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Cut-conditions on sets of multiple-alternative inferences

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I prove that the Boolean Prime Ideal Theorem is equivalent, under some weak set-theoretic assumptions, to what I will call the Cut-for-Formulas to Cut-for-Sets Theorem: for a set F and a binary relation $\vdash$ on $\mathcal{P}(F)$, if $\vdash$ is finitary, monotonic, and satisfies cut for formulas, then it also satisfies cut for sets. I deduce the CF/CS Theorem from the Ultrafilter Theorem twice; each proof uses a different order-theoretic variant of the Tukey-Teichmüller Lemma. I then discuss relationships between various cut-conditions in the absence of finitariness or of monotonicity.

1 Introduction

The Boolean Prime Ideal Theorem (BPI) is weaker than the Axiom of Choice (AC), and has been proved to be equivalent (modulo weak set-theoretic assumptions) to various theorems from diverse corners of mathematics.[1] The main result in this paper supplements that list with what I will call the Cut-for-Formulas to Cut-for-Sets Theorem for multiple-alternative inferences, a version of which was proved by Shoesmith and Smiley [4, p. 37].

Let $\mathcal{P}$ be the power-set operation. For our purposes, an inference on a given set F has the form $\langle \Gamma, \Delta \rangle \in \mathcal{P}(F)^2$. Heuristic: think of members of F as formulas; in an inference $\langle \Gamma, \Delta \rangle$, members of Γ are the assumptions and members of Δ are what I will call the alternatives. Δ should be understood "disjunctively". In [4], Shoesmith and Smiley call the members of Δ conclusions; this strikes me as misleading. (For a review of [4], cf. [2]. According to [4, p. ix], this notion of inference was, in effect, first introduced by Gentzen in his work on his sequent calculi.)

Consider a set $\vdash$ of inferences on F. Following a standard notational convention, for $\Gamma, \Delta \subseteq F$ and $\varphi \in F$, let $\Gamma, \Delta = \Gamma \cup \Delta$, and $\Gamma, \varphi = \Gamma \cup \{\varphi\}$, when they are considered as relata of $\vdash$.

Definition 1.1 1. For $\Psi \subseteq F$, let a splitting of Ψ have the form $\{\Psi_0, \Psi_1\}$ for $\Psi_0 \cup \Psi_1 = \Psi$ and $\Psi_0 \cap \Psi_1 = \{\}$. $Splt(\Psi)$ is the set of splittings of Ψ.[2]

2. For $\Psi \subseteq F$, $\vdash$ satisfies cut for Ψ iff: for every $\Gamma, \Delta \subseteq F$, if

$$\text{for every } \Psi_0, \Psi_1, \text{ if } \{\Psi_0, \Psi_1\} \in Splt(\Psi) \text{ then } \Gamma, \Psi_0 \vdash \Delta, \Psi_1,$$

then $\Gamma \vdash \Delta$.

3. $\vdash$ satisfies cut for sets iff for every $\Psi \subseteq F$, $\vdash$ satisfies cut for Ψ.
4. $\vdash$ satisfies cut for formulas iff for every $\psi \in F$, if $\Gamma, \psi \vdash \Delta$ and $\Gamma \vdash \Delta, \psi$ then $\Gamma \vdash \Delta$.
5. $\vdash$ satisfies overlap iff for every $\Gamma, \Delta \subseteq F$, if $\Gamma \cap \Delta \neq \{\}$ then $\Gamma \vdash \Delta$.
6. $\vdash$ is monotonic (satisfies dilution in the usage of [4]) iff for every $\Gamma, \Gamma', \Delta, \Delta' \subseteq F$, if $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, and $\Gamma \vdash \Delta$, then $\Gamma' \vdash \Delta'$.
7. $\vdash$ is finitary (aka compact) iff for every $\Gamma, \Delta \subseteq F$, if $\Gamma \vdash \Delta$ then for some finite $\Gamma_0 \subseteq \Gamma$ and some finite $\Delta_0 \subseteq \Delta$, $\Gamma_0 \vdash \Delta_0$.[3]

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[1] For more on the variety of theorems from diverse corners of mathematics that are equivalent to the BPI, cf. the lists given in [1, 3] and in several other articles cited in [3].

[2] I have replaced Shoesmith's and Smiley's use of 'partition' by 'splitting', because cells of a partition are usually understood to be non-empty.

[3] For analogous use of 'finitary' regarding single-alternative sets of inferences, cf. [9].

Theorem 1.2 (The Cut-for-Formulas to Cut-for-Sets Theorem; hereafter CF/CS) *For any* $\vdash \subseteq \mathcal{P}(F)^2$, *if* $\vdash$ *is finitary, monotonic, and satisfies cut for formulas, then it satisfies cut for sets.*

In [4, part of Theorem 2.10], Shoesmith and Smiley prove the following slight weakening of CF/CS, using the Tukey-Teichmüller Lemma, which is equivalent to AC.[4]

Theorem 1.3 (CF/CS*) *For any* $\vdash \subseteq \mathcal{P}(F)^2$, *if* $\vdash$ *is finitary, monotonic, and satisfies overlap and cut for formulas, then it satisfies cut for sets.*

Clearly CF/CS entails CF/CS*; this entailment reverses, as we will see.

Notation. Natural numbers will be identified with finite von Neumann ordinals. For a set A, $\mathcal{P}'(A) = \{X \subseteq A \mid X$ is finite but non-empty$\}$, and $card(A)$ is the cardinality of A.

2 The main result

Consider a poset $P = \langle |P|, \preceq \rangle$ (i.e., $\preceq$ is a partial ordering of $|P|$).

Definition 2.1 1. As is standard, let $x \in P$ mean $x \in |P|$ and $X \subseteq P$ mean $X \subseteq |P|$.
2. For $x \in P$, $\downarrow x = \{y \mid y \preceq x\}$.
3. For $x \in P$, let x be P-finite iff $\downarrow x$ is finite.
4. $\mathcal{F}_P = \{x \in P \mid x$ is P-finite$\}$.

Definition 2.2 1. For $X \subseteq |P|$, let x be special for X in P iff: for every P-finite $y \preceq x$, there is finite $X_0 \subseteq X$ such that for every P-finite upper bound u on X_0 we have $y \preceq u$.
2. P is special iff every non-empty subset of $\mathcal{F}_P$ has an upper bound that is special for it in P.

Definition 2.3 A is of P-finite character iff: $A \subseteq P$ and for every $x \in P$,

$$x \in A \text{ iff for every } y \in \mathcal{F}_P \cap \downarrow x, \quad y \in A.$$

Definition 2.4 Consider a Z and T such that $Z : T \to \mathcal{F}_P$.

1. For $S \subseteq T$ and $x \in P$, x makes S-choices from Z iff for every $t \in S$ there is a $z_t \preceq Z(t)$ such that $z_t \preceq x$.
2. For $A \subseteq P$, A makes finite choices from Z iff for every finite $S \subseteq T$ some $x_S \in A \cap \mathcal{F}_P$ makes S-choices from Z.

Lemma 2.5 (The Restricted Tukey-Teichmüller Lemma for Posets; rTT$_{po}$) *Consider any special poset P. For any Z as above and any* $A \subseteq P$, *if A is non-empty, of P-finite character, and makes finite choices from Z, then for some* $b \in A$, *b makes T-choices from Z (i.e., for every* $t \in \mathrm{dom}(Z)$ *there is a* $z \preceq Z(t)$ *so that* $z \preceq b$).[5]

Note 2.6 rTT$_{po}$ is formulated in the second-order language based on one 2-place predicate-constant and 'is finite' as a primitive second-order predicate. It is a distant cousin of [3, Theorem 3.2], which is also a restricted version of the Tukey-Teichmüller Lemma, there called rTT^{++}.

Theorem 2.7 *The Ultrafilter Theorem for power-sets (hereafter UT) entails rTT$_{po}$.*[6]

Note 2.8 UT entails the Axiom of Choice From Finite Sets (for every set $\mathcal{A}$ of finite non-empty sets there is a choice function on $\mathcal{A}$.) For a proof of this, cf. [3, end of § 3].

Proof. (A modification of an argument in [3].) Assume UT. Assume that P is an special poset, $Z : T \to \mathcal{F}_P$, and $A \subseteq P$. Assume that A is non-empty, of P-finite character, and makes finite choices from Z. Let $Y =$

[4] The Tukey-Teichmüller Lemma: every non-empty set of sets of finite character has a maximal element with respect to subsethood. (A set S of sets is of finite character iff for every a, $a \in S$ iff every finite subset of a is in S.) For the original presentations of Tukey-Teichmüller Lemma, cf. [6, 8].

[5] This is trivially true if P has a least member. We will be applying rTT$_{po}$ to posets without least members.

[6] UT is this: for any set X and $F \subseteq \mathcal{P}(X)$, if F has the finite intersection property (i.e., the intersection of any finite number of members of F is non-empty) then there is an ultrafilter U on X (note: so $U \subseteq \mathcal{P}(X)$) such that $F \subseteq U$.

$\prod_{t\in T}\downarrow Z(t)$. Since $\downarrow Z(t)$ is finite for each $t \in T$, by AC from Finite Sets, $Y \neq \{\}$. Fix $g \in \prod_{t\in T}\downarrow Z(t)$. For each finite $S \subseteq T$ let $H_S = \{f \in Y \mid \text{some } u \in A \cap \mathcal{F}_P \text{ is an upper bound on } f[S]\}$.

Claim 1: for each finite $S \subseteq T$, $H_S \neq \{\}$. Since A makes finite choices from Z, we may fix an $u_S \in A \cap \mathcal{F}_P$ that makes S-choices from Z. For each $t \in S$ fix a $z_t \preceq Z(t)$ such that $z_t \preceq u_S$. Let

$$g'(t) = \begin{cases} z_t & \text{if } t \in S, \\ g(t) & \text{otherwise.} \end{cases}$$

So $g' \in Y$. Since u_S is an upper bound on $g'[S] = \{z_{t\in S}\}$, u_S witnesses that $g' \in H_S$, proving Claim 1.

Claim 2: for any finite $S_0, S_1 \subseteq T$, $H_{S_0\cup S_1} \subseteq H_{S_0} \cap H_{S_1}$. Consider an $f \in H_{S_0\cup S_1}$. Fix a u witnessing that $f \in H_{S_0\cup S_1}$. Consider $i \in 2$. Since $f[S_i] \subseteq f[S_0 \cup S_1]$, u also witnesses that $f \in H_{S_i}$. Claim 2 follows.

By Claims 1 & 2, $\{H_S \mid S \subseteq T, S \text{ is finite}\}$ has the finite intersection property. By UT we may fix an ultrafilter U on Y such that for each finite $S \subseteq T$ $H_S \in U$. For $t \in T$ and $z \preceq Z(t)$ let $X_t^z = \{f \in Y \mid f(t) = z\}$.

Claim 3: for each $t \in T$ there is a unique $z_t \preceq Z(t)$ so that $X_t^{z_t} \in U$. Consider a $t \in T$. $\{X_t^z \mid z \preceq Z(t)\}$ is a set of pairwise disjoint sets; also $\bigcup\{X_t^z \mid z \preceq Z(t)\} = Y$. Since U is an ultrafilter there is a unique $z \preceq Z(t)$ so that $X_t^z \in U$. Letting z_t be that z, Claim 3 follows.

Since P is special, we may fix an upper bound b on $\{z_{t\in T}\}$ that is special for $\{z_{t\in T}\}$.

Claim 4: for every $x \preceq b$, if $x \in \mathcal{F}_P$ then $x \in A$. Consider a P-finite $x \preceq b$. Since b is special for $\{z_{t\in T}\}$, we may fix a finite $S \subseteq T$ such that for every P-finite upper bound u on $\{z_{t\in S}\}$, $x \preceq u$. Since $H_S \in U$ and for each $t \in S$ $X_t^{z_t} \in U$, $H_S \cap \bigcap_{t\in S} X_t^{z_t} \in U$. So we may fix an $f \in H_S \cap \bigcap_{t\in S} X_t^{z_t}$. Fix a u witnessing that $f \in H_S$; so $u \in A \cap \mathcal{F}_P$ and $f(t) \preceq u$ for every $t \in S$. For every $t \in S$, $f \in X_t^{z_t}$; so $f(t) = z_t$. Since u is an upper bound on $\{z_{t\in S}\}$, $x \preceq u$. We have $x \in \mathcal{F}_P$, $u \in A$ and A is of P-finite character; so $x \in A$. Claim 4 follows.

Since A has P-finite character, $b \in A$ by Claim 4. For each $t \in T z_t \preceq Z(t)$ and $z_t \preceq b$. So b is as required by rTT_{po}. □

Lemma 2.9 ([4, Theorem 2.2]) *If* $\vdash \subseteq \mathcal{P}(F)^2$ *is monotonic and satisfies cut for* F *then* $\vdash$ *satisfies cut for sets.*

Proof. Assume the if-clause. Given $\Psi, \Gamma, \Delta \subseteq F$, assume that for every splitting $\{\Psi_0, \Psi_1\}$ of Ψ we have $\Gamma, \Psi_0 \vdash \Delta, \Psi_1$. Given a splitting $\{\Phi_0, \Phi_1\}$ of F, let $\Psi_i = \Psi \cap \Phi_i$ for $i \in 2$. Since $\{\Psi_0, \Psi_1\}$ is a splitting of Ψ, $\Gamma, \Psi_0 \vdash \Delta, \Psi_1$. By monotonicity of $\vdash$, $\Gamma, \Phi_0 \vdash \Delta, \Phi_1$. So for every splitting $\{\Phi_0, \Phi_1\}$ of F, $\Gamma, \Phi_0 \vdash \Delta, \Phi_1$. Since $\vdash$ satisfies cut for F, $\Gamma \vdash \Delta$. □

Theorem 2.10 *rTT$_{po}$ entails CF/CS.*

Proof. Assume rTT_{po}. Let F be any set. Assume that $\vdash \subseteq \mathcal{P}(F)^2$ is finitary, monotonic and satisfies cut for formulas. By Lemma 2.9 it suffices to prove that $\vdash$ satisfies cut for F. Consider any $\Gamma, \Delta \subseteq F$. Assume that

$(*)$ for every Ψ_0 and Ψ_1, if $\{\Psi_0, \Psi_1\} \in \mathit{Splt}(F)$ then $\Gamma, \Psi_0 \vdash \Delta, \Psi_1$.

Assume that $F = \{\}$. So $\Gamma = \Delta = \{\}$; so $(*)$ yields that $\{\} \vdash \{\}$; so $\vdash$ trivially satisfies cut for F.

Assume that $F \neq \{\}$. If $\{\} \vdash \{\}$, by monotonicity $\vdash = \mathcal{P}(F)^2$, and so trivially $\vdash$ satisfies cut for F. Assume that $\{\} \nvdash \{\}$.

For $\Phi_0, \Phi_1 \subseteq F$ let $\langle\Phi_0, \Phi_1\rangle \preceq \langle\Phi_0', \Phi_1'\rangle$ iff (i) $\Phi_i \subseteq \Phi_i'$ for both $i \in 2$, and (ii) $\Phi_0 \cup \Phi_1 \neq \{\}$. Let $|P| = \mathcal{P}(F)^2 - \{\langle\{\}, \{\}\rangle\}$ and $P = \langle|P|, \preceq\rangle$. So P is a poset. For any $\langle\Phi_0, \Phi_1\rangle \in P$, $\langle\Phi_0, \Phi_1\rangle$ is P-finite iff both Φ_0 and Φ_1 are finite. For any $X \subseteq \mathcal{P}(F)^2$, let $\bigvee X = \langle\bigcup \mathrm{dom}(X), \bigcup \mathrm{ran}(X)\rangle$; so $\bigvee X$ is the least upper bound on X with respect to $\preceq$. For a finite non-empty $X \subseteq \mathcal{F}_P$, $\bigvee X \in \mathcal{F}_P$.[7]

Claim 1: P is special. Consider any non-empty $X \subseteq \mathcal{F}_P$. Consider a $\langle\Phi_0, \Phi_1\rangle \in \mathcal{F}_P$. Assume that $\langle\Phi_0, \Phi_1\rangle \preceq \bigvee X$; so $\Phi_0 \subseteq \bigcup \mathrm{dom}(X)$ and $\Phi_1 \subseteq \bigcup \mathrm{ran}(X)$. For each $i \in 2$ and $\varphi \in \Phi_i$ select a $\langle\Psi^i_{\varphi,0}, \Psi^i_{\varphi,1}\rangle \in X$ so that $\varphi \in \Psi^i_{\varphi,i}$; let $X_0 = \{\langle\Psi^i_{\varphi,0}, \Psi^i_{\varphi,1}\rangle \mid i \in 2,\ \varphi \in \Phi_i\}$. Since $\Phi_0 \cup \Phi_1$ is finite, X_0 is finite. Also $X_0 \subseteq X$ and $\langle\Phi_0, \Phi_1\rangle \preceq \bigvee X_0$. Since $\bigvee X_0$ is the least upper bound on X_0, $\bigvee X$ is special for X. Claim 1 follows.

Assume for a contradiction that $\Gamma \nvdash \Delta$. Let

$$A = \{\langle\Phi_0, \Phi_1\rangle \in |P| \mid \Phi_0 \cap \Phi_1 = \{\} \text{ and } \Gamma, \Phi_0 \nvdash \Delta, \Phi_1\}.$$

We will consider two cases.

[7] In fact, with $\bigvee$ added, P becomes a complete join-semi-lattice.

Case 1: $\Gamma \cup \Delta \neq \{\}$. So $\langle \Gamma, \Delta \rangle \in |P|$. So $\langle \Gamma, \Delta \rangle \in A$; so $A \neq \{\}$.

Claim 2: for any $\Phi_0, \Phi_1 \subseteq F$, if for every finite $\Phi'_i \subseteq \Phi_i$ for both $i \in 2$ $\langle \Phi'_0, \Phi'_1 \rangle \in A$, then $\langle \Phi_0, \Phi_1 \rangle \in A$. Given such Φ_0 and Φ_1, assume the if-clause, and that $\langle \Phi_0, \Phi_1 \rangle \notin A$. If $\Phi_0 \cap \Phi_1 \neq \{\}$, fix $\varphi \in \Phi_0 \cap \Phi_1$; then $\langle \{\varphi\}, \{\varphi\} \rangle \notin A$ for a contradiction. Assume that $\Gamma, \Phi_0 \vdash \Delta, \Phi_1$; since $\vdash$ is finitary and monotonic there are finite $\Phi'_i \subseteq \Phi_i$ for both $i \in 2$ such that $\Gamma, \Phi'_0 \vdash \Delta, \Phi'_1$, and so $\langle \Phi'_0, \Phi'_1 \rangle \notin A$, for a contradiction. Claim 2 follows.

Claim 3: A is of P-finite character. Consider $\Phi_0, \Phi_1 \subseteq F$. Assume that $\langle \Phi_0, \Phi_1 \rangle \in A$. Since $\vdash$ is monotonic, if $\Phi'_i \subseteq \Phi_i$ for both $i \in 2$, then $\langle \Phi'_0, \Phi'_1 \rangle \in A$. So for any P-finite $x \preceq \langle \Phi_0, \Phi_1 \rangle$, $x \in A$ (regardless of x's P-finitude). Assume that for every P-finite $x \preceq \langle \Phi_0, \Phi_1 \rangle$, $x \in A$. By (2) $\langle \Phi_0, \Phi_1 \rangle \in A$. Claim 3 follows.

For $\varphi \in F$, let $Z(\varphi) = \{\langle \{\varphi\}, \{\} \rangle, \langle \{\}, \{\varphi\} \rangle\}$. So $Z : F \to \mathcal{F}_P$. Fixing a $\varphi \in \Gamma \cup \Delta$, let

$$x = \begin{cases} \langle \{\varphi\}, \{\} \rangle & \text{if } \varphi \in \Gamma, \\ \langle \{\}, \{\varphi\} \rangle & \text{otherwise.} \end{cases}$$

So $x \in A \cap \mathcal{F}_P$; vacuously x makes $\{\}$-choices from Z. Consider an $n \in \omega - \{0\}$ and distinct $\varphi_1, \ldots, \varphi_n \in F$. Since $\vdash$ satisfies cut for formulas and $\Gamma \nvdash \Delta$, either $\Gamma, \varphi_1 \nvdash \Delta$ or $\Gamma \nvdash \Delta, \varphi_1$. In the first case let $\Phi_{1,0} = \{\varphi_1\}$, $\Phi_{1,1} = \{\}$, and $z_1 = \langle \{\varphi_1\}, \{\} \rangle$; in the second case let $\Phi_{1,0} = \{\}$, $\Phi_{1,1} = \{\varphi_1\}$, and $z_1 = \langle \{\}, \{\varphi_1\} \rangle$. So $\Gamma, \Phi_{1,0} \nvdash \Delta, \Phi_{1,1}$ and $z_1 \in A$. Again using cut for formulas, either $\Gamma, \Phi_{1,0}, \varphi_2 \nvdash \Delta, \Phi_{1,1}$ or $\Gamma, \Phi_{1,0} \nvdash \Delta, \Phi_{1,1}, \varphi_2$. In the first case let $\Phi_{2,0} = \Phi_{1,0} \cup \{\varphi_2\}$, $\Phi_{2,1} = \Phi_{1,1}$, and $z_2 = \langle \{\varphi_2\}, \{\} \rangle$; in the second case let $\Phi_{2,0} = \Phi_{1,0}$, $\Phi_{2,1} = \Phi_{1,1} \cup \{\varphi_2\}$ and $z_2 = \langle \{\}, \{\varphi_2\} \rangle$. So $\Gamma, \Phi_{2,0} \nvdash \Delta, \Phi_{2,1}$. Since $\varphi_1 \neq \varphi_2$, $z_2 \in A$. Iterate this. For each $i \in (n)$ $z_i \preceq Z(\varphi_i)$; so $\bigvee\{z_i \mid i \in (n)\}$ makes $\{\varphi_{i \in (n)}\}$-choices from Z. Note that $\bigvee\{z_i \mid i \in (n)\} = \langle \Phi_{n,0}, \Phi_{n,1} \rangle$. Since $\varphi_1, \ldots, \varphi_n$ are distinct, $\Phi_{n,0} \cap \Phi_{n,1} = \{\}$. So $\langle \Phi_{n,0}, \Phi_{n,1} \rangle \in A$. Clearly $\langle \Phi_{n,0}, \Phi_{n,1} \rangle \in \mathcal{F}_P$. So A makes finite choices from Z.

By rTT_{po}, we may fix $\Phi_0, \Phi_1 \subseteq F$ so that $\langle \Phi_0, \Phi_1 \rangle \in A$ and $\langle \Phi_0, \Phi_1 \rangle$ makes F-choices from Z, i.e., for every $\varphi \in F$ there is a $z \preceq Z(\varphi)$ so that $z \preceq \langle \Phi_0, \Phi_1 \rangle$. Given $\varphi \in F$, fix such a z. Since $\langle \{\}, \{\} \rangle \notin |P|$, either $z = \langle \{\varphi\}, \{\} \rangle$, in which case $\varphi \in \Phi_0$, or $z = \langle \{\}, \{\varphi\} \rangle$, in which case $\varphi \in \Phi_1$. So $\varphi \in \Phi_0 \cup \Phi_1$. So $\Phi_0 \cup \Phi_1 = F$. Since $\langle \Phi_0, \Phi_1 \rangle \in A$, $\Phi_0 \cap \Phi_1 = \{\}$. So $\{\Phi_0, \Phi_1\}$ is a splitting of F. Since $\Gamma, \Phi_0 \nvdash \Delta, \Phi_1$, this contradicts $(*)$.

Case 2: $\Gamma = \Delta = \{\}$. $F \neq \{\}$; so fix any $\psi \in F$. Either $\{\} \nvdash \psi$ or $\psi \nvdash \{\}$, since otherwise by cut for formulas $\{\} \vdash \{\}$, contrary to assumptions. If $\{\} \nvdash \psi$ let $\Delta' = \{\psi\}$ and $\Gamma' = \{\}$. If otherwise let $\Gamma' = \{\psi\}$ and $\Delta' = \{\}$. Either way, $\Gamma' \nvdash \Delta'$. Given a splitting $\{\Psi_0, \Psi_1\}$ of F, since $\Gamma, \Psi_0 \vdash \Delta, \Psi_1$ we also have $\Gamma', \Psi_0 \vdash \Delta', \Psi_1$, by the monotonicity of $\vdash$. The argument under Case 1 applies using Γ' and Δ' in place of Γ and Δ, yielding a contradiction.

So $\Gamma \vdash \Delta$. So $\vdash$ satisfies cut for F. □

Theorem 2.11 *CF/CS* entails BPI.*

Proof. Assume CF/CS*. Consider a Boolean algebra $B = \langle |B|, \sqcap, \sqcup, c, \overline{0}, \overline{1} \rangle$. Understand $a \in B$ and $X \subseteq B$ as usual. Let $\sqsubseteq$ be the usual Boolean ordering on B. Form the language L_0 with logical constants $=$, $\supset$ and $\perp$, individual constants $\underline{a}$ for each $a \in B$, two 2-place function constants $\underline{\sqcap}$ and $\underline{\sqcup}$, and a 1-place function-constant $\underline{c}$ (for complementation). Let *Trm* is the set of closed terms of L_0. Form L_1 by adding the 1-place predicate-constant $\underline{I}$ to L_0 (for membership in an ideal). Let F be the set of 0th-order (i.e., propositional) formulas of L_1. Define $\vdash \subseteq \mathcal{P}(F)^2$ using one's favorite classical sequent calculus, e.g., ***G1c*** from [7], applied to L_1; i.e., for $\Psi, \Phi \subseteq F$ let $\Psi \vdash \Phi$ iff for some finite multisets Ψ', Φ' with $set(\Psi') \subseteq \Psi$ and $set(\Phi') \subseteq \Phi$, $\Psi' \Rightarrow \Phi'$ is a theorem of the sequent calculus. So $\vdash$ is finitary, monotonic, and satisfies overlap and cut for formulas. By CF/CS* it satisfies cut for sets.

Relative to B, interpret these non-logical constants in the obvious ways. Define the designation function des : $Trm \to B$ so that for each $a \in B$ we have $des(\underline{a}) = a$, and des is homomorphic with respect to $\sqcap$, $\sqcup$, c and $\underline{\sqcap}$, $\underline{\sqcup}$, $\underline{c}$ respectively. Let Γ_0 be the positive atomic diagram for B in L_0, i.e., for any terms $\tau_{i \in 2}$ of L_0, $\tau_0 = \tau_1 \in \Gamma_0$ iff $B \models \tau_0 = \tau_1$. Let $\Delta_0 = \{\underline{a} = \underline{b} \mid a \neq b\}$. Let

$$\begin{aligned} \Gamma_1 = &\{\underline{I}(\underline{0})\} \cup \{\underline{I}(\underline{a}) \supset \underline{I}(\underline{b}) \mid b \sqsubseteq a\} \cup \\ &\{\underline{I}(\underline{a}) \supset (\underline{I}(\underline{b}) \supset \underline{I}(\underline{a \sqcup b})) \mid a, b \in B\} \cup \\ &\{\underline{I}(\underline{a \sqcap b}) \supset (\neg \underline{I}(\underline{a}) \supset \underline{I}(\underline{b})) \mid a, b \in B\} \end{aligned}$$

Let Γ_2 be the standard axioms for $=$ in L_1. Finally, let $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ and $\Delta = \Delta_0 \cup \{\underline{I}(\underline{1})\}$.

Claim : $\Gamma \nvdash \Delta$. Assume otherwise. Fix finite multisets Ψ', Φ' with $set(\Psi') \subseteq \Gamma$ and $set(\Phi') \subseteq \Delta$, so that $\vdash_{\boldsymbol{G1c}} \Psi' \Rightarrow \Phi'$. Let $\Gamma' = set(\Psi')$ and $\Delta' = set(\Phi')$; so they are finite, and $\Gamma' \vdash \Delta'$. Let

$$Trm' = \{\tau \in Trm \mid \tau \text{ occurs in } \Gamma' \cup \Delta'\},$$

$$C_0 = \{des(\tau) \in B \mid \tau \in Trm'\}.$$

Let B_0 be the smallest sub-algebra of B whose domain is a superset of C_0.

Subclaim 1: B_0 is finite. Let C be the closure of C_0 under disjunctive normal forms, i.e., let

$$C_1 = C_0 \cup \{c(a) \mid a \in C_0\},$$

$$C_2 = \{\sqcap D \mid D \subseteq C_1\},$$

$$C = \{\sqcup D \mid D \subseteq C_2\}.$$

Since C_0 is finite, so are C_1, C_2 and C. Set $\sqcap' = \sqcap|_C$, $\sqcup' = \sqcup|_C$, and $c' = c|_C$. Note that $\langle C, \sqcap', \sqcup', c', \overline{0}, \overline{1}\rangle$ is a Boolean algebra. Also, the domain of any sub-algebra of B whose domain is a superset of C_0 is itself a superset of C. Thus $B_0 = \langle C, \sqcap', \sqcup', c', \overline{0}, \overline{1}\rangle$, proving Subclaim 1.

Subclaim 2: B_0 has a prime ideal. If B_0 is the trivial 2-membered Boolean algebra, $\{\overline{0}\}$ is a prime ideal in B_0. Assume that B_0 has at least three members. Since B_0 is finite, some $b \in C - \{\overline{1}\}$ is maximal in B_0. Fix such a b. The principle ideal that b generates, $\downarrow b$, is an ideal in B_0; since b is maximal in B_0, that ideal is prime, proving Subclaim 2. Let $B_1 = \langle B_0, I\rangle$ for I a prime ideal for B_0. Interpret $\underline{I}$ by I, i.e., for every $a \in C$ $B_1 \models \underline{I}(\underline{a})$ iff $a \in I$, $B_1 \models \Gamma$. Since $\boldsymbol{G1c}$ is sound (with respect to classical propositional logic), for some $\varphi \in \Delta_0$ $B_1 \models \varphi$. Since $\Delta_0 \subseteq \Delta$, every member of Δ_0 is false in B_1, for a contradiction. The Claim follows.

By CF/CS, there is a splitting $\{\Phi_0, \Phi_1\}$ of F such that $\Gamma, \Phi_0 \nvdash \Delta, \Phi_1$. Fix it; so $\Gamma \subseteq \Phi_0$ and $\Delta \subseteq \Phi_1$. Now let $I = \{a \in B \mid \underline{I}(\underline{a}) \in \Phi_0\}$. Check that $\langle B, I\rangle \models \Phi_0$. So $\langle B, I\rangle \models \Gamma_1$. So I is a prime ideal for B.

This proves the weak form of the BPI. The strong form of the BPI (for any Boolean algebra B, if I is an ideal for B and F is a filter for B and $I \cap F = \{\}$, then some prime ideal for B is a superset of I and is disjoint from F) follows from the weak form by factoring B by I. □

3 An alternative approach

In this section, I will present an alternative approach to the equivalence of CF/CS and the BPI Theorem, one that uses a different restricted variation on the Tukey-Teichmüller Lemma. The idea: replace the class of posets by a class of slightly more complex structures; replace specialness for posets by a less complex specialness property that is defined for the latter structures.

Definition 3.1 Consider a poset P.

1. $\bigvee$ is a finite-upper bound (hereafter a fub-) selector for P iff $\bigvee : \mathcal{P}'(\mathcal{F}_P) \to \mathcal{F}_P$ such that, for every $X \in \mathcal{P}'(\mathcal{F}_P)$, $\bigvee X$ is an upper bound on X.
2. A fub-selector $\bigvee$ is monotonic iff for every $X, Y \in \mathcal{P}'(\mathcal{F}_P)$ if $Y \subseteq X$ then $\bigvee Y \preceq \bigvee X$.
3. $\langle P, \bigvee\rangle$ is a fub-selector structure iff: P is a poset and $\bigvee$ is a fub-selector for P. It is monotonic iff $\bigvee$ is.

The following may clarify the previous definitions.

Theorem 3.2 *For any poset P the following are equivalent:*

(i) P has a monotonic fub-selector;
(ii) P has a fub-selector;
(iii) every member of $\mathcal{P}'(\mathcal{F}_P)$ has an upper bound (with respect to $\preceq$) in $\mathcal{F}_P$.

Proof. From (i) to (ii) and from (ii) to (iii) are trivial. Going from (iii) to (i) will use the Axiom of Choice (repeatedly).[8]

[8] In fact, the proposition that if (iii) then (i) is equivalent to AC.

Assume (iii). We define the "height" function $h : \mathcal{F}_P \to \omega$ as follows. If x is minimal in P, $h(x) = 0$. If $x \in \mathcal{F}_P$ is not minimal, $\{y \mid y \prec x\}$ is finite and non-empty; let $h(x) = \max\{h(y) \mid y \prec x\} + 1$. By induction on the maximum lengths of $\prec$-chains in $\mathcal{F}_P$ that terminate at x, h is well-defined. We define a "level" function $L : \mathcal{P}'(\mathcal{F}_P) \to \omega$ thus: for $X \in \mathcal{P}'(\mathcal{F}_P)$, $L(X) = \sup\{h(x) \mid x \in X\}$. Let $\mathcal{P}'_i(\mathcal{F}_P) = \{X \in \mathcal{P}'(\mathcal{F}_P) \mid L(X) = i\}$. Plan: by induction on the cardinality of its arguments, define $\bigvee_i : \mathcal{P}'_i(\mathcal{F}_P) \to \mathcal{F}_P$ for each $i \in \omega$; we will then take $\bigvee = \bigcup_{i\in\omega} \bigvee_i$.

For x such that $h(x) = 0$, let $\bigvee_0\{x\} = x$. Assume that for every $Z \in \mathcal{P}'_0(\mathcal{F}_P)$ with $card(Z) \leq n$, $\bigvee_0 Z$ is defined. For $X \in \mathcal{P}'_0(\mathcal{F}_P)$ such that $card(X) = n + 1$, set $Y_X = \{\bigvee_0 Z \mid \{\} \neq Z \subsetneq X\}$. $Y_X \in \mathcal{P}'(\mathcal{F}_P)$, and thus has an upper bound in $\mathcal{F}_P$; using AC choose one (for each X as described) and let $\bigvee_0 X$ be it. So $\mathrm{dom}(\bigvee_0) = \mathcal{P}'_0(\mathcal{F}_P)$. Check that $\bigvee_0$ is monotonic. Assume that $\bigvee_i$ has been defined. For x such that $h(x) = i + 1$, set

$$Y_{\{x\}} = \{\bigvee_j Z \mid j < i,\ Z \in \mathcal{P}'_j(\mathcal{F}_P),\ Z \subseteq \downarrow x\}.$$

$Y_{\{x\}} \in \mathcal{P}'(\mathcal{F}_P)$, and thus has an upper bound in $\mathcal{F}_P$; using AC choose one (for each x as described) and let $\bigvee_{i+1}\{x\}$ be it. Assume that for every $Z \in \mathcal{P}'_{i+1}(\mathcal{F}_P)$ with $card(Z) \leq n$, $\bigvee_{i+1} Z$ is defined. $X \in \mathcal{P}'_{i+1}(\mathcal{F}_P)$ such that $card(X) = n + 1$, define $\bigvee_{i+1} X$ by imitating the definition at the corresponding step for level 0. So $\mathrm{dom}(\bigvee_{i+1}) = \mathcal{P}'_{i+1}(\mathcal{F}_P)$. Check that $\bigvee_{i+1}$ is monotonic. So $\bigvee$ is a monotonic fub-selector for P, yielding (i). □

Definition 3.3 Consider a fub-selector structure $\langle P, \bigvee\rangle$.

1. For a non-empty $X \subseteq \mathcal{F}_P$, let x be special for X in $\langle P, \bigvee\rangle$ iff: for every $y \in \mathcal{F}_P$, if $y \preceq x$ then there is a finite $X_0 \subseteq X$ such that $y \preceq \bigvee X_0$.
2. $\langle P, \bigvee\rangle$ is special iff every non-empty $X \subseteq \mathcal{F}_P$ has a upper bound x in P that is special for X in $\langle P, \bigvee\rangle$.

Definition 3.4 Consider a $Z : T \to \mathcal{F}_P$ and $A \subseteq P$.

1. For any $F \subseteq T$, let A make F-choices from Z using $\bigvee$ iff: for every $t \in F$ there is a $z_t \preceq Z(t)$ such that $\bigvee\{z_{t\in F}\} \in A$.
2. A makes finite choices from Z using $\bigvee$ iff for every finite $F \subseteq T$, A make F-choices from Z using $\bigvee$.

Lemma 3.5 (The Restricted Tukey-Teichmüller Lemma for Fub-selector Structures (rTT_{fubs})) *Consider a special monotonic fub-selector structure $\langle P, \bigvee\rangle$. For any $Z : T \to \mathcal{F}_P$ and any non-empty $A \subseteq P$, if A is of P-finite character, and makes finite choices from Z using $\bigvee$, then for some $b \in A$, b makes T-choices from Z (as defined in Definition 2.4, i.e., for every $t \in \mathrm{dom}(Z)$ there is a $z \preceq Z(t)$ so that $z \preceq b$).*

Note that rTT_{fubs} is formulated in the second-order language of fub-selector structures, again taking 'is finite' as primitive. It is a cousin of rTT^{++} from [3], closer to it than was rTT_{po}.

Theorem 3.6 *UT entails rTT_{fubs}.*

Proof. (A slight modification of the proof of Theorem 2.7.) Given A, T and Z as above, assume that A is of P-finite character, and makes finite choices from Z using $\bigvee$. We proceed as in Theorem 2.7, with a few changes. For each finite $F \subseteq T$ let $H_F = \{f \in Y \mid \bigvee f[F] \in A\}$.

Claim 1: for each finite $F \subseteq T$, $H_F \neq \{\}$. Since A makes finite choices from Z using $\bigvee$, for each $t \in F$ we can fix a $z_t \preceq Z(t)$ such that $\bigvee\{z_{t\in F}\} \in A$. Define g' from g and $\{\langle t, z_t\rangle \mid t \in T\}$ as in Theorem 2.7. So $g' \in Y$. Since $g'[F] = \{z_{t\in F}\}$, $g' \in H_F$. Claim 1 follows.

Claim 2: for any finite $F_0, F_1 \subseteq T$, $H_{F_0\cup F_1} \subseteq H_{F_0} \cap H_{F_1}$. Consider an $f \in H_{F_0\cup F_1}$. So $\bigvee f[F_0 \cup F_1] \in A$. Consider $i \in 2$. Since $f[F_i] \subseteq f[F_0 \cup F_1]$ and $\bigvee$ is monotonic, $\bigvee f[F_i] \preceq \bigvee f[F_0 \cup F_1]$. Since $\bigvee f[F_i]$ is P-finite and A is of finite-character, $\bigvee f[F_i] \in A$. So $f \in H_{F_i}$.

By Claims 1 & 2, $\{H_F \mid F \subseteq T, F \text{ is finite}\}$ has the finite intersection property. Using UT, fix an ultrafilter U on Y such that for each finite $F \subseteq T$, $H_F \in U$. For $t \in T$ and $z \preceq Z(t)$ let $X^z_t = \{f \in Y \mid f(t) = z\}$. By the argument in Theorem 2.7, we have *Claim* 3: for each $t \in T$ there is a unique $z_t \preceq Z(t)$ so that $X^{z_t}_t \in U$.

Since $\langle P, \bigvee\rangle$ is special, we may fix an upper bound b on $\{z_{t\in T}\}$ that is special for $\{z_{t\in T}\}$ in $\langle P, \bigvee\rangle$.

Claim 4: for every $x \preceq b$, if $x \in \mathcal{F}_P$ then $x \in A$. Consider a P-finite $x \preceq b$. By choice of b, we may fix a finite $F \subseteq T$ such that $x \preceq \bigvee\{z_{t\in F}\}$. Since $H_F \in U$ and for each $t \in F$ $X^{z_t}_t \in U$, $H_F \cap \bigcap_{t\in F} X^{z_t}_t \in U$. So we may fix an $f \in H_F \cap \bigcap_{t\in F} X^{z_t}_t$. Since $f \in \bigcap_{t\in F} X^{z_t}_t$, $f[F] = \{z_{t\in F}\}$. Since $f \in H_F$, $\bigvee f[F] \in A$. So $x \preceq \bigvee\{z_{t\in F}\} = \bigvee f[F] \in A$. Since $x \in \mathcal{F}_P$ and A has finite-character, $x \in A$. Claim 4 follows.

Since A has P-finite character, by Claim 4 we have $b \in A$. For each $t \in T$ $z_t \preceq Z(t)$ and $z_t \preceq b$. So b is as required by rTT_{fubs}. □

Theorem 3.7 *rTT_{fubs} entails CF/CS.*

P r o o f . Assume rTT_{fubs}. Let F be any set. Assume that $\vdash \subseteq \mathcal{P}(F)^2$ is finitary, monotonic and satisfies overlap and cut for formulas. Again, it suffices to prove that $\vdash$ satisfies cut for F. As in Theorem 2.10, we may assume that $F \neq \{\}$ and $\{\} \nvdash \{\}$. Define $\preceq$, $|P|$, P, and $\bigvee$ as in Theorem 2.10, let $\bigvee' = \bigvee |\mathcal{P}'(\mathcal{F}_P)$. $\bigvee'$ is a monotonic fub-selector for P. Check that $\langle P, \bigvee' \rangle$ is special. The rest of the proof is a straightforward modification of the argument in Theorem 2.10. □

So rTT_{po} and rTT_{fubs} are equivalent modulo a weak set-theoretic background.

Observation 3.8 *We can assess the complexity of definitions in Definitions 2.2 & 3.3 by prenexing, taking 'is finite' as a primitive 2nd-order predicate, and counting alternations of second-order quantifiers. Being special in a poset P for an $X \subseteq |P|$ is Π^1_3; so being special is a Π^1_3 property of posets. Being special in a fub-selector structure $\langle P, \bigvee \rangle$ for $X \subseteq |P|$ is Σ^1_1; so being special is a Π^1_2 property of fub-selector structure. So by considering $\langle P, \bigvee \rangle$ in place of P, we gain a simpler notion of specialness.*

Next, we have a brief look at relationships between the concepts defined in Definitions 2.2 to 2.4 and those defined in Definitions 3.1 to 3.4.

Observation 3.9 *If P is a special poset and $\bigvee$ is a fub-selector for P then $\langle P, \bigvee \rangle$ is special.*

P r o o f . Assume the if-clause. Consider a non-empty $X \subseteq \mathcal{F}_P$; fix an upper bound x on X that is special for X in P. Consider any $y \in \mathcal{F}_P$ such that $y \preceq x$; fix a finite $X_0 \subseteq X$ such that for every upper bound u on X_0, $y \preceq u$. In particular, $y \preceq \bigvee X_0$. So x is special for X in $\langle P, \bigvee \rangle$. Note: this did not require that $\bigvee$ be monotonic. □

Observation 3.10 *A special poset need not have a fub-selector. Example: let $|P| = \omega$; let $m \preceq n$ iff $m, n \in \omega$ and either (i) $m = n$ or (ii) $m \in 2$ and $n \notin 2$ or (iii) $m, n \notin 2$ and $n < m$; let $P = \langle |P|, \preceq \rangle$. $\{0, 1\} = \mathcal{F}_P$ has no P-finite upper bound in P. For $m \in 2$, trivially m is special for $\{m\}$ in P; 2 is special for $\{0, 1\}$, since if $y \preceq 2$ is P-finite, $y \in \{0, 1\}$, and y is an upper bound on $\{y\}$. So P is special, but $\{0, 1\}$ has no P-finite upper bound in P.*

Observation 3.11 *A special fub-selector structure need not be based on a special poset. Example: let $|P| = 7$, and let $\preceq$ be the reflexive transitive closure of*

$$\{\langle i, 4 \rangle \mid i \in 3\} \cup \{\langle i + 1, 5 \rangle \mid i \in 3\} \cup \{\langle 4, 6 \rangle, \langle 5, 6 \rangle\}.$$

So $\mathcal{F}_P = 7$; $\bigvee = \{\langle X, 6 \rangle \mid X \subseteq 7\}$ is a fub-selector for P; check that $\langle P, \bigvee \rangle$ is special. The upper bounds on $\{1, 2\}$ in P are 4,5, and 6. But 4 and 6 are not special for $\{1, 2\}$ in P, since $0 \preceq 4$ but $0 \npreceq 5$ and 5 is not special for $\{1, 2\}$ in P, since $3 \preceq 5$ but $3 \npreceq 4$.

4 Further information about cut-conditions

In what follows, let *Even* be the set of even natural numbers, $Odd = \omega - Even$.

We will start by considering sets of single-alternative inferences.

Definition 4.1 Consider a $\vdash \subseteq \mathcal{P}(F) \times F$. The following concepts have been much studied.[9]

1. $\vdash$ satisfies cut for formulas iff for every $\Gamma \subseteq F$ and $\varphi, \delta \in F$, if $\Gamma \vdash \varphi$ and $\Gamma, \varphi \vdash \delta$ then $\Gamma \vdash \delta$.
2. For $\Phi \subseteq F$, $\vdash$ satisfies cut for Φ iff for every $\Gamma \subseteq F$ and $\delta \in F$, if $\Gamma, \Phi \vdash \delta$ and for every $\varphi \in \Phi$ $\Gamma \vdash \varphi$, then $\Gamma \vdash \delta$.
3. $\vdash$ satisfies cut for sets iff for every $\Phi \subseteq F$ it satisfies cut for Φ.
4. $\vdash$ satisfies cut for finite sets iff for every finite $\Phi \subseteq F$ it satisfies cut for Φ.
5. $\vdash$ is monotonic (aka satisfies dilution) iff for every $\Gamma, \Gamma' \subseteq F$ and $\delta \in F$, if $\Gamma \subseteq \Gamma'$ and $\Gamma \vdash \delta$ then $\Gamma' \vdash \delta$.
6. $\vdash$ is finitary (aka compact) iff for every $\Gamma \subseteq F$ and $\delta \in F$, if $\Gamma \vdash \delta$ then for some finite $\Gamma_0 \subseteq \Gamma$ $\Gamma_0 \vdash \delta$.

[9] For a survey cf. [9]. Such a $\vdash$ is usually called a consequence relation on F iff it is reflexive on F, satisfies cut for sets, and is monotonic.

7. $\vdash$ satisfies overlap iff for every $\Gamma \subseteq F$ and $\delta \in F$, if $\delta \in \Gamma$ then $\Gamma \vdash \delta$.

Observation 4.2 *We continue with* $\vdash \subseteq \mathcal{P}(F) \times F$.

1. *If* $\vdash$ *is monotonic and satisfies cut for formulas, then it satisfies cut for finite sets. (This is Theorem 1.2 in [4, p. 17].)*
2. *If* $\vdash$ *is monotonic, finitary, and satisfies cut for formulas, then it satisfies cut for sets. (This is the "single-alternative" analog of CF/CS.)*

P r o o f . For (1), assume the if-clause. It suffices to prove this: for every $n \in \omega$,

(∗) for every $\Phi, \Gamma \subseteq F$ and $\delta \in F$, if $card(\Phi) = n$, $\Gamma, \Phi \vdash \delta$, and for every $\varphi \in \Phi \Gamma \vdash \varphi$, then $\Gamma \vdash \delta$.

If $n = 0$, (∗) is obvious. Given $n \in \omega$, assume (∗). Consider any $\Phi, \Gamma \subseteq F$ and $\delta \in F$; assume that $card(\Phi) = n + 1$ and for every $\varphi \in \Phi \Gamma \vdash \varphi$. Fix $\varphi_0 \in \Phi$ and let $\Phi' = \Phi - \{\varphi_0\}$ and $\Gamma' = \Gamma \cup \{\varphi_0\}$. So $\Gamma', \Phi' \vdash \delta$. By monotonicity, for every $\varphi \in \Phi'\ \Gamma' \vdash \varphi$. By the induction hypothesis, $\Gamma' \vdash \delta$. Since $\Gamma \vdash \varphi_0$ and $\vdash$ satisfies cut for formulas, $\Gamma \vdash \delta$. By induction, for every $n \in \omega$ (∗) is true.

For (2), assume the if-clause. First prove that for every $n \in \omega$,

(∗∗) for every finite $\Phi, \Gamma \subseteq F$ and $\delta \in F$, if $card(\Phi) = n$, $\Gamma, \Phi \vdash \delta$, $\Gamma \cap \Phi = \{\}$, and for every $\varphi \in \Phi\ \Gamma \vdash \varphi$, then $\Gamma \vdash \delta$.

The argument is like that for (1). Then we can use the assumption that $\vdash$ is finitary to complete the argument. □

Remark 4.3 The induction formula for (1) is Π^1_1 in this sense: taking F as the domain and taking union and $\vdash$ as primitive, it starts with a two second-order universal quantifiers prefixed to a first-order formula. For (2) the induction-formula is finite-Π^1_1 in $\vdash$, since the initial two second-order universal quantifiers are restricted to finite subsets of the domain.

I found it somewhat surprising that, in contrast to the CF/CS Theorem, Observation 4.2(2) required merely induction on ω, and a rather simple form at that.

Observation 4.2(1) required use of monotonicity: that $\vdash \subseteq \mathcal{P}(F) \times F$ is finitary and satisfies cut for formulas does not suffice to ensure that $\vdash$ satisfies cut for finite sets.

Example 4.4 Assume that $3 \subseteq F$. For $\Gamma \subseteq F$ and $\delta \in F$, let $\Gamma \vdash \delta$ iff either (i) $\Gamma = 2$ and $\delta = 2$, or (ii) $\Gamma = \{\}$ and $\delta \in 2$.[10] Clearly $\vdash$ is finitary.

Claim: $\vdash$ vacuously satisfies cut for formulas. Assume that (a) $\Gamma, \varphi \vdash \delta$, (b) $\Gamma \vdash \varphi$, and (c) $\varphi \notin \Gamma$. Condition (ii) does not make (a) true; so condition (i) does; so $\Gamma \cup \{\varphi\} = 2$. Fix $i \in 2$ so that $\Gamma = \{i\}$ and $\varphi = 1 - i$. By (b), $i \vdash 1 - i$. But neither (i) nor (ii) makes that true. The claim follows. Since $\nvdash 2$, $\vdash$ does not satisfy cut for finite sets.

We now return to sets of multiple-alternative inferences. For what follows, consider any $\vdash \subseteq \mathcal{P}(F)^2$.

Observation 4.5 *If* $\vdash$ *satisfies cut for formulas, it satisfies cut for finite sets.*[11] *Note: this avoids using monotonicity, in contrast to Observation 4.2(1).*

P r o o f . Assume that $\vdash$ satisfies cut for formulas. It suffices to prove by induction that

(∗) for every $n \in \omega$ for every $\Phi \subseteq F$, if $card(\Phi) = n + 1$ then $\vdash$ satisfies cut for Φ.

For $n = 0$, this is trivial. Given n, assume the obvious induction hypothesis. Given $\Gamma, \Delta \subseteq F$, assume that for every Φ_0 and Φ_1, if $\{\Phi_0, \Phi_1\} \in Splt(\Phi)$ then $\Gamma, \Phi_0 \vdash \Delta, \Phi_1$. Fix $\varphi \in \Phi$ and set $\Phi' = \Phi - \{\varphi\}$. For every Ψ_0 and Ψ_1, if $\{\Psi_0, \Psi_1\} \in Splt(\Phi')$ then $\Gamma, \Psi_0, \varphi \vdash \Delta, \Psi_1$ since $\{\Psi_0 \cup \{\varphi\}, \Psi_1\} \in Splt(\Phi)$; so by the induction hypothesis, $\Gamma, \varphi \vdash \Delta$. Similarly, for every Ψ_0 and Ψ_1, if $\{\Psi_0, \Psi_1\} \in Splt(\Phi')$ then $\Gamma, \Psi_0 \vdash \Delta, \Psi_1, \varphi$; so $\Gamma \vdash \Delta, \varphi$. By one use of cut for formulas, $\Gamma \vdash \Delta$. Hence (∗) follows. So $\vdash$ satisfies cut for Φ. □

Corollary 4.6 *If* F *is finite and* $\vdash$ *satisfies cut for formulas, it satisfies cut for sets.*

Definition 4.7 Consider a set $\vdash$ of inferences on $\mathcal{P}(F)$, and any $\Phi, \Psi \subseteq F$. These definitions are from [4].

[10] Reminder: $2 = \{0, 1\}$.

[11] [4, Theorem 2.3] reads thus: "Cut for formulas is equivalent (granted dilution) to cut for finite sets". Recall: dilution is monotonicity. This might create the impression (well, it did for me) that monotonicity is needed from left to right.

1. $\vdash$ satisfies cut_1 for Φ iff: for every $\Gamma, \Delta \subseteq F$, if $\Gamma, \Phi \vdash \Delta$ and for every $\varphi \in \Phi$ $\Gamma \vdash \Delta, \varphi$, then $\Gamma \vdash \Delta$.
2. $\vdash$ satisfies cut_2 for Ψ iff: for every $\Gamma, \Delta \subseteq F$, if $\Gamma \vdash \Delta, \Psi$ and for every $\psi \in \Psi \Gamma, \psi \vdash \Delta$, then $\Gamma \vdash \Delta$.
3. $\vdash$ satisfies cut_1 [cut_2] iff for every $\Phi \subseteq F$, $\vdash$ satisfies cut_1 [cut_2] for Φ.
4. $\vdash$ satisfies cut_3 for $\langle \Phi, \Psi \rangle$ iff: for every $\Gamma, \Delta \subseteq F$, if (a) $\Gamma, \Phi \vdash \Delta, \Psi$, (b) for every $\psi \in \Psi$ $\Gamma, \psi \vdash \Delta$, and (c) for every $\varphi \in \Phi$ $\Gamma \vdash \Delta, \varphi$, then $\Gamma \vdash \Delta$.
5. $\vdash$ satisfies cut_3 iff for every $\Phi, \Psi \subseteq F$ $\vdash$ satisfies cut_3 for $\langle \Phi, \Psi \rangle$.

Observation 4.8

1. *These are trivially equivalent:*
   (a) *$\vdash$ satisfies cut for formulas;*
   (b) *for every $\varphi \in F$, $\vdash$ satisfies cut_1 for $\{\varphi\}$;*
   (c) *similarly for cut_2;*
   (d) *$\vdash$ satisfies cut_3 for $\langle \{\varphi\}, \{\} \rangle$;*
   (e) *$\vdash$ satisfies cut_3 for $\langle \{\}, \{\varphi\} \rangle$.*
   The following are trivial:
2. *$\vdash$ satisfies cut_3 for $\langle \Phi, \{\} \rangle$ iff $\vdash$ satisfies cut_1 for Φ;*
3. *$\vdash$ satisfies cut_3 for $\langle \{\}, \Psi \rangle$ iff $\vdash$ satisfies cut_2 for Ψ.*
   Somewhat less trivially, if $\vdash$ is monotonic then:
4. *if $\vdash$ satisfies cut_1 and cut_2 then it satisfies cut_3;*
5. *if $\vdash$ satisfies cut for sets then it satisfies cut_3.*

These follow from [4, Theorems 2.6 & 2.7 (p. 32)]. So assuming just monotonicity, cut_1 or cut_2, and then cut_3, are stepping-stones towards cut for sets.

Observation 4.9 *Assume that F is infinite.*

1. *That $\vdash$ is monotonic and satisfies overlap and cut_3 does not suffice to ensure that $\vdash$ satisfies cut for sets. (So in the statements of CF/CS and CF/ CS*, we need the condition that $\vdash$ be finitary.)*
2. *That $\vdash$ is monotonic and satisfies overlap and cut for formulas does not suffice to ensure that $\vdash$ satisfies either cut_1 or cut_2.*

Example 4.10 Let $\omega \subseteq F$.

(1) For $\Gamma, \Delta \subseteq F$, let $\Gamma \vdash \Delta$ iff: either (i) $\Gamma \cap \Delta \neq \{\}$, or (ii) $F - (\Gamma \cup \Delta)$ is finite. So $\vdash$ is monotonic and satisfies overlap.

Claim: $\vdash$ satisfies cut_3. Consider $\Gamma, \Delta, \Phi, \Psi \subseteq F$. Assume that (a) $\Gamma, \Phi \vdash \Delta, \Psi$, (b1) for every $\varphi \in \Phi$ (b1_φ) $\Gamma \vdash \Delta, \varphi$, and (b2) for every $\psi \in \Psi$ (b2_ψ) $\Gamma, \psi \vdash \Delta$. Assume for a contradiction that $\Gamma \nvdash \Delta$. So (c.i) $\Gamma \cap \Delta = \{\}$ and (c.ii) $F - (\Gamma \cup \Delta)$ is infinite. By (a), either $\Phi \nsubseteq \Gamma$ or $\Psi \nsubseteq \Delta$. Case 1: $\Phi \nsubseteq \Gamma$. Fix a $\varphi \in \Phi - \Gamma$. By (c.i) and choice of φ, clause (i) does not make (b1_φ) true; so (ii) does; so $F - (\Gamma \cup \{\varphi\} \cup \Delta)$ is finite; so $F - (\Gamma \cup \Delta)$ is finite, contrary to (c.ii). Case 2: $\Psi \nsubseteq \Delta$. Fix a $\psi \in \Psi - \Delta$. An argument symmetric with the preceding one yields a contradiction. So $\Gamma \vdash \Delta$, proving the claim. Consider any splitting $\{\Psi_0, \Psi_1\}$ of F. By clause (ii), $\Psi_0 \vdash \Psi_1, 0$. But $\{\} \nvdash 0$. So $\vdash$ does not satisfy cut for sets.

(2) For $\Gamma, \Delta \subseteq F$, let $\Gamma \vdash \Delta$ iff: either (i) $\Gamma \cap \Delta \neq \{\}$, or (ii) $F - \Gamma$ is finite, or (iii) $\Delta \cap \mathit{Even} \neq \{\}$, or (iv) $F - \Delta$ is finite, or (v) $\Gamma \cap \mathit{Odd} \neq \{\}$. So $\vdash$ is monotonic and satisfies overlap.

Claim: $\vdash$ satisfies cut for formulas. Consider $\Gamma, \Delta \subseteq F$ and $\vartheta \in F$. Assume that (a) $\Gamma, \vartheta \vdash \Delta$ and (b) $\Gamma \vdash \Delta, \vartheta$. For a contradiction, assume that $\Gamma \nvdash \Delta$. So (c.i) $\Gamma \cap \Delta = \{\}$, (c.ii) $F - \Gamma$ is infinite, (c.iii) $\Delta \cap \mathit{Even} = \{\}$, (c.iv) $F - \Delta$ is infinite, and (c.v) $\Gamma \cap \mathit{Odd} = \{\}$. By (a), $\vartheta \notin \Gamma$; by (b) $\vartheta \notin \Delta$. So by (c.i), clause (i) makes neither (a) nor (b) true. By (c.ii) and (c.iv), neither clause (ii) nor clause (iv) makes (a) true; similarly for (b). By (c.iii), (iii) does not make (a) true; so (v) does; by (c.v), $\vartheta \notin \mathit{Even}$. But by (c.v), clause (v) does not make (b) true; so (iii) does; by (c.iii), $\vartheta \in \mathit{Even}$, a contradiction. So $\Gamma \vdash \Delta$. The claim follows. By clause (v), $F \vdash \{\}$, and for every

$n \in \omega$ by clause (iii) $Even \vdash 2n$; but since $Even \nvdash \{\}$, $\vdash$ does not satisfy cut_1. By clause (iii), $\{\} \vdash F$, and for every $\varphi \in Odd$ by clause (v) $\varphi \vdash Even$; but since $\{\} \nvdash Even$, $\vdash$ does not satisfy cut_2.

Observation 4.11 *For any infinite $\Theta \subseteq F$, that $\vdash$ is finitary and satisfies overlap and cut_3 does not suffice to ensure that $\vdash$ satisfies cut for Θ.*

Example 4.12 For $\Gamma, \Delta \subseteq F$, let $\Gamma \vdash \Delta$ iff: either (i) $\Gamma \cap \Delta \neq \{\}$, or (ii) $\Gamma = \Delta = \{\}$, or (iii) $\Gamma \cup \Delta$ is infinite.

Since $\{\} \vdash \{\}$, $\vdash$ is finitary. (i) ensures that $\vdash$ satisfies overlap. Consider $\Phi, \Psi \subseteq F$. To show that $\vdash$ satisfies cut_3 for $\langle \Phi, \Psi \rangle$, consider any $\Gamma, \Delta \subseteq F$, and assume that (a) $\Gamma, \Phi \vdash \Delta, \Psi$, (b1) for every $\varphi \in \Phi$ $(b1_\varphi)$ $\Gamma \vdash \Delta, \varphi$, and (b2) for every $\psi \in \Psi$ $(b2_\psi)$ $\Gamma, \psi \vdash \Delta$.

Claim: $\Gamma \vdash \Delta$. Assume otherwise. So (c.i) $\Gamma \cap \Delta = \{\}$, (c.ii) either $\Gamma \neq \{\}$ or $\Delta \neq \{\}$, and (c.iii) $\Gamma \cup \Delta$ is finite. By (a), either $\Phi \nsubseteq \Gamma$ or $\Psi \nsubseteq \Delta$. Assume that $\Phi \nsubseteq \Gamma$. Consider $\varphi \in \Phi - \Gamma$. By (c.i) and choice of φ, clause (i) does not make $(b1_\varphi)$ true; by (c.iii) clause (iii) does not make $(b1_\varphi)$ true; trivially clause (ii) does not either; thus a contradiction. By a symmetric argument, the assumption that $\Psi \nsubseteq \Delta$ also yields a contradiction. The claim follows; so $\vdash$ satisfies cut_3 for $\langle \Phi, \Psi \rangle$. So it satisfies cut_3. Consider any infinite $\Theta \subseteq F$. For every $\{\Theta_0, \Theta_1\} \in Splt(\Theta)$, $\Theta_0 \vdash \Theta_1, 0$ by clause (iii); but $\{\} \nvdash 0$. So $\vdash$ does not satisfy cut for Θ.

Corollary 4.13 *1. That $\vdash$ is finitary and satisfies overlap as well as cut_1, cut_2 or both, does not suffice to ensure that it satisfies cut for sets.*

2. That $\vdash$ is finitary and satisfies overlap as well as cut for formulas does not suffice to ensure that it satisfies cut for sets. (So in the statements of CF/CS and CF/CS, we need the condition that $\vdash$ be monotonic.)*

Proof. For (1), note that if it sufficed to ensure satisfaction of cut for sets that $\vdash$ be finitary and satisfy overlap as well as cut_1, cut_2 or both, then, by Observation 4.8(2), adding satisfaction of cut_3 also would suffice, contrary to Observation 4.11.

For (2), note that if that $\vdash$ is finitary and satisfies overlap as well as cut for formulas sufficed to ensure satisfaction of cut for sets, then by Observation 4.8(1) adding satisfaction of cut_1 or cut_2 also would suffice, contrary to (1). □

Observation 4.14 *That $\vdash$ is finitary and satisfies overlap, cut_1 and cut_2 does not suffice to ensure that $\vdash$ satisfies cut_3.*

Example 4.15 Consider any infinite set F, and non-empty Φ and Ψ subsets of F such that $\Phi \cap \Psi = \{\}$. We will define $\vdash$ to be finitary and satisfy overlap, cut_1 and cut_2, but not satisfy cut_3 for $\langle \Phi, \Psi \rangle$.

For $\Gamma, \Delta \subseteq F$, let $\Gamma \vdash \Delta$ iff either (i) $\Gamma \cap \Delta \neq \{\}$, or (ii) $\Delta \cap \Phi \neq \{\}$ and $(\Gamma \cup \Delta) \cap \Psi = \{\}$, or (iii) $\Gamma \cap \Psi \neq \{\}$ and $(\Gamma \cup \Delta) \cap \Phi = \{\}$, or (iv) $\Gamma \cap \Phi \neq \{\}$ and $\Delta \cap \Psi \neq \{\}$.

Check that $\vdash$ is finitary and satisfies overlap. To show that $\vdash$ satisfies cut_1, given $\Gamma, \Delta, \Theta \subseteq F$ assume that (a) $\Gamma, \Theta \vdash \Delta$ and (b) for every $\varphi \in \Theta$ (b_φ) $\Gamma \vdash \Delta, \varphi$. Assume that $\Gamma \nvdash \Delta$. Thus: (c.i) $\Gamma \cap \Delta = \{\}$; (c.ii) either $\Delta \cap \Phi = \{\}$ or $(\Gamma \cup \Delta) \cap \Psi \neq \{\}$; (c.iii) either $\Gamma \cap \Psi = \{\}$ or $(\Gamma \cup \Delta) \cap \Phi \neq \{\}$; (c.iv) either $\Gamma \cap \Phi = \{\}$ or $\Delta \cap \Psi = \{\}$. By (b), (d) $\Delta \cap \Theta = \{\}$. By (c.i) and (d), clause (i) does not make (a) true. By (a), $\Theta \nsubseteq \Gamma$. Fix a $\varphi \in \Theta - \Gamma$. By (c.i), clause (i) does not make (b_φ) true. We now go through the remaining combinations.

Assume that clause (ii) makes (b_φ) true, i.e., $(\Delta \cup \{\varphi\}) \cap \Phi \neq \{\}$ and $(\Gamma \cup \Delta \cup \{\varphi\}) \cap \Psi = \{\}$. So $(\Gamma \cup \Delta) \cap \Psi = \{\}$; by (c.ii), $\Delta \cap \Phi = \{\}$. So $\varphi \in \Phi$. Assume that clause (ii) makes (a) true; so $\Delta \cap \Phi \neq \{\}$, a contradiction. Assume that clause (iii) makes (a) true; so $(\Gamma \cup \Theta \cup \Delta) \cap \Phi = \{\}$; since $\varphi \in \Theta \cap \Phi$, this is a contradiction. Assume that clause (iv) makes (a) true; so $\Delta \cap \Psi \neq \{\}$, contrary to $(\Gamma \cup \Delta) \cap \Psi = \{\}$.

Assume that clause (iii) makes (b_φ) true, i.e., $\Gamma \cap \Psi \neq \{\}$ and $(\Gamma \cup \Delta \cup \{\varphi\}) \cap \Phi = \{\}$. But, by (c.iii), $(\Gamma \cup \Delta) \cap \Phi \neq \{\}$, a contradiction.

Assume that clause (iv) makes (b_φ) true, i.e., $\Gamma \cap \Phi \neq \{\}$ and $(\Delta \cup \{\varphi\}) \cap \Psi \neq \{\}$. By (c.iv), $\Delta \cap \Psi = \{\}$. So $\varphi \in \Psi$. Assume that clause (ii) makes (a) true; so $(\Gamma \cup \Theta \cup \Delta) \cap \Psi = \{\}$; so $\varphi \in \Theta \cap \Psi = \{\}$, a contradiction. Assume that clause (iii) makes (a) true; so $(\Gamma \cup \Theta \cup \Delta) \cap \Phi = \{\}$; so $\Gamma \cap \Phi = \{\}$, a contradiction. Assume that clause (iv) makes (a) true; so $\Delta \cap \Psi \neq \{\}$, a contradiction.

Having exhausted the cases, we have shown that $\Gamma \vdash \Delta$. So $\vdash$ satisfies cut_1. A symmetric argument shows that $\vdash$ satisfies cut_2. But: for each $\varphi \in \Phi$, by clause (ii) $\{\} \vdash \varphi$; for each $\psi \in \Psi$, by clause (iii) $\psi \vdash \{\}$; since $\Phi \neq \{\}$ and $\Psi \neq \{\}$, by clause (iv) $\Phi \vdash \Psi$. Check that $\{\} \nvdash \{\}$. So $\vdash$ does not satisfy cut_3 for $\langle \Phi, \Psi \rangle$.

Observation 4.16 *That* $\vdash$ *is finitary and satisfies overlap and cut for formulas does not suffice to ensure that* $\vdash$ *satisfies* cut_1 *or* cut_2.

Example 4.17 Let $4 \subseteq F$. For $\Gamma, \Delta \subseteq F$, let $\Gamma \vdash \Delta$ iff either (i) $\Gamma \cap \Delta \neq \{\}$, or (ii) $\Gamma \cup \Delta \subseteq Even$ and either (ii.i) $card(\Gamma) + 2 = card(\Delta)$ or (ii.ii) $card(\Gamma) = card(\Delta) + 1$, or (iii) $\Gamma \cup \Delta \subseteq Odd$ and either (iii.i) $card(\Gamma) = card(\Delta) + 2$ or $card(\Gamma) + 1 = card(\Delta)$, or (iv) $\Gamma \cup \Delta$ is infinite.

(i) ensures that $\vdash$ satisfies overlap.

Claim: $\vdash$ is finitary. Assume that $\Gamma \vdash \Delta$. Without loss of generality we may assume that $\Gamma \cup \Delta$ is infinite. So either Γ is infinite or Δ is. Assume that Γ is infinite. Either $\Gamma \cap Even$ or $\Gamma \cap Odd$ is infinite. Assume the former; let $\Delta = \{\}$, and fixing a $\gamma \in \Gamma \cap Even$ let $\Gamma' = \{\gamma\}$; by (ii.ii) $\Gamma' \vdash \Delta'$. Assume the latter; let $\Delta = \{\}$, and fixing distinct $\gamma_0, \gamma_1 \in \Gamma \cap Odd$ let $\Gamma' = \{\gamma_0, \gamma_1\}$; by (iii.i) $\Gamma' \vdash \Delta'$. Assuming that Δ is infinite, the choices of Γ' and Δ' are symmetric, proving the claim.

Given $\Gamma, \Delta \subseteq F$ and $\vartheta \in F$, assume that (a) $\Gamma, \vartheta \vdash \Delta$, and (b) $\Gamma \vdash \Delta, \vartheta$.

Claim: $\Gamma \vdash \Delta$. Assume otherwise; so:

(c.i) $\Gamma \cap \Delta = \{\}$;
(c.ii) either $\Gamma \cup \Delta \nsubseteq Even$ or
both $card(\Gamma) + 2 \neq card(\Delta)$ and $card(\Gamma) \neq card(\Delta) + 1$;
(c.iii) either $\Gamma \cup \Delta \nsubseteq Odd$ or
both $card(\Gamma) \neq card(\Delta) + 2$ and $card(\Gamma) + 1 \neq card(\Delta)$;
(c.iv) $\Gamma \cup \Delta$ is finite.

By (a) and (b), $\vartheta \notin \Gamma \cup \Delta$. Neither clause (i) nor clause (iv) make (a) true; so (a) is made true by either (ii) or (iii). So either (d1) $\Gamma \cup \Delta \cup \{\vartheta\} \subseteq Even$ or (d2) $\Gamma \cup \Delta \cup \{\vartheta\} \subseteq Odd$. Assume (d1). So clause (ii) makes (a) true, and by (c.ii), (e1) $card(\Gamma) + 2 \neq card(\Delta)$ and (e2) $card(\Gamma) \neq card(\Delta) + 1$. Assume that (ii.i) makes (a) true, i.e.,

$$(*) \quad card(\Gamma) + 3 = card(\Gamma \cup \{\vartheta\}) + 2 = card(\Delta).$$

Clause (iii) does not make (b) true; so either (ii.i) or (ii.ii) does. If clause (ii.i) makes (b) true, then

$$card(\Gamma) + 2 = card(\Delta \cup \{\vartheta\}) = card(\Delta) + 1,$$

contradicting (∗). If clause (ii.ii) does,

$$card(\Gamma) = card(\Delta \cup \{\vartheta\}) + 1 = card(\Delta) + 2,$$

contradicting (∗). Assume that clause (ii.ii) makes (a) true, i.e.,

$$(**) \quad card(\Gamma) + 1 = card(\Gamma \cup \{\vartheta\}) = card(\Delta) + 1.$$

So $card(\Gamma) = card(\Delta)$. Again, either clauses (ii.i) or (ii.ii) makes (b) true, and both cases yield contradictions. Assuming (d2) yields a contradiction by symmetric arguments. The claim follows. So $\vdash$ satisfies cut for formulas. Clearly $\{\} \nvdash \{\}$. Since $\{0, 2\} \vdash \{\}$, $\{\} \vdash 0$ and $\{\} \vdash 2$, $\vdash$ does not satisfy cut_1 for $\{0, 2\}$. Since $\{\} \vdash \{1, 3\}$, $\{1\} \vdash \{\}$ and $\{3\} \vdash \{\}$, $\vdash$ does not satisfy cut_2 for $\{1, 3\}$.

Observation 4.18 *That* $\vdash$ *is finitary and satisfies overlap,* cut_3 *and cut for* F *is not sufficient to ensure that it satisfies cut for sets. (Thus for Lemma 2.9 above, i.e., [4, Theorem 2.2, p. 31], we needed that* $\vdash$ *be monotonic.)*

Example 4.19 Let F be such that $\omega \subseteq F$. For $\Gamma, \Delta \subseteq F$ let $\Gamma \vdash \Delta$ iff either (i) $\Gamma \cap \Delta \neq \{\}$, or (ii) $\Gamma = \Delta = \{\}$, or (iii) $\Gamma \cup \Delta \subseteq Even$ and $\Gamma \cup \Delta$ is infinite.

Clearly $\vdash$ is finitary and satisfies overlap. Consider $\Gamma, \Delta, \Phi, \Psi \subseteq F$. Assume that (a) $\Gamma, \Phi \vdash \Delta, \Psi$, (b1) for every $\varphi \in \Phi$ ($b1_\varphi$) $\Gamma \vdash \Delta, \varphi$, and (b2) for every $\psi \in \Psi$ ($b2_\psi$) $\Gamma, \psi \vdash \Delta$.

Claim: $\Gamma \vdash \Delta$. Assume otherwise. So (c.i) $\Gamma \cap \Delta = \{\}$, (c.ii) $\Gamma \cup \Delta \neq \{\}$, and (c.iii) either $\Gamma \cup \Delta$ is finite or $\Gamma \cup \Delta \nsubseteq Even$. Also (d1) $\Phi \cap \Gamma = \{\}$, (d2) $\Psi \cap \Delta = \{\}$, and (e) either $\Phi \nsubseteq \Gamma$ or $\Psi \nsubseteq \Delta$. Assume that $\Phi \nsubseteq \Gamma$. Fix a $\varphi \in \Phi - \Gamma$. By (c.i) and (d1), clause (i) does not make ($b1_\varphi$) true; clearly clause (ii) does not. So clause (iii) does; so $\Gamma \cup \Delta \cup \{\varphi\}$ is an infinite subset of $Even$; but then $\Gamma \cup \Delta$ is infinite, contradicting (c.iii). A symmetric argument applies assuming that $\Psi \nsubseteq \Delta$, using (d2). The claim follows. So $\vdash$ satisfies cut_3. Assume that for every Θ_0 and Θ_1, if $\{\Theta_0, \Theta_1\} \in Splt(F)$ then $\Gamma, \Theta_0 \vdash \Delta, \Theta_1$. If $\Gamma \cap \Delta = \{\}$ then for any such Θ_0 and Θ_1 neither clauses (i), (ii) nor (iii) can make true $\Gamma, \Theta_0 \vdash \Delta, \Theta_1$; so $\Gamma \cap \Delta \neq \{\}$; so $\Gamma \vdash \Delta$. So $\vdash$ satisfies cut for F.

Clearly $\{\} \nvdash \{0\}$. But for any Θ_0 and Θ_1, if $\{\Theta_0, \Theta_1\} \in Splt(Even)$ then $\Gamma, \Theta_0 \vdash \Delta, \Theta_1$ by case (iii). So $\vdash$ does not satisfy cut for *Even*.

5 Appendix

The proof of rTT_{po} made no essential use of the anti-symmetry of partial orderings. So it generalizes from posets to prosets. But the obvious straightforward generalization—simply applying it to prosets which as posets are special—is not the most general generalization. We will not need any such generalization; but the optimal one may be of interest.

Consider a proset $P = \langle |P|, \preceq\rangle$ (i.e., $\preceq$ is a pre-ordering of $|P|$).[12] Let $x \sim y$ iff $x \preceq y \preceq x$. So $\sim$ is an equivalence relation on $|P|$ with respect to which $\preceq$ is compatible. So $P/\sim$ is well-defined, and is a poset.

For readability, let $\preceq_* = (\preceq/\sim)$.

For $X \in P/\sim$ and $y \in P$, let $X \preceq^* y$ iff for every (equivalently, some) $x \in X$, $x \preceq y$.

For an $y \in P$, $\{x \mid x \preceq y\}$ might be infinite even though $\{X \mid X \preceq^* y\}$ is finite; we shall rely on the latter set rather than the former.

Let y be P-finite$_{pro}$ iff: for every $f : \omega \to P$, if for every $i \in \omega$ $f(i+1) \preceq f(i)$ then for some $n \in \omega$ for every $i \in \omega - n$ $f(i) \preceq f(i+1)$. Note: y is P-finite$_{pro}$ iff $\{X \mid X \preceq^* y\}$ is finite. Furthermore $\bigcup \mathcal{F}_{P/\sim} = \{x \in P \mid x \text{ is } P\text{-finite}_{pro}\}$.

Let A be of A is of P-finite character$_{pro}$ iff $A \subseteq P$ and for every $x \in P$,

$$x \in A \text{ iff for every } P\text{-finite}_{pro}\ y,\ \text{ if } y \preceq x \text{ then } y \in A.$$

So A is of P-finite character$_{pro}$ iff for some (equivalantly, any) $A' \subseteq P/\sim$ such that $A = \bigcup A'$, A' is of $P/\sim$-finite character.

Let P be special$_{pro}$ iff $P/\sim$ is special.

Consider a Z and T such that $Z : T \to \mathcal{F}_{P/\sim}$. For $S \subseteq T$ and $x \in P$, x makes S-choices from Z iff for every $t \in S$ there is a $X_t \preceq_* Z(t)$ such that $X_t \preceq^* x$. For $A \subseteq P$, A makes finite choices from Z iff for every finite $S \subseteq T$ some $x_S \in A$ makes S-choices from Z and is P-finite$_{pro}$.

Lemma 5.1 (The Restricted Tukey-Teichmüller Lemma for Prosets; rTT_{pro}) *Assume that P is a special$_{pro}$ proset, and $Z : T \to \mathcal{F}_{P/\sim}$. For any $A \subseteq P$, if A is non-empty, of P-finite character$_{pro}$, and makes finite choices from Z, then for some $b \in A$, b makes T-choices from Z (i.e., for every $t \in \mathrm{dom}(Z)$ there is an $X \preceq_* Z(t)$ so that $X \preceq^* b$).*

Proof. Apply rTT_{po} to $P/\sim$. □

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References

[1] A. Blass, Prime ideals yield almost maximal ideals, Fund. Math. **127**, 57–66 (1987).
[2] A. Blass, Review: D. J. Shoesmith and T. J. Smiley, Multiple-Conclusion Logic, Bull. Amer. Math. Soc. (N.S.) **2**(1), 242–246 (1980).
[3] R. E. Hodel, Restricted versions of the Tukey-Tiechmüller Theorem that are equivalent to the Boolean Prime Ideal Theorem, Arch. Math. Log. **44**(4), 459–472 (2005).
[4] D. J. Shoesmith and T. J. Smiley, Multiple-Conclusion Logic (Cambridge University Press, 1978).
[5] P. Suppes, Axiomatic Set Theory (Van Nostrand, Princeton, 1960).
[6] O. Teichmüller, Braucht der Algebraiker das Auswahlaxiom?, Deutsche Math. **4**, 567–577 (1939).
[7] A. S. Troelstra and H. Schwichtenberg, Basic Proof Theory, 2nd ed. (Cambridge University Press, 2000).
[8] J. Tukey, Convergence and Uniformity in Topology, Annals of Mathematics Studies Vol. 2 (Princeton University Press, 1940).
[9] R. Wojcicki, Theory of Logical Calculi (Kluwer, Dordrecht, 1988).

[12] Pre-orderings were called quasi-orderings in older publications of the recent past; cf. [5], for example.

