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Proof phenomenon as a function of the phenomenology of proving

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ABSTRACT

Kurt Gödel wrote (1964, p. 272), after he had read Husserl, that the notion of objectivity raises a question: “the question of the objective existence of the objects of mathematical intuition (which, incidentally, is an exact replica of the question of the objective existence of the outer world)”. This “exact replica” brings to mind the close analogy Husserl saw between our intuition of essences in *Wesensschau* and of physical objects in perception. What is it like to experience a mathematical proving process? What is the ontological status of a mathematical *proof*? Can *computer assisted provers* output a *proof*? Taking a *naturalized world* account, I will assess the relationship between mathematics, the physical world and consciousness by introducing a significant conceptual distinction between *proving* and *proof*. I will propose that *proving* is a phenomenological conscious experience. This experience involves a combination of what Kurt Gödel called *intuition*, and what Husserl called *intentionality*. In contrast, *proof* is a function of that process — the mathematical phenomenon — that objectively self-presents a property in the world, and that results from a spatiotemporal unity being subject to the exact laws of nature. In this essay, I apply phenomenology to mathematical proving as a performance of consciousness, that is, a lived experience expressed and formalized in language, in which there is the possibility of formulating intersubjectively shareable meanings.

If you will stay close to nature, to its simplicity, to the small things hardly noticeable, those things can unexpectedly become great and immeasurable.

—Rainer Maria Rilke, Letters to a Young Poet.

1. *Naturalizing* mathematics

Husserl wanted to ensure that basic categories employed by natural science were not thought to be products of some merely such contingent features. In fact, he tried to define the limits of what science, or naturalism could inform us of (Gallagher, 2012). He considered that “naturalism is a phenomenon consequent upon the discovery of nature … considered as a unity of spatiotemporal being subject to exact laws of nature. With the gradual realization of this idea in constantly new natural sciences that guarantee strict knowledge regarding many matters, naturalism proceeds to expand more and more” (Husserl, 1965, p. 79). Husserl was not opposed to natural scientific explanation; rather, he considered that an extreme naturalism in formal logic, mathematics, and ideal essences might lead to their reduction to psychological processes of the knowing subject. In his perspective, regarding an extreme version of naturalism, if our brain processes evolve over time (which they do), then the laws of nature may be different in the future. In other words, both psychological processes and laws of nature are subject to biological evolution. Nevertheless, Husserl considered that the results of transcendental phenomenology should not be ignored by science, as “every analysis of theory of transcendental phenomenology—including … the theory of the transcendental constitution of an objective world—can be developed in the natural realm, by giving up the transcendental attitude” (1970, §57).

Certain authors such as De Preester (2002) and Lawlor (2009) consider that naturalizing phenomenology is a contradiction in terms, since phenomenology is, by definition, non-naturalistic. Nevertheless, Merleau-Ponty's work (1942; 1945) seems to contain direct suggestions for naturalizing phenomenology. As reported by Merleau-Ponty, with science, one is expressive in relation to nature (1945, p. 391). This fundamentally changes Husserl's transcendental conception and shifts the focus from the transcendental ego to the body.

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Cognitive Science has grounded this view of the body as a decisive instance in bringing about behavioural and mental capacities (De Preester, 2002, p. 654). The introduction of phenomenology in cognitive science has challenged its basic assumptions and has brought a view that is more consistent with the views of Husserl and Merleau-Ponty concerning intentionality, intersubjectivity action, and embodiment.[1] Contemporary *embodied* cognitive science—contrary to its previous orthodox view—is grounded in the ecological-enactive approach (Bermudez et al., 1995; Clark, 1997; Varela et al., 1991). This perspective claims that cognition is best characterized as belonging to embodied, situated agents, i.e. agents who are in the world (Gallagher and Varela, 2003). In this approach, researchers in artificial intelligence and robotics, phenomenologists and philosophers of mind work together to advance an understanding of the embodied, ecologically situated and enactive mind.

For many authors the difficult question is how *naturalized phenomenology* can be accomplished without losing the specificity of phenomenology. In this regard, it is of great importance to consider what one means by *naturalization*. Naturalization can be defined as to "integrate into an explanatory framework where every acceptable property is made continuous with the properties admitted by the natural science" (Roy et al., 1999, pp. 1–2). Naturalization means, "not being committed to a dualistic kind of ontology" (1999, p. 19). Roy et al. (1999) propose a recategorization of phenomena at a level of abstraction necessary to acknowledge the common properties between phenomenological data and objective data developed in the sciences. As reported by this interdisciplinary group of researchers at the Centre de Recerche en Epistémologie Appliquée (CREA), "It is our general contention … that phenomenological descriptions of any kind can only be naturalized, in the sense of being integrated into the general framework of natural sciences, if they can be mathematized." (1999, p. 42). Accordingly, this idea involves a mathematical interpretation, i.e. a transformation of concepts into algorithms similar to transformations of this kind found in the physical sciences, through a formal language that expresses phenomenological findings. This appeal to mathematics demands formalization and intersubjective meaning verifiable within a common language that is clearly understood by science, namely, mathematics.

Husserl considered mathematical formula as incapable of capturing phenomenological results, as "one cannot define in philosophy as in mathematics; any imitation of mathematical procedure in this respect is not only unfruitful but wrong, and has most injurious consequences" (Husserl, 1976, p. 9). According to Roy et al. (1999), this may have been accurate in mathematics in Husserl's time; however, the development of dynamic systems theory offered new possibilities in this regard (p. 43). In fact, the opposition Husserl introduces between mathematics and phenomenology is "the result of having mistaken certain contingent limitations of the mathematical and material sciences of his time for absolute ones. In our opinion, it is indeed arguable that scientific progress has made Husserl's position on this point largely obsolete and that this *factum rationis* puts into question the properly scientific foundations of his anti-naturalism" (pp. 42–43). In other words, most of Husserl's scientific reasons for opposing naturalism have been invalidated by the progress of science (p. 54). In fact, as illustrated bellow, the editors claim that a genuine mathematical description of experiential consciousness is possible in the construction of a mathematical proof. Therefore one of the major impediments to the naturalization of phenomenology has been removed (pp. 55–56). The essential property in mathematical formalism is its exactness regardless of neurobiological or phenomenological facts (pp. 51, 68). The moment we are in the possession of a mathematical reconstruction of phenomenological descriptions, the only remaining problem is to articulate those reconstructions with the tools of relevant lower-level natural sciences (pp. 48, 63).

This proposal inspired by Marbach's work (1993) who, following Husserl's own proposal[2] for formal notation, suggested a formal symbolic language for phenomenology which developed a formalized notation, and assessed the question of whether it is possible for mathematics to capture the lived experience described by phenomenology. Marbach (1993, 2010) proposed that formalizing language can improve the possibility of formulating intersubjective shareable meanings.

2. Proving as a mathematical description of experiential consciousness

Part of Husserl's work was to provide an adequate phenomenological description of consciousness not contained within any well-established materialistic or naturalistic framework. Moreover, Husserl believed that a proper understanding of the conscious appropriation of the world would provide not only an understanding about consciousness but also about the world. Consciousness is, in his perspective, a place where the world can reveal and articulate itself. Phenomenology is concerned with transcendental subjectivity and not with empirical consciousness. Merleau-Ponty called for a redefinition of transcendental philosophy (1942, p. 241) that does not make us choose between either an external scientific explanation, or an internal phenomenological reflection: one does not unravel the relation between consciousness and nature (Zahavi, 2004). A redefinition that is beyond both objectivism and subjectivism. As reported by Merleau Ponty, "the ultimate task of phenomenology as philosophy of consciousness is to understand its relationship to non-phenomenology. What resists phenomenology within us—natural being, the 'barbarous' source Schelling spoke of—cannot remain outside phenomenology and should have its place within it" (1964, p. 178). In fact, Merleau-Ponty goes a step further since he considers that phenomenology can be changed and modified through its dialogue with the empirical disciplines. The theory of mind and cognition must begin with categories of things in the everyday common sense world—what Husserl called the *life-world*, that world which lies between quarks and the cosmos. This *life-world* is a horizon of all our experiences. In fact, it is that background on which all things appear as themselves and are meaningful. This *life-world* cannot, however, be understood in a purely static manner; but rather a dynamic horizon in which we *live*, and which "lives with us" in the sense that nothing can appear in our *life-world* except as *lived*.

A phenomenology of consciousness cannot begin the ontology of the world-around-us by dealing with bosons and black holes, or neurons and the neural nets, abstracting so far from our familiar concerns that we no longer know where we fit in.

From a phenomenological perspective, mathematics is a performance of consciousness, a mathematical experiential consciousness that involves the notion of *intentionality*. Brentano considered that every mental phenomenon contains the "intentional inexistence" of an object toward which the mental phenomenon is directed. From his perspective, identifying intentionality opens up the possibility of comprehending the mind

[1] See Gallagher and Varela (2003); Thompson (2007); Varela et al. (1991).

[2] See Husserl (2001), 5th Investigation, §39, and Husserl (2005). Text No. 14 (1911–1912), pp. 323–377; Marbach (2010). Marbach (2010) also notes the connection with Frege's *Begriffsschrift*.

in its relatedness level. Husserl studied Brentano and, in fact, intentionality is the core of his work on the phenomenology of consciousness. By intentionality, Husserl meant the directionality of acts of consciousness. Consciousness is constituted by directional acts, not things. Each act of consciousness is directed toward something (1982). In this respect, to be conscious is to be conscious of something, even if the object of consciousness does not exist anywhere outside that particular act of consciousness, which seems to be the case for the process of proving a mathematical proof.

In the light of our considerations, mathematical proving is a directedness of experience toward things in the world. This process is deeply grounded in consciousness as intentionality without reducing consciousness or its intentionality to a causal or computational process along the lines envisioned by cognitive science. This is a process that is an intentional act, and that can be distinguished from its object in the world. In other words, it is a phenomenological conscious experience as experienced from the subjective first person point of view, that is, an epistemological act that uses valid reasoning to achieve knowledge. This process of reasoning implies situated cognition, in other words, an act of knowing that is embedded in the natural context.

Mathematical experience appeals to directedness and attention to actively process specific information present in our environment. According to William James, an attentive process requires "possession of the mind, in clear and vivid form" (1890). Furthermore, this process implies attention on the immediate experience and a state of current awareness, in Gödel's words, "directing our attention in a certain way, namely, onto our own acts in the use of these concepts" (1981).

The experience of mathematical proving is a brain-body-world coupling: cognitive agents bring forth a world by means of the activity of their situated living bodies. In this cognitive experience, knowledge emerges through the primary agent's bodily engagement with the environment, rather than being simply determined by and dependent upon either pre-existent situations or personal construals. Varela et al. (1991) describe this experienced world as portrayed and determined by mutual interactions between the physiology of the organism, and its sensory-body-world constituents.

From Merleau-Ponty's perspective the process of a mathematical proof presupposes an embodied subject, since he considers "the subject of geometry is a motor subject" (1964). Moreover, in his *The Prose of the World* (1973), Merleau-Ponty argues that algebraic proof presupposes the corporeal vectors of temporality such as "next", "succession", and "progression". This means that the body in its perspectival relations with things in the world opens up the con-fusing of spatial meaning-possibilities. Indeed, it is because I am not a transcendental subject, that I find myself having to "make sense" of it through expression (Hass, 2008, p. 158).

Furthermore, the process of mathematical proving implies a purpose or intention in action and a linguistic activity. This is an activity which Husserl called 'presentation'. For Husserl, the content of an act includes only what is *in* the act that makes the act the intentional experience it is; indeed he states that "the object is, properly speaking, nothing at all 'in' a presentation" (2001, §25).

In the light of the discussion so far, it seems accurate to acknowledge that the act of proving in mathematics seems to capture the lived intentional experience described by phenomenology. As reported by Merleau-Ponty (1971), the important thing is to fully understand the nature of a proof as an act. Proving truth is only possible when "one enjoys an absolute self-possession in active thought" (p. 447), which enables further operations to produce a valid result. Mutual exchange between phenomenology and cognitive sciences could result in a closure of the *explanatory gap*,[3] and that a mathematical reconstruction would be of any profitable sense at all or that there might be a way to explain how experiences could be properties of the brain.

2.1. Proving as Husserl's Wesensschau, and Gödel's intuition

The core of phenomenology studies involves the concept of intuition in knowledge, or, as Husserl called it, of categorical intuition. Categorical intuition is the way in which higher order objects are intuited, such as states of affairs. Indeed, in these higher forms of knowledge is what Husserl called apodictic evidence, and it is closely related to the eidetic method, the so-called *Wesensschau*.

According to Husserl, "genuine science and its own genuine freedom from prejudice require, as the foundation of all proof, immediately valid judgements which derive their validity from originally presentive intuitions" (1982, p. 36). Genuine sciences are based on attending to what is immediately given to us in experience. And for Husserl, intuition is the means by which one can trace concepts and our knowledge back to their sources, to what is immediately given to me in my experience. Furthermore, what is given are not the appearances or other representations of reality, but an aspect of reality itself (see Hintikka, 2003, p. 58).

In this perspective, it follows that the intuition of essences is a special case of categorical intuition and this is categorized through apodictic evidence (Lohmar, 2010, p. 77). Nevertheless, the question that one needs to address is how the eidetic method is used in mathematical proof and how this evidence is obtained in formal contexts.

Husserl's theory of seeing essences is contained within his theory of knowledge in the Sixth *Logical Investigation*. The eidetic method of seeing essences involves the results not being restricted to factual empirical matters-of-fact but that they also pertain to universal structures and a priori, necessary laws which are valid for all factual and all possible future cases of acts of consciousness. In fact, phenomenology is the act of achieving a priori insights into universal structures of consciousness. In other words, the structures that are independent from matters-of-fact (Lohmar, 2010, p. 78). This *Wesensschau* method starts with the simple perception of singular objects such as the elementary forms of knowledge. However, seeing these essences also demands actively engaging the mind. These objects are not in a higher "reality"; rather, they belong to the everyday world. Indeed, mathematical objects are objects of thought and we gain intuition of them, since they participate in some way in the only reality. Within this assumption, intuitivity of my intention is what is common in all cases.

Kurt Gödel expressed views on the philosophy of mathematics similar to those of Husserl. Gödel's notion of mathematical intuition is compared to perception. In his *What is Cantor's Continuum Problem* (1964, p. 271), he considers this in a famous passage from his supplement to the second edition:

> "But despite their remoteness from sense experience, we do have something like a perception of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them".

[3] This problem was introduced by Joseph Levine (1983), indicating our incomplete understanding of how consciousness might depend upon a nonconscious substrate, especially a physical substrate. There are many variations in strength. Its weakest form asserts a practical limit to our present explanatory abilities. A stronger version makes an in principle claim about tour human capacities and thus asserts that given human cognitive limits we will never be able to bridge the gap.

Gödel suggests the idea of mathematical intuition about mathematical reality, which is said to be not purely subjective. The intuition proceeds from experience, but not necessarily with the object in a natural world that Gödel was sceptical about. Mathematical objects are considered to exist independently of their construction or individual intuition. Mathematical intuition is supposed to lead us to new axioms. As reported by Føllesdal (1999, p. 397), Gödel outlined four different methods one can use to get insight into the mathematical realm:

(1) *Elementary Consequences*: mathematical objects "are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory or our sense perceptions" (Gödel, 1944, p. 128 of the reprint in Feferman et al., 1990);
(2) *Success*: fruitful and "verifiable" consequences, i.e. consequences demonstrable without a new axiom, the proofs of which with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract many different proofs into one proof;
(3) *Clarification*: Gödel remarks that "it may be conjectured that the continuum problem cannot be solved on the basis of the axioms set up so far, but, on the other hand, may be solvable with the help of some new axioms which could state or imply something about the definability of sets" (Gödel, 1990);
(4) *Systematicity*: "It turns out that in the systematic establishment [*Aufstellen*] of the axioms of mathematics, new axioms, which do not follow by formal logic from those previously established, again and again become evident. It is not at all excluded by the negative results mentioned earlier [incompleteness] that nevertheless every clearly posed mathematical yes-or-no question is solvable in this way. For it is just this becoming evident of more and more new axioms on the basis of the meaning of the primitive notions that a machine cannot imitate (Gödel, 1990).

As it is well known, Husserl had a "reflective equilibrium" that attaches great significance to systematization as a way to clarify concepts. Furthermore, the basic concepts Gödel mentions provide assurance of consistency.

Kurt Gödel's acclaimed first incompleteness theorem proves that any consistent formal system in which a "moderate amount of number theory" can be proven will be incomplete, that is, there will be at least one true mathematical claim that cannot be proven within the system.[4] In recent years, there has been an on-going discussion concerning whether Gödel's incompleteness theorems show that the mind is more than simple machines. These are the anti-mechanist arguments that claim that there is at least one thing that the human mind can do that computers cannot, this is, the human can see that the Gödel Sentence is true but a machine could not have this insight, since the machine must always follow rules as a formal system. There are a considerable number of articles concerning the mechanist and the non-mechanist discussion.[5] Authors of anti-mechanism include J. R. Lucas (1961) and Roger Penrose (1994), the so-called *Lucas–Penrose Argument*. It should be noted, however, that the majority of logicians and experts in this debate consider this argument invalid.[6]

Be that as may, what did Gödel himself think his first incompleteness theorem implied about mechanism and the mind in general? Gödel draws the following inevitable disjunctive conclusion from the incomplete theorems: "either … the human mind (even within the realm of pure mathematics) infinitely surpasses the power of any finite machine, or else there exist absolutely unsolvable diophantine problems" (1951). This claim shows that either:

(1) the human mind is not a Turing machine or
(2) there are certain unsolvable mathematical problems.

As reported by Gödel, the second alternative undecidable mathematical problems "seems to disprove the view that mathematics is only our own creation; for the creator necessarily knows all properties of his creatures … so this alternative seems to imply that mathematical objects and facts … exist objectively and independently of our mental acts and decisions" (1951). However, Gödel tended to reject the possibility of absolutely unsolvable problems (2). For him to support the first alternative, that the human mind infinitely surpasses any finite machine, would mean the possibility of humanly unsolvable problems.

Gödel's considerations in Gibbs Lecture and in his later conversations with Wang, and Turing's *Intelligent Machinery* are evidences of the attempt to scientifically approach mental phenomena. Both Turing and Gödel were convinced that mental processes were present in mathematical experience. On the one hand, Turing noted that for a machine or a brain it is not enough to be converted into a universal (Turing) Machine in order to be intelligent. Therefore, the central scientific task is "to discover the nature of this residue as it occurs in man, and to try and copy it in machines" (Turing, 1948, p. 125).

On the other hand, Gödel considered that there must be a non-mechanical plan for machines, and stated that, "such a state of affairs would show that there is something non mechanical in the sense that the overall plan for the historical development of machines is not mechanical. If the general plan is mechanical, then the whole race can be summarized in one machine." (1990). Taking this in consideration, there is at least one thing that the human mind can do that no computer system can: to understand the validity of a proof, an understanding that no computable system seems to be able to do. Nevertheless, both the human mind and the machines seem to be able to hypnotise some form of reality. In fact, as suggested by Simeonov et al., "Classical computing, framed today in third person descriptions, is often based on unambiguous known algorithmic or rote procedures; it is this lack of ambiguity that makes it precisely suited to modeling mechanisms. A living system is impredicative and self-referential: this is what makes it more than a machine" (2012).

Husserl called his new way "phenomenology", which Gödel described as a method by which we can "focus more sharply on the concepts concerned by directing our attention in a certain way, namely, onto our own acts in the use of these concepts" (1981). If we are successful, Gödel said, we achieve "a new state of consciousness in which we describe in detail the basic concepts we use in out thought" (1981).

3. Proof and the physical world

Mathematical proof is one of the supreme intellectual

[4] First Incompleteness Theorem: "Any effectively generated theory capable of expressing elementary arithmetic cannot be both consistent and complete. In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, there is an arithmetical statement that is true, but not provable in the theory".

[5] Such authors include Benacerraf (1967); Bruni (2006); Chalmers (1996, 1995).

[6] See Post (1922).

accomplishments of humankind. As reported by Gian-Carlo Rota (1997b, p. 156), "an axiom system is a window through which an item, be it a group, a topological space or the real line, can be viewed from a different angle that will reveal heretofore unexpected possibilities".[7] In this assumption, mathematical proofs are inseparable from the constitutive open-endedness of phenomenological research. The world is not constituted by physical objects or ideas but by all real possible identities,[8] since those identities represent the condition of possibility of any being *within-the-world*. From Palombi's perspective (2011, p. 60), the "permanence of identity" is the essential phenomenon upon which our relationship with the world is based.

As reported by Rota (1997a, p. 1), a proof of a mathematical theorem is a sequence of steps, which leads to the desired conclusion. It can be widely acknowledged that a proof is apodictic evidence, which entails that the state of affairs given in this evidence is true and that it cannot be otherwise. A proof is a chain of valid operation that may be repeated infinitely. Furthermore, a proof is important to us since it brings clarity, rigor, and order to our thought. Even under the lens of a simple metaphysical conjecture that survived, until today, the falsifiability test.

As discussed in the previous section, proving is a phenomenological reflection from the first person that leads to the proof. The proof is a function of the proving process; it is, the mathematical object that self-presents a property in the world. In this sense, the proof results from the unity of a spatio-temporal being subject to exact laws in nature. It is itself an idea that comes into physical being, belonging to the unified totality of physical nature, and therefore, a mathematical proof must be acknowledged as an entity, an object that can be assessed through natural experience.

Consider proving $a \times b = b \times a$ in the natural numbers. To prove the identity, for instance, between 3×2 and 2×3, one needs to match two groups of three with three groups of two. Such that,

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versus
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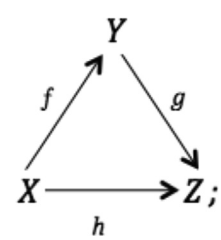
This intuitive necessity is an epistemological ability to deduct mathematics from entities spatiotemporally or causally connected to us. The ability to prove the argument is possible since the human inquiry is rooted in a first person perspective of a scientific naturalism to reveal what is the case. This is the temporal nature of the experience as *experienced*. In fact, the object of enquiry is the phenomenal appearing itself, considered as a natural process. Other examples of this natural experience are the Lorenzen dialogue games: a pragmatic approach to meaning that is learned in a non axiomatic form, but in a rule following method. Lorenzen's philosophical programme was to show how as much of mathematics and the natural sciences as possible could be produced by the process of construction and abstraction.

Christian Thiel discusses Lorenzen's construction and abstraction process on his *Philosophie und Mathematik* (1995). Thiel gives an example of the concept of structure by giving a model of a natural sequence (pp. 114–121) in terms of tallies (|) and a model in terms of circles (in which zero is blank, one is two horizontally adjacent circles, and to form the successor of a number one adds a circle in front of the number and one above it). The tallies and the circles, which form a "number", can be put into one-to-one correspondence, and the successor function for tallies corresponds to the successor function for circles (Powell, 1997). The natural number structure is then the equivalence relation between such models of the natural numbers.

Certainly, what a mathematician needs depend on the level and language of the desired proof. Category theory, for example, deals with systems of structures in a conceptual framework, allowing us to see the universal components of a family of structures of a given kind, and how structures of a different kind are interrelated. Category theory is an interesting object to phenomenology studies since it is a tool for investigations of concepts such as space, system and even truth.

There are different definitions of a category. The fundamental notion of category theory is, according with Mac Lane's (1971), that of a monoid—"a set with a binary operation of multiplication that is associative and that has a unit; a category itself can be regarded as a sort of generalized monoid … its close connection to pairs of adjoint functors illuminates the ideas of universal algebra and culminates in Beck's theorem characterizing categories of algebras; on the other hand, categories with a monoidal structure (given by a tensor product) lead inter alia to the study of more convenient categories of topological spaces".

Category theory starts with the observation that many properties of mathematical systems can be unified and simplified by a presentation with diagrams of arrows, such that:



Each arrow $f: X \to Y$ represents a function; that is, a set X, a set Y, and a rule $x \mapsto f\,x$ which assigns to each element $x \in X$ an element $f\,x \in Y$; whenever possible we write $f\,x$ and not $f\,(x)$, omitting unnecessary parenthesis. This typical diagram of sets and functions is commutative when h is $h = g \circ f$, where $g \circ f$ is the usual composite function $g \circ f$: X $\to Z$, defined by $x \mapsto$ g $(f\,x)$.

This theory is of great interest to phenomenology since it (1) proceeds in terms of mappings and objects; (2) unifies mathematical structures; (3) almost every set theoretically defined mathematical structure with the appropriate notion of homomorphism, yields a category; and (4) once a type structure has been defined, it is imperative to determine how new structures can be constructed out of a given one. Thus, category theory allows revealing certain objects as having a "universal property" and how different kinds of structures are related to one another, via morphisms between categories.

In this respect, Humberto Eco, in *The Name of the Rose*, interestingly seems to well apprehend this connection between phenomenology and category theory, "I have no doubt about the truth of signs, (…) they are the only things man has with which to orient himself in the world" (Eco, 2006). Category theory sheds light on how to understand this relation among signs; it organizes and unifies much of mathematics and the nature of mathematical objects. From a phenomenological perspective, on the one hand, category theory's mathematical object exists in and depends upon an ambient category, and on the other hand, objects are always characterized up to isomorphism. This is a categorical intuition

[7] See Rota (1997b), p. 156. In this regard, see Section 4.2 and Cellucci (2002), pp. 195–196.
[8] Rota (1997c), p. 112; see (1997), pp. 185–186.

present where there is a simple apprehension of the categorical that seems to be above all the demonstration since it is invested in the most everyday of perceptions and in everyday experience.

The theory of mind and cognition must begin with categories of things in the everyday common sense world—what Husserl called the *life-world*, that is, a horizon of all our experiences. In Husserl's work, a mathematical object is to be understood in terms of the 'invariants' or 'identities' in our experience. These physical objects are identities that emerge for us through various sensory experiences or observations. Mathematical objects express axioms and theorems that constitute invariants across our experience with these objects.[9] Invariants may emerge through conscious and systematic efforts of embodied methods of sciences.

According with Roger Shepard's work (1999), through natural selection, the mind has come to reflect long-enduring properties of the world. Perception, contrary to intuitions, involves many inferences that go beyond the "sense data". Such inferences must track the way the world is, so they reflect properties of the physical world. This means that fundamental invariances of physics emerge in pure mathematically expressible form (1978). Moreover, he claims that some features of the environment, such as universal principles, in which animals (including human beings) live will have been 'internalized' (Shepard, 2001).

Mathematical proof, being a function or a result of a conscious experience, can no longer be objects of sense experience—as in their proving process. The proof is an ideal entity because it is no longer subject to space or time. On the other hand, it can no longer be mental in nature since what is mental occurs and changes in time. Would a mathematical proof be subject to changeability, there would be no stability or constancy in mathematics. Moreover, as we discussed in the previous section about the nature of the phenomenological experience of proving, if proving is a first person perspective (an act of the *inner)*, its result, the proof, is the third person perspective, object of the *inner*.

A mathematical proof is an object that surpasses variability, and such property is the condition to shareable meanings. Since it is constructed in a neutral language of natural science, the properties that are delivered by the proof are available for further use. This further use by another phenomenological mind is possible by virtue of the intrinsic features of the proof, which enhances intersubjectively sharable meanings. However, proving to a colleague, in a third person sharable language, is handled only by []p. It is incompleteness which assures that the logic of []p & p is different from the logic of []p.

This is possible because although mathematical proofs start from an act of subjective intention; they arrive at a public, objective language with meaning. This meaning is intersubjectively shareable, and, indeed, this intersubjective meaning is taken as a purpose to which scientific knowledge should aspire, since intersubjective agreement is certainly an outcome of the *objectivity* of knowledge.

3.1. Phenomenology meets artificial intelligence

The great enterprise of Artificial Intelligence is to find out what sort of rules could possibly capture intelligent reasoning. A computer-assisted proof is a proof in which every logical inference has been checked all the way back to the fundamental axioms of mathematics (Hales, 2008). Furthermore, it is written in a precise artificial language that admits only a fixed repertoire of stylized steps (Harrison, 2008). A part of mathematics can be presented through algorithm.

Proof assistants or *computer theorem provers* are artificial intelligent agents that mechanically verify, in a formal language, the correctness of a proof. In fact, with this artificial system, the user is allowed to set up a mathematical theory, define properties and undertake logical reasoning (Geuvers, 2009).

A computer-assisted proof implies that a programmed intelligent agent with symbolic language is able to demonstrate a proof: a computer-assisted proof which uses strings of symbols produced by typographical rules. From a symbolic language, an inanimate, inflexible, and mind-independent agent accomplishes a proof that is not subject to spatio-temporal causality. The phenomenological consequence is an objective, non-intuitive proof that is so trivial that it is beyond reproach.

Nonetheless, in this artificial proving system, what seems to be absent and lost is the proving process—the phenomenological experience—since the computer-assisted proof is far too extensive for a human mind to follow, although its results are of such accuracy and sometimes surpass human ability to obtain a proof. In this computer-assisted mathematical endeavour, the embodied cognitive act required to grasp the proof is somehow obsolete, and the proof comes to a halt without having had intentional or linguistic activity. In other words, the result of a computer-assisted proof involves the absence of the brain-body-world coupling that utilises the *eidetic method* of seeing essences. Moreover, what is lost is the mathematical intuition, to use Gödel's words, which is supposed to lead us to new axioms.

The deep problem underlying the philosophical and mathematical reflection on the computer-assisted proof does not only concern the fact that the proof is produced by a machine and the means by which the proof is computed; rather, that the human mind is not able to attribute consistency to that computer-assisted proof. The modes by which a mathematician interprets a computer-assisted proof could be seen as a tacit disposition to accept the validity of the proof in Carnap's epistemological terms.[10]

What makes this issue even more challenging to the phenomenology of mathematics is the fact that these artificial intelligent agents are able to prove previously unprovable theorems, and, consequently, to compute an ideal entity. The philosophical implications of this reality seem to lead us to the Church-Turing thesis, which claims that any computer as powerful as a Turing machine can, in principle, calculate anything that a human can calculate, given enough time. According with Marchal (2015) "since Gödel, we know that Truth, even just the Arithmetical Truth, is vastly bigger than what the machine can rationally justify, yet, with Church's thesis, and the mechanizability of the diagonalizations involved, machines can apprehend this and can justify their limitations, and get some sense of what might be true beyond what they can prove or justify rationally". Löbian Universal Machine is an example of higher cognitive ability.[11]

Also, nowadays there are unique ontological identifiers for associated sets of items in areas of formalized knowledge and intelligent systems that perceive and act in an environment, such as machine-learning, which is a learning system that uses prior knowledge, handles complex environments, forms new concepts, active explores, and so on. The well-known basic problem in machine learning has been inducing a representation of a function from

[9] In this regard, see Tieszen (2009). Phenomenology, Logic, and the Philosophy of Mathematics. Chapter 1 Reason Science and Mathematics, pp. 21–69.

[10] Carnap questions, in his *Philosophy and Logical Syntax*, about how can we become certain as to the truth or falsehood of a proposition. He considers that "the function of logical analysis is to analyse all knowledge, all assertions of science and everyday life, in order to make clear the sense of each such assertion and the connections between them". In the absence of the possibility of a logical analysis, one needs to tacitly accept the truth of a proposition.

[11] See Marchal, The Universal Numbers. From Biology to Physics, in this Special Issue; and Calculabilité, Physique et Cognition, 1998.

examples. Hypothesis might be represented in four basic categories, such as (1) *attribute-based representations*, which includes Boolean functions, decision trees. This could also include *neural networks* and *belief networks*. (2) *First-order logic*; (3) neural networks, continuous, nonlinear functions represented by a parameterized network of simple computing elements; and (4) probabilistic functions, which involves *belief networks*. Furthermore, the task of determining a scientific hypothesis on the basis of current evidence is similar to the task of determining a model on the basis of a given data, this is, the model selection in machine learning. In the hypothesis construction, also known as inductive reasoning, conclusions made are based on current knowledge and predictions. The availability heuristic causes the reasoned to depend primarily upon information that is readily available to him/her.

Learning, whether in artificial or in natural agents, is essential both as a construction process and as a way to deal with unknown environments. With minimal inductive inference ability, machines can be aware of much more than they can justify. Nevertheless, machines cannot believe rationally, or justify that they are machines: in fact, this requires Husserl's *Wesensschau,* and Gödel's *intuition*. If we can naturalize consciousness and phenomenology, then machines have no phenomenological consciousness.

4. Conclusions

It was my concern in this paper to understand the conditions of possibility within the *eidetic* domain that is called mathematics. I applied phenomenology to mathematical proving as a performance of consciousness, a lived experience expressed and formalized in language.

Mathematics is the means by which an individual phenomenological act of consciousness reveals itself and gives birth to an ideal entity. In fact, we are certain of this phenomenological experience of consciousness by “these acts of experience … within which we live as human beings” (Weyl, 1994). It is through this mathematical experience that consciousness proposes descriptions and deductions, not randomly, but as a function of a plurality and variability of experiences from a subjectivity that is subject to time and space. Subjectivity gives birth to objectivity in a neutral language and spatial formalizations, in virtue of a phenomenological act that captures a form in the world, and consequently enhances our original intuitions.

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References

Bermudez, J., Marcel, A.J., Eilan, N., 1995. The Body and the Self. MIT Press, Cambridge, MA.
Bruni, R., 2006. Gödel, Turing, the Undecidability Results and the Nature of Human Mind, p. 37.
Chalmers, D.J., 1995. Facing up to the problem of consciousness. J. Conscious. Stud. 2 (3), 200–219.
Chalmers, David J., 1996. Facing up to the problem of consciousness. In: Toward a Science of Consciousness: the First Tucson Discussions and Debates, vol. 1, p. 5.
Cellucci, C., 2002. *Filosofia e matematica*. Monograph Collection (Matt – Pseudo).
Clark, A., 1997. Being There: Putting Brain, Body, and World Together Again. MIT Press, Cambridge, MA.
De Preester, H., 2002. Naturalizing Husserlian phenomenology: an introduction. Psychoanal. Perspect. 4, 633–647.
Eco, U., 2006. The Name of the Rose. Everyman's Library, London.
Føllesdal, D., 1999. Gödel and Husserl. In: Naturalizing Phenomenology. Stanford University Press.
Gallagher, S., 2012. On the possibility of naturalizing phenomenology. In: The Oxford Handbook of Contemporary Phenomenology. Oxford University Press.
Gallagher, S., Varela, F., 2003. Redrawing the map and resetting the time: phenomenology and the cognitive sciences. Can. J. Philos. 29 (Suppl.), 93–132.
Geuvers, J.H., 2009. Proof assistants: history, ideas and future. Sadhana Acad. Proc. Eng. Sc. – Indian Acad. Sci. 34 (1), 3–25.
Gödel, K., 1944. Russell's mathematical logic. In: Feferman, Dawson, Kleene (Eds.), The Philosophy of Bertrand Russell. Northwestern University Press, pp. 119–141.
Gödel, K., 1951. Some basic theorems on the foundations of mathematics and their implications. In: Feferman, S., et al. (Eds.), Collected Works, vol. III. Oxford University Press.
Gödel, K., 1964. What is Cantor's continuum problem?. In: Benacerraf and Putnam, pp. 258–273.
Gödel, K., 1981. Collected Works III, 1961. Oxford University Press.
Gödel, K., 1990. Collected Works II. Publications 1938–1974. Oxford University Press.
Hales, T.C., 2008. Formal proof. Not. AMS 55 (11), 1370–1380.
Harrison, J., 2008. Formal proof – theory and practice. Notices AMS 55, 11.
Hass, L., 2008. Merleau-Ponty's Philosophy. Indiana University Press, Bloomington.
Hintikka, J., 2003. The notion of intuition in Husserl. Rev. Int. Philos. 2, 57–79.
Husserl, E., 1965. Philosophy as a rigorous science. In: Lauer, Q. (Ed.), Phenomenology and the Crisis of Philosophy. Harper and Row, New York, pp. 71–147 (Original work published 1910–1911).
Husserl, E., 1970. Cartesian Meditations (trans: Cairns D.). Martinus Nijhoff, The Hague.
Husserl, E., 1976. Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie I. Husserliana III/1–2. Martinus Nijhoff, The Hague.
Husserl, E., 1982. Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy. First Book. General Introduction to a Pure Phenomenology. Martinus Nijhoff, The Hague. Trans. F. Kersten.
Husserl, E., 2001. Logical Investigations, 3 vols.. Routledge, London trans. J.N. Findlay.
Husserl, Edmund, 2005. Phantasy, Image Consciousness, and Memory (1898–1925). In: Collected Works, vol. XI. Springer, Dordrecht. Trans. J. B. Brough.
James, W., 1890. The Principles of Psychology. Holt, New York.
Lawlor, L., 2009. In: . ‘Becoming and Auto-affection (Part II): Who Are We?’ Invited Lecture, ICNAP, 2009. Published at: http://www.icnap.org/meetings.htm (accessed 15.01.15.).
Levine, J., 1983. Materialism and qualia: the explanatory gap. Pac. Philos. Q. 64 (4), 354–361.
Lohmar, D., 2010. Intuition in mathematics: on the function of eidetic variation in mathematical proofs. In: Phenmonenology and Mathematics. Springer.
Lucas, J.R., 1961. Minds, machines and gödel. Philosophy XXXVI, 112–127.
Mac Lane, S., 1971. Categories for the Workinh Mathematician. Springer-Verlag, New York.
Marbach, E., 1993. Mental Representation and Consciousness: towards a Phenomenological Theory of Representation and Reference. Kluwer Academic Publishers, Dordrecht.
Marbach, E., 2010. Towards a formalism for expressing structures of consciousness. In: Gallagher, S., Schmicking, D. (Eds.), Handbook of Phenomenology and Cognitive Science. Springer, Dordrecht.
Marchal, B., 2015. The universal numbers. From biology to physics. Special issue biomathics: life sciences J. Prog. Biophys. Mol. Biol 119 (3), 368–381.
Merleau-Ponty, M., 1942. La structure du comportement. PUF, Paris.
Merleau-Ponty, M., 1964. Signs (trans: McCleary RC). Northwestern University Press, Evanston, IL.
Merleau-Ponty, M., 1971[1945]. Phénoménologie de la perception, English ed. Gallimard, Paris. first published 1962 by Routledge & Kegan Paul.
Merleau-Ponty, M., 1973. The Prose of the World. Northwestern University Press, Evanston. John O'Neill (trans.).
Palombi, F., 2011. The Star and the Whole: Gian-Carlo on Mathematics and Phenomenology. CRC Press, Boca Raton.
Penrose, R., 1994. Shadows of the Mind. Oxford University Press, Oxford.
Post, E., 1922. Absolutely Unsolvable Problems and Relatively Undecidable Propositions: Account of an Anticipation. In [Davis, 1965], pp. 338–433.
Powell, A., 1997. Philosophy and mathematics. Theorema XVI (2), 97–108.
Rota, G., 1997a. The phenomenology of mathematical proof. Synthese 111, 183–196.
Rota, G., 1997b. In: Palombi, F. (Ed.), Indiscrete Thought. Birkäuser, Boston, MA.
Rota, G., 1997c. In: . The Primacy of Identity, pp. 195–203. In: 1997b.
Roy, J.-M., Petitot, J., Pachoud, B., Varela, F.J., 1999. Beyond the gap: an introduction to naturalizing phenomenology. In: Varela, Pachoud & Petitot, Naturalizing Phenomenology. Stanford University Press.
Shepard, R., 1978. On the status of “direct” psychophysical measurement. In: Savage (Ed.), Perception and Cognition Issues in the Foundations of Psychology. University of Minnesota, USA.
Shepard, R., 1999. Lecture 1: Perception, Imagination, and Science. April 12 at 2:15,

Engineering Pavillion II.
Shepard, R., 2001. Perceptual-cognitive universals as reflections of the World. Behav. Brain Sci. 24, 581–601.
Simeonov, P., Brezina, E., Cottam, R., Ehresmann, A., Gare, A., Goranson, T., Gomez-Ramirez, J., Josephson, B., Marcha, B., Matsuno, K., Root-Bernstein, R., Rössler, Salthe, S., Schroeder, M., Seaman, B., Siregar, P., Smith, L., 2012. Stepping beyond the Newtonian paradigm in biology towards an integrable model of life: accelarating discovery in the biological foundations of science. In: Integral Biomathics. Springer, Heidelberg.
Thiel, C., 1995. Philosophie und Mathematik. Eine Einführung in ihre Wechselwirkungen und in die Philosophie der Mathematik. Wissenschaftliche Buchgesellschaft, Darmstadt.
Thompson, E., 2007. Mind in Life: Biology, Phenomenology, and the Sciences of Mind. Harvard University Press, Cambrige, MA.
Tieszen, R., 2009. Phenomenology, Logic, and the Philosophy of Mathematics. Cambridge University Press.
Turing, A., 1948. ‘Intelligent machinery’, report for National Physical Laboratory. In: Meltzer, B., Michie, D. (Eds.), Machine Intelligence 7.
Varela, F., Thompson, E., Rosch, E., 1991. The Embodied Mind: Cognitive Science and Human Experience. MIT Press, Cambridge, MA.
Weyl, H., 1994. The Continuum. A Critical Examination of the Foundation of Analysis. Dover, New York.
Zahavi, D., 2004. Phenomenology and the project of naturalization. Phenomenol. Cognit. Sci. 3, 331–347. Kluger Academic Publishers.