Apophatic finitism and infinitism

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Abstract

This article is about the ontological dispute between finitists, who claim that only finitely many numbers exist, and infinitists, who claim that infinitely many numbers exist. Van Bendegem set out to solve the 'general problem' for finitism: how can one recast finite fragments of classical mathematics in finitist terms? To solve this problem Van Bendegem comes up with a new brand of finitism, namely so-called 'apophatic finitism'. In this article it will be argued that apophatic finitism is unable to represent the negative ontological commitments of infinitism or, in other words, that which does not exist according to infinitism. However, there is a brand of infinitism, so-called 'apophatic infinitism', that is able to represent both the positive and the negative ontological commitments of apophatic finitism. Unfortunately, apophatic finitism cannot adopt that way without losing the ability to represent the positive ontological commitments of infinitism.

Keywords: Finitism · Infinitism · Apophatic finitism · Apophatic infinitism · Ontological commitments

Strict finitism is the philosophical doctrine that there exist only finitely many mathematical objects. This doctrine conflicts with all the main mathematical theories: set theory postulates the existence of infinitely many sets, arithmetic entails the existence of infinitely many natural numbers, analysis studies an infinity of real numbers, and so on. In this article we will focus on strict finitism with respect to natural numbers, i.e. the doctrine that there are only finitely many natural numbers and, consequently, that there is a largest natural number. Henceforth, the qualifier 'strict' will be dropped. Let us use the label 'infinitism' for the position that there are infinitely many natural numbers and, therefore, that there is no largest natural number.

In section 1 a challenge to finitism, due to Van Bendegem (1999; 2012), will be presented. In section 2 the solution offered by Van Bendegem (1999; 2012) will be given. This solution leads Van Bendegem to adopt so-called 'apopathic finitism'. It will then be argued in section 3 that apophatic finitism is confronted with a new problem. Next, so-called 'apophatic infinitism' will be introduced in section 4. It will be shown that this brand of infinitism can solve the problem that apophatic finitism is faced with in a way that is unavailable to the latter.

1 The general challenge to finitism

To repeat, finitism conflicts with all the main mathematical theories: set theory postulates the existence of infinitely many sets, arithmetic entails the existence of infinitely many natural numbers, analysis studies an infinity of real numbers, and so on. This raises the following question: can one recast mathematics in finitist terms?

This question is related to what is called the 'the general problem' or the 'argument from poverty' in (Van Bendegem 1999, pp. 120-121) and (Van Bendegem 2012, p. 146).[1] The challenge to finitism is described by Jean Paul Van Bendegem as follows:

1. Consider any *finite* set S of sentences of the language of first-order logic with identity.

2. Is it always possible to give an interpretation of those sentences in the form of a model with a *finite* domain?

At first sight, the answer is clear: no, it is not always possible. For consider the finite set that contains exactly the following three sentences.

$$\forall x \neg Rxx \tag{1}$$

$$\forall x \exists y Rxy \tag{2}$$

$$\forall x \forall y \forall z \left((Rxy \wedge Ryz) \to Rxz \right) \tag{3}$$

It is easily provable that any model for (1)-(3) has an infinite domain.[2]

Note that R can be interpreted as the successor relation of arithmetic: 'Rxy' means that y succeeds x. Under this interpretation (1) means that no natural number succeeds itself, (2) means that every natural number has a natural number that succeeds it, and (3) means that, if a natural number is succeeded by another natural number and that second natural number is in turn succeeded by a third natural number, then that third natural number succeeds the first natural number. The question is then: can one recast finite fragments of arithmetic, including (1)-(3) under their arithmetical interpretation, in finitist terms?

[1] The paper has been derived from (Van Bendegem 2010).

[2] Suppose that there is a model for (1)-(3) that has a domain D with only n elements ($n \in \mathbb{N}$). Pick an element, d_1. By (2), there is an element $d_2 \in D$ such that $\langle d_1, d_2 \rangle \in I(R)$ (with I the interpretation function of the model, which assigns pairs from D to R). By (1), $d_1 \neq d_2$, because otherwise $\langle d_1, d_1 \rangle \in I(R)$. By (2) again, there is an element $d_3 \in D$ such that $\langle d_2, d_3 \rangle \in I(R)$. Again by (1), $d_2 \neq d_3$. Moreover, by (3), $\langle d_1, d_3 \rangle \in I(R)$ and, therefore, by (1), $d_1 \neq d_3$. Continue this reasoning until you have reached the n-th element of the domain, d_n. By (2), there has to be an element $d_i \in D$ such that $\langle d_n, d_i \rangle \in I(R)$. By (1), $i \neq n$. But then by (3) and by what has been proved before, it follows that $\langle d_i, d_n \rangle \in I(R)$. By (3) again, it follows that $\langle d_n, d_n \rangle \in D$, which contradicts (1).

2 Van Bendegem's solution to the general problem

Van Bendegem has a strategy for dealing with the above challenge — see (Van Bendegem 1999, pp. 120–123) and (Van Bendegem 2012, pp. 146–147). In section 2.1 I will introduce his translation. In Subsection 2.2 I will describe his models. In section 2.3 I will show that all 'truths' of infinitism are kept. Finally, in section 2.4 I will show that some 'truths' are gained.

2.1 The translation

The first step consists in giving a *translation* of the sentences of S.

1. Write each sentence ϕ of S in an equivalent *prenex normal form*, i.e. a sentence ψ that is of the form

$$Q_1 v_1 \ldots Q_n v_n \theta, \tag{4}$$

with each Q_i $(1 \leq i \leq n)$ either $\exists$ or $\forall$,[3] with v_i meta-variables (i.e. variables ranging over variables), and with θ a quantifier-free formula. The sequence

$$Q_1 v_1 \ldots Q_n v_n$$

is called the prefix and θ is called the matrix. The prenex normal form theorem says that, for every sentence, there is a logically equivalent sentence that is in prenex normal form (Boolos et al. 2003, p. 246).

2. If there are multiple sentences in prenex normal form that are logically equivalent to ϕ, then pick the first according to the lexicographical order of those sentences in their LaTeX-notation.

3. Pick a $k \in \mathbb{N}$ and replace θ in (4) by the following:[4]

$$((v_1 < k \wedge \cdots \wedge v_n < k) \rightarrow \theta),$$

with $v_1, \ldots, v_n$ the free variables in θ (if any). The result is the following:

$$Q_1 v_1 \ldots Q_n v_n ((v_1 < k \wedge \cdots \wedge v_n < k) \rightarrow \theta). \tag{5}$$

4. Introduce k individual constants that do not appear in any of the sentences of S. These are, say, c_j, ..., c_{j+k-1}. Next, for each $1 \leq i \leq n$, replace $v_i < k$ in (5) by:

$$(v_i = c_j \vee \cdots \vee v_i = c_{j+k-1}) \tag{6}$$

Call the result ϕ^τ.

[3] So-called 'bounded quantifiers', i.e. $\forall v \leq u$ and $\exists v \leq u$, are defined away — see (Boolos et al. 2003, p. 76).

[4] Here I use $<$ rather than $\leq$, because I will include 0 in the set of natural numbers. Therefore, number k is the $k+1$-th number. For instance, 0 is the first number, 1 is the second number, and so on.

For the application of the above procedure to the set that contains exactly (1)-(3) one needs to fix a natural number k. Let's pick the number two. Note that step one is redundant, because (1)-(3) are already in prenex normal form. Note also that we can pick individual constants c_0 and c_1 $(= 0 + 2 - 1)$, because there are no individual constants in (1)-(3). The result of the application of the translation procedure to (1)-(3) is then the following:

$$\forall x \left((x = c_0 \vee x = c_1) \rightarrow \neg Rxx \right) \tag{7}$$

$$\forall x \exists y \left(((x = c_0 \vee x = c_1) \wedge (y = c_0 \vee y = c_1)) \rightarrow Rxy \right) \tag{8}$$

$$\forall x \forall y \forall z ((((x = c_0 \vee x = c_1) \wedge (y = c_0 \vee y = c_1)) \wedge (z = c_0 \vee z = c_1)) \rightarrow ((Rxy \wedge Ryz) \rightarrow Rxz)) \tag{9}$$

So, (7) is (1^{τ}), (8) is (2^{τ}), and (9) is (3^{τ}).

2.2 Apophatic finitism

The second step consists in defining a model with a finite domain that makes translations of all the sentences in S true. What follows, is a variation on the strategy illustrated in (Van Bendegem 1999). Given the canonical-domains theorem (Boolos et al. 2003, p. 147), any set of sentences that has a model, has a model whose domain is a subset of the natural numbers ($\mathbb{N}$). We have seen that the domain of any set that makes (1)-(3) true, is a set with an infinite domain. So, there is a model $\mathcal{M}$ of (1)-(3) with the set of natural numbers as its domain. Now define a model $\mathcal{M}^*$ as follows:

- the domain D^* of $\mathcal{M}^*$ is $\{n \in \mathbb{N} \mid n \leq k\}$;[5]
- for the non-logical vocabulary of the sentences in S, let the interpretation function I^* of $\mathcal{M}^*$ be the same as the interpretation function I of $\mathcal{M}$, except that it is restricted to $\{n \in \mathbb{N} \mid n \leq k - 1\}$;
- $I^*(c_j) = 0$, $I^*(c_{j+1}) = 1, \ldots, I^*(c_{j+k-1}) = k - 1$.

It is necessary that $k - 1$ is not smaller than the greatest natural number denoted by an individual constant occurring in S. Consequently, all the natural numbers denoted by the individual constants in sentences in S belong to D^*.

This gives us a model $\mathcal{M}^*$ with a finite domain. Van Bendegem (1999, p. 121) refers to models such as $\mathcal{M}^*$ as 'finite (quasi-)models'. Since no atomic formulas are true of k in the model, Horsten (2010) speaks of 'apophatic finitism', i.e. the doctrine that holds that no (primitive) predicate is true of the largest number. The reason for restricting the interpretation function I^* to all elements of D^* except k will become clear in what follows.

[5] Van Bendegem (1999, p. 122) opts for a domain consisting of the individual constants themselves. From the logical point of view, nothing essential seems to depend on that.

2.3 Keeping the 'truths' of infinitism

The upshot is that a finite (quasi-)model makes (1^τ), (2^τ) and (3^τ) true. Suppose again that $k = 2$. Then the domain D^* of $\mathcal{M}^*$ is $\{0, 1, 2\}$. Suppose that in model $\mathcal{M}$ the interpretation function assigns to R the same extension as the successor relation, i.e. the infinite set the following of pairs:

$$\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, \ldots$$

Consequently, $I^*(R) = \{\langle 0, 1\rangle\}$. Furthermore, $I^*(c_0) = 0$ and $I^*(c_1) = 1$. It is easy to check that (7)-(9) are true in $\mathcal{M}^*$. Take (7). Neither $\langle 0, 0\rangle$ nor $\langle 1, 1\rangle$ belong to the interpretation of R. Sentence (9) is left to the reader. Finally, take (8). The only relevant cases are those in which x is assigned 0 or 1. (In case it is assigned 2, the antecedent is not satisfied.) In case x is assigned 0, pick 1 for y, because $\langle 0, 1\rangle \in I^*(R)$. In case x is assigned 1, pick 2 for y, because then the antecedent is not satisfied. Here the importance of choosing for a *conditional* restriction even on existential quantifiers, cf. (5), becomes clear. The proof of the theorem below generalizes this reasoning.

THEOREM 1 Let S be a finite set of sentences belonging to the language of first-order logic with identity. Suppose that S has a model $\mathcal{M} = \langle D, I\rangle$, with $D = \mathbb{N}$. Pick a $k \in \mathbb{N}$, with $k - 1 \geq I(t)$, for any individual constant t (if any) occurring in S. Then the finite (quasi-)model $\mathcal{M}^* = \langle D^*, I^*\rangle$ (relative to $\mathcal{M}$ and k) is such that: for any $\phi \in S$, if $\mathcal{M} \models \phi$, then $\mathcal{M}^* \models \phi^\tau$ (with ϕ^τ the Van Bendegem translation of ϕ relative to S and k).

PROOF Recall that ϕ is logically equivalent to a sentence ψ that is of the form (4), i.e.

$$Q_1 v_1 \ldots Q_n v_n \theta,$$

with each Q_i $(1 \leq i \leq n)$ either $\exists$ or $\forall$, with v_i meta-variables, and with θ a quantifier-free formula. Also recall that ϕ^τ is of the form

$$Q_1 v_1 \ldots Q_n v_n (((v_1 = c_j \vee \cdots \vee v_1 = c_{j+k-1}) \wedge \cdots \wedge (v_n = c_j \vee \cdots \vee v_n = c_{j+k-1})) \rightarrow \theta). \quad (10)$$

By the definition of the interpretation function I^* (and the choice of k), for any of the constants t in θ, $I(t) = I^*(t)$, which belongs to $\{n \mid n \leq k - 1\}$. Also by the definition of I^*, for any of the m-place predicates Φ in θ, for any m-tuple $\langle d_1, \ldots, d_m\rangle$ of elements of $\{n \mid n \leq k - 1\}$, $\langle d_1, \ldots, d_m\rangle \in I(\Phi)$ if and only if $\langle d_1, \ldots, d_m\rangle \in I^*(\Phi)$. Next, consider any two variable assignment functions s and s^*, defined respectively over D and D^*, which agree on $v_1, \ldots, v_n$, which are all the free variables that may occur in θ. Given that θ is quantifier-free, if $\mathcal{M}, s \models \theta$, then $\mathcal{M}^*, s^* \models \theta$. This can easily be proved by induction on the complexity of θ, given that I and I^* agree on the constants and predicates in θ and given that s and s^* agree on $v_1, \ldots, v_n$, which are all the free variables that may occur in θ.[6] It follows by tautological reasoning that, if $\mathcal{M}, s \models \theta$, then $\mathcal{M}^*, s^* \models (11)$, with (11) the following:

$$((v_1 = c_j \vee \cdots \vee v_1 = c_{j+k-1}) \wedge \cdots \wedge (v_n = c_j \vee \cdots \vee v_n = c_{j+k-1})) \\ \to \theta. \quad (11)$$

Next, consider Q_n. We will now suppose that the two variable assignment functions s and s^*, defined respectively over D and D^*, agree on $v_1, \ldots, v_{n-1}$, which are all the free variables that may occur in $Q_n v_n \theta$.

First, suppose that Q_n is $\forall$. Assume that $\mathcal{M}, s \models \forall_n v_n \theta$. By the definition of satisfaction, $\mathcal{M}, s' \models \theta$, for all v_n-variants s' of s. Note that all these variants assign elements of $\{n \mid n \leq k-1\}$ to $v_1, \ldots, v_{n-1}$. Moreover, the set of all v_n-variants of s includes the set of all v_n-variants of s that assign elements of $\{n \mid n \leq k-1\}$ to v_n. Therefore, it follows that $\mathcal{M}, s' \models \theta$, for all s' that assign elements of $\{n \mid n \leq k-1\}$ to $v_1, \ldots, v_n$, which are all the free variables that may occur in θ. By what has been proved above, it follows that $\mathcal{M}^*, s^{**} \models (11)$, for all v_n-variants s^{**} of s^* that assign elements of $\{n \mid n \leq k-1\}$ to $v_1, \ldots, v_n$. For any variable assignment function s^{**} that assigns to v_n an element of D^* outside $\{n \mid n \leq k-1\}$, i.e. that assigns k to v_n, $s^{**}(v_n) \neq c_j, \ldots, s^{**}(v_n) \neq c_{j+k-1}$. For those variable assignment functions, it follows that

$$\mathcal{M}^*, s^{**} \not\models v_n = c_j \vee \cdots \vee v_n = c_{j+k-1}$$

and, therefore,

$$\mathcal{M}^*, s^{**} \not\models ((v_1 = c_j \vee \cdots \vee v_1 = c_{j+k-1}) \wedge \cdots \wedge (v_n = c_j \vee \cdots \vee v_n = c_{j+k-1}))$$

and, hence, $\mathcal{M}^*, s^{**} \models (11)$. Putting these two facts together, it follows that $\mathcal{M}^*, s^{**} \models (11)$, for all v_n-variants s^{**} of s^*. Consequently, $\mathcal{M}^*, s^* \models \forall v_n (11)$.

Second, suppose that Q_n is $\exists$. Then $\mathcal{M}^*, s^* \models \exists v_n (11)$, since there is a v_n-variant s^{**} of s^* that assigns k to v_n, for which the antecedent of (11) is not satisfied relative to $\mathcal{M}^*$, whence it follows that (11) is satisfied relative to $\mathcal{M}^*$. It is a tautological consequence that, if $\mathcal{M}, s \models \exists v_n \theta$, then $\mathcal{M}^*, s^* \models \exists v_n (11)$. So, if $\mathcal{M}, s \models Q v_n \theta$, then $\mathcal{M}^*, s^* \models Q v_n (11)$. To repeat, the latter holds for any two variable assignment functions s and s^*, defined respectively over D and D^*, which agree on $v_1, \ldots, v_{n-1}$.

Next, consider Q_{n-1}. We will now suppose that the two variable assignment functions s and s^*, defined respectively over D and D^*, agree on $v_1, \ldots, v_{n-2}$, which are all the free variables that may occur in $Q_{n-1} v_{n-1} Q_n v_n \theta$.

[6] Compare to the extensionality lemma (Boolos et al. 2003, p. 118-119).

First, suppose that Q_{n-1} is $\forall$. Assume that $\mathcal{M}, s \models \forall_n v_{n-1} Q_n v_n \theta$. By the definition of satisfaction, $\mathcal{M}, s' \models Q_n v_n \theta$, for all v_{n-1}-variants s' of s. Note that all these variants assign elements of $\{n \mid n \leq k-1\}$ to $v_1, \ldots, v_{n-2}$. Moreover, the set of all v_{n-1}-variants of s includes the set of all v_{n-1}-variants of s that assign elements of $\{n \mid n \leq k-1\}$ to v_{n-1}. Therefore, it follows that $\mathcal{M}, s' \models Q_n v_n \theta$, for all s' that assign elements of $\{n \mid n \leq k-1\}$ to $v_1, \ldots, v_{n-2}$, which are all the free variables that may occur in θ. By what has been proved above, it follows that $\mathcal{M}^*, s^{**} \models Qv_n(11)$, for all v_{n-1}-variants s^{**} of s^* that assign elements of $\{n \mid n \leq k-1\}$ to $v_1, \ldots, v_{n-2}$. For any variable assignment function s^{**} that assigns to v_{n-1} an element of D^* outside $\{n \mid n \leq k-1\}$, i.e. that assigns k to v_{n-1}, $s^{**}(v_{n-1}) \neq c_j$, ..., $s^{**}(v_{n-1}) \neq c_{j+k-1}$. For those variable assignment functions, it follows that

$$\mathcal{M}^*, s^{**} \not\models v_{n-1} = c_j \vee \cdots \vee v_{n-1} = c_{j+k-1}$$

and, therefore,

$$\mathcal{M}^*, s^{**} \not\models ((v_1 = c_j \vee \cdots \vee v_1 = c_{j+k-1}) \wedge \cdots \wedge (v_n = c_j \vee \cdots \vee v_n = c_{j+k-1}))$$

and, hence, $\mathcal{M}^*, s^{**} \models (11)$. For any v_n-variant s^{***} of s^{**}, it also holds that $\mathcal{M}^*, s^{***} \models (11)$, since $s^{***}(v_{n-1}) = k$. So, by the definition of satisfaction, $\mathcal{M}^*, s^{**} \models \forall v_n(11)$. It follows from the latter that $\mathcal{M}^*, s^{**} \models \exists v_n(11)$. Therefore, $\mathcal{M}^*, s^{**} \models Qv_n(11)$. Putting these two facts together, it follows that

$$\mathcal{M}^*, s^{**} \models (11),$$

for all v_{n-1}-variants s^{**} of s^*. Consequently, $\mathcal{M}^*, s^* \models \forall v_{n-1} Q v_n(11)$.

Second, suppose that Q_{n-1} is $\exists$. Then $\mathcal{M}^*, s^* \models \exists v_{n-1} Q v_n(11)$ for the following reason. There is a v_{n-1}-variant s^{**} of s^* that assigns k to v_{n-1}, for which the antecedent of (11) is not satisfied relative to $\mathcal{M}^*$, whence it follows that (11) is satisfied relative to $\mathcal{M}^*$. The latter holds for all v_n-variants s^{***} of s^{**} as well and, therefore, $\mathcal{M}^*, s^{**} \models \forall v_n(11)$, from which it also follows that $\mathcal{M}^*, s^{**} \models \exists v_n(11)$. So, $\mathcal{M}^*, s^{**} \models Qv_n(11)$, for at least one v_{n-1}-variant s^{**} of s^*. By definition, this means that $\mathcal{M}^*, s^* \models \exists v_{n-1} Q v_n(11)$. It is a tautological consequence that, if $\mathcal{M}, s \models \exists v_{n-1} Q v_n \theta$, then $\mathcal{M}^*, s^* \models \exists v_{n-1} Q v_n(11)$.

So, if $\mathcal{M}, s \models Qv_{n-1} Q v_n \theta$, then $\mathcal{M}^*, s^* \models Qv_{n-1} Q v_n(11)$. To repeat, the latter holds for any two variable assignment functions s and s^*, defined respectively over D and D^*, which agree on $v_1, \ldots, v_{n-2}$.

Continue the above line of reasoning in an analogous way until it has been established that, if $\mathcal{M}, s \models Qv_1 \ldots Qv_n \theta$, then $\mathcal{M}^*, s^* \models Qv_1 \ldots Qv_n(11)$, for any two variable assignment functions s and s^*, defined respectively over D and D^*. Hence, if $\mathcal{M}, s \models Qv_1 \ldots Qv_n \theta$, for any variable assignment s, defined over D, then $\mathcal{M}^*, s^* \models Qv_1 \ldots Qv_n(11)$, for any variable assignment s^*, defined over D^*. By definition this means that, if $\mathcal{M} \models Qv_1 \ldots Qv_n\theta$, then $\mathcal{M}^* \models Qv_1 \ldots Qv_n(11)$. In other words, if $\mathcal{M} \models \phi$, then $\mathcal{M}^* \models (10)$. □

The conclusion of Theorem 1 is that all 'truths' of infinitism are kept (Van Bendegem 1999, p. 122). This is Van Bendegem's solution to the general problem (section 1).

2.4 Gaining 'truths' relative to infinitism

It has been argued that all 'truths' of infinitism are kept. In addition, some 'truths' will be gained (Van Bendegem 1999, p. 122). Note that the following is false in any model with as its domain the set of natural numbers and with an interpretation of R as the standard successor relation, given that (1) is true in every such model:

$$\exists x Rxx. \tag{12}$$

Yet, in the example of finite (quasi-)model given above the following is true:

$$\exists x \left(((x = c_0 \vee x = c_1)) \rightarrow Rxx\right). \tag{13}$$

The above sentence is true, because there is an element of the domain, namely 2, that when assigned to x does not satisfy the antecedent.

More generally, there is at least one sentence ϕ such that $\mathcal{M} \not\models \phi$ (with $\mathcal{M}$ a model with as its domain the set of natural numbers) but $\mathcal{M}^* \models \phi^\tau$ (with $\mathcal{M}^*$ the associated finite (quasi-)model and with τ the Van Bendegem translation). It follows from $\mathcal{M} \not\models \phi$ by definition that $\mathcal{M} \models \neg\phi$, which by Theorem 1 implies that $\mathcal{M}^* \models (\neg\phi)^\tau$ as well. So, there is at least one sentence such that $\mathcal{M}^* \models \phi^\tau$ and $\mathcal{M}^* \models (\neg\phi)^\tau$. Van Bendegem (1999, p. 124) says that

> we get pretty close in the neighbourhood of paraconsistency[.]

Within the context of τ, ϕ and $\neg\phi$ are no longer contradictory sentences.

Van Bendegem (1999, p. 124) uses $\forall^*$ and $\exists^*$ as abbreviations for respectively the reinterpreted universal quantifier and the reinterpreted existential quantifier. With the help of that notation one can rewrite (13) as follows:

$$\exists^* x Rxx. \tag{14}$$

Given the truth of (1) and Theorem 1 it is also the case that the following is true in finite (quasi-)models:

$$\forall^* x \neg Rxx. \tag{15}$$

Note that, if (14) is an abbreviation of the form ϕ^τ, then (15) is an abbreviation of the form $(\neg\phi)^\tau$. This means that the reinterpreted quantifiers are no longer so-called 'duals', i.e. there are formulas ϕ such that both $\mathcal{M}^* \models \forall^* v_i \neg\phi$ ($\mathcal{M}^* \not\models \neg\forall^* v_i \neg\phi$) and $\mathcal{M}^* \models \exists^* v_i \phi$. In contrast, in classical logic $\forall$ and $\exists$ are duals (i.e. $\exists v_i \phi$ is equivalent to $\neg\forall v_i \neg\phi$).

As we will see in the next section, we already have here one element of a new problem.

3 The problem of internalizing negative ontological commitments

I will introduce some concepts and distinctions that will be very useful for the evaluation of Van Bendegem's solution to the general problem.

The first concept is that of ontological commitment. According to Quine's famous criterion (Quine 1948, Bricker 2016), a theory T has (explicit) ontological commitment to K's if and only if T logically entails $\exists x Kx$ (with $\exists$ the objectual existential quantifier). Let us call the latter a criterion for *positive* (explicit) ontological commitment. From it one can easily derive a criterion for *lack* of positive ontological commitment: a theory T does not have (explicit) ontological commitment to K's if and only if T does not logically entails $\exists x Kx$ (with $\exists$ the objectual existential quantifier). In the latter negation has wide scope of each of the sides of the equivalence. If one gives negation narrow scope in each of the sides of the equivalence one gets a criterion for *negative* ontological commitment: a theory T has (explicit) negative ontological commitment to K's if and only if T logically entails $\neg\exists x Kx$ (with $\exists$ the objectual existential quantifier). For example, classical number theory has positive (explicit) ontological commitment to prime numbers and has negative (explicit) commitment to even prime numbers larger than two. This is a first useful conceptual distinction.[7] Ontology is not only about what there is but also about what there is not. The ontology of monotheism does not only recognize the existence of a god but also the denial of the existence of any other gods.

Another useful conceptual distinction is that between internal and external ontological questions Carnap (1950). An example of an ontological question is: 'Do natural numbers exist?' This question can be asked *internal* to a framework, e.g. classical number theory. Since it is provable within the latter that five is a natural number, it follows logically within the framework of classical number theory that numbers exist. Alternatively, one could ask the same question relative to a kind of nominalistic framework in which it is false that, for instance, three is a prime number, let alone that natural numbers exist. What Carnap denies, is that it makes sense to ask the ontological question about the

[7] Asay (2010) also makes this distinction.

existence of numbers *external* to frameworks.[8] Of course, this leaves open that the ontological commitments of one framework can be *internalized* within the other framework via a reinterpretation or translation. Whether Van Bendegem's translation strategy is adequate is very important, because it is about how the two parties to the debate *internalize the ontological commitments* of each other.

Let's start from a infinite model, $\mathcal{M}$, with the set of natural numbers as its domain, and the associated theory, T, namely the set of sentences that are true in $\mathcal{M}$. After a choice of the largest natural number, we then also have a finite (quasi-)model, $\mathcal{M}^*$, and its associated theory, T^*, namely the set of sentences that are true in $\mathcal{M}^*$. The question before us is how T^* internalizes the ontological commitments of T relative to Van Bendegem's translation.

Let us begin with the positive (explicit) ontological commitments of T. First, going by those sentences of the form $\exists^* v\phi$ that belong to T^* is not going to fly, because it was already established in section 2.4 that $\mathcal{M}^* \models \exists^* xRxx$, whereas $\mathcal{M} \not\models \exists xRxx$. In other words, as a criterion it would *overshoot*. Second, going by sentences of the form $\neg\forall^* v\neg\phi$ does work. If $\mathcal{M}^* \models \neg\forall^* v\neg\phi$, then $\mathcal{M}^* \not\models \forall^* v\neg\phi$, whence it follows by Theorem 1 that $\mathcal{M} \not\models \forall v\neg\phi$, which entails that $\mathcal{M} \models \neg\forall v\neg\phi$ and, therefore, $\mathcal{M} \models \exists v\phi$. In other words, there is a way of internalizing the positive (explicit) ontological commitments of T in T^*.

Next, let us consider the negative (explicit) ontological commitments of T. First, going by those sentences of the form $\neg\exists^* v\phi$ that belong to T^* is not going to fly, because $\mathcal{M}^* \models \exists^* v\phi$, *for all* ϕ, and therefore, there are *no* sentences of the form $\neg\exists^* v\phi$ that belong to T^*. In other words, as a criterion it would *undershoot*. Second, going by sentences of the form $\forall^* v\neg\phi$ would also not work, because there are formulas ϕ such that $\mathcal{M}^* \models \forall^* v\neg\phi$ but $\mathcal{M}^* \not\models \forall v\neg\phi$. Take $\forall^* x\neg\neg x < k$ (with $x < k$ eliminated as in Van Bendegem's translation). Clearly, $\forall x\,(x < k \rightarrow \neg\neg x < k)$ is logically true and, therefore, also true in $\mathcal{M}^*$. Yet, in $\mathcal{M}$ with as its domain the set of natural numbers, it is false that $\forall x\neg\neg x < k$. In other words, as a criterion, it *overshoots*. So, relative to the Van Bendegem translation strategy there is no way of internalizing the the negative (explicit) ontological commitments of T in T^*.

To sum up, relative to Van Bendegem's translation the apophatic finitist can internalize the positive but not the negative (explicit) ontological commitments of the infinitist. This is the new problem that the apophatic finitist is facing. In the next section a new approach is considered, one in which both the positive and the negative (explicit) ontological commitments can be internalized in one of the frameworks.

[8]For a contemporary defense of a broadly Carnapian position in meta-metaphysics, see (Chalmers 2009).

4 Relative interpretation and absolute infinity

In section 4.1 I will briefly introduce the notion of a relative interpretation and compare it to Van Bendegem's translation. In section 4.2 a particular relative interpretation will be introduced and it will be connected to a variety of infinitism called 'apophatic infinitism' that is related to apophatic finitism. In section 4.3 it will be shown that the ontological commitments, both positive and negative, of apophatic finitism can be internalized within apophatic infinitism via the relative interpretation introduced earlier.

4.1 Relative interpretation and the Van Bendegem translation

The key property of Van Bendegem's translation strategy is the reinterpretation of the existential quantifier ($\exists v_i$). In a so-called 'relative interpretation' an existential quantifier $\exists v$ is replaced by:

$$\exists v(\alpha\,(v) \wedge \ldots).$$

In constrast, in Van Bendegem's translation an existential quantifier $\exists v_i$ was reinterpreted as something of the following form:

$$\exists v\,(\alpha\,(v) \rightarrow \ldots)\,.$$

Van Bendegem (2003, p. 243, fn. 13) is aware that his translation strategy deviates from classical relative interpretation in this respect.

In addition, within relative interpretations the reinterpreted existential quantifier, which is of the form $\exists v\,(\alpha\,(v) \wedge \ldots)$, is the dual of the reinterpreted universal quantifier, which is of the form $\forall v\,(\alpha\,(v) \rightarrow \ldots)$: $\exists v\,(\alpha\,(v) \wedge \ldots)$ is equivalent to $\neg\forall v\,(\alpha\,(v) \rightarrow \neg\ldots)$. Recall that Van Bendegem's reinterpreted universal quantifier $\forall v$ is not the dual of his reinterpreted existential quantifier: i.e. there are formulas ϕ such that in a finite (quasi-)model $\mathcal{M}^*$ both $\mathcal{M}^* \models \forall^* v \neg\phi$ ($\mathcal{M}^* \not\models \neg\forall^* v \neg\phi$) and $\mathcal{M}^* \models \exists^* v\phi$.

Given two models, $\mathcal{M}'$ and $\mathcal{M}''$, let us call a translation π of $\mathcal{M}'$ into $\mathcal{M}''$ faithful if and only if, for every sentence ϕ, $\mathcal{M}' \models \phi$ if and only if $\mathcal{M}'' \models \phi^\pi$ (with ϕ^π the translation π of ϕ). We have already seen in Subsection 2.4 that there is at least one sentence ϕ, namely (12), such that $\mathcal{M} \not\models \phi$ (with $\mathcal{M}$ a model with as its domain the set of natural numbers) but $\mathcal{M}^* \models \phi^\tau$ (with $\mathcal{M}^*$ the associated finite (quasi-)model and with τ the Van Bendegem translation, which translates (12) into (13)). The question that we will answer in the next section is whether there is a faithful relative interpretation of a finite (quasi)-model into a infinite model of a certain kind.

The connection with the problem of how to internalize the negative (explicit) ontological commitments of a theory is as follows. If the relative interpretation is faithful, then the truth/falsity within the kind of infinite model of the reinterpreted existentially quantified sentences that are of the form

$$\exists v \left(\alpha\left(v\right) \wedge \phi\right),$$

is equivalent to the truth/falsity within the finite (quasi-)model of the existentially quantified sentences that are of the form $\exists v\phi$. So an infinitist of the variety alluded to would be able to internalize both the positive and the negative (explicit) ontological commitments of the apophatic finitist. In the next section more will be said about the kind of infinitist alluded to and about the relative interpretation that goes with it.

4.2 Apophatic infinitism

Before I define a relative interpretation, let me comment on a choice that can be made. One option is to use individual constants, as before. In our working example these were c_0 and c_1. Another option is to use individual descriptions. In a variation on the example we make use of the distinguished relation symbol R, which is informally interpreted as before as 'is succeeded by' and which is formally interpreted as the standard successor relation on natural numbers. We could then use the following individual descriptions:[9]

1. $Zero\left(v\right) := \neg\exists yRyv \wedge \exists yRvy$
2. $One\left(v\right) := \exists y_1\left(Zero\left(y_1\right) \wedge Ry_1v\right)$

Of course, the above could be extended to any finite natural number. The choice between using individual constants and using individual descriptions is more interesting once one introduces a non-standard number. Think for a moment about Van Bendegem's largest natural number that does not satisfy any primitive non-logical predicate. We could introduce an individual constant for it, e.g. c_3. Alternatively, we could define it (in a negative way):

3. $Abs\left(v\right) := \neg\exists y_2Ry_2v \wedge \neg\exists y_2Rvy_2$

The second conjunct is needed to distinguish the non-standard number from zero, which does have a successor, and the first conjunct is needed to distinghuish it from the second-largest number, which does have a predecessor.

[9]Note that on the interpretation of R as the standard successor relation it is true that (1) and, hence, that successors are distinct from their predecessors. As a result, $Zero\left(v\right)$ and $One\left(v\right)$ have to be satisfied by different elements of the domain. Furthermore, it is true on that same interpretation that $\forall x\forall y\forall z\left(\left(Rxy \wedge Rxz\right) \rightarrow y = z\right)$ or, in other words, successors are unique.

Whichever option one prefers may depend on whether one thinks that one can have knowledge by acquaintance or merely knowledge by description of the largest number. (For the distinction between those two types of knowledge, see (Russell 1910).) Relatedly, it may depend on the conditions under which one can introduce a name by description (Kripke 1980; 2011). Here I will choose the definitional, descriptive approach.

Let a finite set S of sentences belonging to the language of first-order logic with identity be given. Let there be a finite (quasi-)model $M^* = \langle D^*, I^* \rangle$ with $D^* = \{n \in \mathbb{N} \mid n \leq k\}$, which makes all the sentences in S true. If R belongs to the language of S, then it is furthermore presupposed that the interpretation I^* of R is the standard successor relation on the natural numbers restricted to $\{n \in \mathbb{N} \mid n \leq k-1\}$. (Otherwise some relettering would be required.) Then for all formulas ϕ belonging to the language of S, define ϕ^π as follows:

1. if ϕ is an atomic formula, then $\phi^\pi = \phi$;
2. if $\phi = \neg\psi$, then $\phi^\pi = \neg\psi^\pi$;
3. if $\phi = (\psi \wedge \theta)$, then $\phi^\pi = (\psi^\pi \wedge \theta^\pi)$;
4. if $\phi = \forall v \psi$, then $\phi^\pi = \forall v \left((v < k \vee Abs\,(v)) \to \psi^\pi\right)$.

Note that in the above $v < k$ can be replaced by:

$$Zero\,(v) \vee One\,(v)\,,$$

assuming that $k = 2$ (as in our working example). Then define a infinite model $\mathcal{M}^\dagger$ as follows:

- the domain $D^\dagger$ is $\mathbb{N} \cup \{\mathfrak{g}\}$;[10]
- for the non-logical vocabulary of the sentences in S, let the interpretation function I^* of $\mathcal{M}^*$ be the restriction of the interpretation function $I^\dagger$ of $\mathcal{M}^\dagger$ to $\{n \in \mathbb{N} \mid n \leq k-1\}$;[11]
- let $I^\dagger\,(R)$ be the standard successor relation on $\mathbb{N}$.

The symbol $\mathfrak{g}$ could be taken to stand for God, if one accepts negative theism, i.e. the doctrine that no (primitive) predicate applies to God. Indeed, in the model no atomic formula is true of $\mathfrak{g}$. There is a clear analogy with apophatic finitism, which is the doctrine that holds that no (primitive) predicate is true of the largest number (see section 2.2). One could also link it to Cantor's notion of the 'absolute infinite', which he associates with God and which he considers in some sense beyond description. As we will see, only $\mathfrak{g}$ can satisfy the predicate Abs, which could be taken to stand for 'absolute infinity'. Let us talk about 'apophatic infinitism' in this context.

[10] The cardinality of this domain is the same as the cardinality of the set of natural numbers.

[11] There are many such $I^\dagger$, but that does not matter: any interpretation function that satisfies the condition will do.

4.3 Internalizing all ontological commitments

As we are about to see, the ontological commitments, both positive and negative, of apophatic finitism can be internalized within apophatic infinitism. A key role is played by 'absolute infinity', which has the same role as the 'largest number'.

THEOREM 2 Let a finite set S of sentences belonging to the language of first-order logic with identity be given. Let there be a finite (quasi-)model $M^* = \langle D^*, I^* \rangle$ with $D^* = \{n \in \mathbb{N} \mid n \leq k\}$, which makes all the sentences in S true and, if R occurs in S, then the interpretation I^* of R is the standard successor relation on the natural numbers restricted to $\{n \in \mathbb{N} \mid n \leq k - 1\}$. Then, for all sentences $\phi \in S$, $\mathcal{M}^* \models \phi$ if and only if $\mathcal{M}^\dagger \models \phi^\pi$.

PROOF I am going to prove this via a slightly more general result: for all formulas ϕ of the language of S, for all variable assignment functions s^* defined over D^*, for all variable assignment functions $s^\dagger$ that are defined over $D^\dagger$ and that assign elements of $\{n \in \mathbb{N} \mid n \leq k - 1\} \cup \{\mathfrak{g}\}$ to any variable v that occurs freely in ϕ (if any) such that

1. $s^*(v) = s^\dagger(v)$ or
2. $s^*(v) = k$ and $s^\dagger(v) = \mathfrak{g}$,

it is the case that $\mathcal{M}^*, s^* \models \phi$ if and only if $\mathcal{M}^\dagger, s^\dagger \models \phi^\pi$. The proof is by induction on the complexity of ϕ.

First, suppose that ϕ is an atomic formula. Note that by definition $\phi^\pi = \phi$. The interpretations of the predicates and individual constants occurring in S are by the definition of $\mathcal{M}^\dagger$ exactly the same in $\mathcal{M}^*$ and $\mathcal{M}^\dagger$. Furthermore, for any variable v that occurs freely in ϕ,

1. $s^*(v) = s^\dagger(v)$ or
2. $s^*(v) = k$ and $s^\dagger(v) = \mathfrak{g}$,

If s^* and $s^\dagger$ agree on all the free variables in ϕ (if any), then there is agreement on the denotations of all the terms (variables or individual constants) in ϕ. Given that there is also agreement on the extensions of the predicates, it follows then from the definition of satisfaction of atomic formulas that $\mathcal{M}^*, s^* \models \phi$ if and only if $\mathcal{M}^\dagger, s^\dagger \models \phi^\pi$. If s^* and $s^\dagger$ disagree on at least one free variable in ϕ (if any), then s^* assigns k to that free variable and $s^\dagger$ assigns $\mathfrak{g}$. By the stipulations on $\mathcal{M}^*$ and $\mathcal{M}^\dagger$ neither element is within the extension of any of the predicates that occur in S. Therefore, if s^* and $s^\dagger$ disagree on at least one free variable in ϕ (if any), then it follows from the definition of satisfaction of atomic formulas that $\mathcal{M}^*, s^* \not\models \phi$ and $\mathcal{M}^\dagger, s^\dagger \not\models \phi$, whence it follows that $\mathcal{M}^*, s^* \not\models \phi$ if and only if $\mathcal{M}^\dagger, s^\dagger \not\models \phi$. Either way, it follows that $\mathcal{M}^*, s^* \not\models \phi$ if and only if $\mathcal{M}^\dagger, s^\dagger \not\models \phi$.

Second, the cases of negation and conjunction are left to the reader.

Third, suppose that $\phi = \forall v \psi$. Recall that $I^\dagger$ interprets R as the standard successor relation on the natural numbers. Since on the standard interpretation only zero has no predecessor, if $Zero\,(v)$ is satisfied by a variable assignment s, then $s\,(v) = 0$. Moreover, due to the fact that on the standard interpretation the successor of zero is one, it is also the case that, if $One\,(v)$ is satisfied by a variable assignment s, then $s\,(v) = 1$. This can be continued. Finally, given that that under the interpretation of R as the standard successor relation on the natural numbers every natural number has a successor, if $Abs\,(v)$ is satisfied by a variable assignment s, then $s\,(v) \notin \mathbb{N}$ and, therefore, $s\,(v) = \mathfrak{g}$. Now suppose that $\mathcal{M}^*, s^* \models \forall v \psi$. By the definition of satisfaction of universally quantified formulas the latter is equivalent to the claim that, for all v-variants s^{**} of s^*, $\mathcal{M}^*, s^{**} \models \psi$. Given the induction hypothesis, the latter is in turn equivalent to $\mathcal{M}^\dagger, s^{\dagger\dagger} \models \psi^\pi$, for all variable assignments $s^{\dagger\dagger}$ such that, for any variable v' that occurs freely in ψ (if any),

1. $s^{**}\,(v') = s^{\dagger\dagger}\,(v')$ or
2. $s^{**}\,(v') = k$ and $s^{\dagger\dagger}\,(v') = \mathfrak{g}$.

Recall that, if $v' \neq v$, then

1. $s^*\,(v') = s^{**}\,(v') = s^{\dagger\dagger}\,(v')$ or
2. $s^*\,(v') = s^{**}\,(v') = k$ and $s^{\dagger\dagger}\,(v') = \mathfrak{g}$.

Note that all the variable assignments $s^{\dagger\dagger}$ assign elements from $\{n \in \mathbb{N} \mid n \leq k-1\} \cup \{\mathfrak{g}\}$ to v. Given what has been said above about the satisfaction of $Zero$, One and Abs and the satisfaction clause for material implication, the latter is equivalent to

$$\mathcal{M}^\dagger, s^{\dagger\dagger} \models (Zero\,(v) \vee One\,(v) \vee Abs\,(v)) \rightarrow \psi^\pi,$$

for all variable assignment functions $s^{\dagger\dagger}$ that are v-variants of $s^\dagger$ in light of the enumerated conditions. Based on the satisfaction clause of universally quantified formulas and the stipulation, the latter is equivalent to

$$\mathcal{M}^\dagger, s^\dagger \models \forall v\,((Zero\,(v) \vee One\,(v) \vee Abs\,(v)) \rightarrow \psi^\pi)\,.$$

Now that the general result has been proven, the theorem can be derived as a special case, since sentences do not contain any free variables. □

It is an immediate consequence of Theorem 2 that, under the conditions listed in the theorem, for all sentences $\exists v\psi \in S$, $\mathcal{M}^* \models \exists v\psi$ if and only if $\mathcal{M}^\dagger \models (\exists v\psi)^\pi$. In other words, apophatic infinitism can via the relative interpretation defined in section 4.2 internalize the ontological commitments, both positive and negative, of apophatic finitism. This is important for two reasons.

First, as we have seen, the Van Bendegem translation does not enable the apophatic finitist to internalize the negative ontological commitments of infinitism. This is like a polytheist grasping that a monotheist believes in the existence of a god without grasping that that person rejects the existence of any other gods. The relative interpretation defined in section 4.2 does enable the apophatic infinist to internalize the negative ontological commitments of apophatic finitism.

Second, the apophatic finist cannot use the same relative interpretation to internalize the negative ontological commitments of apophatic infinitism. Take the following example:

$$\exists x R c_1 x, \tag{16}$$

with c_1 the name of the second largest natural number, which in our working example is the number one. Clearly, (16) is true in any of the infinitist models that have a domain that includes the natural numbers and that interpret R as the standard successor relation on the natural numbers. Indeed, those models have the number two in their domain and two is the successor of one. Yet, the following is false in a finite (quasi-)model with as its domain $\{0, 1, 2\}$ and with an interpretation of R as the standard successor relation, restricted to $\{0, 1\}$:

$$\exists x \left(\left(Zero\left(x\right) \vee One\left(x\right) \vee Abs\left(x\right) \right) \wedge R c_1 x \right). \tag{17}$$

So, the relative interpretation fails to give the apophatic finitist the means to internalize all the *positive* ontological commitments of the apophatic infinitist. This is like the Greek polytheist grasping that the Jewish monotheist repudiates the existence of Ares but failing to grasp that that person accepts the existence of Yahweh.

5 Conclusion

Finitism is the ontological doctrine that there exist only finitely many numbers, whereas infinitism is the ontological doctrine that there exist infinitely many numbers. The general problem for finitism is that even finite sets of sentences from the language of first-order logic are sometimes only true in models with infinite domains. This is connected to the problem of recasting finite fragments of classical mathematics in finitist terms. To solve this problem Van Bendegem has proposed a translation strategy and a new type of models, namely finite (quasi-)models. A striking feature of the latter is that they contain a 'largest number' that does not satisfy any of the primitive predicates. Hence why Horsten dubbed this 'apophatic finitism'. As it turns out, the 'truths' of infinitism, namely the sentences that are true in a given model with an infinite domain that are translated according to Van Bendegem's translation strategy, are also 'truths' of finitism in the sense that they are true in an appropriate (quasi-)model.

With Van Bendegem's solution to the general problem there came a new problem on the horizon: Van Bendegem's translation strategy is adequate when it comes to representing within apophatic finitism the positive ontological commitments of infinitism but not its negative ontological commitments. To repeat, this is like a polytheist grasping that a monotheist believes in the existence of a god without grasping that that person rejects the existence of any other gods. It was then shown that there is a variety of infinitism, namely apophatic infinitism, for which it holds that, under a certain relative interpretation, it can represent both the positive and the negative ontological commitments of apophatic finitism. This makes that relative interpretation superior to Van Bendegem's translation. Alas, it turns out that it cannot be used by the apophatic finitist to represent the positive ontological commitments of infinitism (whether it is apophatic or not). To repeat, this is like the Greek polytheist grasping that the Jewish monotheist repudiates the existence of Ares but failing to grasp that that person accepts the existence of Yahweh.

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