A NOTE ON GABRIEL UZQUIANO'S 'VARIETIES OF INDEFINITE EXTENSIBILITY'

SIMON HEWITT

1

Following Dummett, a number of philosophers have responded to the set-theoretic paradoxes by claiming that the concept *set* is indefinitely extensible [1]. One way of understanding this is in terms of ontological extensibility: however many sets there are, it is always possible that there be more. In line with ontological extensibility, given that the sets are amongst the entities, and given that whether or not a given entity is a set is modally invariant: however many entities there are, it is always possible that there be more. The challenge for the proponent of this view is to provide a satisfactory account of the operative modality. Linnebo has recently proposed that an individuative modality can underwrite a modal set theory, and similar themes can be identified in Rayo's work on ontology [3] [5]. In rough outline, for both views, there will always be new ways of 'carving up' reality linguistically, so as to yield new objects.

In his *Varieties of Indefinite Extensibility*, which draws inspiration from [8], Uzquiano proposes an alternative understanding of indefinite extensibility. Working in a fixed domain modal logic, he lays out a modalised account of set formation. The modality is interpreted in terms of linguistic extensibility. Whilst the existence of entities is fixed, the extensions of the predicates 'is a set' and '$\in$' are extensible, such that however many entities (pairs) fall within their extension, it is always possible to extend the language so that new entities (pairs) fall within their extension.

As an example, consider a given interpretation of the language, $\mathcal{I}$. There will be some xx such that xx are all and only the sets according to $\mathcal{I}$. The motivation behind indefinite extensibility is the thought that the process of collecting objects together to form sets is unbounded above. Informally, take any things you like, yy, there is some way of getting the set of yy. So in particular, there is some way of getting the set of all those entities xx that are sets according to $\mathcal{I}$. But disaster lurks nearby. For the following principle is plausible:

(COLLECTION) $\forall xx(\forall x(x \prec xx \rightarrow \alpha(x)) \rightarrow \exists x \equiv xx)$

Where $\alpha(x)$ reads 'x is available to form a set' and '$x \equiv xx$' reads 'x is the set of xx'.

Now if we capture the intuition that all the sets of $\mathcal{I}$ can be formed into a set remembering that our choice of $\mathcal{I}$ is arbitrary as,

(AVAILIBILITY) $$\forall x((\mathtt{Set}(x) \rightarrow \alpha(x))$$

then a proof of Russell's paradox is immediate: consider the xx which are all x such that $(\mathtt{Set}(x) \wedge x \notin x)$. There are such xx by plural comprehension, so by (AVAILIBILITY) and (COLLECTION) we have the existence of the Russell set. Uzquiano's solution is to modalise (AVAILIBILITY), thus blocking the derivation,

(AVAILIBILITY$^\Diamond$) $$\forall x(\mathtt{Set}(x) \rightarrow \Diamond\alpha(x))$$

The modality indicated by the Diamond is interpretational,

> ... $\Diamond\phi$ tells us that ϕ is true on some subsequent reinterpretation of the set-theoretic vocabulary – while $\Box\phi$ tells us that ϕ remains true on all subsequent reinterpretations of the set-theoretic vocabulary. [7,]

According to (AVAILIBILITY$^\Diamond$) anything that is a set on one interpretation is available to form a set according to a subsequent interpretation. So if all of yy are sets on one interpretation, it follows from (COLLECTION) that there is a subsequent interpretation on which yy form a set.

Now consider xx, which are all and only the sets on $\mathcal{I}$. Whilst there is no set of xx according to $\mathcal{I}$ there *is* a subsequent reinterpretation $\mathcal{I}^*$ of the set-theoretic vocabulary on which xx form a set[1]. Paradox is avoided, and due acknowledgement given to indefinite extensibility.

2. I

A certain metaphysical picture of sets underlies Uzquiano's system. There is, in general, no interpretation-independent fact of the matter whether some entity is a set. We do have that once the extension of 'Set()' has been extended to include an entity, all subsequent interpretations will have that object satisfying the predicate:

($\Box$Set) $$\forall x(\mathtt{Set}(x) \rightarrow \Box\mathtt{Set}(x))$$

But not that anything that could be included in the interpretation of Set() is a set,

($\Diamond$Set) $$\forall x(\Diamond\mathtt{Set}(x) \rightarrow \mathtt{Set}(x))$$

[1]And, since Uzquiano's system contains the closures of $x \prec xx \rightarrow \Box x \prec xx$ and $\neg x \prec xx \rightarrow \Box\neg x \prec xx$, those xx that form a set according $\mathcal{I}$ are all and only xx according to $\mathcal{I}*$.

The failure of ($\Diamond$Set) invites us to contemplate the possibility that there is some object which, given our current interpretation of set-theoretic language, we truthfully describe as a non-set but which, on a legitimate reinterpretation of the language, is truthfully described as a set[2]. Uzquiano discusses how this consequence might be motivated metaphysically and anticipates objections,

> Perhaps we should think of a set as a mere node in a structure that satisfies certain formal conditions imposed by the axioms at the outset. The set-theoretic universe could perhaps be reduced to a domain of objects related by a formally appropriate relation that satisfies the relevant axioms... But one may well object to this that there is much more to the element-set relation than to stand in a relation that satisfies certain structural conditions, one may be tempted to dismiss the linguistic model of indefinite extensibility as a non-starter. [7]

The structuralist flavour of this will receive attention in due course. Immediately, however, it is clear that Uzquiano is correct in anticipating metaphysical concerns about the project. The following principle commands wide assent:

> **NMC** No mathematical object is identical with any concretum.

It is far from apparent that Uzquiano's theory delivers (NMC). What stops the interpreter of set theoretic language from re-interpreting her vocabulary so that a certain cat is a set? If the answer is 'nothing', then there are Lewisian worries: if Tiddles meets a sad end, this should not have mathematical implications [2, 13]. If the answer is 'something', then more is required of the account of interpretation to cash that out. This issues aside, (NMC) is surely part of our working understanding of mathematics. It is open to the supporter of the re-interpretation account to acknowledge this but to argue that our working understanding needs revising. In this eventuality, the task is to assess the benefits of revision against the costs.

3

Those costs are grave. What is on offer is a version of structuralism about set theory. There is no more to being a set than being a satisfier of the predicate $Set()$ in an interpretation of the language which yields a 'formally appropriate' membership relation. In order for the account to serve its stated purpose of explicating indefinite extensibility, it is important that the structuralism in question be *in rebus* rather than *ante rem*[3]. For suppose that any adequate extension of the language satisying Uzquiano's constraints should

[2] It is part of Uzquiano's view that the *urelemente* are a proper subplurality of the non-sets.

[3] On structuralism, see *e.g.* [6]

be understood as instancing an initial segment of an abstract structure. Then the question about indefinite extensibility arises about the abstract structure, and we are no further forward. Alternatively, one might propose an indefinitely extensible series of such structures, each corresponding to a possible extension of the language. But in this case, the explication of extensibility in terms of linguistic extensions is not genuine: for what ultimately stands in need of explanation here is the existence of the indefinitely extensible series, which being *ante rem* is not susceptible to explanation in terms of language use.

In rebus structuralism about mathematical theories faces what we might call the *not enough objects* worry. Suppose I am a structuralist about analysis, that I accept only physical objects into my ontology, that spacetime is not continuous, and that mereological composition is restricted[4] Then there aren't going to be enough objects to instantiate the structure of $\mathbb{R}$ under $<$. The moral of the story: *in rebus* structuralism is hostage to reality supplying enough objects to instantiate the structures of our mathematical theories.

Uzquiano gives no indication of supporting an ontology as sparse as that of our imagined physicalist. Successive re-interpretations of the language might, then, extend the extension of '$Set()$' so that it is satisfied by an ever increasing number of abstracta (or of abstracta together with concreta). However, the cardinality demands made by set theory are considerable. Whilst Uzquiano flags that his 'axioms do not, by themselves, tell us how far we should proceed in the cumulative process of reinterpretation of the set theoretic vocabulary', the fact that he is working in a plural logic yields a lower bound. Assuming that the intention is to be hermenutic, rather than revisionary of set theory as practiced, we will want to validate ZFC. Plural resources allow us to express Separation and Replacement as axioms, rather than schemata (in the case of Replacement, we simulate quantification over functions by plural quantification over tuples). This gives us a variant of *second-order* ZF. On the standard semantics the smallest model of this is strongly inaccessible.

Why should we believe that there are strongly inaccessible many objects? Absent an answer to this question, there is no reason to believe that Uzquiano's project provides a workable account of indefinite extensibility. We need assurance that there is a sufficiently sized domain over which the language can be re-interpreted. Of course, one excellent reason to believe that there are at least strongly inaccessibly many objects is that one believes that set theory is true. But appeal to set theoretic ontology at this point would be fatally circular. We need to be justified in accepting a sufficiently sized ontology for reasons independent of set-theory in order to

[4]If the composition of physical objects satisfies classical extensional mereology then, given $\aleph_0$ physical objects as atoms, we get $2^{\aleph_0}$ objects as required.

motivate an understanding of indefinite extensibility in terms of linguistic re-interpretation.

Some years ago James Mayberry wrote,

> The universe of sets is not a structure: it is the world that all mathematical structures inhabit, the sea in which they swim. [4, 34]

The metaphors are compelling. If we but avail ourselves of set theory, understood in a non-structuralist fashion, we can – if we like – be structuralists about other mathematical theories[5], reassured that the universe of sets contains enough objects to instantiate the structures described by these theories. Those who would be *in rebus* structuralists about set theory itself need to offer an alternative route to the requisite ontology. That task, in particular, awaits those who would understand indefinite extensibility in terms of linguistic re-interpretation.

*

REFERENCES

[1] Michael Dummett, *Frege : Philosophy of Mathematics*, Duckworth, London, 1991.

[2] David Lewis, *Parts of classes*, Blackwell, Oxford, 1991.

[3] Øystein Linnebo, *The Potential Hierarchy of Sets*, Review of Symbolic Logic **6** (2013), no. 2, 205–29.

[4] J.P. Mayberry, *What is required of a foundation for mathematics?*, Philosophia Mathematica **3** (1994), no. 2, 16–35.

[5] Agustin Rayo, *The Construction of Logical Space*, Oxford University Press, Oxford, 2013.

[6] Stewart Shapiro, *Philosophy of mathematics : Structure and ontology*, Oxford University Press, Oxford, 1997.

[7] Gabriel Uzquiano, *Varieties of Indefinite Extensibility*, Notre Dame Journal of Formal Logic **56** (2015), no. 1, 147–166.

[8] Timothy Williamson, *Indefinite Extensibility*, Grazer Philosphische Studien **55** (1998), 1–24.

*

[5] Or at least, *most* other mathematical theories. There are familiar problems around category theory