends $\mathcal{C}$. Let $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}\rangle$ be the canonical frame for $\mathcal{S}$. Assume that $\Gamma \in W$. It follows that $\Box(\phi \leftrightarrow \psi) \in \Gamma$ iff $S_\Gamma \cap [\phi] = S_\Gamma \cap [\psi]$.*

Proof. Assume that $\Gamma \in W$ and $\Box(\phi \leftrightarrow \psi) \in \Gamma$. It follows that $\phi \leftrightarrow \psi \in \Gamma'$ for all $\Gamma' \in S_\Gamma$. Hence, $(\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi) \in \Gamma'$ for all $\Gamma' \in S_\Gamma$. Thus, by Corollary F.3, if $\Gamma' \in S_\Gamma$, then $\phi \wedge \psi \in \Gamma'$ or $\neg\phi \wedge \neg\psi \in \Gamma'$. From Lemma F.2, it follows that for all $\Gamma' \in S_\Gamma$, either $\phi, \psi \in \Gamma$ of $\phi, \psi \notin \Gamma$. The desired conclusion follows immediately. Assume that $\Gamma \in W$ and $S_\Gamma \cap [\phi] = S_\Gamma \cap [\psi]$. It follows that, for all $\Gamma' \in S_\Gamma$, either $\phi, \psi \in \Gamma'$ or $\neg\phi, \neg\psi \in \Gamma'$. Hence, by classical, propositional reasoning, if $\Gamma' \in S_\Gamma$, then $(\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi) \in \Gamma'$. Thus, $\phi \leftrightarrow \psi \in \Gamma'$ for all $\Gamma' \in S_\Gamma$. From Lemma G.1, it follows that $\Box(\phi \leftrightarrow \psi) \in \Gamma$. □

Proposition G.4. *Let $\mathcal{S}$ be a system that extends $\mathcal{C}$. Let $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}\rangle$ be the canonical frame for $\mathcal{S}$. C_Γ is a choice function on $\mathcal{P}_\omega(\mathcal{X}_\Gamma)$ for each $\Gamma \in W$.*

Proof. We must show that if $V_1, ..., V_n \in \mathcal{X}_\Gamma$, where $n \geq 1$, then there is a U such that $U \in C_\Gamma(\{V_1, ..., V_n\})$. Suppose that $V_1, ..., V_n \in \mathcal{X}_\Gamma$. It follows that, for all $i \in \{1, ..., n\}$, $V_i \neq \emptyset$, and it follows that there are L-formulas $\phi_1, ..., \phi_n$ such that $V_i = S_\Gamma \cap [\phi_i]$ for all $i \in \{1, ..., n\}$. By Lemma G.2, $\Diamond\phi_i \in \Gamma$ for all $i \in \{1, ..., n\}$. Hence, by Lemma F.2, $\bigwedge_{i=1}^{n} \Diamond\phi_i \in \Gamma$. Since, by Lemma F.2, Γ contain all instances of **C2**, it follows that Γ contains $\bigwedge_{i=1}^{n} \Diamond\phi_i \to \bigvee_{i=1}^{n} A(\phi_i \mid \phi_1, ..., \phi_n)$. From Lemma F.2, Γ is closed under **MP**. Hence, Γ contains $\bigvee_{i=1}^{n} A(\phi_i \mid \phi_1, ..., \phi_n)$. Thus, by Corollary F.3, $A(\phi_j \mid \phi_1, ..., \phi_n) \in \Gamma$ for some $j \in \{1, ..., n\}$. We claim that $V_j \in C_\Gamma(\{V_1, ..., V_n\})$. By assumption, Conditions (1) and (2) in the specification of C_Γ are satisfied. Now, suppose that $\phi'_1, ..., \phi'_n$ are L-formulas such that $V_i = S_\Gamma \cap [\phi'_i]$ for all $i \in \{1, ..., n\}$. From Lemma G.3, it follows that $\Box(\phi_i \leftrightarrow \phi'_i) \in \Gamma$ for all

$i \in \{1, ..., n\}$. By Lemma F.2, Γ contains all instances of **C4**. In particular, Γ contains the following formula:

$$(\Box(\phi_j \leftrightarrow \phi'_j) \wedge \bigwedge_{i=1}^{n} \Box(\phi_i \leftrightarrow \phi'_i) \wedge A(\phi_j \mid \phi_1, ..., \phi_n)) \to A(\phi'_j \mid \phi'_1, ..., \phi'_n).$$

Since $A(\phi_j \mid \phi_1, ..., \phi_n) \in \Gamma$ and $\Box(\phi_i \leftrightarrow \phi'_i) \in \Gamma$ for $i \in \{1, ..., n\}$, it follows from Lemma F.2 that $\Box(\phi_j \leftrightarrow \phi'_j) \wedge \bigwedge_{i=1}^{n} \Box(\phi_i \leftrightarrow \phi'_i) \wedge A(\phi_j \mid \phi_1, ..., \phi_n) \in \Gamma$. By Lemma F.2, Γ is closed under **MP**. Thus, Γ contains $A(\phi'_j \mid \phi'_1, ..., \phi'_n)$, and we have shown that Condition (3) in the specification of C_Γ is satisfied. $\Box$

Proposition G.5. *If $\mathcal{S}$ is a system that extends $\mathcal{C}$ and $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}\rangle$ is the canonical frame for $\mathcal{S}$, then the following hold for all $\phi, \psi_1, ..., \psi_n \in L$.*

- $[\neg\phi] = W - [\phi]$
- $[\phi \wedge \psi] = [\phi] \cap [\psi]$
- $[\Box\phi] = \{\Gamma \mid S_\Gamma \subseteq [\phi]\}$
- $[A(\phi \mid \psi_1, ..., \psi_n)] = \{\Gamma \mid S_\Gamma \cap [\phi] \in C_\Gamma(\{S_\Gamma \cap [\psi_1], ..., S_\Gamma \cap [\psi_n]\})\}$

Proof. Suppose that $\Gamma \in [\neg\phi]$. Since Γ is consistent, it follows that $\Gamma \notin [\phi]$. Hence, $\Gamma \in W - [\phi]$. Conversely, suppose that $\Gamma \in W - [\phi]$. Hence, $\phi \notin \Gamma$. From Lemma F.2, it follows that $\neg\phi \in \Gamma$. Thus, $\Gamma \in [\neg\phi]$. Suppose that $\Gamma \in [\phi \wedge \psi]$. Hence, $\phi \wedge \psi \in \Gamma$. From Lemma F.2, it follows that $\phi, \psi \in \Gamma$. Thus, $\Gamma \in [\phi] \cap [\psi]$. Conversely, suppose that $\Gamma \in [\phi] \cap [\psi]$. It follows that $\phi, \psi \in \Gamma$. Thus, by Lemma F.2, $\phi \wedge \psi \in \Gamma$. Hence, $\Gamma \in [\phi \wedge \psi]$. Suppose that $\Gamma \in [\Box\phi]$. Thus, $\Box\phi \in \Gamma$, and it follows immediately that $S_\Gamma \subseteq [\phi]$. Conversely, suppose that $S_\Gamma \subseteq [\phi]$. From Lemma G.1, it follows that $\Box\phi \in \Gamma$. Thus, $\Gamma \in [\Box\phi]$. Suppose that $\Gamma \in [A(\phi \mid \psi_1, ..., \psi_n)]$. Hence, $A(\phi \mid \psi_1, ..., \psi_n) \in \Gamma$. By Lemma F.2, Γ contains all instances of **C1**. In particular, Γ contains $A(\phi \mid \psi_1, ..., \psi_n) \to (\Diamond\phi \wedge \bigwedge_{i=1}^{n} \Diamond\psi_i)$. By Lemma F.2, Γ is closed under **MP**. Thus, Γ contains $\Diamond\phi \wedge \bigwedge_{i=1}^{n} \Diamond\psi_i$. By Lemma F.2, $\Diamond\phi \in \Gamma$ and $\Diamond\psi_i \in \Gamma$ for all $i \in \{1, ..., n\}$. Thus, by Lemma G.2, $S_\Gamma \cap [\phi] \neq \emptyset$ and $S_\Gamma \cap [\psi_i] \neq \emptyset$ for all $i \in \{1, ..., n\}$. We have verified the first condition in the specification of C_Γ. By Lemma F.2, Γ contains all instances of **C3**. In particular, Γ contains $A(\phi \mid \psi_1, ..., \psi_n) \to \bigvee_{i=1}^{n} \Box(\phi \leftrightarrow \psi_i)$. By Lemma F.2, Γ is closed under **MP**. Thus, Γ contains $\bigvee_{i=1}^{n} \Box(\phi \leftrightarrow \psi_i)$. By Corollary F.3, there is a $j \in \{1, ..., n\}$ such that $\Box(\phi \leftrightarrow \psi_j) \in \Gamma$. Thus, by Lemma G.3, there is a $j \in \{1, ..., n\}$ such that $S_\Gamma \cap [\phi] = S_\Gamma \cap [\psi_j]$. We have verified the second condition in the specification of C_Γ. Suppose that $\phi', \psi'_1, ..., \psi'_n$ are L-formulas such that $S_\Gamma \cap [\phi] = S_\Gamma \cap [\phi']$ and $S_\Gamma \cap [\psi_i] = S_\Gamma \cap [\psi'_i]$ for all $i \in \{1, ..., n\}$. It follows from Lemma G.3 that $\Box(\phi \leftrightarrow \phi') \in \Gamma$ and $\Box(\psi_i \leftrightarrow \psi'_i) \in \Gamma$ for all $i \in \{1, ..., n\}$. Thus, by Lemma F.2, Γ contains the formula $\Box(\phi \leftrightarrow \phi') \wedge \bigwedge_{i=1}^{n} \Box(\psi_i \leftrightarrow \psi'_i) \wedge A(\phi \mid \psi_1, ..., \psi_n)$. By Lemma F.2, Γ contains all instances of **C4**. In particular, Γ contains $(\Box(\phi \leftrightarrow \phi') \wedge \bigwedge_{i=1}^{n} \Box(\psi_i \leftrightarrow \psi'_i) \wedge A(\phi \mid \psi_1, ..., \psi_n)) \to A(\phi' \mid \psi'_1, ..., \psi'_n)$. By Lemma F.2, Γ is closed under **MP**. Thus, Γ contains $A(\phi' \mid \psi'_1, ..., \psi'_n)$. We have verified the third condition in the specification of C_Γ. Hence, $S_\Gamma \cap [\phi] \in C_\Gamma(\{S_\Gamma \cap [\psi_1], ..., S_\Gamma \cap [\psi_n]\})$. Conversely, suppose that $S_\Gamma \cap [\phi] \in C_\Gamma(\{S_\Gamma \cap [\psi_1], ..., S_\Gamma \cap [\psi_n]\})$. It follows immediately from the third condition in the specification of C_Γ that $A(\phi \mid \psi_1, ..., \psi_n) \in \Gamma$. $\Box$

Proposition G.6. *Let $\mathcal{S}$ be a system that extends $\mathcal{C}$. Let $\langle W, R, \mathcal{X}, \{C_w\}_{w \in W}\rangle$ be the canonical frame for $\mathcal{S}$. $\mathcal{X}$ satisfies the closure requirements of **F5**.*

Proof. Follows easily from Proposition G.5. $\Box$

Appendix H. Results from Section 8

Proposition H.1. *For all $\phi \in L$, $\pi^*(\phi) = [\phi]$.*

Proof. The result follows from Proposition G.5 via a straightforward induction on L. *Atom*: If ϕ is an atom, then the result follows immediately. *Negation*: Assume that the result holds for ϕ. From the negation clause for interpretations, $\pi^*(\neg\phi) = W - \pi^*(\phi)$. Thus, from the inductive hypothesis, $\pi^*(\neg\phi) = W - [\phi]$. Since, by Proposition G.5, $[\neg\phi] = W - [\phi]$, it follows that $\pi^*(\neg\phi) = [\neg\phi]$. *Conjunction*: Assume that the result holds for ϕ and ψ. From the negation clause for interpretations, $\pi^*(\phi \wedge \psi) = \pi^*(\phi) \cap \pi^*(\psi)$. Thus, from the inductive hypothesis, $\pi^*(\phi \wedge \psi) = [\phi] \cap [\psi]$. Since, by Proposition G.5, $[\phi \wedge \psi] = [\phi] \cap [\psi]$, it follows that $\pi^*(\phi \wedge \psi) = [\phi \wedge \psi]$. *Necessity*: Assume that the result holds for ϕ. From the necessity clause for interpretations, $\pi^*(\Box\phi) = \{\Gamma \in W \mid S_\Gamma \subseteq \pi^*(\phi)\}$. Thus, from the inductive hypothesis, $\pi^*(\Box\phi) = \{\Gamma \in W \mid S_\Gamma \subseteq [\phi]\}$. Since, by Proposition G.5, $[\Box\phi] = \{\Gamma \in W \mid S_\Gamma \subseteq [\phi]\}$, it follows that $\pi^*(\Box\phi) = [\Box\phi]$. *Admissibility*: Assume that the result holds for ϕ and $\psi_1, ..., \psi_n$. From the admissibility clause for interpretations, $\pi^*(A(\phi \mid \psi_1, ..., \psi_n))$ is equal to

$$\{\Gamma \mid S_\Gamma \cap \pi^*(\phi)) \in C_\Gamma(\{S_\Gamma \cap \pi^*(\psi_1), ..., S_\Gamma \cap \pi^*(\psi_n)\})\}$$

which, by the inductive hypothesis, is equal to

$$\{\Gamma \mid S_\Gamma \cap [\phi] \in C_\Gamma(\{S_\Gamma \cap [\psi_1], ..., S_\Gamma \cap [\psi_n]\})\}.$$

which, by Proposition G.5, is equal to $[A(\phi \mid \psi_1, ..., \psi_n)]$. □

Proof of 8.1. We prove the contrapositive. Suppose that $\mathcal{S} \nvdash \phi$. It follows that $\mathcal{S} \nvdash \neg\neg\phi$, since $\mathcal{S}$ extends $\mathcal{L}$. Hence, $\neg\phi$ is consistent with respect to $\mathcal{S}$. Let $\mathcal{I}_\mathcal{S} = \langle W, \{R_w\}_{w\in W}, \mathcal{X}, \{C_w\}_{w\in W}, \pi\rangle$ be the canonical interpretation for $\mathcal{S}$. By Lemma F.1 there is a $\Gamma \in W$ such that $\neg\phi \in \Gamma$. Hence, $\Gamma \in [\neg\phi]$. By Proposition H.1, $\pi^*(\neg\phi) = [\neg\phi]$. Thus, $\Gamma \in \pi^*(\neg\phi)$, i.e. $(\mathcal{I}_\mathcal{S}, \Gamma) \models \neg\phi$. Thus, it is not the case that $\mathcal{I}_\mathcal{S} \models \phi$. □

Proof of Proposition 8.3. Suppose that $\Gamma \in W$ and that $Y_1, ..., Y_m, Z_1, ..., Z_n \in \mathcal{X}_\Gamma$. Hence, there are L-formulas $\phi_1, ..., \phi_m$ such that $S_\Gamma \cap [\phi_i] = Y_i \neq \emptyset$ for all $i \in \{1, ..., m\}$ and L-formulas $\psi_1, ..., \psi_n$ such that $S_\Gamma \cap [\psi_i] = Z_i \neq \emptyset$ for all $i \in \{1, ..., n\}$. Assume that $Y_j \in C_\Gamma(\{Y_1, ..., Y_m, Z_1, ..., Z_n\})$, where $j \in \{1, ..., m\}$. It follows from the specification of C_Γ that $A(\phi_j \mid \phi_1, ..., \phi_m, \psi_1, ..., \psi_n)$ is in Γ. Since $\mathcal{S}$ extends $\mathcal{C}_\alpha$ it follows from Lemma F.2 that Γ contains all instances of $\mathbf{C}_\alpha$. In particular, Γ contains the following formula:

$$(A(\phi_j \mid \phi_1, ..., \phi_m, \psi_1, ..., \psi_n) \wedge \bigvee_{i=1}^{m} \Box(\phi_j \leftrightarrow \phi_i)) \rightarrow A(\phi_j \mid \phi_1, ..., \phi_m).$$

By Lemma G.3, it is clear that $\Box(\phi_j \leftrightarrow \phi_j) \in \Gamma$. Hence, by Lemma F.3, $\bigvee_{i=1}^{m} \Box(\phi_j \leftrightarrow \phi_i)$ is in Γ. Thus, by Lemma F.2,

$$A(\phi_j \mid \phi_1, ..., \phi_m, \psi_1, ..., \psi_n) \wedge \bigvee_{i=1}^{m} \Box(\phi_j \leftrightarrow \phi_i)$$

is in Γ. Since, by Lemma F.2, Γ is closed under **MP**, it follows that $A(\phi_j \mid \phi_1, ..., \phi_m)$ is in Γ. Now we must show that $Y_j \in C_\Gamma(Y_1, ..., Y_m)$. Of the three conditions in the specification of C_Γ, all but Condition (3) follow immediately from what has already been assumed. Thus, suppose that $\phi'_1, ..., \phi'_m$ are L-formulas such that $S_\Gamma \cap [\phi'_i] = Y_i$ for all $i \in \{1, ..., m\}$. It follows from Lemma G.3 that $\Box(\phi_i \leftrightarrow \phi'_i) \in \Gamma$ for all $i \in \{1, ..., m\}$. Since $\mathcal{S}$ extends $\mathcal{C}$ it follows from Lemma F.2 that Γ contains all instances of **C4**. In particular, Γ contains the following formula:

$$(\Box(\phi_j \leftrightarrow \phi'_j) \wedge \bigwedge_{i=1}^{m} \Box(\phi_i \leftrightarrow \phi'_i) \wedge A(\phi_j \mid \phi_1, ..., \phi_m)) \rightarrow A(\phi'_j \mid \phi'_1, ..., \phi'_m).$$

We have shown that Γ contains each of the conjuncts from the antecedent of this formula. Hence, by Lemma F.2, Γ contains $\Box(\phi_j \leftrightarrow \phi'_j) \wedge \bigwedge_{i=1}^{m} \Box(\phi_i \leftrightarrow \phi'_i) \wedge A(\phi_j \mid \phi_1, ..., \phi_m)$. By Lemma F.2, Γ is closed under **MP**. Thus, Γ contains $A(\phi'_j \mid \phi'_1, ..., \phi'_m)$. This completes the verification of Condition (3) in the specification of C_Γ. Thus, $Y_j \in C_{\{\Gamma}(Y_1, ..., Y_m\})$. □

Proof of 8.4. Suppose that $\Gamma \in W$ and that $Y_1, ..., Y_m, Z_1, ..., Z_n \in \mathcal{X}_\Gamma$. Hence, there are L-formulas $\phi_1, ..., \phi_m$ such that $S_\Gamma \cap [\phi_i] = Y_i \neq \emptyset$ for all $i \in \{1, ..., m\}$ and L-formulas $\psi_1, ..., \psi_n$ such that $S_\Gamma \cap [\psi_i] = Z_i \neq \emptyset$ for all $i \in \{1, ..., n\}$. Assume that $Y_j, Y_k \in C_\Gamma(\{Y_1, ..., Y_m\})$ and $Y_j \in C_\Gamma(\{Y_1, ..., Y_m, Z_1, ..., Z_n\})$, where $j, k \in \{1, ..., m\}$. It follows from the specification of C_Γ that Γ contains $A(\phi_j \mid \phi_1, ..., \phi_m)$, $A(\phi_k \mid \phi_1, ..., \phi_m)$, and $A(\phi_j \mid \phi_1, ..., \phi_m, \psi_1, ..., \psi_n)$. Hence, by Lemma F.2,

$$A(\phi_j \mid \phi_1, ..., \phi_m) \wedge A(\phi_k \mid \phi_1, ..., \phi_m) \wedge A(\phi_j \mid \phi_1, ..., \phi_m, \psi_1, ..., \psi_n)$$

is contained in Γ. Since $\mathcal{S}$ extends $\mathcal{C}_\beta$ it follows from Lemma F.2 that Γ contains all instances of $\mathbf{C}_\alpha$. In particular, Γ contains the following formula:

$$(A(\phi_j \mid \phi_1, ..., \phi_m) \wedge A(\phi_k \mid \phi_1, ..., \phi_m) \wedge A(\phi_j \mid \phi_1, ..., \phi_m, \psi_1, ..., \psi_n)) \rightarrow$$
$$A(\phi_k \mid \phi_1, ..., \phi_m, \psi_1, ..., \psi_n)).$$

By Lemma F.2, Γ is closed under **MP**. Thus, Γ contains $A(\phi_k \mid \phi_1, ..., \phi_m, \psi_1, ..., \psi_n)$. Now we must show that $Y_k \in C_\Gamma(\{Y_1, ..., Y_m, Z_1, ..., Z_n\})$. Of the three conditions in the specification of C_Γ, all but Condition (3) follow immediately from what has already been assumed. Thus, suppose that $\phi'_1, ..., \phi'_m$ are L-formulas such that $S_\Gamma \cap [\phi'_i] = Y_i$ for all $i \in \{1, ..., m\}$, and suppose that $\psi'_1, ..., \psi'_n$ are L-formulas such that $S_\Gamma \cap [\psi'_i] = Z_i$ for all $i \in \{1, ..., n\}$. It follows from Lemma G.3 that $\Box(\phi_i \leftrightarrow \phi'_i) \in \Gamma$ for all $i \in \{1, ..., m\}$ and $\Box(\psi_i \leftrightarrow \psi'_i) \in \Gamma$ for all $i \in \{1, ..., n\}$. Thus, by Lemma F.2, Γ contains

$$\Box(\phi_k \leftrightarrow \phi'_k) \wedge \bigwedge_{i=1}^{m} \Box(\phi_i \leftrightarrow \phi'_i) \wedge \bigwedge_{i=1}^{n} \Box(\psi_i \leftrightarrow \psi'_i) \wedge A(\phi_k \mid \phi_1, ..., \phi_m, \psi_1, ..., \psi_n).$$

Since $\mathcal{S}$ extends $\mathcal{C}$ it follows from Lemma F.2 that Γ contains all instances of **C4**. In particular, Γ contains the following formula:

$$(\Box(\phi_k \leftrightarrow \phi'_k) \wedge \bigwedge_{i=1}^{m} \Box(\phi_i \leftrightarrow \phi'_i) \wedge \bigwedge_{i=1}^{n} \Box(\psi_i \leftrightarrow \psi'_i) \wedge A(\phi_k \mid \phi_1, ..., \phi_m, \psi_1, ..., \psi_n)) \rightarrow$$
$$A(\phi'_k \mid \phi'_1, ..., \phi'_m, \psi'_1, ..., \psi'_n).$$

By Lemma F.2, Γ is closed under **MP**. Hence, Γ contains $A(\phi'_k \mid \phi'_1, ..., \phi'_m, \psi'_1, ..., \psi'_n)$. This completes the verification of Condition (3). $Y_k \in C_\Gamma(\{Y_1, ..., Y_m, Z_1, ..., Z_n\})$. □

References

1. M.A. Aizerman, *New problems in the general choice theory*, Social Choice and Welfare **2** (1985), 235–282.
2. M.A. Aizerman and A.V. Malishevski, *General theory of best varients of choice: Some aspects*, IEEE Transactions on Automatic Control **26** (1981), 1030–1040.
3. M. Allais, *Le comportement de l'homme rationnel devant le risque: Critique des postulats et axiomes de l'ecole americaine*, Econometrica **21** (1953), 503–546.
4. W. Bossert, *Opportunity sets and uncertain consequences*, Journal of Mathematical Economics **33** (2000), 475–496.
5. B.F. Chellas, *Modal logic: an introduction*, Cambridge University Press, 1980.
6. H. Arlo Costa, *Rationality and value: The epistemological role of indeterminate and agent-dependent values*, Philosophical Studies **128** (2006), no. 1, 7–48.
7. D. Ellsberg, *Risk, ambiguity, and the Savage axioms*, The Quarterly Journal of Economics **75** (1961), 643–669.
8. ______, *Risk, ambiguity and decision*, Garland Publishing, 2001.
9. Ronald Fagin, Joeseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi, *Reasoning about knowledge*, MIT Press, 1995.
10. P. Gardenfors and N.E. Sahlin, *Unreliable probabilities, risk taking, and decision making*, Synthese **53** (1982), 361–386.
11. J. Halpern, *Reasoning about uncertainty*, MIT Press, 2003.
12. S.O. Hansson, *Handbook of philosophical logic*, vol. 4, ch. Preference Logic, pp. 319–393, Kluwer, 2001.
13. J. Helzner, *On the application of multiattribute utility theory to models of choice*, Theory and Decision **66** (2009), no. 4, 301–315.
14. J.-J.Ch.Meyer and W. van der Hoek, *Epistemic logic for ai and computer science*, Cambridge University Press, 1995.
15. D. Kahneman and A. Tversky, *Prospect theory: An analysis of decision under risk*, Econometrica **47** (1979), 263–291.
16. G. Kalai, A. Rubinstein, and R. Spiegler, *Rationalizing choice functions by multiple rationales*, Econometrica **70** (2002), no. 6, 2481–2488.
17. J.M. Keynes, *A treatise on probability*, MacMillan, 1921.
18. F.H. Knight, *Risk, uncertainty and profit*, Houghton-Mifflin, 1921.
19. D. Kreps, *Notes on the theory of choice*, Westview, 1988.
20. H.E. Kyburg, *Bets and beliefs*, American Philosophical Quarterly **5** (1968), 63–78.
21. I. Levi, *On indeterminate probabilities*, Journal of Philosophy **71** (1974), 391–418.
22. ______, *The enterprise of knowledge*, MIT Press, 1980.
23. ______, *Hard choices: Decision making under unresolved conflict*, Cambridge University Press, 1986.
24. ______, *The paradoxes of Allais and Ellsberg*, Economics and Philosophy **2** (1986), 23–53.
25. R.D. Luce and H. Raiffa, *Games and decisions: Introduction and critical survey*, Dover Publications, 1989, The Dover edition is a republication of the 1957 work.
26. H. Moulin, *Choice functions over a finite set: a summary*, Social Choice and Welfare **2** (1985), 147–160.
27. K. Nehring, *Rational choice and revealed preference without binariness*, Social Choice and Welfare **14** (1997), 403–425.
28. K. Nehring and C. Puppe, *Extending partial orders: A unifying structure for abstract choice theory*, Annals of Operations Research **80** (1998), 27–48.
29. H. Rott, *Belief contraction in the context of the general theory of rational choice*, Journal of Symbolic Logic **58** (1993), 1426–1450.
30. ______, *Knowledge contributors*, Synthese Library: Studies in Epistemology, Logic, Methodology, and Philosophy of Science, vol. 322, ch. Economics and Economy in the Theory of Belief Revision, pp. 57–86, Kluwer, 2003.
31. A. Rubinstein, *Lecture notes in microeconomic theory*, Princeton University Press, 2006.
32. L.J. Savage, *The foundations of statistics*, Dover Publications, 1972, The Dover edition is a republication of the 1954 work.

33. M. Schervish, T. Seidenfeld, J. Kadane, and I. Levi, *Extensions of expected utility theory and some limitations of pairwise comparisons*, Proceedings of the Third International Symposium on Imprecise Probability: Theories and Applications, 2003.
34. T. Seidenfeld, *Decision theory without 'independence' or without 'ordering': What is the difference?*, Economics and Philosophy **4** (1988), 267–290.
35. T. Seidenfeld, J. Kadane, and M. Schervish, *On the shared preferences of two bayesian decision makers*, Journal of Philosophy **86** (1989), no. 5, 225–244.
36. T. Seidenfeld, M.J. Schervish, and J.B. Kadane, *Coherent choice functions under uncertainty*, Proceedings of the Fifth International Symposium on Imprecise Probability: Theories and Applications, 2007.
37. A. Sen, *Choice functions and revealed preference*, Review of Economic Studies **38** (1971), no. 3, 307–317.
38. ______, *Internal consistency of choice*, Econometrica **61** (1993), no. 3, 495–521.
39. ______, *Maximization and the act of choice*, Econometrica **65** (1997), no. 4, 745–779.
40. E. Ullmann-Margalit and S. Morgenbesser, *Picking and choosing*, Social Research **Winter** (1977), 757–785.

Columbia University
E-mail address: jh2239@columbia.edu