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ON MODAL LOGICS WHICH ENRICH FIRST-ORDER S5

A logic is determined by:

(a) a set of logical constants and a set of "types" of variables, which determine a class of associated languages sharing this "logical" lexicon;

(b) an assignment to each associated language of a class of models, each model "interpreting" (in the weak, extensional "model-theoretic" sense) the non-logical lexical items;

(c) an assignment to each model for each language and to each "type" of variable of a variable-assignment in that model;

(d) a relation of satisfaction, between formulae in each language, models for that language, and variable assignments in that model for each type of variable.

These items provide the raw material for the definition of implication in the sense of the given logic: Γ implies ϕ iff for any model for the relevant language and any choice of variable assignments for the relevant types of variables: if these satisfy all members of Γ then they satisfy ϕ; ϕ is valid iff the empty set implies ϕ. One logic is an enrichment of another if it is obtained by expanding the latter's set of logical constants or of "types" of variables, and extending the satisfaction relation to apply to the new formulae. This paper presents several enrichments of the first-order modal logic S5.

1. SYNTAX AND SEMANTICS

Fix an infinite set **Var** of individual variables, and the logical constants '$\perp$' (the absurd), '$\approx$' (identity), '$\supset$', '$\forall$' and '$\Box$'. Let **Pred** be a set of predicate constants, each associated with an $n < \omega$ (its number of places); let **C** be a set of individual constants. Of course **Var**, **Pred**, **C** and the set of logical constants are mutually disjoint. A term of $L = L(\mathbf{Pred}, \mathbf{C})$ is a member of **Var** or of **C**. The set Fml(L) of formulae of L is defined by the usual induction. Let '$\perp$' be an atomic formula and $\neg\phi$ abbreviate $(\phi \supset \perp)$. (I'll ignore the

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distinction between use and mention where confusion seems unlikely.) Other abbreviations are as usual; where τ is a term of L, $E\tau$ abbreviates $(\exists\nu)(\nu \approx \tau)$, where ν is any variable distinct from τ. Hereafter, I'll use ν, ν', ν_0, etc., as meta-variables ranging over **Var**, τ, σ, etc., as meta-variables ranging over the set of terms of L, and ϕ, ψ, θ, etc., as meta-variables ranging over the formulae of L and of the enrichments of L to be introduced.

Since our underlying modal logic is S5, we may let a structure for L be an ordered triple $\mathfrak{A} = (W, A, V)$, where W and $\bar{A} = \cup\{A(w): w \in W\}$ are non-empty sets, A is a function from W into power$(\bar{A})$, and V is a function on $\mathbf{Pred} \cup \mathbf{C}$ so that:

$$V(\tau) \in \bar{A} \qquad \text{for } \tau \in \mathbf{C};$$

$$V(\mathbf{P}) \subseteq W \times \bar{A}^n \qquad \text{for } \mathbf{P} \in \mathbf{Pred}, \mathbf{P}\ n\text{-place},$$

where $W \times \bar{A}^0 = W$. An individual-assignment in $\mathfrak{A}$ is a function α from **Var** into $\bar{A}$. Where $w \in W$, $(\mathfrak{A}, w)$ is a model for L. We define $(\mathfrak{A}, w) \models \phi[\alpha]$ ("α satisfies ϕ in $(\mathfrak{A}, w)$") as usual:

$$\text{den}(\mathfrak{A}, \alpha, \tau) = \begin{cases} V(\tau) & \text{if } \tau \in \mathbf{C}; \\ \alpha(\tau) & \text{if } \tau \in \mathbf{Var}; \end{cases}$$

$$(\mathfrak{A}, w) \not\models \bot[\alpha];$$

$$(\mathfrak{A}, w) \models (\sigma \approx \tau)[\alpha] \text{ iff } \text{den}(\mathfrak{A}, \alpha, \sigma) = \text{den}(\mathfrak{A}, \alpha, \tau);$$

$$(\mathfrak{A}, w) \models \mathbf{P}\tau_1 \ldots \tau_n[\alpha] \text{ iff } (w, \text{den}(\mathfrak{A}, \alpha, \tau_1), \ldots, \text{den}(\mathfrak{A}, \alpha, \tau_n)) \in V(\mathbf{P});$$

$$(\mathfrak{A}, w) \models (\phi \supset \psi)[\alpha] \text{ iff either } (\mathfrak{A}, w) \not\models \phi[\alpha] \text{ or } (\mathfrak{A}, w) \models \psi[\alpha];$$

$$(\mathfrak{A}, w) \models (\forall\nu)\phi[\alpha] \text{ iff for every } a \in A(w), (\mathfrak{A}, w) \models \phi[\alpha^\nu_a];$$

(notice that "$\forall$" is an actualistic quantifier; as usual, $\alpha^\nu_a(\nu') = \alpha(\nu')$ if ν' is not ν, and $\alpha^\nu_a(\nu) = a$);

$$(\mathfrak{A}, w) \models \Box\phi[\alpha] \text{ iff for every } v \in W, (\mathfrak{A}, v) \models \phi[\alpha].$$

As usual, $(\mathfrak{A}, w) \models \phi$ iff for all α, $(\mathfrak{A}, w) \models \phi[\alpha]$; for T a set of formulae, $(\mathfrak{A}, w) \models T$ iff for all $\phi \in T$, $(\mathfrak{A}, w) \models \phi$.

We now present several enrichments of L. In all cases, a structure for one of these enrichments of L is a structure for L; similarly for models. Let $L^{\dot{\forall}}$ be the result of adding '$\dot{\forall}$' (the possibilist universal quantifier) to the logical vocabulary of L, with the obvious formation rule:

$$\text{if } \phi \text{ is a formula of } L^{\dot{\forall}} \text{ then so is } (\dot{\forall}\nu)\phi.$$

Since $(\mathfrak{A}, w) \models E\nu[\alpha]$ if $\alpha(\nu) \in A(w)$, we could regard $E\nu$ as atomic and regard $(\forall\nu)\ldots$ as abbreviating $(\dot{\forall}\nu)(E\nu \supset \ldots)$; we'll call this "regarding '$\forall$' as defined". To define satisfaction for $\phi \in \text{Fml}(L^{\dot{\forall}})$, add the clause:

$$(\mathfrak{A}, w) \models (\dot{\forall}\nu)\phi[\alpha] \text{ iff for every } a \in \bar{A}, (\mathfrak{A}, w) \models \phi[\alpha^{\nu}_{a}].$$

Let $L^{@}$ and $L^{\downarrow}$ be the result of adding the operators '@' and '$\downarrow$' respectively to the logical vocabulary of L, with the rules:

$$\text{if } \phi \text{ is a formula of } L^{@} \text{ then so is } @\phi;$$

$$\text{if } \phi \text{ is a formula of } L^{\downarrow} \text{ then so is } \downarrow\phi.$$

For $\phi \in \text{Fml}(L^{@})$, satisfaction is defined relative to the sequence $(\mathfrak{A}, w, v)$, for $w, v \in W$, and to an individual-assignment α in $\mathfrak{A}$. Think of w as the actual world and of v as the world under scrutiny; for the most part, v here plays the role that w played in the definition for L. Here are several key clauses:

$$(\mathfrak{A}, w, v) \models \mathbf{P}\tau_1 \ldots \tau_n[\alpha] \text{ iff } (v, \text{den}(\mathfrak{A}, \alpha, \tau_1), \ldots, \text{den}(\mathfrak{A}, \alpha, \tau_n)) \in V(\mathbf{P});$$

$$(\mathfrak{A}, w, v) \models (\forall\nu)\phi[\alpha] \text{ iff for every } a \in A(v), (\mathfrak{A}, w, v) \models \phi[\alpha^{\nu}_{a}];$$

$$(\mathfrak{A}, w, v) \models \Box\phi[\alpha] \text{ iff for every } u \in W, (\mathfrak{A}, w, u) \models \phi[\alpha];$$

$$(\mathfrak{A}, w, v) \models @\phi[\alpha] \text{ iff } (\mathfrak{A}, w, w) \models \phi[\alpha].$$

As we unpack the satisfaction conditions of $\phi \in \text{Fml}(L^{@})$, hitting '@' "resets" the world under scrutiny to be the actual world of the model $(\mathfrak{A}, w)$. Satisfaction in a model is now defined as:

$$(\mathfrak{A}, w) \models \phi[\alpha] \text{ iff } (\mathfrak{A}, w, w) \models \phi[\alpha].$$

$L^{@}$ is more expressive than L: "there could be something non-actual", expressible as '$\Diamond(\exists x)@\neg Ex$', is not expressible in L; see [4] for a proof of this. However even in $L^{@}$ the necessity of that condition cannot be expressed: there is no set T of sentences of $L^{@}$ so that for any structure $\mathfrak{A}$ for $L^{@}$:

$$(\mathfrak{A}, w) \models T \text{ if for every } u \in W \text{ there is a } v \in W \text{ so that } A(v) \nsubseteq A(u).$$

Notice that '$\Box(\exists x)@\neg Ex$' misses the mark: in assessing whether $(\mathfrak{A}, w) \models$ '$\Box\Diamond(\exists x)@\neg Ex$', after '$\Box$' "moves us" from w to an arbitrary world u, '$\Diamond$' "moves us" to an arbitrary world v, and then '@' "moves us" back to w, not to u as would be desired.

$L^{\downarrow}$ is designed to allow more delicate world traveling than does $L^{@}$: '$\downarrow$' is to '@' as on a typewriter "back-space" is to "carriage-return".

Define a function on $\mathrm{Fml}(L^{\downarrow})$ as follows:

$$d(\phi) = 0 \text{ if } \phi \text{ is atomic;}$$

$$d(\phi \supset \psi) = \max\{d(\phi), d(\psi)\};$$

$$d((\forall \nu)\phi) = d(\phi);$$

$$d(\downarrow\phi) = 1 + d(\phi);$$

$$d(\Box\phi) = d(\phi) \dot{-} 1.$$

(If $n > 0, n \dot{-} 1 = n - 1; 0 \dot{-} 1 = 0$.) Where $\mathbf{w} = (w_0, \ldots, w_{n-1})$ for $w_0, \ldots, w_{n-1} \in W$, and $d(\phi) < n$, we define $(\mathfrak{A}, \mathbf{w}) \models \phi[\alpha]$. Here are the non-obvious clauses.

$$(\mathfrak{A}, \mathbf{w}) \models \mathbf{P}\tau_1 \ldots \tau_n[\alpha] \text{ iff } (w_{n-1}, \mathrm{den}(\mathfrak{A}, \alpha, \tau_1), \ldots, \mathrm{den}(\mathfrak{A}, \alpha, \tau_n)) \in V(\mathbf{P});$$

$$(\mathfrak{A}, \mathbf{w}) \models (\forall \nu)\phi[\alpha] \text{ iff for every } a \in A(w_{n-1}), (\mathfrak{A}, \mathbf{w}) \models \phi[\alpha_a^\nu];$$

$$(\mathfrak{A}, \mathbf{w}) \models \Box\phi[\alpha] \text{ iff for every } w \in W, (\mathfrak{A}, \mathbf{w}^\frown(w)) \models \phi[\alpha], \text{ where } \mathbf{w}^\frown(w) = (w_0, \ldots, w_{n-1}, w);$$

$$(\mathfrak{A}, \mathbf{w}) \models \downarrow\phi[\alpha] \text{ iff } (\mathfrak{A}, (w_0, \ldots, w_{n-2})) \models \phi[\alpha];$$

notice that in the last case we have $1 \leqslant d(\downarrow\phi) < n$, so $n - 2 \geqslant 0$. Thus '$\Box\Diamond(\exists x)\downarrow\neg Ex$' expresses the condition discussed in the previous paragraph.

We now introduce a family of logics with a common class of languages, but different notions of a variable-assignment for type 1 variables (or, alternatively, different notions of satisfaction). For expository convenience, we'll blur the distinction between a logic and its associated class of languages. To form L^1 we introduce an infinite set $\mathbf{Var}^1$ of type 1 variables, disjoint from all other lexical categories: 'Υ', 'Υ^1', 'Υ_0', etc., shall be meta-variables varying over $\mathbf{Var}^1$; we add the formation rules:

> if τ is a term of L then $\Upsilon\tau$ is a formula of L^1; if ϕ is a formula of L^1 then so is $(\forall\Upsilon)\phi$.

The languages $L^{\mathbf{max}\cdot e}$, $L^{\mathbf{min}\cdot e}$, $L^{i\cdot\mathbf{max}\cdot e}$ and $L^{i\cdot\mathbf{min}\cdot e}$ shall be syntactically identical to L^1, though they'll have different semantics.

B is a monadic attribute in $\mathfrak{A}$ iff $B \subseteq W \times \bar{A}$. B is a maximal essence in $\mathfrak{A}$ if B is a monadic attribute and for all $w, w' \in W$ and $a \in \bar{A}$: if $(w, a) \in B$ then $(w', a) \in B$. B is a minimal essence in $\mathfrak{A}$ iff B is a monadic attribute and for all $w, w' \in W$ and $a \in \bar{A}$:

> if $(w, a) \in B$ and $a \in A(w')$ then $(w', a) \in B$;
> if $(w, a) \in B$ then $a \in A(w)$.

B is an individual maximal essence in $\mathfrak{A}$ iff B is a non-empty maximal essence in $\mathfrak{A}$ and for any $w \in W$ there is at most one $a \in \bar{A}$ such that $(w, a) \in B$; B is an individual minimal essence in $\mathfrak{A}$ iff B is a non-empty minimal essence in $\mathfrak{A}$ and satisfies the previous second conjunct. Satisfaction in $(\mathfrak{A}, w)$ for formulae of L^1, $L^{\mathbf{max}\cdot e}$, $L^{\mathbf{min}\cdot e}$, $L^{i\cdot\mathbf{max}\cdot e}$ and $L^{i\cdot\mathbf{min}\cdot e}$ is defined by letting type 1 variables range over monadic attributes, maximal essences, minimal essences, individual maximal essences and individual minimal essences in $\mathfrak{A}$ respectively. Let β be a monadic attribute assignment in $\mathfrak{A}$ if β maps $\mathbf{Var}^1$ into the set of monadic attributes in $\mathfrak{A}$; satisfaction for $\phi \in \mathrm{Fml}(L^1)$ is relative both to an individual assignment α and such a β; similarly for maximal essence, minimal essence, individual maximal essence, and individual minimal essence assignments in $\mathfrak{A}$, and the languages $L^{\mathbf{max}\cdot e}$, $L^{\mathbf{min}\cdot e}$, $L^{i\cdot\mathbf{max}\cdot e}$ and $L^{i\cdot\mathbf{min}\cdot e}$. In all cases we add these clauses to the definition of satisfaction.

> $(\mathfrak{A}, w) \models \Upsilon\tau[\alpha, \beta]$ iff $(w, \mathrm{den}(\mathfrak{A}, \alpha, \tau)) \in \beta(\Upsilon)$;
>
> $(\mathfrak{A}, w) \models (\forall\Upsilon)\phi[\alpha, \beta]$ iff for all monadic attributes B in $\mathfrak{A}$ (maximal essences B in $\mathfrak{A}$, etc.), $(\mathfrak{A}, w) \models \phi[\alpha, \beta^{\Upsilon}_{B}]$.

The semantics of $L^{\mathbf{max}\cdot e}$, $L^{\mathbf{min}\cdot e}$, $L^{i\cdot\mathbf{max}\cdot e}$ and $L^{i\cdot\mathbf{min}\cdot e}$ may be presented somewhat differently. Let an essence in $\mathfrak{A}$ be a subset of $\bar{A}$; then where β maps $\mathbf{Var}^1$ into the set of essences in $\mathfrak{A}$, we could let:

$$(\mathfrak{A}, w) \stackrel{\mathbf{max}}{\models} \Upsilon\tau[\alpha, \beta] \text{ iff } \operatorname{den}(\mathfrak{A}, \alpha, \tau) \in \beta(\Upsilon);$$

$$(\mathfrak{A}, w) \stackrel{\mathbf{min}}{\models} \Upsilon\tau[\alpha, \beta] \text{ iff } \operatorname{den}(\mathfrak{A}, \alpha, \tau) \in \beta(\Upsilon) \cap A(w);$$

$$(\mathfrak{A}, w) \stackrel{\mathbf{max}}{\models} (\forall \Upsilon)\phi[\alpha, \beta] \text{ iff for every essence } B \text{ in } \mathfrak{A},$$

$$(\mathfrak{A}, w) \stackrel{\mathbf{max}}{\models} \phi[\alpha, \beta^{\Upsilon}_{B}];$$

similarly for $(\mathfrak{A}, w) \stackrel{\mathbf{min}}{\models} (\forall \Upsilon)\phi[\alpha, \beta]$.

Where $B' = W \times B$ and $\beta'(\Upsilon) = \beta(\Upsilon)'$, for ϕ a formula of $L^{\mathbf{max}\cdot e}$ we have:

$$(\mathfrak{A}, w) \stackrel{\mathbf{max}}{\models} \phi[\alpha, \beta] \text{ iff } (\mathfrak{A}, w) \models \phi[\alpha, \beta'].$$

Where $B^* = \{(w, a) \mid a \in B \cap A(w)\}$ and $\beta^*(\Upsilon) = \beta(\Upsilon)^*$, for ϕ a formula of $L^{\mathbf{min}\cdot e}$ we have:

$$(\mathfrak{A}, w) \stackrel{\mathbf{min}}{\models} \phi[\alpha, \beta] \text{ iff } (\mathfrak{A}, w) \models \phi[\alpha, \beta^*].$$

Similar definitions and remarks apply to $L^{i\cdot\mathbf{max}\cdot e}$ and $L^{i\cdot\mathbf{min}\cdot e}$ where B is an individual essence in $\mathfrak{A}$ iff $B = \{a\}$ for $a \in \bar{A}$. For Section 3 (3), our first definitions of satisfaction are preferable; for other purposes, the second definitions are simpler and preferable.

Note: in all these languages, '$\approx$' is definable: $(\tau \approx \sigma)$ may abbreviate $(\forall \Upsilon)\Box(\Upsilon\tau \equiv \Upsilon\sigma)$.

Finally, the devices introduced in this section may be combined in various ways to produce languages $L^{i\cdot\mathbf{max}\cdot e, @}$, $L^{@, \dot{\forall}}$, $L^{\mathbf{max}\cdot e, \dot{\forall}}$, $L^{1, \dot{\forall}}$, $L^{1, \downarrow}$, etc. The appropriate definitions of satisfaction for formulae of these languages are the obvious results of combining the semantics for the languages introduced above.

2. AXIOMATICS

Complete axiomatizations of the valid formulae of L, $L^{\dot{\forall}}$, $L^{@}$ and $L^{\downarrow}$ are provided in [1], [8], [5] and [6] respectively. We'll present axiomatizations of the valid formulae of $L^{i\cdot\mathbf{max}\cdot e}$ and $L^{i\cdot\mathbf{min}\cdot e}$. As our axioms for the logic of individual maximal essences, we take all formulae of the forms (1)–(11) listed in [1] on p. 3, as well as all those of the following forms:

$\diamond(\exists\nu)\Upsilon\nu$;

$\Box(\forall\nu)(\Upsilon\nu \supset \Box(\forall\nu')(\Upsilon\nu' \supset \nu \approx \nu'))$;

$\Box(\forall\nu)(\Upsilon\nu \supset \Box\Upsilon\nu)$;

$\Box(\forall\nu)(\exists\Upsilon)\Upsilon\nu$;

$(\forall\Upsilon)(\phi \supset \psi) \supset ((\forall\Upsilon)\phi \supset (\forall\Upsilon)\psi)$;

$\phi \supset (\forall\Upsilon)\phi$ where Υ is not free in ϕ;

$(\forall\Upsilon)\phi \supset \phi(\Upsilon/\Upsilon')$ where Υ' is substitutable for Υ in ϕ.

Then theoremhood is defined by the familar induction:

if ϕ is an axiom then $\vdash \phi$;

if $\vdash \phi \supset \psi$ and $\vdash \phi$ then $\vdash \psi$;

if $\vdash \phi$ then $\vdash (\forall\nu)\phi$;

if $\vdash \phi$ then $\vdash (\forall\Upsilon)\phi$;

if $\vdash \phi$ then $\vdash \Box\phi$.

Clearly this axiomatization is sound: if $\vdash \phi$ then ϕ is valid in $L^{i\cdot\mathbf{max}\cdot e.}$. Using the method of diagrams (with the notion of $\exists$-completeness extended to handle type 1 variables, and with a set **Pred*** of Henkin one-place predicate-constants in addition to the set $\mathbf{C}^*$ of Henkin individual constants) we may prove completeness via Henkin's lemma: if a set Γ of formulae of $L^{i\cdot\mathbf{max}\cdot e.}$ is consistent (i.e., $\Gamma \nvdash \bot$) then there is a structure $\mathfrak{A}$, a world w, an individual assignment α and an individual maximal-essence assignment β so that $(\mathfrak{A}, w) \models \Gamma[\alpha, \beta]$. It should suffice to note: the first four schemata ensure that, in the Henkin structure $\mathfrak{A}$ associated with a complete consistent diagram $\Delta \supseteq \Gamma$ (Δ a diagram in L $(\mathbf{Pred} \cup \mathbf{Pred}^*, \mathbf{C} \cup \mathbf{C}^*)$), the members of **Pred*** are assigned to all and only the individual maximal essences of $\mathfrak{A}$.

To axiomatize the validities of $L^{i\cdot\mathbf{min}\cdot e}$, replace the third new schema listed above by

$$\Box(\forall\nu)(\Upsilon\nu \supset \Box(\Upsilon\nu \equiv E\nu)).$$

Then in the Henkin model $\mathfrak{A}$, members of **Pred*** are assigned to all and only the individual minimal essences of $\mathfrak{A}$.

Even with **Pred** empty, the validities of $L^{\dot{\mathbf{max}}\cdot e}$ and $L^{\mathbf{min}\cdot e}$ are very non-axiomatizable, in fact, these sets encode full second-order validity for a language $K(\mathbf{R})$ with a single 2-place predicate constant '**R**', and quantifiers of the type 1 interpreted as ranging over all subsets of the universe of a model. This result for $L^{\mathbf{max}\cdot e}$ is a straightforward extension of Theorem 1 of [7] to a second-order language; by (5) of Section 3, this extends to $L^{\mathbf{min}\cdot e}$. Here is a capsule of the argument. Satisfiability of a formulae of $K(\mathbf{R})$ is many-one reducible to satisfiability by a symmetric irreflexive binary (s.i.b.) relation of formula of $K(\mathbf{S})$. Let a binary structure (B, S) match a structure $\mathfrak{A} = (W, A)$ iff $\bar{A} = B$ and for all $a, b \in \bar{A}$, $(a, b) \in S$ iff $a \neq b$ and for some $w \in W, a, b \notin A(w)$; if (B, S) matches $\mathfrak{A}$ then (B, S) is an s.i.b.; if (B, S) is an s.i.b. then some $\mathfrak{A}$ matches (B, S); we transform $\phi \in \mathrm{Fml}(K(\mathbf{S}))$ to $\phi' \in \mathrm{Fml}(L^{\mathbf{max}\cdot e})$ by replacing each occurrence of Svv' by $\Diamond((\neg Ev) \,\&\, (\neg Ev'))$. ϕ is satisfiable by an s.i.b. iff ϕ' is satisfiable in the sense of $L^{\mathbf{max}\cdot e}$.

3. INCLUSIONS

One language includes another iff each sentence of the latter translates into a sentence of the former, the two sentences having exactly the same models. Here are a number of such inclusions which hold for any choice of **Pred** and **C**.

(1) $L^{\downarrow}$ includes $L^{\dot{\forall}}$; view $(\dot{\forall} v)\phi$ as abbreviating $\Box(\forall v){\downarrow}\phi$.

(2) $L^{\downarrow}$ includes $L^{@}$. Suppose $\phi \in \mathrm{Fml}(L^{@})$ contains no nested occurrences of '@'; form ϕ' from ϕ by replacing each occurrence of '@' by '${\downarrow}^k$' where that occurrence is in the scope of exactly k occurrences of '$\Box$'; it's easy to see that $(\mathfrak{A}, w) \models \phi[\alpha]$ iff $(\mathfrak{A}, w) \models \phi'[\alpha]$. If ϕ contains nested occurrences of '@', eliminate outermost occurrences as above; for any occurrence of $@\psi$ as a proper subformula of $@\theta$, which is in turn a subformula of ϕ, if within θ, that occurrence of $@\psi$ is not in the scope of any occurrence of @ and is in the scope of exactly k occurrences of '$\Box$', replace that occurrence of $@\psi$ by ${\downarrow}^k\psi$. It's easy to see that, where ϕ' is the result of eliminating all occurrences of '@' from ϕ in this manner, $(\mathfrak{A}, w) \models \phi[\alpha]$ iff $(\mathfrak{A}, w) \models \phi'[\alpha]$.

(3) L^1 includes both $L^{\mathbf{max}\cdot e.}$ and $L^{\mathbf{min}\cdot e.}$. Let Max$E(\Upsilon)$ and Min$E(\Upsilon)$ abbreviate $\Box(\forall\nu)(\Upsilon\nu \supset \Box\Upsilon\nu)$ and $\Box(\forall\nu)(\Upsilon\nu \supset \Box(\Upsilon\nu \equiv E\nu))$ respectively (for any choice of ν). Where β is a monadic attribute assignment in $\mathfrak{A}$:

$$(\mathfrak{A}, w) \models \text{Max}E(\Upsilon)[\beta] \text{ iff } \beta(\Upsilon) \text{ is a maximal essence in } \mathfrak{A};$$

$$(\mathfrak{A}, w) \models \text{Min}E(\Upsilon)[\beta] \text{ iff } \beta(\Upsilon) \text{ is a minimal essence in } \mathfrak{A}.$$

Translate a formula ϕ of $L^{\mathbf{max}\cdot e.}$ or of $L^{\mathbf{min}\cdot e.}$ into L^1 by restricting all occurrences of '$(\forall\Upsilon)$' by Max$E(\Upsilon)$ or Min$E(\Upsilon)$, yielding ϕ_0 and ϕ_1 respectively. Where β is a maximal essence assignment in $\mathfrak{A}$ and $\phi \in \text{Fml}(L^{\mathbf{max}\cdot e.})$,

$$(\mathfrak{A}, w) \models \phi[\alpha, \beta] \text{ iff } (\mathfrak{A}, w) \models \phi_0[\alpha, \beta].$$

Where β is a minimal essence assignment in $\mathfrak{A}$ and $\phi \in \text{Fml}(L^{\mathbf{min}\cdot e.})$,

$$(\mathfrak{A}, w) \models \phi[\alpha, \beta] \text{ iff } (\mathfrak{A}, w) \models \phi_1[\alpha, \beta].$$

(4) $L^{\mathbf{max}\cdot e}$ includes $L^{i\cdot\mathbf{max}\cdot e}$; $L^{\mathbf{min}\cdot e}$ includes $L^{i\cdot\mathbf{min}\cdot e}$. Let Ind$(\Upsilon)$ abbreviate

$$\Diamond(\exists\nu)\Upsilon\nu \;\&\; \Box(\forall\nu)(\Upsilon\nu \supset \Box(\forall\nu')(\Upsilon\nu' \supset \nu \approx \nu')),$$

where ν and ν' are distinct. To translate $\phi \in \text{Fml}(L^{i\cdot\mathbf{max}\cdot e})$ into $L^{\mathbf{max}\cdot e}$ simply restrict all occurrences of '$(\forall\Upsilon)$' in ϕ by Ind(Υ); similarly for translating $\phi \in \text{Fml}(L^{i\cdot\mathbf{min}\cdot e})$ into $L^{\mathbf{min}\cdot e}$.

(5) $L^{\mathbf{max}\cdot e}$ and $L^{\mathbf{min}\cdot e}$ are mutually inclusive; similarly for $L^{i\cdot\mathbf{max}\cdot e}$ and $L^{i\cdot\mathbf{min}\cdot e}$. We'll use the second definition of satisfaction for such formulae. For $\phi \in \text{Fml}(L^{\mathbf{max}\cdot e})$ form $f(\phi) \in \text{Fml}(L^{\mathbf{min}\cdot e})$ by replacing each occurrence of $\Upsilon\tau$ in ϕ by $\Diamond\Upsilon\tau$; clearly:

$$(\mathfrak{A}, w) \overset{\mathbf{max}}{\models} \Upsilon\tau[\alpha, \beta] \text{ iff } (\mathfrak{A}, w) \overset{\mathbf{min}}{\models} \Diamond\Upsilon\tau[\alpha, \beta];$$

by induction on ϕ, we have:

$$(\mathfrak{A}, w) \overset{\mathbf{max}}{\models} \phi[\alpha, \beta] \text{ iff } (\mathfrak{A}, w) \overset{\mathbf{min}}{\models} f(\phi)[\alpha, \beta].$$

For $\phi \in \text{Fml}(L^{\mathbf{min}\cdot e})$ form $g(\phi) \in \text{Fml}(L^{\mathbf{max}\cdot e})$ by replacing each occurrence of $\Upsilon\tau$ in ϕ by $(\Upsilon\tau \;\&\; E\tau)$; clearly:

$$(\mathfrak{A}, w) \overset{\mathbf{min}}{\models} \Upsilon\tau[\alpha, \beta] \text{ iff } (\mathfrak{A}, w) \overset{\mathbf{max}}{\models} (\Upsilon\tau \;\&\; E\tau)[\alpha, \beta];$$

by induction on ϕ we have:

$$(\mathfrak{A}, w) \overset{\min}{\models} \phi[\alpha, \beta] \text{ iff } (\mathfrak{A}, w) \overset{\max}{\models} g(\phi)[\alpha, \beta].$$

Similarly f translates $L^{i\cdot\max\cdot e}$ into $L^{i\cdot\min\cdot e}$ and g translates $L^{i\cdot\min\cdot e}$ into $L^{i\cdot\max\cdot e}$. These results show that as far as considerations of expressibility go, the difference between maximal and minimal essences has no importance; for definiteness, I'll consider only maximal essences and refer to $L^{\max\cdot e}$ and $L^{i\cdot\max\cdot e}$ as "L^e" and "$L^{i\cdot e}$".

(6) $L^{\dot\forall}$ includes $L^{i\cdot e}$. Associate with each type 1 variable Υ_i occurring in ϕ a distinct individual variable ν_i not occurring in ϕ; form ϕ' from ϕ by replacing each occurrence of $\Upsilon_i\tau$ in ϕ by $(\nu_i \approx \tau)$ and each occurrence of $(\forall\Upsilon_i)$ by $(\dot\forall\nu_i)$; where β is an individual essence assignment in $\mathfrak{A}$ and $\beta(\Upsilon_i) = \{\alpha(\nu_i)\}$:

$$(\mathfrak{A}, w) \models \Upsilon_i\tau[\alpha, \beta] \text{ iff } (\mathfrak{A}, w) \models (\nu_i \approx \tau)[\alpha, \beta].$$

Thus if $\Upsilon_0, \ldots, \Upsilon_{n-1}$ are the free type 1 variables in ϕ, where $\beta(\Upsilon_i) = \{a_i\}$ for all $i < n$ we have by induction on ϕ:

$$(\mathfrak{A}, w) \models \phi[\alpha, \beta] \text{ iff } (\mathfrak{A}, w) \models \phi'\begin{bmatrix} \alpha\nu_0, \ldots, \nu_{n-1} \\ a_0, \ldots, a_{n-1} \end{bmatrix}.$$

Languages associated with logics which combine the devices introduced in Section 1 provide some less obvious inclusions.

(7) $L^{1,@}$ includes $L^{1,@,\dot\forall}$, which are thus mutually inclusive. Regard '$\dot\forall$', '$\dot\exists$', '$\forall$', '$\exists$', '$\Box$' and 'E' as primitive. We'll consider two normal forms for formulae of $L^{1,@,\dot\forall}$. For $Z \subseteq \mathrm{Fml}(L^{1,@,\dot\forall})$, let:

$$cZ = \left\{ \bigvee_{i<r} \bigwedge_{j<q} \{\phi_{ij} \mid \text{either } \phi_{ij} \in Z \text{ or } \phi_{ij} = \neg\psi_{ij} \text{ for } \psi_{ij} \in Z\} \right\};$$

$$Z_0 = \{\phi, \neg\phi, @\phi, \neg@\phi \mid \phi \text{ is atomic}\};$$

$$Z_1 = \{\Box\phi \mid \phi \in cZ_0\};$$

$$Z_{n+2} = \left\{ \Box(Q_0\xi_0)\ldots(Q_{m-1}\xi_{m-1})\phi \mid \phi \mathrel{\dot{\in}} c\left(\bigcup_{j\leqslant n+1} Z_j\right), m \geqslant 1 \right\};$$

ϕ is pre-prenex iff ϕ is of the form $(Q_0\xi_0)\ldots(Q_{m-1}\xi_{m-1})\psi$ for $\psi \in c(\bigcup_{j<\omega} Z_j)$ and $m \geqslant 0$. Claim: any formula of $L^{1,@,\dot{\forall}}$ may be transformed into an equivalent pre-prenex formula. First, eliminate occurrences of '$(\forall\nu)$' and '$(\exists\nu)$' using '$(\dot{\forall}\nu)$', '$(\dot{\exists}\nu)$' and 'E'; secondly, drive occurrences of '@' inward until all such occurrences govern atomic or negations of atomic formula, using the facts that '@□' reduces to '□', and '@' commutes with truth-functions, '$(\dot{\forall}\nu)$', '$(\forall\Upsilon)$', etc. Thirdly, drive negations inward so that '$\neg$' only governs atomic subformulae or those of the form $\Box\psi$. Fourthly, using the technique which convert any quantities-free formula of L into one containing no nested occurrences of '□', make sure that any nested occurrences of '□' are separated by some quantifiers; simultaneously pull quantifiers outward, using familiar prenexing equivalences. It is not hard to see that eventually the given formula is transformed into a pre-prenex formula.

A formula of $L^{1,@,\dot{\forall}}$ is clean iff no occurrence of '□' in it governs an occurrence of '$\dot{\forall}$' or '$\dot{\exists}$'.

LEMMA. Any formula of $L^{1,@,\dot{\forall}}$ may be transformed into an equivalent clean formula.

Proof. Let ψ be a pre-prenex formula. We'll transform ψ into a clean formula by working from the inside outward on the subformulae of the form $\Box(Q'_0\xi_0)\ldots(Q'_{m-1}\xi_{m-1})\theta$ for $\theta \in c(\bigcup_{j<\omega} Z_j)$. By induction hypothesis, suppose θ has been replaced by a clean formula $\phi \in c(\bigcup_{j<\omega} Z_j)$. (These assumptions imply that ϕ contains *no* occurrences of '$\dot{\forall}$' or '$\dot{\exists}$'.) We may also assume that if ξ_i is an individual variable then Q'_i is '$\dot{\forall}$' or '$\dot{\exists}$'. By inserting double negations and pulling negations inward through quantifiers it will suffice to transform

$$(*) \qquad \Diamond(Q_0\xi_0)\ldots(Q_{m-1}\xi_{m-1})\phi$$

into

$$(**) \qquad (\exists\Upsilon)\Diamond(\exists\nu')(\Diamond\Upsilon\nu' \,\&\, (\hat{Q}_0\xi_0)\ldots(\hat{Q}_{m-1}\xi_{m-1})\Box(\Upsilon\nu' \supset \phi)),$$

where ν' and Υ do not occur free in ϕ and where:

$$(\hat{Q}_i\xi_i) \text{ if } \begin{cases} (Q_i\xi_i) & \text{if } \xi_i \text{ is of type } 1; \\ \Box(\forall\xi_i) & \text{if } Q_i \text{ is '}\dot{\forall}\text{'}; \\ \Diamond(\exists\xi_i) & \text{if } Q_i \text{ is '}\dot{\exists}\text{'}. \end{cases}$$

Suppose that in $(\mathfrak{A}, w, v)$, α, β satisfy $(*)$. Then for some $u \in W$:

$$(\mathfrak{A}, w, u) \models (Q_0\xi_0) \ldots (Q_{m-1}\xi_{m-1})\phi[\alpha, \beta].$$

Pick $a \in \bar{A}$; let $B = \{(u, a)\}$, $\alpha' = \alpha_a^{\nu'}$, $\beta' = \beta_B^{\Upsilon}$; so for any $w' \in W$, $(\mathfrak{A}, w, w') \models \Upsilon\nu'[\alpha', \beta']$ iff $w' = u$. Therefore:

$$(\mathfrak{A}, w, w') \models (\hat{Q}_0\xi_0) \ldots (\hat{Q}_{m-1}\xi_{m-1})\Box(\Upsilon\nu' \supset \phi)[\alpha', \beta'].$$

Thus in $(\mathfrak{A}, w, v)$, $(**)$ is satisfied by α, β.

Conversely, suppose that in $(\mathfrak{A}, w, v)$, $(**)$ is satisfied by α, β. Select a B, w' and $a \in A'$ so that for $\alpha' = \alpha_a^{\nu'}$, $\beta' = \beta_B^{\Upsilon}$:

$$(\mathfrak{A}, w, w') \models \Diamond\Upsilon\nu' \,\&\, (\hat{Q}_0\xi_0) \ldots (\hat{Q}_{m-1}\xi_{m-1}) \Box(\Upsilon\nu' \supset \phi)[\alpha', \beta'];$$

fix a u so that $(u, a) \in B$. It's not hard to see that:

$$(\mathfrak{A}, w, u) \models (Q_0\xi_0) \ldots (Q_{m-1}\xi_{m-1})\phi[\alpha', \beta'].$$

Since ν' and Υ are not free in ϕ, in $(\mathfrak{A}, w, v)$, α, β satisfies $(*)$. Note that $(**)$ is clean.

In a clean formula, all occurrences of '$(\dot{\forall}\nu)$' or '$(\dot{\exists}\nu')$' are not in the scope of '$\Box$', and so may be replaced by '$\Box(\forall\nu)@$' and '$\Diamond(\exists\nu)@$'. Thus we may eliminate all occurrences of '$\dot{\forall}$' or '$\dot{\exists}$' from a clean formula, proving (7).

(8) $L^{1,\dot{\forall}}$ includes $L^{@,\dot{\forall}}$. Let $\phi \in \mathrm{Fml}(L^{1,@,\dot{\forall}})$ be interesting iff it has the form

$$\Box(Q_0\nu_0) \ldots (Q_{n-1}\nu_{n-1})\left[\bigvee_{i<k} (@\psi_i \,\&\, \theta_i)\right],$$

where Q_0 is '$\dot{\exists}$' and Q_i is '$\dot{\exists}$' or '$\dot{\forall}$' for $1 \leqslant i < n$, $n \geqslant 0$, ψ_i and θ_i containing no occurrences of '@' and no type 1 variables. Our goal is to eliminate '@' from interesting formulae.

Let $i \in I$ iff Q_i is '$\dot{\exists}$', $I = \{i_0 < \ldots < i_{q-1}\}$; introducing distinct type 1 variables $\Upsilon_0, \ldots, \Upsilon_{q-1}$, form the quantifier prefex $\mathbf{Q}$ from $(Q_0\nu_0) \ldots (Q_{n-1}\nu_{n-1})$ by replacing each $(Q_{i_j}\nu_{i_j})$ by $(\exists\Upsilon_j)(\dot{\forall}\nu_{i_j})$. Let $g_0(\phi)$, $g_1(\phi)$ and $g_2(\phi)$ be the following formulae (respectively):

$$\bigwedge_{j<q} \Box(\dot{\exists}\nu)\Upsilon_j\nu;$$

$$\Diamond\left(\bigwedge_{j<q} \Upsilon_j \nu_{i_j}\right) \supset \left(\bigvee_{i<k} \psi_i\right);$$

$$\bigwedge_{s\in S}\left(J(\psi_0, \ldots, \psi_{k-1}, s) \supset \Box\left(\left(\bigwedge_{j<q} \psi_{j_1}\nu_{i_j}\right) \supset \right.\right.$$

$$\left.\left.\supset K(\theta_0, \ldots, \theta_{k-1}, s)\right)\right),$$

where:

$$S = \{s \in {}^k 2 \mid \text{for some } i < k, s(i) = 1\},$$

$$J(\psi_0, \ldots, \psi_{k-1}, s) \text{ is } \bigwedge_{i<k} \psi_i^{s(i)},$$

$$K(\theta_0, \ldots, \theta_{k-1}, s) \text{ is } \bigvee\{\theta_i \mid s(i) = 1\},$$

and where ψ^0 is ψ, ψ^1 is $\neg\psi$. *Claim*: ϕ is equivalent to $\mathbf{Q}(g_0(\phi) \,\&\, g_1(\phi) \,\&\, g_2(\phi))$. It should suffice to prove this equivalence for a particular quantifier prefex; suppose $n = 4$, $Q_0 = Q_2 =$ '$\dot\exists$', $Q_1 = Q_3 =$ '$\dot\forall$'; so $\mathbf{Q}$ is $(\exists\Upsilon_0)(\dot\forall\nu_0)(\dot\forall\nu_1)(\exists\Upsilon_1)(\dot\forall\nu_2)(\dot\forall\nu_3)$.

First, suppose $(\mathfrak{A}, w_0) \models \phi[\alpha, \beta]$. (Where no type 1 variable is free in a formula, we'll omit mention of β when we use '$\models$'.) For notation simplification, let $\alpha\{c_0\}$ be $\alpha^{\nu_0}_{c_0}$, $\alpha\{c_0, c_1\}$ be $\alpha\{c_0\}^{\nu_1}_{c_1}$, etc. For each $w \in W$ select $f_0(w) \in \bar{A}$ so that

$$(\mathfrak{A}, w_0, w) \models (\dot\forall\nu_1)(\dot\exists\nu_2)(\dot\forall\nu_3)\left(\bigvee_{i<k} (@\psi_1 \,\&\, \theta_i)\right)[\alpha\{f_0(w)\}].$$

Given $a_0 \in \bar{A}$ select $f_1(w, a_0) \in \bar{A}$ so that

$$(\mathfrak{A}, w_0, w) \models (\dot\forall\nu_3)\left(\bigvee_{i<k} (@\psi_i \,\&\, \theta_i)\right)[\alpha\{f_0(w), a_0, f_1(w, a_0)\}].$$

Thus given $a_1 \in \bar{A}$,

$$(\mathfrak{A}, w_0, w) \models \bigvee_{i<k} (@\psi_i \,\&\, \theta_i)[\alpha\{f_0(w), a_0, f_1(w, a_0), a_1\}].$$

Let $t = t_{b_0, a_0, b_1, a_1}$ be defined by:

$$t(i) = \begin{cases} 1 & \text{if } (\mathfrak{A}, w_0) \models \psi_i[\alpha\{b_0, a_0, b_1, a_1\}], \\ 0 & \text{otherwise.} \end{cases}$$

For $t = t_{f_0(w), a_0, f_1(w, a_0), a_1}$, $t \in S$; since '@' doesn't occur in $\theta_0, \ldots, \theta_{k-1}$, we also have:

$$(\mathfrak{A}, w) \models K(\theta_0, \ldots, \theta_{k-1}, t)[\alpha\{f_0(w), a_0, f_1(w, a_0), a_1\}].$$

Let $B_0 = \{(w, f_0(w)) \mid w \in W\}$; suppose we're given $b_0, a_0 \in \bar{A}$; let $B_1 = \{(w, f_1(w, a_0)) \mid w \in W\}$; let $\beta' = \beta^{\Upsilon_0}_{B_0}$, $\beta'' = \beta'^{\Upsilon_1}_{B_1}$; so $(\mathfrak{A}, w_0) \models g_0(\phi)[\beta'']$. Suppose we're given $b_1, a_1 \in \bar{A}$; for $\alpha' = \alpha\{b_0, a_0, b_1, a_1\}$ $(\mathfrak{A}, w_0) \models g_1(\phi)[\alpha']$. Let $t = t_{b_0, a_0, b_1, a_1}$; for $s \in {}^k 2$, $s = t$ iff

$$(\mathfrak{A}, w_0) \models J(\psi_0, \ldots, \psi_{k-1}, s)[\alpha'].$$

Most importantly,

$$(\mathfrak{A}, w_0) \models \Box((\Upsilon_0 \nu_0 \mathbin{\&} \Upsilon_1 \nu_2) \supset K(\theta_0, \ldots, \theta_{k-1}, t))[\alpha', \beta''];$$

For suppose $(\mathfrak{A}, w_0, w) \models \Upsilon_0 \nu_0 \mathbin{\&} \Upsilon \nu_2 [\alpha', \beta'']$; then $b_0 = f_0(w)$, $b_1 = f_1(w, a_0)$, so $\alpha' = \alpha\{f_0(w), a_0, f_1(w, a_0), a_1\}$, $t = t_{f_0(w), a_0, f_1(w, a_0), a_1}$; so by previous remarks, $(\mathfrak{A}, w_0, w) \models K(\theta_0, \ldots, \theta_{k-1}, t)[\alpha']$. Since b_1 and a_1 were arbitrary,

$$(\mathfrak{A}, w_0) \models (\dot{\forall}\nu_2)(\dot{\forall}\nu_3)(g_0(\phi) \mathbin{\&} g_1(\phi) \mathbin{\&} g_2(\phi))[\alpha\{b_0, b_1\}, \beta''];$$

thus:

$$(\mathfrak{A}, w_0) \models (\exists \Upsilon_1)(\dot{\forall}\nu_2)(\dot{\forall}\nu_3)(g_0(\phi) \mathbin{\&} g_1(\phi) \mathbin{\&} g_2(\phi)) [\alpha\{b_0, b_1\}, \beta'];$$

iterating this we get:

$$(\mathfrak{A}, w_0) \models \mathbf{Q}(g_0(\phi) \mathbin{\&} g_1(\phi) g_2(\phi))[\alpha, \beta].$$

Secondly, suppose that $(\mathfrak{A}, w_0) \models \mathbf{Q}(g_0(\phi) \mathbin{\&} g_1(\phi) \mathbin{\&} g_2(\phi))[\alpha]$. We're given $w \in W$, and wish to show:

$$(\mathfrak{A}, w_0, w) \models (\dot{\exists}\nu_0)(\dot{\forall}\nu_1)(\dot{\exists}\nu_2)(\dot{\forall}\nu_3)\left(\bigvee_{i<k} (@\psi_i \mathbin{\&} \theta_i)\right)[\alpha].$$

Fix B_0 so that for $\beta' = \beta^{\Upsilon_0}_{B_0}$,

$$(\mathfrak{A}, w_0) \models (\dot{\forall}\nu_0)(\dot{\forall}\nu_1)(\exists \Upsilon_1)(\dot{\forall}\nu_2)(\dot{\forall}\nu_3)(g_0(\phi) \mathbin{\&} g_1(\phi) \mathbin{\&} g_2(\phi))\ [\alpha, \beta'].$$

Thus $(\mathfrak{A}, w_0) \models \Box(\dot{\exists}\nu)\Upsilon_0\nu[\beta']$; we may pick b_0 so that $(w, b_0) \in B_0$; given $a_0 \in \bar{A}$,

$$(\mathfrak{A}, w_0) \models (\exists \Upsilon_1)(\dot{\forall}\nu_2)(\dot{\forall}\nu_3)(g_0(\phi) \mathbin{\&} g_1(\phi) \mathbin{\&} g_2(\phi)) [\alpha\{b_0, a_0\}, \beta'].$$

Now fix B_1 so that for $\beta'' = \beta'^{\Upsilon_1}_{B_1}$,

$$(\mathfrak{A}, w_0) \models (\dot\forall\nu_2)(\dot\forall\nu_3)(g_0(\phi) \mathbin{\&} g_1(\phi) \mathbin{\&} g_2(\phi))[\alpha\{b_0, a_0\}, \beta''],$$

again $(\mathfrak{A}, w_0) \models \Box(\dot\exists\nu)\Upsilon_1[\beta'']$; so we may pick b_1 with $(w, b_1) \in B_1$; given $a \in \bar{A}$ and $\alpha' = \alpha\{b_0, a_0, b_1, a_1\}$,

$$(\mathfrak{A}, w_0) \models g_1(\phi) \mathbin{\&} g_2(\phi)[\alpha', \beta''].$$

By choice of b_0 and b_1, and using $g_1(\phi)$, there is a unique $t \in S$ so that $(\mathfrak{A}, w_0) \models J(\psi_0, \ldots, \psi_{k-1}, t)[\alpha']$; using $g_2(\phi)$,

$$(\mathfrak{A}, w_0) \models \Box((\dot\Upsilon_0\nu_0 \mathbin{\&} \dot\Upsilon_1\nu_2) \supset K(\theta_0, \ldots, \theta_{k-1}, t))[\alpha', \beta''].$$

So $(\mathfrak{A}, w_0, w) \models K(\theta_0, \ldots, \theta_{k-1}, t)[\alpha']$. Thus:

$$(\mathfrak{A}, w_0, w) \models \bigvee_{i<k} (@\psi_i \mathbin{\&} \theta_i)[\alpha'].$$

Since a_1 was given after choice of b_1,

$$(\mathfrak{A}, w_0, w) \models (\dot\exists\nu_2)(\dot\forall\nu_3)\left(\bigvee_{i<k} (@\psi_i \mathbin{\&} \theta_i)\right)[\alpha\{b_0, a_0\}].$$

Since a_0 was given after choice of b_0,

$$(\mathfrak{A}, w_0, w) \models (\dot\exists\nu_0)(\dot\forall\nu_1)(\dot\exists\nu_2)(\dot\forall\nu_3)\left(\bigvee_{i<k} (@\psi_i \mathbin{\&} \theta_i)\right)[\alpha].$$

Thus $(\mathfrak{A}, w_0) \models \phi[\alpha]$, proving the claim.

Given $\psi \in \mathrm{Fml}(L^{@,\dot\forall})$, suppose ψ is in pre-prenex form; we wish to pull '@' outward; using the fact that '□@' may be replaced by '@', we need only consider subformula of ψ in Z_j for $j \geqslant 2$. Where ϕ is an interesting subformula of ψ, replace ϕ by $@\mathbf{Q}(g_0(\phi) \mathbin{\&} g_1(\phi) \mathbin{\&} g_2(\phi)) = \phi'$; since $(\mathfrak{A}, w_0, w) \models (\phi \equiv \phi')[\alpha]$, and '@' doesn't occur in $\mathbf{Q}(g_0(\phi) \mathbin{\&} g_1(\phi) \mathbin{\&} g_2(\phi))$, iterating this process yields a formula ψ' equivalent to ψ with no occurrence of '@' in the scope of '□'; then all occurrences of '@' may be deleted, yielding $\psi^* \in \mathrm{Fml}(L^{1,\dot\forall})$ so that $(\mathfrak{A}, w_0) \models (\psi' \equiv \psi^*)[\alpha]$.

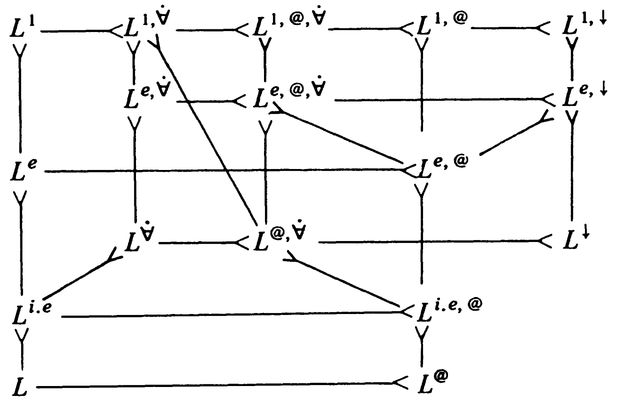
Further inclusions require special constraints on **Pred**.

(9) If all members of **Pred** are 0-place, then $L^{i.e.}$ includes $L^{\dot\forall}$, which are then mutually inclusive; similarly for $L^{i.e.,@}$ and $L^{@,\dot\forall}$; similarly for L^{e} and $L^{e,\dot\forall}$.

Proof. Given $\phi \in \mathrm{Fml}(L^{\dot\forall})$, regard '$\dot\forall$' as defined. For each individual variable ν occurring bound in ϕ, introduce a corresponding distinct type 1 variable Υ_ν; replace all occurrences of '$(\dot\forall\nu)$', '$E\nu$', '$\nu \approx \nu'$' and '$\nu \approx \sigma$' for $\sigma \in \mathbf{C}$ or σ free in ϕ by '$(\forall\Upsilon_\nu)$', '$(\exists x)\Upsilon_\nu x$', '$\Diamond(\exists x)(\Upsilon_\nu x \mathbin{\&} \Upsilon_{\nu'} x)$', and '$\Upsilon_\nu\sigma$' respectively. Given $\mathfrak{A}$ and α, fix an individual essence assignment β so that $\beta(\Upsilon_\nu) = \{\alpha(\nu)\}$. Then $(\mathfrak{A}, w) \models \phi[\alpha]$ iff $(\mathfrak{A}, w) \overset{\max}{\models} \phi[\alpha, \beta]$. For $\phi \in \mathrm{Fml}(L^{e,\dot\forall})$, make sure that Υ_ν doesn't occur in ϕ, and revise the above translation by replacing '$(\dot\forall\nu)$' by '$(\forall\Upsilon_\nu)\,\mathrm{Ind}(\Upsilon_\nu)\ldots)$'.

(10) If each member of **Pred** is 0-place or 1-place, then $L^{e,\dot\forall}$ includes $L^{e,\downarrow}$, in this sense: if $\phi \in \mathrm{Fml}(L^{e,\downarrow})$ and $d(\phi) = 0$ then ϕ is expressible in $L^{e,\dot\forall}$; thus in this case $L^{e,\dot\forall}$, $L^{e,@,\dot\forall}$ and $L^{e,\downarrow}$ are mutually inclusive. This will be proved at the end of Section 4.

The following picture shows inclusions (1)–(8) and some obvious consequences of (1)–(6), where '——<' represents 'is included in'.



4. WORLD TRAVELING AND EXTENSIONALIZATIONS

We associate with L a two-sorted non-modal language $L_{\mathbf{ext}}$, the extensionalization L, as follows. Let $\mathbf{Var}_w$ be an infinite set of world variables disjoint from all other lexical categories, 'μ', 'μ'', 'μ_0', etc. shall be used as metavariables ranging over $\mathbf{Var}_w$. The logical constants of $L_{\mathbf{ext}}$ are '$\perp$', '$\approx$', '**E**', '$\supset$' and '$\forall$'; to each $\mathbf{P} \in \mathbf{Pred}$ we associate $\mathbf{P}^{\mathbf{ext}}$; the formation rules are:

if $\tau_1, \ldots, \tau_n$ are terms of L then $\mathbf{P}^{\mathbf{ext}}\mu\tau_1 \ldots \tau_n$ is a formula;

if τ and σ are terms of L, $(\sigma \approx \tau)$ is a formula;

$\perp$ is a formula;

if τ is a term of L then $\mathbf{E}\mu\tau$ is a formula.

Other formation rules are as usual.

We transform a structure $\mathfrak{A}$ for L into a two-sorted classical model $\mathfrak{A}_{\mathbf{ext}} = (W, \bar{A}, V_{\mathbf{ext}})$ for $L_{\mathbf{ext}}$, where $V_{\mathbf{ext}} \supseteq V$ by $V_{\mathbf{ext}}(\mathbf{E}) = \{(w, a) \mid a \in A(w)\}$. A world assignment in $\mathfrak{A}$ is a function γ from $\mathbf{Var}_w$ into W. For $\phi \in \mathrm{Fml}(L_{\mathbf{ext}})$, let $\mathfrak{A}_{\mathbf{ext}} \models \phi[\alpha, \gamma]$ be defined in the obvious way. We expand $L_{\mathbf{ext}}$ to $L^e_{\mathbf{ext}}$ by adding an infinite class of type 1 variables, the formation rule:

if τ is a term of L then $\Upsilon\tau$ is a formula of $L^e_{\mathbf{ext}}$,

and the satisfaction clause:

$$\mathfrak{A}_{\mathbf{ext}} \models \Upsilon\tau[\alpha, \beta, \gamma] \text{ iff } \mathrm{den}(\mathfrak{A}, \alpha, \tau) \in \beta(\Upsilon),$$

where β is an essence assignment in $\mathfrak{A}$. We expand $L_{\mathbf{ext}}$ to $L^1_{\mathbf{ext}}$ by adding the formation rule:

if τ is a term of L then $\Upsilon\mu\tau$ is a formula of $L^1_{\mathbf{ext}}$,

and the satisfaction clause:

$$\mathfrak{A}_{\mathbf{ext}} \models \Upsilon\mu\tau[\alpha, \beta, \gamma] \text{ iff } (\gamma(\mu), \mathrm{den}(\mathfrak{A}, \alpha, \tau)) \in \beta(\Upsilon).$$

(11) $L^{\downarrow}$ and $L_{\mathbf{ext}}$ are mutually inclusive; similarly for $L^{e,\downarrow}$ and $L^e_{\mathbf{ext}}$, and for $L^{1,\downarrow}$ and $L^1_{\mathbf{ext}}$. In other words, we may associate with each sequence $\mu_0, \ldots, \mu_{n-1}$ of world variables functions $g = g_{\mu_0, \ldots, \mu_{n-1}}$ and $h = h_{\mu_0, \ldots, \mu_{n-1}}$ meeting these conditions. For $\phi \in \mathrm{Fml}(L^{\downarrow})$ and $d(\phi) < n$, $g(\phi) \in \mathrm{Fml}(L_{\mathbf{ext}})$ has free world variables among $\mu_0, \ldots, \mu_{n-1}$, and for any $\mathfrak{A}$, α, and γ:

$$(\mathfrak{A}, (\gamma(\mu_0), \ldots, \gamma(\mu_{n-1}))) \models \phi[\alpha] \text{ iff } \mathfrak{A}_{\mathbf{ext}} \models g(\phi)[\alpha, \gamma].$$

For $\psi \in \mathrm{Fml}(L_{\mathbf{ext}})$ with free world variables among $\mu_0, \ldots, \mu_{n-1}$, $h(\psi) \in \mathrm{Fml}(L^{\downarrow})$ and for any $\mathfrak{A}$, α, and γ:

$$(\mathfrak{A}, (\gamma(\mu_0), \ldots, \gamma(\mu_{n-1}))) \models h(\psi)[\alpha] \text{ iff } \mathfrak{A}_{\mathbf{ext}} \models \psi[\alpha, \gamma].$$

Furthermore, both g and h extend appropriately to $L^{e,\downarrow}$ and $L^1_{\mathbf{ext}}$.

Proof. It will be convenient to regard '$\forall$' as defined from '$\dot{\forall}$' and 'E'. Define g and h as follows:

$$g(\mathbf{P}\tau_1 \ldots \tau_m) = \mathbf{P}^{\mathbf{ext}}\mu_{n-1}\tau_1 \ldots \tau_m;$$

$$g(E\tau) = \mathbf{E}\mu_{n-1}\tau;$$

$$g(\sigma \approx \tau) = (\sigma \approx \tau);$$

$$g(\bot) = \bot;$$

$$g(\phi \supset \phi') = (g(\phi) \supset g(\phi'));$$

$$g((\dot{\forall}\nu)\phi) = (\forall\nu)g(\phi);$$

$$g(\Box\phi) = (\forall\mu)g_{\mu_0,\ldots,\mu_{n-1},\mu}(\phi) \text{ where } \mu \text{ is distinct from } \mu_0, \ldots, \mu_{n-1};$$

$$g(\downarrow\phi) = g_{\mu_0,\ldots,\mu_{n-2}}(\phi), \text{ which is well-defined, since } 1 \leqslant d(\downarrow\phi) < n \text{ implies } 2 \leqslant n.$$

$$h(\mathbf{P}^{\mathbf{ext}}\mu_i\tau_1 \ldots \tau_m) = \downarrow^{n-i-1}\mathbf{P}\tau_1 \ldots \tau_m;$$

$$h(\mathbf{E}\mu_i\tau) = \downarrow^{n-i-1}E\tau;$$

$$h(\sigma \approx \tau) = (\sigma \approx \tau);$$

$$h(\bot) = \bot;$$

$$h(\psi \supset \psi') = (h(\psi) \supset h(\psi'));$$

$$h((\forall\nu)\psi) = (\dot{\forall}\nu)h(\psi) = \Box(\forall\nu)\downarrow h(\psi);$$

$$h((\forall\mu)\psi) = \Box h_{\mu_0,\ldots,\mu_{n-1},\mu}(\psi), \text{ which is well-defined since } \mu \text{ is not free in } (\forall\mu)\psi, \text{ and thus is distinct from } \mu_0, \ldots, \mu_{n-1}.$$

If we're concerned with $L^{e,\downarrow}$ and $L^{e}_{\mathbf{ext}}$, let $g(\Upsilon\tau) = \Upsilon\tau$, $h(\Upsilon\tau) = \Upsilon\tau$. If we're concerned with $L^{1,\downarrow}$ and $L^{1}_{\mathbf{ext}}$, let $g(\Upsilon\tau)$ be $g(\Upsilon\mu_{n-1}\tau)$, and $h(\Upsilon\mu_i\tau))$ be $\downarrow^{n-i-1}\Upsilon\tau$.

A straightforward induction on the construction of $\phi \in \mathrm{Fml}(L^{\downarrow})$ shows that $g(\phi)$ is as desired; similarly for $\psi \in \mathrm{Fml}(L_{\mathbf{ext}})$ and $h(\phi)$. Similarly when we consider quantification over essences and monadic attributes. Notice that $gh(\psi) = \psi$, although $hg(\phi)$ needn't be ϕ.

Where $\phi \in \mathrm{Fml}(L_{\mathbf{ext}})$, let ϕ be proper iff for each subformula ψ of ϕ there is at most one world variable free in ψ. For $\phi \in \mathrm{Fml}(L^{\dot\forall})$, and any μ, $g_\mu(\phi)$ is proper; in fact, the converse holds: if ψ is proper and contains at most μ free, $h_\mu(\psi) \in \mathrm{Fml}(L^{\dot\forall})$. This result extends to $L^{e,\dot\forall}$ and $L^e_{\mathbf{ext}}$, and to $L^{1,\dot\forall}$ and $L^1_{\mathbf{ext}}$; in an obvious way these facts express in terms of the syntax of $L_{\mathbf{ext}}$ the feature of L and $L^{\dot\forall}$ which $L^{@}$ partially and $L^{\downarrow}$ totally overcome: their failure to permit "backwards world travelling".

We'll use the extensionalization of $L^{e,\downarrow}$ to prove (10). Suppose **Pred** = $\{\mathbf{P}_0, \ldots, \mathbf{P}_{k-1}, \mathbf{P}_k, \ldots, \mathbf{P}_{q-1}\}$, where $\mathbf{P}_i$ is 0-place for $i < k$, 1-place for $k \leqslant i < q$. Where $s \in {}^k2$, let $P(s) = \bigwedge_{i<k} \mathbf{P}_i^{s(i)}$, where '$\mathbf{P}^0$' is '$\mathbf{P}$' and '$\mathbf{P}^1$' is '$\neg\mathbf{P}$'. Let $\psi \in \mathrm{Fml}(L_{\mathbf{ext}})$ have free world variables $\mu_0, \ldots, \mu_{n-1}$; let $s_0, \ldots, s_{n-1}$ be a sequence of members of ${}^k2$. It suffices to define a function $f = f_{\mu_0, \ldots, \mu_{n-1}, s_0, \ldots, s_{n-1}}$ so that $f(\psi) \in L^{e,\dot\forall}$ and for any structure $\mathfrak{A}$ and any α, β, γ, individual, essence and world assignments in $\mathfrak{A}$, if for all $i < n$:

$$\gamma(\mu_i) \in V(\mathbf{P}_j) \text{ iff } s_i(j) = 1 \text{ for } j < k,$$

$$\beta(\Upsilon_{i,j}) = \{a \mid (\gamma(\mu_i), a) \in V(\mathbf{P}_j)\} \text{ for } k \leqslant j < q,$$

$$\beta(\Upsilon_{i,q}) = A(\gamma(\mu_i)),$$

where the $\Upsilon_{i,j}$ are distinct type 1 variables not occurring in ψ, then: $\mathfrak{A}_{\mathbf{ext}} \models \psi[\alpha, \beta, \gamma]$ iff $(\mathfrak{A}, w) \models f(\psi)[\alpha, \beta]$ for any $w \in W$. Define f as follows:

$$f(P_j^{\mathbf{ext}}\mu_i) = \mathbf{P}_j^{s_i(j)} \quad \text{for } j < k;$$

$$f(P_j^{\mathbf{ext}}\mu_i\tau) = \Upsilon_{i,j}\tau \quad \text{for } k \leqslant j < q;$$

$$f(\mathbf{E}\mu_i\tau) = \Upsilon_{i,q}\tau;$$

$$f(\Upsilon\tau) = \Upsilon\tau;$$

$$f(\sigma \approx \tau) = (\sigma \approx \tau);$$

$$f(\bot) = \bot;$$

$$f(\psi_0 \supset \psi_1) = (f(\psi_0) \supset f(\psi_1));$$

$$f((\forall\nu)\psi) = (\dot\forall\nu)f(\psi);$$

$$f((\forall\Upsilon)\psi) = (\forall\Upsilon)f(\psi);$$

$$f((\forall\mu)\psi) = \Box(\exists\Upsilon_{n,k}) \ldots (\exists\Upsilon_{n,q-1})\Bigg(\bigwedge_{k \leqslant j < q} (\dot\forall x)(\Upsilon_{n,j} \equiv \mathbf{P}_j x) \mathbin{\&} (\dot\forall x)(\Upsilon_{n,q}x \equiv Ex) \mathbin{\&} \bigvee_{s \in {}^k2} (P(s) \mathbin{\&} f_s(\psi))\Bigg),$$

where $f_s = f_{\mu_0,\ldots,\mu_{n-1},\mu,s_0,\ldots,s_{n-1},s}$ and the $\Upsilon_{n,j}$ are distinct type 1 variables different from the $\Upsilon_{i,j}$ for $i < n$ and not occurring in ψ; this is well-defined because μ is not free in $(\forall\mu)\psi$, and so is distinct from μ_i for $i < n$.

An easy induction shows that f is as required. For example, consider the case of $(\forall\mu)\psi$. Suppose $s_0, \ldots, s_{n-1}, \alpha, \beta$, and γ meet the given conditions. Suppose that $\mathfrak{A}_{\mathbf{ext}} \models (\forall\mu)\psi[\alpha,\beta,\gamma]$; given any $u \in W$ let $\beta'(\Upsilon_{n,j}) = \{a \mid (u,a) \in V(\mathbf{P}_j)\}$ for $k \leqslant j < q$, $\beta'(\Upsilon_{n,q}) = A(u)$, and $\beta'(\Upsilon) = \beta(\Upsilon)$ for all other Υ; since $\mathfrak{A}_{\mathbf{ext}} \models \psi[\alpha,\beta,\gamma^{\mu}_{u}]$ and no $\Upsilon_{n,j}$ occurs in ψ, $\mathfrak{A}_{\mathbf{ext}} \models \psi[\alpha, \beta', \gamma^{\mu}_{u}]$; fix s so that $(\mathfrak{A},u) \models P(s)$; since now $s_0, \ldots, s_{n-1}, s, \alpha, \beta'$, and γ^{μ}_{u} meet the required conditions, we can apply the induction hypothesis to get $(\mathfrak{A},u) \models f_s(\psi)[\alpha,\beta']$; so letting $f(\psi)$ be $\Box\theta$, we have $(\mathfrak{A},u) \models \theta[\alpha,\beta]$; so $(\mathfrak{A},w) \models f(\psi)[\alpha,\beta]$. Now suppose that $(\mathfrak{A},w) \models f(\psi)[\alpha,\beta]$; given any $u \in W$, $(\mathfrak{A},u) \models \theta[\alpha,\beta]$; for β' as above, $(\mathfrak{A},u) \models (P(s) \,\&\, f_s(\psi))[\alpha,\beta']$ for some $s \in {}^{k}2$; then $s_0, \ldots, s_{n-1}, s, \alpha, \beta'$, and γ^{μ}_{u} meet the required conditions for applying the induction hypothesis to conclude that $\mathfrak{A}_{\mathbf{ext}} \models \psi[\alpha,\beta',\gamma^{\mu}_{u}]$; since none of the $\Upsilon_{n,j}$ occur in ψ, $\mathfrak{A}_{\mathbf{ext}} \models \psi[\alpha,\beta,\gamma^{\mu}_{u}]$; so $\mathfrak{A}_{\mathbf{ext}} \models (\forall\mu)\psi[\alpha, \beta,\gamma]$.

If $\phi \in \mathrm{Fml}(L^{e,\downarrow})$ and $d(\phi) = 0$ then $g(\phi)$ has no free world variables, and has the same free variables as ϕ; but then $f(g(\phi))$ and $g(\phi)$ have the same free variables; for any $\mathfrak{A}$, w, α and β:

$$(\mathfrak{A},(w)) \models \phi[\alpha,\beta] \text{ iff } (\mathfrak{A},w) \models f(g(\phi))[\alpha,\beta].$$

5. FAILURES OF INCLUSION

(12) As mentioned in Section 1, '$\Diamond(\exists x)@\neg Ex$' is not expressible in L; since this sentence is expressed in $L^{i.e.}$ by '$(\exists X)\neg(\exists x)Xx$', L doesn't include $L^{@}$ or $L^{i.e}$.

(13) Using the encoding of an arbitrary binary structure by an s.i.b., there is a sentence θ of $K(\mathbf{S})$ so that for all s.i.b.'s (B,S), if $(B,S) \models \theta$ then B is finite; but there are such s.i.b.'s (B,S) with $\mathrm{card}(B)$ arbitrarily large. Using our encoding of an s.i.b. into a structure for L, we find a sentence θ' of L^{e} so that if $(\mathfrak{A},w) \models \theta'$ then $\bar{A}$ is finite, but there are such structures $\mathfrak{A}$ with $\bar{A}$ arbitrarily large. Since the logic of $L^{i.e.}$ is axiomatizable, and thus (since a derivation can use only finitely many premises) compact, θ' can't be translated into $L^{i.e}$; similarly for $L^{\dot\forall}, L^{@}, L^{@,\dot\forall}$ and $L^{\downarrow}$.

(14) Let θ_0 be '$(\dot\exists x)(\mathbf{P}x \,\&\, \neg Ex)$'; θ_0 is not translatable into L^1. Similarly for $\mathbf{P}$ n-place, $n \geqslant 2$ and $(\exists x_1) \ldots (\exists x_n)(\mathbf{P}x_1 \ldots x_n \,\&\, \neg Ex_1)$. Thus if **Pred** contains at least one n-place predicate for $n \geqslant 1$, neither $L^{i.e}, L^e$, nor L^1 includes $L^{\dot\forall}$; nor does L^e or L^1 include $L^{e,\dot\forall}$; nor does L^1 include $L^{1,\dot\forall}$.

Proof. Let $W = \{0, 1, 2\}, A(0) = A(1) = \{a\}, A(2) = \{a, b\}, V(\mathbf{P}) = \{(0, b)\}$, $V(\mathbf{P}')$ is empty for all $\mathbf{P}'$ different from $\mathbf{P}$; $\mathfrak{A} = (W, A, V)$. Where B is a monadic attribute in $\mathfrak{A}$ let B' be

$$\{(2,x) \mid (2,x) \in B\} \cup \{(0,x) \mid (1,x) \in B\} \\ \cup \{(1,x) \mid (0,x) \in B\};$$

where β is a monadic attribute assignment in $\mathfrak{A}$ let $\beta'(\Upsilon) = \beta(\Upsilon)'$. *Claim*: if $\alpha(\nu) = a$ for all ν, then for all $\phi \in \mathrm{Fml}(L^1)$:

$$(\mathfrak{A}, 0) \models \phi[\alpha, \beta] \text{ iff } (\mathfrak{A}, 1) \models \phi[\alpha, \beta'].$$

This holds for atomic ϕ; the induction on ϕ is straightforward; notice that $(\mathfrak{A}, 0) \models (\forall\nu)\phi[\alpha, \beta]$ iff $(\mathfrak{A}, 0) \models \phi[\alpha, \beta]$, and $(\mathfrak{A}, 1) \models (\forall\nu)\phi[\alpha, \beta']$ iff $(\mathfrak{A}, 1) \models \phi[\alpha, \beta']$. Since $(\mathfrak{A}, 0) \models \phi_0$ and $(\mathfrak{A}, 1) \not\models \phi_0$, ϕ_0 is not equivalent to any formula of L^1.

(15) Let θ_1 be '$\Box(\dot\exists x)(\mathbf{P}x \,\&\, \neg Ex)$'; θ_1 is not expressible in $L^{e,@}$. Thus if **Pred** contains an n-place predicate for $n \geqslant 1$, $L^{\dot\forall}$ is not included in $L^{i.e,@}$, nor in $L^{e,@}$.

Proof. Let $W = \{(N, P) \mid N$ and P are disjoint infinite subsets of ω and $\omega - (N \cup P)$ is infinite$\}$; let $W^1 = W \cup \{0\}, A((N, P)) = A'((N, P)) = N$; select $w_1 \in W$ and let $A'(0) = A(w_1)$; let $V(\mathbf{P}) = \{((N, P), a) \mid a \in P\}$; let $P((N, P)) = P, P(0)$ is empty; let $\mathfrak{A} = (W, A, V), \mathfrak{B} = (W', A', V)$. Select $w_0 \in W$; then $(\mathfrak{A}, w_0) \models \theta_1$, but $(\mathfrak{B}, w_0) \not\models \theta_1$. *Claim*: $(\mathfrak{A}, w_0)$ and $(\mathfrak{B}, w_0)$ satisfy the same sentences of $L^{e,@}$.

Let $\mathbf{b} = (b_0, \ldots, b_{n-1}), \mathbf{c} = (c_0, \ldots, c_{n-1}) \in \omega^n$; let $\mathbf{b} \sim \mathbf{c}$ iff for all $i, j < n$:

$$b_i = b_j \text{ iff } c_i = c_j;$$

$$b_i \in A(w_0) \text{ iff } c_i \in A'(w_0);$$

$$b_i \in P(w_0) \text{ iff } c_i \in P(w_0).$$

For $w, u \in W'$, let $(w, \mathbf{b}) \sim (u, \mathbf{c})$ iff $\mathbf{b} \sim \mathbf{c}$ and for all $i < n$:

$$b_i \in A'(w) \text{ iff } c_i \in A'(u);$$

$$b_i \in P(w) \text{ iff } c_i \in P(u).$$

Let $\mathbf{B} = (B_0, \ldots, B_{m-1})$, $B_i \subseteq \omega$ for $i < m$, $s \in {}^{m+2}2$; let:

$$g(\mathbf{B}, s) = \bigcap_{i<m} B_i^{s(i)} \cap A(w_0)^{s(m)} \cap P(w_0)^{s(m+1)},$$

where for $X \subseteq \omega$, $X^0 = X$, $X^1 = \omega - X$. Clearly $\{g(B, s) \mid s \in {}^{m+2}2\}$ is a partition of ω. Let $(w, \mathbf{b}, \mathbf{B}) \sim (u, \mathbf{c}, \mathbf{C})$ iff $(w, \mathbf{b}) \sim (u, \mathbf{c})$ and for all $i < n, j < m$:

$$b_i \in B_j \text{ iff } c_i \in C_j;$$

$$\text{for all } s \in {}^{m+2}2, \operatorname{card}(A'(w) \cap g(\mathbf{B}, s)) = \operatorname{card}(A'(u) \cap g(\mathbf{C}, s)).$$

For $\phi \in \mathrm{Fml}(L^{e,@})$ with free variables among $\nu_0, \ldots, \nu_{n-1}, \Upsilon_0, \ldots, \Upsilon_{m-1}$, and $w \in W$, let $(\mathfrak{A}, w_0, w) \models \phi[\mathbf{b}, \mathbf{B}]$ iff $(\mathfrak{A}, w_0, w) \models \phi[\alpha, \beta]$ for any α and β with $\alpha(\nu_i) = b_i$ for $i < n$ and $\beta(\Upsilon_i) = B_i$ for $i < m$; similarly for $\mathfrak{B}$ and $w \in W'$.

LEMMA 1. If $(w, \mathbf{b}, \mathbf{B}) \sim (u, \mathbf{c}, \mathbf{C})$ then:

$$(\mathfrak{A}, w_0, w) \models \phi[\mathbf{b}, \mathbf{B}] \text{ iff } (\mathfrak{B}, w_0, u) \models \phi[\mathbf{c}, \mathbf{C}].$$

Where ϕ is atomic, this is clear. The only non-trivial induction step is where ϕ is $(\forall \nu)\psi$ or $(\forall \Upsilon)\psi$.

Suppose ϕ is $(\forall \nu)\psi$. Then for any $b \in A'(w)$ there is a $c \in A'(u)$ (and for any $c \in A'(u)$ there is a $b \in A'(w)$) so that $(w, \mathbf{b}^\frown(b), \mathbf{B}) \sim (u, \mathbf{c}^\frown(c), \mathbf{C})$. Given $b \in A'(w)$, if $b = b_i$, let $c = c_i$. If $b \neq b_i$ for all $i < n$, there is a unique $s \in {}^{m+2}2$ so that $b \in g(\mathbf{B}, s)$; any $c \in A'(u) \cap g(\mathbf{C}, s)$, $c \neq c_i$ for all $i < n$, will be as desired; since $\operatorname{card}(A'(w) \cap g(\mathbf{B}, s)) = \operatorname{card}(A'(u) \cap g(\mathbf{C}, s))$, such a c exists. A similar argument produces a $b \in A'(w)$ given $c \in A'(u)$.

Suppose ϕ is $(\forall \Upsilon)\psi$. Then for any $B \subseteq \omega$ there is a $C \subseteq \omega$ (and for any $C \subseteq \omega$ there is a $B \subseteq \omega$), so that $(w, \mathbf{b}, \mathbf{B}^\frown(B)) \sim (u, \mathbf{c}, \mathbf{C}^\frown(C))$. Given B and $s \in {}^{m+2}2$, let $B^s = B \cap g(\mathbf{B}, s)$. $\{B^s \mid s \in {}^{m+2}2\}$ partitions B. For each s, select $C^s \subseteq g(\mathbf{C}, s)$ so that:

$$\text{for all } b_i \in g(\mathbf{B}, s), b_i \in B^s \text{ iff } c_i \in C^s;$$

$$\text{card}(A'(w) \cap B^s) = \text{card}(A'(u) \cap C^s).$$

Because $b_i \in g(\mathbf{B}, s)$ iff $c_i \in g(\mathbf{C}, s)$, we can satisfy the first condition; because $\text{card}(A'(w) \cap g(\mathbf{B}, s)) = \text{card}(A'(u) \cap g(\mathbf{C}, s))$ we can satisfy the second conjunct. Let $C = \bigcup \{C^s \mid s \in {}^{m+2}2\}$; C is as desired. A similar construction produces B from a given C. Lemma 1 follows.

Suppose $w_1 = (N_1, P_1), P \subseteq \omega - N_1, \omega - (N_1 \cup P)$ is infinite, and for all $i < n, b_i \notin P$; then for $w = (N_1, P)$, $(w, \mathbf{b}, \mathbf{B}) \sim (0, \mathbf{b}, \mathbf{B})$. Thus for any $\mathbf{b}$ and $\mathbf{B}$ there is a $w \in W$ so that for all $\phi \in \text{Fml}(L^{e,@})$:

$$(\mathfrak{B}, w_0, w) \models \phi[\mathbf{b}, \mathbf{B}] \text{ iff } (\mathfrak{B}, w_0, 0) \models \phi[\mathbf{b}, \mathbf{B}].$$

LEMMA 2. For any $\phi \in \text{Fml}(L^{e,@})$ and $w \in W$:

$$(\mathfrak{A}, w_0, w) \models \phi[\mathbf{b}, \mathbf{B}] \text{ iff } (\mathfrak{B}, w_0, w) \models \phi[\mathbf{b}, \mathbf{B}].$$

Where ϕ is atomic, this is clear; the only non-trivial induction step is where ϕ is $\Box\psi$; since $W' = W \cup \{0\}$, by the preceding remark, if $(\mathfrak{B}, w_0, w) \not\models \Box\psi[\mathbf{b}, \mathbf{B}]$; then $(\mathfrak{A}, w_0, w) \not\models \Box\psi[\mathbf{b}, \mathbf{B}]$. This proves the lemma, letting $w = w_0$, we get

$$(\mathfrak{A}, w_0) \models \phi[\mathbf{b}, \mathbf{B}] \text{ iff } (\mathfrak{B}, w_0) \models \phi[\mathbf{b}, \mathbf{B}].$$

(16) Let $(\exists^{\geqslant 2}\nu)\phi$ abbreviate

$$(\exists\nu)(\exists\nu')(\nu \not\approx \nu' \ \&\ \phi \ \&\ \phi(\nu/\nu')),$$

where ν' is distinct from ν and doesn't occur in ϕ; let θ_2 be

$$\Box((\exists x)@\neg Ex \supset (\exists^{\geqslant 2}x)@\neg Ex).$$

Then θ_2 is not expressible in $L^{\dot\forall}$. So even if **Pred** is empty, $L^{\dot\forall}$ does not include $L^{@}$.

Proof. Regard '$\forall$' as defined in $L^{\dot\forall}$. Let $W = \{0\} \cup \{(n, m) \mid n \neq m, n, m \in \omega\}$, $W' = W \cup \{1\}$. Let $A(0) = B(0) = \omega$, $A((n, m)) = B((n, m)) = (\omega - \{n, m\}) \cup \{-n \mid n \in \omega\}$, let $V(\mathbf{P})$ be empty for all $\mathbf{P} \in \mathbf{Pred}$, $\mathfrak{A} = (W, A, V)$, $\mathfrak{B} = (W', B, V)$. Clearly $(\mathfrak{A}, 0) \models \theta_2$ and $(\mathfrak{B}, 0) \not\models \theta_2$. *Claim*: $(\mathfrak{A}, 0)$ and $(\mathfrak{B}, 0)$ satisfy the same sentences of $L^{\dot\forall}$.

For $\mathbf{a} = (a_0, \ldots, a_{k-1}), \mathbf{b} = (b_0, \ldots, b_{k-1}) \in \bar{B}^k$, and $w, v \in W'$, let $(w, \mathbf{a}) \sim (v, \mathbf{b})$ iff for all $i, j < k$:

$$a_i = a_j \text{ iff } b_i = b_j;$$

$$a_i \in B(w) \text{ iff } b_i \in B(v).$$

Where $(w, \mathbf{a}) \sim (v, \mathbf{b})$, for every $a \in \bar{B}$ there is a $b \in \bar{B}$ (and for every $b \in \bar{B}$ there is an $a \in \bar{B}$) so that $(w, \mathbf{a}^\frown(a)) \sim (v, \mathbf{b}^\frown(b))$. As usual, for $\phi \in \mathrm{Fml}(L^{\dot{\forall}})$ with free variables among $v_0, \ldots, v_{k-1}$, let $(\mathfrak{A}, w) \models \phi[\mathbf{a}]$ iff $(\mathfrak{A}, w) \models \phi[\alpha]$ for $\alpha(v_i) = a_i$ for all $i < k$. *Claim*: if $(w, \mathbf{a}) \sim (v, \mathbf{b})$ then $(\mathfrak{B}, w) \models \phi[\mathbf{a}]$ iff $(\mathfrak{B}, v) \models \phi[\mathbf{b}]$. This follows by an easy induction. For example, suppose ϕ is $(\dot{\forall} v)\psi$; if $(\mathfrak{B}, w) \models \phi[\mathbf{a}]$ and $b \in \bar{B}$, select a as above; since $(\mathfrak{B}, w) \models \psi[\mathbf{a}, a]$, by induction hypothesis $(\mathfrak{B}, v) \models \psi[\mathbf{b}, b]$; so $(\mathfrak{B}, v) \models \phi[\mathbf{b}]$; similarly in the other direction.

LEMMA 3. For $\phi \in \mathrm{Fml}(L^{\dot{\forall}})$ with free variables among $v_0, \ldots, v_{k-1}$, and $w \in W$:

$$(\mathfrak{A}, w) \models \phi[\mathbf{a}] \text{ iff } (\mathfrak{B}, w) \models \phi[\mathbf{a}].$$

The only non-trivial induction step is where ϕ is $\Box\psi$. It suffices to show that if $(\mathfrak{B}, 1) \models \psi[\mathbf{a}]$ then for some $v \in W$, $(\mathfrak{A}, v) \models \psi[\mathbf{a}]$; but clearly there is a $v \in W$ so that $(1, \mathbf{a}) \sim (v, \mathbf{a})$; so the lemma follows.

(17) Let θ_3 be '$\Box((\exists x)\!\downarrow\!\neg Ex \supset (\exists^{\geq 2} x)\!\downarrow\!\neg Ex)$'. Then θ_3 is not expressible in $L^{@,\dot{\forall}}$. Where $\mathfrak{A}$ and $\mathfrak{B}$ are as in (14) and $w_0 = (2, 3)$, clearly $(\mathfrak{A}, w_0) \models \theta_3$; however $(\mathfrak{A}, w_0) \not\models \theta_3$. It suffices to show that $(\mathfrak{A}, w_0)$ and $(\mathfrak{B}, w_0)$ satisfy the same formulae of $L^{@,\dot{\forall}}$. For $\mathbf{a}$ and $\mathbf{b}$ as before, and $w, v \in W'$, let $(w, \mathbf{a}) \sim (v, \mathbf{b})$ iff the conditions defining '$\sim$' in (14) are satisfied, and for all $i < k$:

$$a_i \in B(w_0) \text{ iff } b_i \in B(w_0).$$

If $(w, \mathbf{a}) \sim (v, \mathbf{b})$ then for any $\phi \in \mathrm{Fml}(L^{@,\dot{\forall}})$ with free variables among $v_0, \ldots, v_{k-1}$:

$$(\mathfrak{B}, w_0, w) \models \phi[\mathbf{a}] \text{ iff } (\mathfrak{B}, w_0, v) \models \phi[\mathbf{b}].$$

The argument is analogous to that used in (14). For any $\mathbf{a}$ is a $v \in W$ with $(1, \mathbf{a}) \sim (v, \mathbf{a})$. So as in (14) we can show that for any $w \in W$ and any $\phi \in \mathrm{Fml}(L^{@,\dot{\forall}})$ with free variables from $v_0, \ldots, v_{k-1}$:

$$(\mathfrak{A}, w_0, w) \models \phi[\mathbf{a}] \text{ iff } (\mathfrak{B}, w_0, v) \models \phi[\mathbf{a}].$$

Letting $w = w_0$, the claim follows.

Let $\mathfrak{A} = (W, A, V)$ and $\mathfrak{B} = (W', A', V')$ be structures for L, $w \in W$ and $w' \in W'$; w and w' are redundancy-matched for $\mathfrak{A}$ and $\mathfrak{B}$ iff $\bar{A} = \bar{A}'$, $A(w) = A'(w')$, and for all n-place $\mathbf{P} \in \mathbf{Pred}$ and $a_1, \ldots, a_n \in \bar{A}$: $(w, a_1, \ldots, a_n) \in V(\mathbf{P})$ iff $(w', a_1, \ldots, a_n) \in V'(\mathbf{P})$. $U \subseteq W$ is a redundancy set for $\mathfrak{A}$ iff for all $w, u \in U$, w and u are redundancy-matched for $\mathfrak{A}$ and $\mathfrak{A}$. $\mathfrak{A}$ and $\mathfrak{B}$ are redundancy-equivalent iff for each $w \in W$ there is a $w' \in W'$ (and for each $w' \in W'$ there is a $w \in W$) so that w and w' are redundancy-matched for $\mathfrak{A}$ and $\mathfrak{B}$; $(\mathfrak{A}, w)$ and $(\mathfrak{B}, w')$ are redundancy-equivalent iff $\mathfrak{A}$ and $\mathfrak{B}$ are redundancy equivalent and w and w' are redundancy-matched for $\mathfrak{A}$ and $\mathfrak{B}$. A formula ϕ containing no free type 1 variables ranging over all monadic attributes is preserved under redundancy (hereafter r-preserved) iff for any $(\mathfrak{A}, w)$ and $(\mathfrak{B}, w')$ which are redundancy-equivalent, the same assignments satisfy ϕ in $(\mathfrak{A}, w)$ and in $(\mathfrak{B}, w')$. (Since members of W are "built into" monadic attributes in $\mathfrak{A}$, and similarly for W' and $\mathfrak{B}$, this definition would make sense for a formula involving free variables for monadic attributes only if $W = W'$; thus such formula are excluded in the preceding definition.) Fact: all formula of $L, L^{@}, L^{\downarrow}, L^{\dot\forall}, L^{i.e.}$ and $L^{e.}$ are r-preserved.

Let $\mathfrak{A}$ be redundant iff at least one redundancy set for $\mathfrak{A}$ has more than one member.

THEOREM. If **Pred** is finite and all members of **Pred** are 0-place, then there is a sentence θ of L^1 so that $(\mathfrak{A}, w) \models \theta$ iff $\mathfrak{A}$ is redundant. Furthermore, if **Pred** is finite and all members of **Pred** are 0-place or 1-place, then there is such a sentence of $L^{1,\dot\forall}$. Thus some formulae of L^1 containing no free type 1 variables are not r-preserved.

Proof. Let World(Υ) abbreviate:

$$\text{Max}E(\Upsilon) \;\&\; (\forall x)\Upsilon x \;\&\; (\forall \Upsilon')$$

$$((\text{Max}E(\Upsilon') \;\&\; \text{Ind}(\Upsilon') \;\&\; \Diamond(\exists x)(\Upsilon x \;\&\; \Upsilon' x)) \supset (\exists x)\Upsilon' x),$$

where Υ' is distinct from Υ. Then $(\mathfrak{A}, w) \models \text{World}(\Upsilon)[\alpha, \beta]$ iff for any $v \in W$ and $a \in \bar{A}$, $(v, a) \in \beta(\Upsilon)$ iff $a \in A(w)$. Suppose $\mathbf{Pred} = \{\mathbf{P}_0, \ldots, \mathbf{P}_{k-1}\}$, all $\mathbf{P}_i$ are 0-place. For $s \in {}^k2$, let $P(s)$ be $\bigwedge_{i<k} \mathbf{P}_i^{s(i)}$, where $\mathbf{P}^0$ is $\mathbf{P}$, $\mathbf{P}^1$ is $\neg\mathbf{P}$. Let $\hat\theta(s, \Upsilon, \nu)$ abbreviate:

$$(\exists \Upsilon')(\Diamond(P(s) \,\&\, \text{World}(\Upsilon') \,\&\, \Upsilon\nu) \,\&\, \Diamond(P(s) \,\&\, \text{World}(\Upsilon') \,\&\, \neg\Upsilon\nu)),$$

where Υ' is distinct from Υ. Then $(\mathfrak{A}, w) \models \hat{\theta}(s, \Upsilon, \nu)[\alpha, \beta]$ iff there are u and $v \in W$ so that $\{u, v\}$ is a redundancy set in $\mathfrak{A}$ and $(u, \alpha(\nu)) \in \beta(\Upsilon)$ though $(v, \alpha(\nu)) \notin \beta(\Upsilon)$; so $u \neq v$. Let θ be $\dot{\bigvee}\{(\exists\Upsilon)(\exists\nu)\hat{\theta}(s, \Upsilon, \nu) \mid s \in {}^k 2\}$. θ is as claimed.

If **Pred** $= \{\mathbf{P}_0, \ldots, \mathbf{P}_{k-1}, \mathbf{P}_k, \ldots, \mathbf{P}_{q-1}\}$ for $\mathbf{P}_i$ 0-place where $i < k$ and $\mathbf{P}_i$ 1-place where $k \leqslant i < q$; for $k \leqslant i < q$ let $R_i(\Upsilon)$ abbreviate $(\dot{\forall}x)(\mathbf{P}_i x \equiv \Upsilon x)$; now let $\hat{\theta}(s, \Upsilon, \nu)$ abbreviate

$$(\exists\Upsilon')(\exists\Upsilon_k)\ldots(\exists\Upsilon_{q-1})(\Diamond(P(s) \,\&\, \text{World}(\Upsilon') \,\&\, R_k(\Upsilon_k) \,\&\, \ldots \,\&\, R_{q-1}(\Upsilon_{q-1}) \,\&\, \Upsilon\nu) \,\&\, \Diamond(P(s) \,\&\, \text{World}(\Upsilon') \,\&\, R_k(\Upsilon_k) \,\&\, \ldots \,\&\, R_{q-1}(\Upsilon_{q-1}) \,\&\, \neg\Upsilon\nu)),$$

and construct θ as before.

COROLLARY. For any choice of **Pred**: L^e does not include L^1, $L^{e,\dot{\forall}}$ does not include $L^{1,\dot{\forall}}$, and $L^{e,@}$ does not include $L^{1,@}$. Let θ be as constructed above for the case in which **Pred** is empty. Even if **Pred** is non-empty, θ cannot be expressed in L^e, $L^{e,\dot{\forall}}$ or $L^{e,@}$ since θ is not r-preserved.

CONJECTURES

(A) '$\Box(\dot{\exists}x)(\dot{\exists}y)(Rxy \,\&\, @Rxy)$' is not expressible in $L^{e,\dot{\forall}}$.

(B) '$\Box^2(\dot{\exists}x)(\dot{\exists}y)(Rxy \,\&\, \downarrow Rxy)$' is not expressible in $L^{e,@,\dot{\forall}}$.

(C) Even if **Pred** is empty, $L^{1,@}$ is not included in $L^{1,\dot{\forall}}$.

(D) Even if **Pred** is empty, $L^{1,\downarrow}$ is not included in $L^{1,@}$.

(E) If **Pred** contains an n-place for $n \geqslant 2$ then $L^{\downarrow}$ is not included in $L^{1,\dot{\forall}}$.

Finally, a question: Does every formula of L^1 which is without free type 1 variables and is r-preserved translate into L^e? If not, does the set of such formulae have a nice syntactic characterization?

6. REMARKS ON THE ONTOLOGICAL COMMITMENTS OF MODAL LANGUAGES

Consider a language $\mathscr{L}$ which is interpreted (in the strong sense of having sense assigned to its well-formed formulae) and whose "semantic form" is modelled by one of the disinterpreted languages described in Section 1. Does such a language carry commitment to possible worlds? The semantics of our disinterpreted languages does not settle that of $\mathscr{L}$; truth in a model for a disinterpreted language must be distinguished from truth for an interpreted language. The former relation models (in the ordinary "engineering" sense) the latter property; and as with models in physics or engineering, we must handle the relation between a model and what it models carefully. This relation may be straightforward in the case of extensional first-order languages (though even here it's not identity); for modal languages this relation may well be not so straightforward. Suppose L is the disinterpretation of $\mathscr{L}$. Our notion of truth (or satisfaction) in a model for L permits us to give a first-order set-theoretical definition of implication for L; when transferred to $\mathscr{L}$ via the parsing relation, the result coincides with our pre-analytical notion of logical consequence. Thus the notion of truth in a model for L has heuristic value.

At one extreme, it might be urged that this is the sole value of that notion, that the coincidence remarked above is mere coincidence. On this line, parsing statements of $\mathscr{L}$ by formulae of L is "appropriate" only because the parsing carries implication on L into the right relation on $\mathscr{L}$, rather than the latter being the case because the semantics of L models something fundamental about the semantic facts underlying $\mathscr{L}$.

At the other extreme, one might maintain that the relation between truth and truth in a model in the modal case is not different from that in the extensional case, that in $\mathscr{L}$, whatever does the work of '$\Diamond$' is a notationally novel expression of existential quantification over possible worlds, and carries as much commitment as any other expression of existential quantification; modal discourse posits a domain of very peculiar objects.

If we don't like the first "thin" view of the relation between truth in $\mathscr{L}$ and truth in a model for L, how might we resist robust modal realism? Allan Hazen has suggested that the comparative expressive weakness of a language is relevant to the nature of its ontological commitments:

In view of the comparative weakness of modal languages, compared to the explicitly quantificational ones Quine takes as canonical, there is surely a sense in which the

concept of existence embodied in that disguised existential quantifier, the possibility operator, is defective [3].

This argument merits careful scrutiny. L is expressively weaker than $L_{\mathbf{ext}}$ in two respects: its object-quantifier is actualistic, and it permits no "backward world travelling" (i.e., the image of Fml(L) under g from Section 4 is contained in the set of proper formulae of $L_{\mathbf{ext}}$). These respects are not totally independent: if we eliminate the first "weakness" by considering $L^{\dot{\forall}}$, we don't thereby eliminate the second; but if we eliminate the second by considering $L^{\downarrow}$, we also eliminate the first. (Of course we can partially eliminate the second weakness by considering $L^{@}$ without eliminating the first.) The first weakness seems to bear on commitment to possible objects; Hazen seems to be concerned with the second weakness, which he takes to show that within $\mathscr{L}$, what corresponds to '$\Diamond$' embodies "a pre-individuative concept of existence".

What does this mean? Presumably at least this: for entities to which a pre-individuative conception of existence is appropriate, the Quinean dictum "No entity without identity" does not apply; when quantifying over or referring to such entities, '$\approx$' is out of place. This feature of quantification over possible worlds is reflected in the formation rules of $L_{\mathbf{ext}}$: '$\approx$' only occurs between terms of L. We could easily change this: let $L^{\approx}_{\mathbf{ext}}$ be the result of adding the formation rule:

$$(\mu \approx \mu') \text{ is a formula of } L^{\approx}_{\mathbf{ext}},$$

and the satisfied clause

$$\mathfrak{A}_{\mathbf{ext}} \models (\mu \approx \mu')[\alpha, \gamma] \text{ iff } \gamma(\mu) = \gamma(\mu').$$

Obviously there are sentences of $L^{\approx}_{\mathbf{ext}}$ which are not expressible in $L_{\mathbf{ext}}$. One construal of Hazen's conclusion is then this: within a language whose disinterpretation is $L^{\approx}_{\mathbf{ext}}$, what corresponds to '$(\exists\mu)$' embodies a post-individuative conception of existence; within an interpreted version of $L_{\mathbf{ext}}$, it embodies a pre-induative conception. It may seem ad-hoc to so glorify the difference between $L_{\mathbf{ext}}$ and $L^{\approx}_{\mathbf{ext}}$. But if **Pred** is finite, this mere difference in choice of formation-rules corresponds to a more interesting model-theoretic distinction. For $\phi \in$ Fml($L^{\approx}_{\mathbf{ext}}$), let ϕ be r-preserved iff for any redundancy-equivalent structures $\mathfrak{A}$ and $\mathfrak{B}$ for L, any individual assignment α, and world assignments γ and γ' in $\mathfrak{A}$ and $\mathfrak{B}$ respectively, if for all μ free in ϕ $\gamma(\mu)$ matches $\gamma'(\mu)$ in $\mathfrak{A}$ and $\mathfrak{B}$, then

$$\mathfrak{A}_{\mathbf{ext}} \models \phi[\alpha, \gamma] \text{ iff } \mathfrak{B}_{\mathbf{ext}} \models \phi[\alpha, \gamma'].$$

Thus if **Pred** is finite, ϕ is r-preserved iff ϕ is equivalent to a formula of $L_{\mathbf{ext}}$. This suggests the following construal of Hazen's conclusion: within a language whose disinterpretation is $L^{\approx}_{\mathbf{ext}}$, the quantifier expressions corresponding to '$(\exists\mu)$' in a given sentence embody a pre-individuative conception of existence iff its parsing in $L_{\mathbf{ext}}$ is r-preserved. And if the only conception of existence appropriate to possible worlds is pre-individuative, then only such sentences are really significant.

Unfortunately, this construal of Hazen's conclusion is not what is supported by his argument: for there he is impressed by the difference between L and $L_{\mathbf{ext}}$, by "the comparative weakness of modal languages, compared to the explicitly quantificational ones", and *not* by the difference between $L_{\mathbf{ext}}$ and $L^{\approx}_{\mathbf{ext}}$. The result of Section 4 shows that '$\Diamond$' is as much an explicit quantifier in $L^{\downarrow}$ as '$(\exists\mu)$' is in $L_{\mathbf{ext}}$. Hazen thinks that there is a sense in which within L and $L^{\downarrow}$ '$\Diamond$' embodies different "concepts of existence". (If so, certainly the term 'pre-individuative' is a red-herring. For its steers us towards consideration of identity on possible worlds.) To make this claim respectable, it would suffice to find a model-theoretic property which isolates the class of equivalents of proper formulae of $L_{\mathbf{ext}}$, just as being r-preserved isolates the class of equivalents of formula of $L_{\mathbf{ext}}$.

Regardless of what Hazen had in mind, the thesis developed above is somewhat plausible. However, if our interpreted language involves quantification over monadic relations, then we must either abandon the thesis or rule out as non-significant those statements whose parsings in L^1 are not r-preserved. Is there a construal of Hazen's conclusion which does not have such consequences? I'll leave this question to those who better grasp a pre-individuative concept of existence.

Plantinga has suggested that modal discourse, at least if it involves only actualistic object-quantification, carries commitment to individual essences rather than to non-actual possible objects. Is this plausible? And if so, are such commitments preferable? Here again, appeal to comparative expressive power may be relevant: the fact that $L^{i.e}$ is properly included in $L^{\dot{\forall}}$ suggests that commitment to individual essences is "lighter" than commitment to possible objects. On the other hand, one might object that our only grasp of the concept of an individual essence is: the property of being a given possible object; if so, commitment to individual essences seems to presuppose commitment to possible objects. Certainly this is suggested by our model-theory: one can't have singletons without the objects of which they are singletons. Once again, this consideration is not decisive: the bearing of truth in a model for $L^{i,e}$ on truth for an interpreted language need not be straightforward.

Suppose one refuses to posit possible objects on the grounds that the question "Where are they?" requires an answer, and the answer "In other possible worlds" is unsatisfactory. Plantinga's case relies on this fact: The analogous question and answer about individual essences are both incoherent; individual essences do not exist in worlds, but are merely instanced in worlds; failure of $L^{i,e}$ to in general include $L^{\dot{\forall}}$ points to the reality of this distinction. The appropriate question about an individual essence is not "Where is it?" but "What has it?", and the answer may well be "Nothing".

I'm not persuaded that this defense of Plantinga *is* completely adequate. It doesn't address the other question: do we lose anything by retreating from possible objects to individual essences? The fact that $L^{i,e}$ is properly included in $L^{\dot{\forall}}$ may cut both ways. However, if we accept the doctrine that Fine labels "predicate actualism", we may rest assured that nothing is lost.

Predicate actualism is the doctrine:

> that in no possible world is there a genuine relation among the non-existents of that world or between the non-existents and the existents [2].

The predicate actualist recognizes the apparent need to consider non-actual possible objects in determining the conditions for satisfaction of a formula in a model; but she regards the need to do this for atomic formulae as an "artifact" of the model theory. This doctrine can be formulated model-theoretically.

Where $\mathfrak{A} = \langle W, A, V\rangle$ and $\mathfrak{B} = \langle W, A, V'\rangle$ are structures for L, $\mathfrak{A}$ and $\mathfrak{B}$ are internally indistinguishable iff for every n-place $\mathbf{P} \in \mathbf{Pred}$ and $w \in W$:

$$V(\mathbf{P}) \cap (\{w\} \times \bar{A}^n) = V'(\mathbf{P}) \cap (\{w\} \times \bar{A}^n).$$

A formula ϕ is preserved under internal indistinguishability iff for any internally indistinguishable structures $\mathfrak{A}$ and $\mathfrak{B}$, any $w \in W$ and any assignments $\alpha, \ldots$:

$$(\mathfrak{A}, w) \models \phi[\alpha, \ldots] \text{ iff } (\mathfrak{B}, w) \models \phi[\alpha, \ldots].$$

For the predicate actualist, differences between internally indistinguishable structures are irrelevant to their task of modelling truth. Predicate actualism may then be viewed as the doctrine: an interpreted sentence is significant only if it's parsed by a formula preserved under internal indistinguishability. A formula ϕ is restricted iff for each n-place $\mathbf{P} \in \mathbf{Pred}$, $P\tau_1 \ldots \tau_n$ occurs in ϕ only in the context $(P\tau_1 \ldots \tau_n \,\&\, E\tau_1 \ldots \tau_n)$. In [2] Fine shows that for $\phi \in \mathrm{Fml}(L)$, ϕ is preserved under internal indistinguishability iff it's equivalent to a restricted formula. This result easily extends to the enrichments of L described in Section 1. (Perhaps the predicate actualist would favor $L^{\mathrm{min}\cdot e.}$ and $L^{i\cdot \mathrm{min}\cdot e.}$ over L^e and $L^{i.e.}$.)

THEOREM. If $\phi \in \mathrm{Fml}(L^{\dot{\forall}})$ is restricted, then ϕ translates into $L^{i.e.}$.

Proof. Regard '$\dot{\forall}$' as defined; associate with each variable ν occurring in ϕ a distinct type 1 variable Υ_ν; form ϕ' from ϕ by replacing subformula of the form $(\mathbf{P}\tau_1 \ldots \tau_n \,\&\, E\tau_1 \,\&\, \ldots \,\&\, E\tau_n)$ by:

$$(\exists\nu_1) \ldots (\exists\nu_m)(P\tau_1 \ldots \tau_n \,\&\, \Upsilon_{\nu_1}\nu_1 \,\&\, \ldots \,\&\, \Upsilon_{\nu_m}\nu_m \,\&\, E\sigma_1 \,\&\, \ldots \,\&\, E\sigma_k),$$

where $\{\nu_1, \ldots, \nu_m\} = \mathbf{Var} \cap \{\tau_1, \ldots, \tau_n\}$ and $\{\sigma_1, \ldots, \sigma_k\} = \mathbf{C} \cap \{\tau_1, \ldots, \tau_n\}$. Then form ϕ^* from ϕ' by replacing occurrences of '$\nu \approx \nu'$', '$\nu \approx \tau$', '$\tau \approx \nu$' for $\tau \in \mathbf{C}$, and all occurrences of '$E\nu$' not in contexts just considered, by '$\Diamond(\exists x)(\Upsilon_\nu x \,\&\, \Upsilon_{\nu'} x)$', by '$\Upsilon_\nu\tau$', '$\Upsilon_\nu\tau$' and '$(\exists x)\Upsilon_\nu x$' respectively. Notice that:

$$(\mathfrak{A}, w) \models (\mathbf{P}\tau_1 \ldots \tau_n \,\&\, E\tau_1 \,\&\, \ldots \,\&\, E\tau_n)[\alpha] \text{ iff}$$

$$(\mathfrak{A}, w) \models (\exists\nu_1) \ldots (\exists\nu_m)(P\tau_1 \ldots \tau_n \,\&\, \Upsilon_{\nu_1}\nu_1 \,\&\, \ldots \,\&\, \Upsilon_{\nu_m}\nu_m \,\&\, E\sigma_1 \,\&\, \ldots \,\&\, E\sigma_k)[\beta],$$

where β is an individual essence assignment such that $\beta(\Upsilon_\nu) = \{\alpha(\nu)\}$. Since ϕ is restricted, by an easy induction, for such β we have

$$(\mathfrak{A}, w) \models \phi[\alpha] \text{ iff } (\mathfrak{A}, w) \models \phi^*[\beta],$$

proving the theorem. For the predicate actualist, the advantages of an ontology of non-actual possibles are available for the price of commitment to individual essences. But as usual, this bargain is double-edged: a skeptic about Plantinga's move could interpret the previous result as showing that the predicate actualist who claims to only posit individual essences, has really adopted an alternative notation, covering an underlying commitment to non-actual possibles with a haze of type 1 variables. Though model theory may help us formulate a good theory of reference, it is no substitute for one; and only a theory of reference will really make clear the commitments involved in modal discourse.

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