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THREE-VALUED LOGICS: AN INTRODUCTION, A COMPARISON OF VARIOUS LOGICAL LEXICA, AND SOME PHILOSOPHICAL REMARKS

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The thesis that certain sentences or statements are neither true nor false has been repeatedly proposed through the history of logic. According to some commentators, Aristotle proposed this status for certain statements about the future. Frege, and more recently Strawson and many others, have proposed this status for certain statements containing non-designating singular terms. Others have proposed this status for indicative conditionals with false antecedents, troublesome counterfactual conditionals, some statements involving vague predicates, category errors, or self-reference, and so on. These suggestions are mutually independent and not of equal value. But if you think that there are some cases in which statements are neither true nor false, then you have reason to take seriously at least some logic which accommodates this phenomenon.

Such logics have been rather ignored by mathematical logicians. In part this is because there is so much left to learn about two-valued logic; in part it's because mathematical logicians are most interested in mathematical applications of logic, and most think that in mathematics truth-value gaps do not arise. Unlike intuitionistic logic, three-valued logic is not a new ball-game: rather it's a "rounding off" of classical logic. The classical logician wants his discourse to be two-valued, and usually presupposes that it is; a three-valued logic is a default logic to which the classical logician may fall back when that presupposition fails, because of reference failure, an undetermined future, or whatever.

Another reason for avoidance of three-valued logics is the fear, illustrated by the remark of Dana Scott quoted in [6], that no such logic is "pleasant to work with" or even "really workable". Of course three-valued logics will be somewhat more complicated than classical two-valued logic. In fact, proof-theoretically they are at least twice as complicated: the non-structural natural-deduction rules from two-valued logic split into a weak and a strong version for three-valued logics (and some of our logics require a further definedness rule for '$\supset$' and '$\exists$'): see Section 4. But model-theoretically they are only 50% more complicated, since we have three, rather than two, truth-values. When it comes to what's "pleasant" and "workable", it's different strokes for different folks; but I hope that the reader will find these additional complications interesting rather than off-putting.

The three-valued option does, however, confronts us with some choices that we wouldn't otherwise face. The logical lexicon of two-valued propositional logic extends to the three-valued setting in several ways; similarly for predicate-logic. Negation and the biconditional carry over uniquely; but conjunction and disjunction can be extended to yield strong connectives (in this paper '&', '$\vee$') or weak connectives ('**&**', '$\boldsymbol{\vee}$'); similarly the existential quantifier yields '$\exists$' and '$\boldsymbol{\exists}$'. The material conditional extends in four ways: on the pattern of conjunction and disjunction there is a strong ('$\supset$') and a weak ('$\boldsymbol{\supset}$') conditional; and there is a further strengthening ('$\supset_s$') and weakening ('$\supset_w$') of the strong conditional. Use of '=' extends in three ways. I'll continue to use '=' in our object-languages to represent the identity relation. (In Butler's words "the relation each object bears to itself and to no other".) In mathematical writing we find use of a strong bivalent notion of equality, to be represented in our object-languages by '$=_b$'. And an intermediate sort of equality, to be represented by '$=_s$', is also of interest. These choices generate the logical lexica to be discussed in this paper.

After presenting basic model-theoretic definitions, I'll map out the inclusions between these lexica, and then work towards an "algebraic" characterization of their expressive power, guided by Keisler and Shelah's characterization of the elementary and basic elementary classes of models under the two-valued semantics; the main result here is in Section 8. Some, e.g. Michael Dummett, think that there is a distinction between allowing for truth-value gaps (i.e. allowing statements to be neither true nor false) and introducing a third truth-value. In the preceding remarks my use of the phrase 'three-valued' was intended to apply in both situations. I try in Section 11 to give content to the question: which of our lexica merely allow for truth-valuelessness?

I became interested in three-valued semantics because they offer the non-ad-hoc way to handle non-designating singular terms. After searching unsuccessfully for a survey paper on such logics. I found myself writing Sections 1 to 4. The material presented in Sections 3 and 4 is tangential to the main drift of this paper and may already be in the literature.

In places I have ignored the distinction between use and mention, e.g. in use of variables (always Greek letters) ranging over syntactic objects, in speaking of subscripts for 'lex', and in what follows; my usage in these cases should be sufficiently clear.

After a singular term, read '$\downarrow$' as 'is defined' or 'stands for someting', '$\uparrow$' as 'is undefined' or 'doesn't stand for anything'. Read '$\cdots = ---$' and '$\cdots \in ---$' as imply that $\cdots$ and $---$ exist. A statment or definition of the form '$\cdots \simeq ---$' means that either $\cdots$ and $---$ both exist and are identical, or else both don't exist. Throughout this paper I work in standard set-theory with proper classes and assuming the Axiom of Choice.

1. Basic definitions

Our logical lexica will be as follows:

$$\text{lex}_0 = \{`\boldsymbol{\supset}', `\bot', `\boldsymbol{\exists}', `='\};$$

$$\text{lex}_{0,\text{x}} = \{`\boldsymbol{\supset}', `\bot', \text{x}, `\boldsymbol{\exists}', `='\};$$

$$\text{lex}_{0,\text{T},\text{u}} = \{`\boldsymbol{\supset}', `\text{T}', `\text{u}', `\boldsymbol{\exists}', `='\};$$

$$\text{lex}_1 = \{`\supset', `\bot', `\exists', `='\};$$

$$\text{lex}_{1,\text{x}} = \{`\supset', `\bot', \text{x}, `\exists', `='\};$$

$$\text{lex}_{1,\text{T},\text{u}} = \{`\supset', `\text{T}', `\text{u}', `\exists', `='\},$$

where 'x' is replaced by 'T' or 'u'. For $\text{lex}_{\dots}$ one of the above lexica, form $\text{lex}_{\dots,\text{s}}$ by replacing '=' in $\text{lex}_{\dots}$ by '$=_\text{s}$'; form $\text{lex}_{\dots,\text{b}}$ similarly with '$=_\text{b}$'. Fix a countable set *Var* of variables. For each $n < \omega$ fix proper classes PRED(n) and FUNCT(n) of n-place predicate-constants and function-constants respectively. Needless to say these lexical classes are pairwise disjoint. For $\mathit{Pred} \subseteq \bigcup_n \text{PRED}(n)$ and $\mathit{Funct} \subseteq \bigcup_n \text{FUNCT}(n)$, let $\mathit{Pred}(n) = \mathit{Pred} \cap \text{PRED}(n)$, $\mathit{Funct}(n) = \mathit{Funct} \cap \text{FUNCT}(n)$. Replacing 'x' by one of the above subscripts on 'lex', let $L_\text{x}(\mathit{Pred}, \mathit{Funct})$ be first-order language based on lex_x generated by the non-logical vocabulary $\mathit{Pred} \cup \mathit{Funct}$; with the latter fixed we write this as L_x.

The class of terms of L_x is defined by the usual induction. The class $\text{Atfml}(L_\text{x})$ of atomic formulae of L_x consists of whichever of the following contains logical constants from lex_x:

$$\text{u}, \quad \bot, \quad \tau_0 = \tau_1, \quad \tau_0 =_\text{s} \tau_1, \quad \tau_0 =_\text{b} \tau_1,$$

together with strings of the form $\zeta(\tau_0, \dots, \tau_n)$ for $\zeta \in \mathit{Pred}(n)$ and any terms $\tau_0, \dots, \tau_n$ of L_x. The class $\text{Fml}(L_\text{x})$ of formulae of L_x is formed by closing $\text{Atfml}(L_\text{x})$ under those among the following induction clauses that govern logical constants in lex_x:

If φ and ψ are formulae then so are $\text{T}\varphi$, $(\varphi \boldsymbol{\supset} \psi)$ and $(\varphi \supset \psi)$;

if φ is a formula and $v \in \mathit{Var}$ then $(\boldsymbol{\exists} v)\varphi$ and $(\exists v)\varphi$ are formulae.

As usual, the class $\text{Sent}(L_\text{x})$ of sentences of L_x consits of the formulae of L_x in which no variable occurs free. Obviously these are equivalent: $\mathit{Pred} \cup \mathit{Funct}$ is a set; $\text{Fml}(L_\text{x})$ is a set; $\text{Sent}(L_\text{x})$ is a set. L_x is a fragment of a proper-class-size language $L_\text{x}(\bigcup_n \text{PRED}(n), \bigcup_n \text{FUNCT}(n))$, hereafter called $\mathbb{L}_\text{x}$. For $\Delta \subseteq \text{Atfml}(L_\text{x})$ let $\text{Fml}_\text{x}(\Delta)$ be the class of formulae of L_x all of whose atomic subformulae belong to Δ; let $\text{Sent}_\text{x}(\Delta)$ be the class of sentences in $\text{Fml}_\text{x}(\Delta)$.

Let 1 [0] represent truth [falsehood], $2 = \{0, 1\}$. A partial model for L_x (also for $\mathit{Pred} \cup \mathit{Funct}$) shall be an ordered triple $\mathscr{A} = (|\mathscr{A}|, \mathscr{E}, \mathscr{N})$, where:

$|\mathscr{A}|$ is a set;

$\mathcal{N}$ is a set-function with $\mathrm{dom}(\mathcal{N}) \subseteq Funct$ so that for each $n < \omega$ and $\xi \in Funct(n) \cap \mathrm{dom}(\mathcal{N})$:

if $n = 0$, then $\mathcal{N}(\xi) \in |\mathcal{A}|$;

if $n \geq 1$, then $\mathcal{N}(\xi)$ is a function with $\mathrm{dom}(\mathcal{N}(\xi)) \subseteq |\mathcal{A}|^n$ and into $|\mathcal{A}|$;

$\mathcal{E}$ is a set-function with $\mathrm{dom}(\mathcal{E}) \subseteq Pred$ so that for each $n < \omega$ and $\zeta \in Pred(n) \cap \mathrm{dom}(\mathcal{E})$:

if $n = 0$, then $\mathcal{E}(\zeta) \in 2$;

if $n \geq 1$, then $\mathcal{E}(\zeta)$ is a function with $\mathrm{dom}(\mathcal{E}(\zeta)) \subseteq |\mathcal{A}|^n$ and into 2.

Let $\zeta^{\mathcal{A}} \simeq \mathcal{E}(\zeta)$, $\xi^{\mathcal{A}} \simeq \mathcal{N}(\xi)$ for $\zeta \in$ PRED, $\xi \in$ FUNCT.

Hereafter by 'model' we'll mean 'partial model'. Notice that models are sets, not proper classes. Let MOD be the class of all partial models. Note: in two-valued model-theory, a model is a model for uniquely determined sets *Pred* and *Funct*. This is not the case here; indeed if $\mathcal{A}$ is a model for $Pred \cup Funct$, $Pred \subseteq Pred'$ and $Funct \subseteq Funct'$, then $\mathcal{A}$ is a model for $Pred' \cup Funct'$.

For a model $\mathcal{A}$ as above, we adopt threse definitions: $\mathcal{E}$ is total on *Pred* iff for any $n < \omega$ and $\zeta \in Pred(n)$:

$$\text{if } n = 0, \quad \text{then } \mathcal{E}(\zeta)\downarrow;$$
$$\text{if } 0 < n, \quad \text{then } \mathrm{dom}(\mathcal{E}(\zeta)) = |\mathcal{A}|^n.$$

$\mathcal{N}$ is total on *Funct* iff for any $n < \omega$ and $\xi \in Funct(n)$:

$$\text{if } n = 0, \quad \text{then } \mathcal{N}(\xi)\downarrow;$$
$$\text{if } n > 0, \quad \text{then } \mathrm{dom}(\mathcal{N}(\xi)) = |\mathcal{A}|^n.$$

$\mathcal{A}$ is total on $Pred \cup Funct$ iff $\mathcal{E}$ and $\mathcal{N}$ are: $\mathcal{A}$ is a total model for $Pred \cup Funct$ iff it is total on $Pred \cup Funct$, $\mathrm{dom}(\mathcal{E}) = Pred$ and $\mathrm{dom}(\mathcal{N}) = Funct$. For a fixed choice of *Pred* and *Funct*, let $\mathcal{A}$ be extensionwise total, hereafter et, iff $\mathcal{E}$ is total on *Pred* and $\mathrm{dom}(\mathcal{E}) = Pred$. $\mathcal{A}$ is non-null, herafter nn, iff $|\mathcal{A}|$ is non-empty. Let α be a partial $\mathcal{A}$-assignment iff α is a function with $\mathrm{dom}(\alpha) \subseteq Var$ into $|\mathcal{A}|$; hereafter we'll drop the 'partial'. Let α be a total $\mathcal{A}$-assignment iff α is an $\mathcal{A}$-assignment with $\mathrm{dom}(\alpha) = Var$.

Fix a model $\mathcal{A}$ for $L_\varkappa$ and an $\mathcal{A}$-assignment α. We define the partial denotation function $\mathrm{den}(\mathcal{A}, \alpha, \cdot)$ on the terms of $L_\varkappa$ as usual:

if $\tau \in Funct(0)$, then

$$\mathrm{den}(\mathcal{A}, \alpha, \tau) \simeq \mathcal{N}(\tau);$$

if $v \in Var$, then

$$\mathrm{den}(\mathcal{A}, \alpha, v) \simeq \alpha(v);$$

if $\xi \in Funct(n)$, then

$$\mathrm{den}(\mathcal{A}, \alpha, \xi(\tau_1, \ldots, \tau_n)) \simeq \mathcal{N}(\xi)(\mathrm{den}(\mathcal{A}, \alpha, \tau_1), \ldots, \mathrm{den}(\mathcal{A}, \alpha, \tau_n))$$

As usual, we'll frequently write den($\mathscr{A}$, α, τ) as $\tau^{\mathscr{A},\alpha}$. We define $\vDash$ (satisfaction) and $\Dashv$ (frustration) as follows.

$\mathscr{A} \vDash \tau_0 = \tau_1\,[\alpha]$ iff for both $i \in 2$, $\tau_i^{\mathscr{A},\alpha}\!\downarrow$ and $\tau_0^{\mathscr{A},\alpha} = \tau_i^{\mathscr{A},\alpha}$;

$\mathscr{A} \Dashv \tau_0 = \tau_1\,[\alpha]$ iff for both $i \in 2$, $\tau_i^{\mathscr{A},\alpha}\!\downarrow$ and $\tau_0^{\mathscr{A},\alpha} \neq \tau_1^{\mathscr{A},\alpha}$;

$\mathscr{A} \vDash \tau_0 =_{\mathrm{s}} \tau_1\,[\alpha]$ iff $\mathscr{A} \vDash \tau_0 = \tau_1\,[\alpha]$;

$\mathscr{A} \Dashv \tau_0 =_{\mathrm{s}} \tau_1\,[\alpha]$ iff either $\mathscr{A} \Dashv \tau_0 = \tau_1\,[\alpha]$ or for some $i \in 2$, $\tau_i^{\mathscr{A},\alpha}\!\uparrow$ and $\tau_{1-i}^{\mathscr{A},\alpha}\!\downarrow$;

$\mathscr{A} \vDash \tau_0 =_{\mathrm{b}} \tau_1\,[\alpha]$ iff $\tau_0^{\mathscr{A},\alpha} \simeq \tau_1^{\mathscr{A},\alpha}$;

$\mathscr{A} \Dashv \tau_0 =_{\mathrm{b}} \tau_1\,[\alpha]$ iff $\mathscr{A} \nvDash \tau_0 =_{\mathrm{b}} \tau_1\,[\alpha]$;

$\mathscr{A} \Dashv$ '$\bot$' $[\alpha]$;

$\mathscr{A} \nvDash$ 'u' $[\alpha]$; $\mathscr{A} \not\Dashv$ 'u' $[\alpha]$;

$\mathscr{A} \vDash \zeta(\tau_1, \ldots)\,[\alpha]$ iff $\mathscr{E}(\zeta)(\tau_1^{\mathscr{A},\alpha}, \ldots) \simeq 1$;

$\mathscr{A} \Dashv \zeta(\tau_1, \ldots)\,[\alpha]$ iff $\mathscr{E}(\zeta)(\tau_1^{\mathscr{A},\alpha}, \ldots) \simeq 0$;

$\mathscr{A} \vDash \top\varphi\,[\alpha]$ iff $\mathscr{A} \vDash \varphi\,[\alpha]$;

$\mathscr{A} \Dashv \top\varphi\,[\alpha]$ iff $\mathscr{A} \nvDash \varphi\,[\alpha]$;

$\mathscr{A} \vDash (\varphi \supset \psi)\,[\alpha]$ iff either $\mathscr{A} \Dashv \varphi\,[\alpha]$ or $\mathscr{A} \vDash \varphi\,[\alpha]$;

$\mathscr{A} \Dashv (\varphi \supset \psi)\,[\alpha]$ iff $\mathscr{A} \vDash \varphi\,[\alpha]$ and $\mathscr{A} \Dashv \psi\,[\alpha]$;

$\mathscr{A} \vDash (\varphi \boldsymbol{\supset} \psi)\,[\alpha]$ iff either $\mathscr{A} \Dashv \varphi\,[\alpha]$ and $\mathscr{A} \vDash \psi\,[\alpha]$ or $\mathscr{A} \Dashv \psi\,[\alpha]$,
or $\mathscr{A} \vDash \varphi\,[\alpha]$ and $\mathscr{A} \vDash \psi\,[\alpha]$;

$\mathscr{A} \Dashv (\varphi \boldsymbol{\supset} \psi)\,[\alpha]$ iff $\mathscr{A} \Dashv (\varphi \supset \psi)\,[\alpha]$;

$\mathscr{A} \vDash (\exists v)\varphi\,[\alpha]$ iff for some $a \in |\mathscr{A}|$, $\mathscr{A} \vDash \varphi\,[\alpha_a^v]$;

$\mathscr{A} \Dashv (\exists v)\varphi\,[\alpha]$ iff for all $a \in |\mathscr{A}|$, $\mathscr{A} \Dashv \varphi\,[\alpha_a^v]$;

$\mathscr{A} \vDash (\boldsymbol{\exists} v)\varphi\,[\alpha]$ iff $\mathscr{A} \vDash (\exists v)\varphi\,[\alpha]$ and for every
$a \in |\mathscr{A}|$ either $\mathscr{A} \vDash \varphi\,[\alpha_a^v]$ or $\mathscr{A} \Dashv \varphi\,[\alpha_a^v]$;

$\mathscr{A} \Dashv (\boldsymbol{\exists} v)\varphi\,[\alpha]$ iff $\mathscr{A} \Dashv (\exists v)\varphi\,[\alpha]$.

For $\varphi \in \mathrm{Fml}(L_{\mathrm{x}})$ let:

$\mathscr{A} \mid \varphi\,[\alpha]$ iff $\mathscr{A} \nvDash \varphi\,[\alpha]$ and $\mathscr{A} \not\Dashv \varphi\,[\alpha]$;

$\mathscr{A} \vDash^{\mathrm{w}} \varphi\,[\alpha]$ iff $\mathscr{A} \not\Dashv \varphi\,[\alpha]$;

$\mathscr{A} \vDash \varphi$ iff for all $\mathscr{A}$-assignments α, $\mathscr{A} \vDash \varphi\,[\alpha]$;

similarly for $\mathscr{A} \Dashv \varphi$, $\mathscr{A} \vDash^{\mathrm{w}} \varphi$ and $\mathscr{A} \mid \varphi$.

'$\vDash^{\mathrm{w}}$' represents weak satisfaction or weak truth. Extend this notation to $\Gamma \subseteq \mathrm{Fml}(L_{\mathrm{x}})$ as usual, e.g:

$\mathscr{A} \vDash^{\mathrm{w}} \Gamma\,[\alpha]$ iff for every $\psi \in \Gamma$, $\mathscr{A} \vDash^{\mathrm{w}} \psi\,[\alpha]$.

As usual, for a formula φ with free variables among those in a fixed list $v_0, \ldots, v_{n-1}$ and $\vec{a} \in |\mathscr{A}|^n$, $\mathscr{A} \models \varphi[\vec{a}]$ iff $\mathscr{A} \models \varphi[\alpha]$ for any (equivalently some) $\mathscr{A}$-assignment α with $\alpha(v_i) = a_i$ for all $i < n$. Similarly for $\dashv$, $\models^{w}$ and $|$.

In the following, we take the string on the left to abbreviate any of the strings to its right:

$\neg\varphi$: $\varphi \supset \bot$ or $\varphi \boldsymbol{\supset} \bot$;

$\varphi \boldsymbol{\vee} \psi$: $(\neg\varphi) \boldsymbol{\supset} \psi$; $\varphi \vee \psi$: $(\neg\varphi) \supset \psi$;

$\varphi \boldsymbol{\&} \psi$: $\neg(\varphi \boldsymbol{\supset} (\neg\psi))$; $\varphi \& \psi$: $\neg(\varphi \supset (\neg\psi))$;

$\mathsf{F}\varphi$: $\mathsf{T}(\neg\varphi)$; $\mathsf{U}\varphi$: $\neg\mathsf{T}(\varphi \supset \varphi)$ or $\neg\mathsf{T}(\varphi \boldsymbol{\supset} \varphi)$;

$\varphi \equiv \psi$: $(\varphi \supset \psi) \& (\psi \supset \varphi)$ or $(\varphi \boldsymbol{\supset} \psi) \boldsymbol{\&} (\psi \boldsymbol{\supset} \varphi)$;

$\varphi \supset_{w} \psi$: $(\mathsf{T}\varphi) \supset \psi$; $\varphi \supset_{s} \psi$: $(\mathsf{F}\varphi) \vee \psi$;

$(\boldsymbol{\forall} v)\varphi$: $\neg(\boldsymbol{\exists} v)\neg\varphi$; $(\forall v)\varphi$: $\neg(\exists v)\neg\varphi$;

$\mathsf{E}(\tau)$: $(\exists v)v = \tau$ or $(\boldsymbol{\exists} v)v = \tau$;

$\mathsf{E}_s(\tau)$: $(\exists v)v =_s \tau$ or $(\boldsymbol{\exists} v)v =_s \tau$ or these with '$=_b$' replacing '$=_s$'.

In the last two clauses v is any variable not occurring in τ.

Thus we have the following 'truth-tables' (Tables 1–3).

Note that '$\boldsymbol{\supset}$', '$\boldsymbol{\&}$' and '$\boldsymbol{\vee}$' have the weak-Kleene (alias the Bochvar) semantics; '$\supset$', '$\&$' and '$\vee$' have the strong-Kleene semantics; '$\supset_w$' is a weakening of '$\supset$', since $\varphi \supset_w \psi$ is more easily satisfied that is $\varphi \supset \psi$; '$\supset_s$' is a

Table 1

φ	$\neg\varphi$	$\mathsf{T}\varphi$	$\mathsf{F}\varphi$	$\mathsf{U}\varphi$
$\models$	$\dashv$	$\models$	$\dashv$	$\dashv$
$\|$	$\|$	$\dashv$	$\dashv$	$\models$
$\dashv$	$\models$	$\dashv$	$\models$	$\dashv$

Table 2

φ	ψ	$\varphi \boldsymbol{\&} \psi$	$\varphi \& \psi$	$\varphi \boldsymbol{\vee} \psi$	$\varphi \vee \psi$
$\models$	$\models$	$\models$	$\models$	$\models$	$\models$
$\models$	$\|$	$\|$	$\|$	$\|$	$\models$
$\models$	$\dashv$	$\dashv$	$\dashv$	$\models$	$\models$
$\|$	$\models$	$\|$	$\|$	$\|$	$\models$
$\|$	$\|$	$\|$	$\|$	$\|$	$\|$
$\|$	$\dashv$	$\|$	$\dashv$	$\|$	$\|$
$\dashv$	$\models$	$\dashv$	$\dashv$	$\models$	$\models$
$\dashv$	$\|$	$\|$	$\dashv$	$\|$	$\|$
$\dashv$	$\dashv$	$\dashv$	$\dashv$	$\dashv$	$\dashv$

Table 3

$\varphi \supset \psi$	$\varphi \supset \psi$	$\varphi \supset_w \psi$	$\varphi \supset_s \psi$	$\varphi \equiv \psi$
$\vDash$	$\vDash$	$\vDash$	$\vDash$	$\vDash$
$\mid$	$\mid$	$\mid$	$\mid$	$\mid$
$\Dashv$	$\Dashv$	$\Dashv$	$\Dashv$	$\Dashv$
$\mid$	$\vDash$	$\vdash$	$\vDash$	$\mid$
$\mid$	$\mid$	$\vDash$	$\mid$	$\mid$
$\mid$	$\mid$	$\vDash$	$\Dashv$	$\mid$
$\vDash$	$\vDash$	$\vDash$	$\vDash$	$\Dashv$
$\mid$	$\vDash$	$\vDash$	$\vDash$	$\mid$
$\vDash$	$\vDash$	$\vDash$	$\vDash$	$\vDash$

strengthening of '$\supset$', since $\varphi \supset_s \psi$ is more easily frustrated than is $\varphi \supset \psi$. We'll use this convention: φ^0 is φ; φ^1 is $\neg\varphi$. Note that '$=$' and '$=_s$' have the same satisfaction conditions, and '$=_s$' and '$=_b$' the same frustration conditions. Finally, note that the sentential connectives in $\text{lex}_{1,\mathrm{T},\mathrm{u}}$ suffice to define all three-valued truth-functions.

A sequence for L_x is an ordered triple $(\Gamma, \Delta, \varphi)$ for $\Gamma \subseteq \Delta \subseteq \mathrm{Fml}(L_x)$ and $\varphi \in \mathrm{Fml}(L_x)$. The most distinctive feature of three-valued logics are that (1) there are two basic notions of validity: strong validity (hereafter validity) and weak validity, and (2) these notions apply to sequents of the sort just defined. Both features arise from the need to consider $\vDash^w$ as well as $\vDash$.

For a sequent $(\Gamma, \Delta, \varphi)$ these are our fundamental logical concepts:

$(\Gamma, \Delta, \varphi)$ is valid [nn-valid, et-valid, nn&et-valid] iff for any model [nn model, et model, nn and et model] $\mathscr{A}$ and any $\mathscr{A}$-assignment α:

$$\text{if } \mathscr{A} \vDash \Gamma[\alpha] \text{ and } \mathscr{A} \vDash^w \Delta[\alpha] \quad \text{then } \mathscr{A} \vDash \varphi[\alpha];$$

$(\Gamma, \Delta, \varphi)$ is weakly valid [weakly nn-valid, weakly et-valid, weakly nn&et valid] iff for any model [nn model, et model, nn and et model] $\mathscr{A}$ and any $\mathscr{A}$-assignment α:

$$\text{if } \mathscr{A} \vDash \Gamma[\alpha] \text{ and } \mathscr{A} \vDash^w \Delta[\alpha] \quad \text{then } \mathscr{A} \vDash^w \varphi[\alpha].$$

For $\psi \in \mathrm{Sent}(L_x)$ we'll adopt these definitions. φ entails$^+$ [nn-entails$^+$, etc.] ψ iff $(\{\varphi\}, \{\varphi\}, \psi)$ is valid [nn-valid, etc.]. φ entails [nn-entails, etc.] ψ iff φ entails$^+$ [nn-entails$^+$, etc.] ψ and $\neg\psi$ entails$^+$ [nn-entails$^+$, etc.] $\neg\varphi$ (i.e. for any model [nn-model, etc.] $\mathscr{A}$ and $\mathscr{A}$-assignment α; if $\mathscr{A} \Dashv \psi$ then $\mathscr{A} \Dashv \varphi$). φ is valid [nn-valid, etc.] iff $\{\ \}$ entails [nn-entails, etc.] φ. φ is weakly valid [weakly nn-valid, etc.] iff $(\{\ \}, \{\ \}, \varphi)$ is weakly valid [weakly nn-valid, etc.]. φ and ψ are equivalent$^+$ [nn-equivalent$^+$, etc.] iff each entails$^+$ [nn-entails$^+$] the other. φ and ψ are equivalent [nn-equivalent, etc.] iff φ and ψ are equivalent$^+$ [nn-equivalent$^+$, etc.] and so are $\neg\varphi$ and $\neg\psi$ (i.e. for any $\mathscr{A}$ and α as above: $\mathscr{A} \Dashv \varphi[\alpha]$ iff $\mathscr{A} \Dashv \psi[\alpha]$). Example: for '$\mathbb{P}$' $\in$ PRED(0): '$\mathbb{P} \supset \mathbb{P}$' is weakly valid but not valid; '$\mathbb{P}$ & $\mathbb{Q}$' entails$^+$ but doesn't entail '$\mathbb{P}$'; '$\mathbb{P}$' and '$\top\mathbb{P}$' are equivalent$^+$ but not equivalent, and similarly for 'u' and '$\neg\bot$'.

'$\supset_s$' and '$\supset_w$' are introduced because of these deduction theorems:

$(\Gamma \cup \{\varphi\}, \Delta \cup \{\varphi\}, \psi)$ is valid iff $(\Gamma, \Delta, \varphi \supset_w \psi)$ is valid;

$(\Gamma, \Delta \cup \{\varphi\}, \psi)$ is weakly valid iff $(\Gamma, \Delta, \varphi \supset_s \psi)$ is weakly valid.

In addition, the following are equivalent:

$(\Gamma \cup \{\varphi\}, \Delta \cup \{\varphi\}, \psi)$ is weakly valid;

$(\Gamma, \Delta, \varphi \supset \psi)$ is weakly valid;

$(\Gamma, \Delta, \varphi \boldsymbol{\supset} \psi)$ is weakly valid.

Furthermore $(\Gamma, \Delta \cup \{\varphi\}, \psi)$ is valid iff $(\Gamma, \Delta, \varphi \supset \psi)$ is valid.

For $\Gamma = \{\varphi_0, \ldots, \varphi_n\}$, we adopt these abbreviations:

$$\boldsymbol{\&}\Gamma : \varphi_0 \boldsymbol{\&} \cdots \boldsymbol{\&} \varphi_n; \qquad \&\Gamma : \varphi_0 \& \cdots \& \varphi_n;$$
$$\boldsymbol{\vee}\Gamma : \varphi_0 \boldsymbol{\vee} \cdots \boldsymbol{\vee} \varphi_n; \qquad \vee\Gamma : \varphi_0 \vee \cdots \vee \varphi_n.$$

Thus these are equivalent: $\mathscr{A} \vDash \Gamma[\alpha]$; $\mathscr{A} \vDash \&\Gamma[\alpha]$; $\mathscr{A} \vDash \boldsymbol{\&}\Gamma[\alpha]$. Also $\mathscr{A} \vDash^w \&\Gamma[\alpha]$ iff $\mathscr{A} \vDash^w \Gamma[\alpha]$. Also if $\mathscr{A} \vDash^w \Gamma[\alpha]$, then $\mathscr{A} \vDash^w \boldsymbol{\&}\Gamma[\alpha]$; but the converse fails.

Note that if $\tau_i^{\mathscr{A},\alpha}\uparrow$ and $\tau_{1-i}^{\mathscr{A},\alpha}\downarrow$, then $\mathscr{A} \mid \tau_0 = \tau_1[\alpha]$, $\mathscr{A} \dashv \tau_0 =_s \tau_1[\alpha]$, and $\mathscr{A} \dashv \tau_0 =_b \tau_1[\alpha]$. Also $\tau^{\mathscr{A},\alpha}\uparrow$ iff $\mathscr{A} \dashv \mathsf{E}_s(\tau)[\alpha]$. Furthermore, if $\mathscr{A}$ is nn, these are equivalent: $\tau^{\mathscr{A},\alpha}\uparrow$; $\mathscr{A} \mid \mathsf{E}(\tau)[\alpha]$; $\mathscr{A} \nvDash \mathsf{E}(\tau)[\alpha]$. In fact, $\mathscr{A} \dashv \mathsf{E}(\tau)[\alpha]$ iff $|\mathscr{A}|$ is empty. One might argue that the first and third of these facts show '=' and 'E' to be defective as parsings of 'is identical to' and 'exists' respectively. For example, 'Ronald Reagan is identical to the Tooth Fairy' and 'The Tooth Fairy exists' are, so one might insist, false, not truth-valueless. If this is accepted, '$=_s$', or perhaps '$=_b$', is a better parsing for 'is identical to', and 'E_s' is better for 'exists'. Of course '$=_b$' is bivalent; '$=_s$' is stronger than '=' and 'E_s' is stronger than 'E', since they're easier to frustrate; thus the choice of subscripts.

The above argument "against" '=' and 'E' is not conclusive. One might argue that the only datum behind the previous argument was the incorrectness of assertive use of 'Ronald Reagan is identical to the Tooth Fairy'; and as semantic theorists we are not required to count it as false. Indeed, there is reason not to so count it. Sentences of the form $\mathsf{E}(\tau)$ [$\neg\mathsf{E}(\tau)$] can't be false [true] in an nn model; so the only semantic contrast for such sentences is between truth [falsity] and undefinedness. Thus it would be natural for the distinction between assertoric correctness and incorrectness for statements of the form $\mathsf{E}(\tau)$ [$\neg\mathsf{E}(\tau)$] to align with the true/undefined [false/undefined] distinction, rather than with the true/false distinction. If this story is accepted, 'E' is an adequate parsing for 'exists', and a similar story can be told for '='. I'll take no position on this matter, but instead consider all three ways of handling identity.

One further enrichment deserves mention. Let $\text{lex}_{\ldots,\iota} = \text{lex}_{\ldots} \cup \{\text{'ι'}\}$. For $L_{\ldots,\iota}$ terms and formulae are defined simultaneously with the new clause:

if φ is a formula and $v \in Var$ then $(\iota v)\varphi$ is a term.

Also den, $\vDash$ and $\dashv$ are defined simultaneously with the new clause:

$$\text{den}(\mathscr{A}, \alpha, (\iota v)\varphi) \simeq \text{ the } a \in |\mathscr{A}| \text{ so that } \mathscr{A} \vDash \varphi[\alpha_a^v].$$

2. Some lemmas and facts about inclusions

First, some definitions. Suppose that for $i \in 2$, $\mathscr{A}_i = (A, \mathscr{E}_i, \mathscr{N}_i)$ is a model for *Pred* ∪ *Funct*. Let $\mathscr{E}_0 \sqsubseteq \mathscr{E}_1$ iff for any $n < \omega$, $\zeta \in Pred(n)$:

$$\text{if } n = 0 \text{ and } \mathscr{E}_0(\zeta)\downarrow, \quad \text{then } \mathscr{E}_1(\zeta) = \mathscr{E}_0(\zeta);$$

$$\text{if } n \geq 1,\ \vec{a} \in A^n \text{ and } \mathscr{E}_0(\zeta)(\vec{a})\downarrow, \quad \text{then } \mathscr{E}_0(\zeta)(\vec{a}) = \mathscr{E}_1(\zeta)(\vec{a}).$$

Define $\mathscr{N}_0 \sqsubseteq \mathscr{N}_1$ analogously. Let $\mathscr{A}_0 \sqsubseteq \mathscr{A}_1$ iff $\mathscr{E}_0 \sqsubseteq \mathscr{E}_1$ and $\mathscr{N}_0 \sqsubseteq \mathscr{N}_1$; $\mathscr{A}_0 \sqsubseteq_s \mathscr{A}_1$ iff $\mathscr{E}_0 \sqsubseteq \mathscr{E}_1$ and $\mathscr{N}_0 = \mathscr{N}_1$. $\mathscr{E}_0$ and $\mathscr{E}_1$ are compatible iff for any $\zeta \in Pred(n)$:

$$\text{if } n = 0,\ \mathscr{E}_0(\zeta)\downarrow \text{ and } \mathscr{E}_1(\zeta)\downarrow, \quad \text{then } \mathscr{E}_0(\zeta) = \mathscr{E}_1(\zeta):$$

$$\text{if } n \geq 1,\ \vec{a} \in A^n,\ \mathscr{E}_0(\zeta)(\vec{a})\downarrow \text{ and } \mathscr{E}_1(\zeta)(\vec{a})\downarrow, \quad \text{then } \mathscr{E}_0(\zeta)(\vec{a}) = \mathscr{E}_1(\zeta)(\vec{a}).$$

Define the compatibility of $\mathscr{N}_0$ and $\mathscr{N}_1$ analogously. $\mathscr{A}_0$ and $\mathscr{A}_1$ are compatible, abbreviated as $\mathscr{A}_0 * \mathscr{A}_1$, iff $\mathscr{E}_0$ and $\mathscr{E}_1$ are compatible and so are $\mathscr{N}_0$ and $\mathscr{N}_1$. $\mathscr{A}_0$ and $\mathscr{A}_1$ are strongly compatible, abberviated as $\mathscr{A}_0 *_s \mathscr{A}_1$, iff $\mathscr{E}_0$ and $\mathscr{E}_1$ are compatible and $\mathscr{N}_0 = \mathscr{N}_1$.

Consider models $\mathscr{A}_i = (|\mathscr{A}_i|, \mathscr{E}_i, \mathscr{N}_i)$ (perhaps with different universes, unlike as above). Let π be an isomorphism from $\mathscr{A}_0$ to $\mathscr{A}_1$, abbreviated $\pi : \mathscr{A}_0 \cong \mathscr{A}_1$, iff π is a one–one function from $|\mathscr{A}_0|$ onto $|\mathscr{A}_1|$ so that for any $n < \omega$: for any $\zeta \in Pred(n)$,

$$\text{if } n = 0, \quad \text{then } \mathscr{E}_0(\zeta) = \mathscr{E}_1(\zeta);$$

$$\text{if } 0 < n \text{ and } \vec{a} \in |\mathscr{A}_0|^n, \quad \text{then } \mathscr{E}_0(\zeta)(\vec{a}) = \mathscr{E}_1(\zeta)(\vec{a});$$

for any $\xi \in Funct(n)$ the analogous condition holds.

Note: if $\mathscr{E}_i(\zeta)(\vec{a})\uparrow$ this can be because $\vec{a} \notin \text{dom}(\mathscr{E}_i(\zeta))$ or because $\zeta \notin \text{dom}(\mathscr{E}_i)$. Similarly for $\mathscr{N}_i$. This permits a slight anomaly: we can have $|\mathscr{A}_0| = |\mathscr{A}_1|$ and π = identity on $|\mathscr{A}_0|$ though $\mathscr{A}_0 \neq \mathscr{A}_1$. Of course $\mathscr{A}_0$ is isomorphic to $\mathscr{A}_1$, abbreviated $\mathscr{A}_0 \cong \mathscr{A}_1$, iff for some π, $\pi : \mathscr{A}_0 \cong \mathscr{A}_0$. Let $\pi : \mathscr{B} \sqsubseteq\!\!\!\to \mathscr{A}$. $[\pi : \mathscr{B} \sqsubseteq\!\!\!\to_s \mathscr{A}_1]$ iff for some $\mathscr{A}_0 \sqsubseteq \mathscr{A}_1$ $[\mathscr{A}_0 \sqsubseteq_s \mathscr{A}_1]$, $\pi : \mathscr{B} \cong \mathscr{A}_0$; $\mathscr{B} \sqsubseteq \mathscr{A}_1$ $[\mathscr{B} \sqsubseteq_s \mathscr{A}_1]$ iff for some π, $\pi : \mathscr{B} \sqsubseteq\!\!\!\to \mathscr{A}_1$ $[\pi : \mathscr{B} \sqsubseteq\!\!\!\to_s \mathscr{A}_1]$. Let $\pi : \mathscr{B} \xrightarrow{*} \mathscr{A}_1$ $[\pi : \mathscr{B} \xrightarrow{*} \mathscr{A}_1]$ iff for some $\mathscr{A}_0$ compatible [strongly compatible] with $\mathscr{A}_1$, $\pi : \mathscr{B} \cong \mathscr{A}_0$; $\mathscr{B} \overset{*}{\sim} \mathscr{A}_1$ iff for some π, $\pi : \mathscr{B} \xrightarrow{*} \mathscr{A}_0$.

Lemma 1. *Let $i \in 2$, $x = i$ or i, u $[i$, s or i, u, s or i, b or i, u, b$]$. For models $\mathscr{A}_0$ and $\mathscr{A}_1$, α and $\mathscr{A}_0$-assignment and $\varphi \in \text{Fml}(L_x)$:*

(i) *if $\pi : \mathscr{A}_0 \sqsubseteq\!\!\!\to \mathscr{A}_1$ $[\pi : \mathscr{A}_0 \sqsubseteq\!\!\!\to_s \mathscr{A}_1]$ then:*

$$\text{if } \mathscr{A}_0 \vDash \varphi\,[\alpha], \quad \text{then } \mathscr{A}_1 \vDash \varphi\,[\pi \circ \alpha];$$

$$\text{if } \mathscr{A}_0 \mathrel{=\!\!|} \varphi\,[\alpha], \quad \text{then } \mathscr{A}_1 \mathrel{=\!\!|} \varphi\,[\pi \circ \alpha];$$

(ii) *if $\pi : \mathscr{A}_0 \xrightarrow{*} \mathscr{A}_1$ $[\pi : \mathscr{A}_0 \xrightarrow{*}_s \mathscr{A}_1]$ then:*

$$\text{if } \mathscr{A}_0 \vDash \varphi\,[\alpha], \quad \text{then } \mathscr{A}_1 \vDash^{w} \varphi\,[\pi \circ \alpha];$$

$$\text{if } \mathscr{A}_0 \mathrel{=\!\!|} \varphi\,[\alpha], \quad \text{then } \mathscr{A}_1 \nvDash \varphi\,[\pi \circ \alpha].$$

These follow by a straight forward induction on φ.

Lemma 2. *For any $i \in 2$ and $\varphi \in \text{Fml}(L_{i,\text{T}})$ or $\text{Fml}(L_{i,\text{T,s}})$ $[\text{Fml}(L_{i,\text{T,b}})]$, a total [et-total] model $\mathscr{A}$ and a total [partial] $\mathscr{A}$-assignment α: either $A \vDash \varphi\,[\alpha]$ or $\mathscr{A} \mathrel{=\!\!|} \varphi\,[\alpha]$.* Proof by induction of construction of φ.

Let an occurrence of a formula θ within a formula φ be exposed in φ iff it is not in the scope of an occurrence of 'T' in φ.

Lemma 3. *Let* $\varphi \in \mathrm{Fml}(L_{0,\mathsf{T},\mathsf{u}})$.

(1) *For any exposed occurrence of a subformula θ in φ, suppose that $\vec{v}$ is a list without repetition of the variables occurring free in θ that are bound in φ; for any model $\mathscr{A}$, $\mathscr{A}$-assignment α and $\vec{a} \in |\mathscr{A}|$, if $\mathscr{A} \mid \theta\,[\alpha^{\vec{a}}_{\vec{v}}]$ then $\mathscr{A} \mid \varphi\,[\alpha]$; so if θ is* 'u' *then φ is equivalent to* 'u'.

(2) *If φ contains no exposed occurrence of* 'u' *then some formula of $L_{0,\mathsf{T}}$ is equivalent to φ.*

(1) follows from the semantics for '$\supset$' and '$\exists$'. For (2): if $\mathsf{T}\psi$ is a subformula of φ and ψ contains an exposed occurrence of 'u' then $\mathsf{T}\psi$ is equivalent to '$\perp$'; so replacing $\mathsf{T}\psi$ by '$\perp$' in φ preserves equivalence to φ. Doing this for all such $\mathsf{T}\psi$ yields the desired formula of $L_{0,\mathsf{T}}$.

In what follows, 'y' is to be replaced by blank, 'nn', 'et', or 'nn&et'. For languages L and L' let $L \overset{\mathrm{y}}{\prec} L'$ iff for every $\varphi \in \mathrm{Fml}(L)$ there is a $\varphi' \in \mathrm{Fml}(L')$ y-equivalent to φ; let $L \overset{\mathrm{y}^*}{\prec} L'$ iff for every $\varphi \in \mathrm{Fml}(L)$ there is a $\varphi' \in \mathrm{Fml}(L')$ y-equivalent* to φ.

Let $\mathrm{lex}_x \overset{\mathrm{y}}{\prec} \mathrm{lex}_{x'}$ iff for any *Pred, Funct,*

$$L_x(\mathit{Pred}, \mathit{Funct}) \overset{\mathrm{y}}{\prec} L_{x'}(\mathit{Pred}, \mathit{Funct});$$

define $\mathrm{lex}_x \overset{\mathrm{y}^*}{\prec} \mathrm{lex}_{x'}$ analogously. We'll say that a logical constant is expressible$^{\mathrm{y}}$ [expressible$^{\mathrm{y}^*}$] using lex_x iff for any $\mathrm{lex}_{x'}$ containing that constant $\mathrm{lex}_{x'} \prec \mathrm{lex}_x$ [$\mathrm{lex}_{x'} \overset{\mathrm{y}^*}{\prec} \mathrm{lex}_x$].

In the following list, each entry on the right could be used for the entry on the left preserving all semantic facts; thus we'll freely treat the left entries as abbreviations of the right entries when the left entries are not in the lexicon under discussion.

$\perp$	:	$\mathsf{T}\mathsf{u}$;
u	:	$(\mathrm{t}v)\perp = (\mathrm{t}v)\perp$ or $(\mathrm{t}v)\perp =_{\mathrm{s}} (\mathrm{t}v)\perp$;
$\varphi \supset \psi$	:	$(\varphi \supset \psi) \,\&\, (\varphi \supset \varphi) \,\&\, (\psi \supset \psi)$;
$(\exists v)\varphi$	:	$(\exists v)\varphi \,\&\, (\forall v)(\varphi \supset \varphi)$;
$\tau_0 = \tau_1$	:	$\tau_0 =_{\mathrm{s}} \tau_1 \,\&\, \tau_0 =_{\mathrm{s}} \tau_0 \,\&\, \tau_1 =_{\mathrm{s}} \tau_1$;
$\tau_0 =_{\mathrm{s}} \tau_1$	:	$\tau_0 = \tau_1 \,\&\, (\mathsf{T}\mathsf{E}(\tau_0) \equiv \mathsf{T}\mathsf{E}(\tau_1))$,
		or $\tau_0 =_{\mathrm{b}} \tau_1 \,\&\, ([\neg \mathsf{E}_{\mathrm{s}}(\tau_0) \,\&\, \neg \mathsf{E}_{\mathrm{s}}(\tau_1)] \supset \mathsf{u})$;
$\tau_0 =_{\mathrm{b}} \tau_1$	:	$(\forall v)(v =_{\mathrm{s}} \tau_0 \equiv v =_{\mathrm{s}} \tau_1)$, or $(\mathsf{T}\mathsf{E}(\tau_0) \vee \mathsf{T}\mathsf{E}(\tau_1)) \supset \mathsf{T}(\tau_0 = \tau_1)$,
		or either of these with '$\forall$', '$\vee$', and '$\supset$' replacing
		'$\forall$', '$\vee$', and '$\supset$'.

Observation 1. *Let* $i \in 2$.

(i) $\text{lex}_{0,\ldots} \prec \text{lex}_{1,\ldots}$.

(ii) $\text{lex}_i \prec \text{lex}_{i,s}$.

(iii) $\text{lex}_{1,s} \prec \text{lex}_{1,\top}$ *and* $\text{lex}_{1,u,s} \prec \text{lex}_{1,u,b}$.

(iv) $\text{lex}_{i,b} \prec \text{lex}_{i,s}$ *and* $\text{lex}_{i,b} \prec \text{lex}_{i,\top}$.

(v) $\text{lex}_{i,u} \prec \text{lex}_{i,t}$; $\text{lex}_{i,u,s} \prec \text{lex}_{i,s,t}$.

(vi) $\text{lex}_{1,\top,u,t} \prec \text{lex}_{1,\top,u}$.

(i) through (v) all follow using the above abbreviations.

(vi) For any term τ, variable v and formula φ, let $(!v)(\varphi, \tau)$ be $(\forall v)(\top\varphi \equiv v = \tau)$. Thus for any nn-model $\mathscr{A}$ and $\mathscr{A}$-assignment α, $\mathscr{A} \vDash (!v)(\varphi, \tau)\,[\alpha]$ iff $\text{den}(\mathscr{A}, \alpha, \tau)\downarrow$ and is the unique $a \in |\mathscr{A}|$ so that $\mathscr{A} \vDash \varphi\,[\alpha^a_v]$. For $\zeta(\tau_1, \ldots, \tau_n)$, suppose τ_j is $(\text{t}v)\varphi$. Picking a variable μ not occuring in any of these terms, $\zeta(\tau_1, \ldots, \tau_n)$ is equivalent to:

$$(\exists u)((!v)(\varphi, \mu) \,\&\, \zeta(\tau_1, \ldots, \mu, \ldots, \tau_n)) \vee (\neg(\exists\mu)(!v)(\varphi, \mu) \,\&\, \mathsf{u}).$$

It's easy to do the same sort of thing for equations containing $(\text{t}v)\varphi$. By iterating this procedure on atomic subformulae, a given formula of $L_{1,\top,u,t}$ transforms to an equivalent formula of $L_{1,\top,u}$.

Unlike the classic Russellian elimination of 't', the above approach does not produce scope ambiguities: E.g. for 'P', 'Q' $\in$ *Pred*(1), '$\neg\text{P}((\text{t}x)\text{Q}x)$' is equivalent to both of these:

$$\neg((\exists x)[(!y)(\text{Q}x, y) \,\&\, \text{P}x] \vee (\neg(\exists x)(!y)(\text{Q}x, y) \,\&\, \mathsf{u}));$$

$$(\exists x)[(!y)(\text{Q}x, y) \,\&\, \neg\text{P}x] \vee (\neg(\exists x)(!y)(\text{Q}x, y) \,\&\, \mathsf{u})).$$

Observation 2. *Let* $i \in 2$.

(i) $\text{lex}_{i,\top} \overset{\text{et}}{\prec} \text{lex}_{i,s}$; $\text{lex}_{i,\top,u} \overset{\text{et}}{\prec} \text{lex}_{i,u,b}$.

(ii) $\text{lex}_{i,\top} \overset{\text{nn}}{\prec} \text{lex}_{i,s,t}$; $\text{lex}_{i,\top} \overset{\text{nn}}{\prec} \text{lex}_{i,b,t}$.

(iii) *If for* $n > 0$, $Pred(n) \neq \{\ \}$ *then* 'u' *is* nn-*equivalent to a sentence of* $L_{i,b,t}$*(Pred, Funct)*.

Temporarily treat 'F' as primitive. Let $\varphi \in \text{Fml}(L_{i,\top})$. For $i = 1$ we drive occurrences of '$\top$' and 'F' in φ inward preserving et-equivalence, using these et-equivalences;

$\top(\psi \supset \theta)$	$: (\neg\text{F}\psi) \supset \top\theta$;	$\text{F}(\psi \supset \theta)$	$: \neg(\top\psi \supset \theta)$;
$\top\top\psi$	$: \top\psi$;	$\top\text{F}\psi$	$: \text{F}\psi$;
$\text{F}\top\psi$	$: \neg\top\psi$;	$\text{FF}\psi$	$: \neg\text{F}\varphi$;
$\top(\exists v)\psi$	$: (\exists v)\top\psi$;	$\text{F}(\exists v)\psi$	$: (\forall v)\text{F}\psi$.

Then $\mathsf{T}\zeta(\tau_0, \ldots, \tau_n)$ and $\mathsf{F}\zeta(\tau_0, \ldots, \tau_n)$ may be replaced by:

$$\zeta(\tau_0, \ldots, \tau_n) \,\&\, \mathsf{E}_s(\tau_0) \,\&\, \cdots \,\&\, \mathsf{E}_s(\tau_n) \quad \text{and}$$

$$\neg\zeta(\tau_0, \ldots, \tau_n) \,\&\, \mathsf{E}_s(\tau_0) \,\&\, \cdots \,\&\, \mathsf{E}_s(\tau_n),$$

respectively. $\mathsf{T}(\tau_0 = \tau_1)$ and $\mathsf{F}(\tau_0 = \tau_1)$ can also be replaced using '$=_s$'; for remaining occurrences of $\tau_0 = \tau_1$ use the above-given abbreviations. The second inclusion follows analogously.

For $i = 0$, we first associate each $\psi \in \mathrm{Fml}(L_{0,\mathsf{T}})$ with a $\psi' \in \mathrm{Fml}(L_{0,s})$ et-equivalent to $\mathsf{T}(\psi \supset \psi)$:

$$\text{for } \psi \text{ atomic, } \psi' \text{ is } \psi \supset \psi; \qquad (\psi \supset \theta)' \text{ is } \psi' \,\&\, \theta';$$

$$(\mathsf{T}\psi)' \text{ is } \neg\bot; \qquad ((\exists v)\psi)' \text{ is } (\forall v)(\psi').$$

Then we drive 'T' and 'F' in φ inward using the previous et-equivalences for $\mathsf{T}\mathsf{T}\psi$, etc. and these:

$$\begin{aligned}
\mathsf{T}(\psi \supset \theta) &: (\neg\mathsf{F}\psi \supset \mathsf{T}\theta) \,\&\, \psi' \,\&\, \theta';\\
\mathsf{F}(\psi \supset \theta) &: \neg(\mathsf{T}\psi \supset \neg\mathsf{F}\theta) \,\&\, \psi' \,\&\, \theta';\\
\mathsf{T}(\exists v)\psi &: (\exists v)\mathsf{T}\psi \,\&\, (\forall v)(\psi');\\
\mathsf{F}(\exists v)\psi &: (\forall v)\mathsf{F}\psi \,\&\, (\forall v)(\psi').
\end{aligned}$$

On atomic formulae we eliminate 'T' and 'F' as before. For $\varphi \in \mathrm{Fml}(L_{0,\mathsf{T},u})$ this procedure yields an et-equivalent in $\mathrm{Fml}(L_{0,u,s})$. To get an et-equivalent in $\mathrm{Fml}(L_{0,u,b})$, just use '$=_b$' in place of '$=_s$', together with the fact that '$=$' is expressible in $L_{0,u,b}$; notice that for $\varphi \in \mathrm{Fml}(L_{0,\mathsf{T}})$ this procedure need not yield a result in $L_{0,b}$, because 'u' is needed to handle "remaining occurrences of $\tau_0 = \tau_1$".

(ii) Given $\varphi \in \mathrm{Fml}(L_{1,\mathsf{T}})$ fix distinct variables v, μ not free in φ; $\mathsf{T}\varphi$ is nn-equivalent to $(\exists v)\mathsf{E}((\mathsf{t}\mu)(\varphi \,\&\, v = \mu))$. For $L_{0,\mathsf{T}}$ use '&' and '∃' in place of '&' and '∃'. This also holds with '$=_b$' in place of '$=_s$'.

(iii) Say '$\mathbf{P}$' $\in \mathit{Pred}(1)$; let φ be '$(\forall y)\mathbf{P}((\mathsf{t}x)x \neq_b y)$' and φ be '$(\forall y)(\forall z)(y \neq_b z \supset \mathbf{P}((\mathsf{t}x)(x \neq_b y \,\&\, x \neq_b y)))$'. For any nn-model $\mathscr{A}$: if $\mathrm{card}(\mathscr{A}) \neq 2$, $\mathscr{A} \mid \varphi$; otherwise $\mathscr{A} \vDash \varphi$; if $\mathrm{card}(\mathscr{A}) \neq 3$, $A \mid \psi$; otherwise $\mathscr{A} \vDash \psi$; thus $\mathscr{A} \,\dot{\mid}\, \varphi \,\&\, \psi$. Similarly with '⊃' in place of '⊃', etc.

Observation 3. *Let* $i \in 2$.

(i) $\mathrm{lex}_{\ldots u} \overset{+}{\prec} \mathrm{lex}_{\ldots}$.

(ii) $\mathrm{lex}_{1,\mathsf{T},\ldots} \overset{+}{\prec} \mathrm{lex}_{0,\mathsf{T},\ldots}$.

(iii) $\mathrm{lex}_{i,s} \overset{+}{\prec} \mathrm{lex}_{i,b}$, and so $\mathrm{lex}_i \overset{+}{\prec} \mathrm{lex}_{i,b}$;

(iv) $\mathrm{lex}_{i,\mathsf{T},t} \overset{+}{\prec} \mathrm{lex}_{i,\mathsf{T}}$.

(i) For $\varphi \in \mathrm{Fml}(L_{\ldots u})$, we form φ^+ and $\varphi^- \in \mathrm{Fml}(L_{\ldots})$ so that (1) for any

$\mathscr{A} \in \mathrm{MOD}$ and any $\mathscr{A}$-assignment α:

(1) $\mathscr{A} \models \varphi[\alpha]$ iff $\mathscr{A} \models \varphi^{+}[\alpha]$;

$\mathscr{A} \dashv \varphi[\alpha]$ iff $\mathscr{A} \dashv \varphi^{-}[\alpha]$.

Let u^{+} be $\bot$, u^{-} be $\neg\bot$, $(\psi \supset \theta)^{+}$ be $\psi^{-} \supset \theta^{+}$, $((\exists v)\theta)^{+}$ be $(\exists v)(\theta^{+})$, $((\exists v)(\theta)^{-}$ be $(\exists v)(\theta^{-})$, $(\psi \supset \theta)^{+}$ be $(\mathsf{T}\psi^{+} \mathbin{\&} \mathsf{T}\theta^{+}) \vee (\mathsf{F}\psi^{-} \mathbin{\&} \mathsf{T}\theta^{+}) \vee (\mathsf{F}\psi^{-} \mathbin{\&} \mathsf{F}\theta^{-})$, $(\psi \supset \theta)^{-}$ be $\mathsf{T}\psi^{+} \mathbin{\&} \mathsf{F}\theta^{-}$, $(\mathsf{T}\theta)^{+}$ and $(\mathsf{T}\theta)^{-}$ be $\mathsf{T}(\theta^{+})$, with θ^{+} and θ^{-} being θ for other atomic θ. By an easy induction (1) holds.

(ii) For $\varphi \in \mathrm{Fml}(L_{1,\mathsf{T},\ldots})$ form φ^{+}, $\varphi^{-} \in \mathrm{Fml}(L_{0,\mathsf{T},\ldots})$ satisfying (1), by making these changes in the preceding: $(\psi \supset \theta)^{+}$ is $(\neg\mathsf{F}\psi^{-}) \supset \mathsf{T}\theta^{+}$, $(\psi \supset \theta)^{-}$ is $(\psi^{+} \supset \theta^{-})$, $((\exists v)\theta)^{+}$ is $(\exists v)\mathsf{T}(\theta^{+})$, $((\exists v)\theta)^{-}$ is $(\exists v)(\theta^{-})$.

(iii) For $\varphi \in \mathrm{Fml}(L_{0,\mathrm{s}})$ form φ^{+}, $\varphi^{-} \in \mathrm{Fml}(L_{0,\mathrm{b}})$ so that (1) holds. Let $(\tau_0 =_{\mathrm{s}} \tau_1)^{+}$ be $\tau_0 =_{\mathrm{b}} \tau_1 \mathbin{\&} \mathsf{E}_{\mathrm{s}}(\tau_0) \mathbin{\&} \mathsf{E}_{\mathrm{s}}(\tau_1)$; $(\tau_0 =_{\mathrm{s}} \tau_1)^{-}$ is $\tau_0 =_{\mathrm{b}} \tau_1$; other atomic formulae remain the same; other clauses as in (i). A similar construction applies for $i = 1$.

(iv) In $\varphi \in \mathrm{Fml}(L_{1,\mathsf{T},\mathrm{t}})$ replace positive occurrences of $\zeta(\ldots, (\iota v)\theta, \ldots)$ by $(\exists\mu)((!v)(\theta, \mu) \mathbin{\&} \zeta(\ldots, \mu, \ldots))$, negative occurrences by $(\forall\mu)((!v)(\theta, \mu) \supset \zeta(\ldots, \mu, \ldots))$; handle '=' similary. Similarly for $i = 0$.

These inclusions, and others following trivially from them, are summarized in Fig. 1; there $\mathrm{lex}_x \overset{Y\;Y'}{\succ\!\!-\!\!\prec} \mathrm{lex}_{x'}$ iff $\mathrm{lex}_x \overset{Y'}{-\!\!\prec} \mathrm{lex}_{x'}$ and $\mathrm{lex}_{x'} \overset{Y}{-\!\!\prec} \mathrm{lex}_x$.

Usually we'll state failures of expressibility for only the strongest relevent lexica; obviously what's not expressible in it is also not expressible in equivalent or weaker lexica.

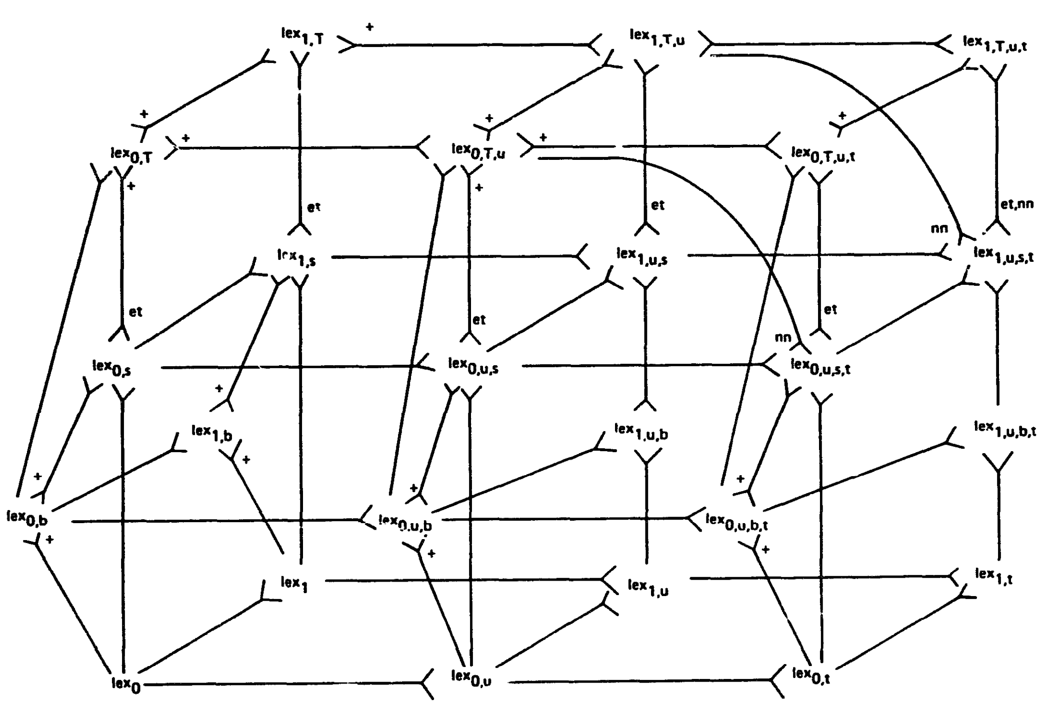

Fig. 1. '$\succ\!\prec$' represents '$\prec$ & $\succ$'.

Observation 4. *In what follows, replace* 'γ' *as above.*

(i) 'T' *is not expressible*$^{+\gamma}$ *using* $\mathrm{lex}_{1,u,b}$ (*or* $\mathrm{lex}_{1,u,s}$, $\mathrm{lex}_{1,u}$, *etc.*).

(ii) 'u' *is not expressible*$^{\gamma}$ *using* $\mathrm{lex}_{i,\mathsf{T}}$.

(iii) *Neither* '$=_s$' *nor* '$=_b$' *is expressible*$^{\gamma}$ *using* $\mathrm{lex}_{1,u}$; *furthermore, neither is expressible*$^{+\gamma}$ *using* $\mathrm{lex}_{0,u}$.

(iv) *Neither* '$=$' *nor* '$=_s$' *is expressible*$^{\gamma}$ *using* $\mathrm{lex}_{1,\mathsf{T},b}$.

(v) '$=_s$' *is not expressible*$^{\gamma}$ *using* $\mathrm{lex}_{0,\mathsf{T},u}$.

(vi) 't' *is not expressible*$^{\gamma}$ *using* $\mathrm{lex}_{0,\mathsf{T},u,s}$.

(vii) 't' *is not expressible*$^{\gamma}$ *using* $\mathrm{lex}_{1,\mathsf{T}}$ *or* $\mathrm{lex}_{1,u}$.

(i) For '$\mathbf{P}$' $\in \mathit{Pred}(0)$ and '$\mathbf{c}$' $\in \mathit{Funct}(0)$ '$\mathsf{U}\mathbf{P}$' and '$\neg\mathsf{TE}(\mathbf{c})$' have no γ-equivalents$^+$, using Lemma 1(i) on appropriate models. (Similar constructions apply for an n-place predicate- or function-constant if $n>1$. But note that for both $i\in 2$, $L_{i,\mathsf{T}}(\{\ \},\{\ \})\prec L_{i,b}(\{\ \},\{\ \})$.) What follows applies for any choice of *Pred* and *Funct*.

(ii) By Lemma 2, 'u' has no γ-equivalent in $L_{1,\mathsf{T}}$.

(iii) By Lemma 1(i), $\mathsf{E}_s(v)$ has no γ-equivalent in $\mathrm{Fml}(L_{1,u})$. Since '$=_s$' and '$=_b$' both generate $\mathsf{E}_s(v)$, they are not expressible$^{\gamma}$. Suppose $\varphi\in\mathrm{Fml}(L_{0,u})$ is equivalent$^+$ to '$\mathbf{a}\neq_s\mathbf{b}$'; without loss of generality $\varphi\in\mathrm{Sent}(L_0)$; for a model $\mathscr{A}$ with '$\mathbf{a}$'$^{\mathscr{A}}\!\downarrow$ and '$\mathbf{b}$'$^{\mathscr{A}}\!\uparrow$, $\mathscr{A}\vDash\varphi$; so by Lemma 3, '$\mathbf{b}$' doesn't occur in φ; but then we may surely get $\mathscr{A}\sqsubseteq\mathscr{B}$, $\mathscr{B}\dashv$ '$\mathbf{a}\neq_s\mathbf{b}$', so $\mathscr{B}\dashv\varphi$, which violates Lemma 1(i). Similarly for '$\mathbf{a}\neq_b\mathbf{b}$'.

(iv) $v=v$ has no γ-equivalent in $\mathrm{Fml}(L_{1,\mathsf{T},b})$, by Lemma 2 and the fact that if $\alpha(v)\!\downarrow$ then $\mathscr{A}\mid v=v\,[\alpha]$; similarly with $v=_s v$.

To prove (v) we show that for any $\varphi\in\mathrm{Fml}(L_{0,\mathsf{T}})$, φ is not equivalent to $v_0=_s v_1$. For $i\in 2$ let an occurrence of v_i in φ be exposed in φ iff it's within an occurrence of an atomic formula that's exposed in φ. Fix a model $\mathscr{A}$ and $\mathscr{A}$-assignments α_0 and α_1 with $\alpha_i(v_i)\!\uparrow$. If there is an exposed occurrence of v_i in φ, then $\mathscr{A}\mid\varphi\,[\alpha_i]$: for any such occurrence would be in an exposed occurrence of $v_i=\tau$ or $\tau=v_i$; where θ is that formula, $\mathscr{A}\mid\theta\,[\alpha_i]$; by Lemma 3, $\mathscr{A}\mid\varphi\,[\alpha]$; but by taking $\alpha(v_{1-i})\!\downarrow$, $\mathscr{A}\dashv v_0=_s v_1\,[\alpha]$. We now show that if no occurrences of v_0 or v_1 are exposed in φ and $\alpha(v)\!\downarrow$ for all variables v other than v_0 and v_1, either $\mathscr{A}\vDash\varphi\,[\alpha]$ or $\mathscr{A}\dashv\varphi[\alpha]$. The easiest way to see that is to prenex φ; the usual prenexing rules with '$\exists$' and '$\forall$' preserve equivalence; then note that it holds for the matrix, and then for the prenexed formula. By taking such an α with $\alpha(v_0)\!\uparrow$ and $\alpha(v_1)\!\uparrow$ we have $\mathscr{A}\mid v_0=_s v_1\,[\alpha]$.

We can come rather close to expressing '$=_s$' with $\mathrm{lex}_{0,\mathsf{T},u}$. Let:

$$\Gamma_0=\{\neg\mathsf{F}(\tau_0=\tau_1),\ \mathsf{TE}(\tau_0)\equiv\mathsf{TE}(\tau_1),\ \mathsf{E}(\tau_0),\ \mathsf{E}(\tau_1)\};$$
$$\Gamma_1=\{\neg\mathsf{T}(\tau_0=\tau_1),\ \neg(\mathsf{TE}(\tau_0)\equiv\mathsf{TE}(\tau_1)),\ \mathsf{E}(\tau_0),\ \mathsf{E}(\tau_1)\}.$$

Then for any $\mathscr{A}$ and α as usual:

$$\mathscr{A}\vDash\tau_0=_s\tau_1\,[\alpha]\quad\text{iff}\quad\mathscr{A}\vDash\Gamma_0\,[\alpha];$$
$$\mathscr{A}\vDash^{w}\tau_0=_s\tau_1\,[\alpha]\quad\text{iff}\quad\mathscr{A}\vDash^{w}\Gamma_0\,[\alpha];$$

similarly for $\tau_0\neq\tau_1$ and Γ_1.

But neither $\&\Gamma_0$ nor $\neg\&\Gamma_1$ is equivalent$^{\text{v}}$ to $\tau_0 =_{\text{s}} \tau_1$!

(vi) Let φ be '$(\exists y)((\text{t}x)x \neq_{\text{s}} y =_{\text{s}} (\text{t}x)x \neq_{\text{s}} y)$'. By Lemma 3 any $\theta \in \text{Sent}(L_{0,\text{T},\text{s}}(\{\ \}, \{\ \}))$ is either equivalent to 'u' or to a sentence of $L_{0,\text{T},\text{s}}(\{\ \}, \{\ \})$, in which case by Lemma 2 for a model $\mathscr{A}$ of cardinality $\neq 2$ either $\mathscr{A} \vDash \theta$ or $\mathscr{A} \dashv \theta$. Either way θ isn't y-equivalent to φ.

(vii) Similarly '$(\exists y)(y = (\text{t}x)x \neq y)$' has no equivalent in $\text{Fml}(L_{1,\text{T}})$. For '$\mathbb{P}$' $\in$ *Pred*(1) it's easy to find models $\mathscr{A}_0$ and $\mathscr{A}_1$ with den($\mathscr{A}_0$, α, '$(\text{t}x)\mathbb{P}x$')$\downarrow$ and den($\mathscr{A}_1$, α, '$(\text{t}x)\mathbb{P}x$')$\uparrow$; so $\mathscr{A}_0 \vDash$ '$\text{E}((\text{t}x)\mathbb{P}x)$' and $\mathscr{A}_1 \mid$ '$\text{E}((\text{t}x)\mathbb{P}x)$'; by Lemma 1 no formula of $L_{1,\text{u}}$ is equivalent or even equivalent$^+$, to '$\text{E}((\text{t}x)\mathbb{P}x)$'. A similar construction applies to '$\mathbb{f}$' $\in$ *Funct*(1), using '$(\text{t}x)(x = \mathbb{f}x)$'. And similarly for predicate- or function-constants with more places. If *Pred* = *Pred*(0) and *Funct* = *Funct*(0), $L_{1,\text{u},\text{t}} \prec L_{1,\text{u}}$, by a normal form argument that's too tedious to consider here.

3. Collapsing to two-valued semantics

In order to avoid the terrors of a three-valued semantics, some logicians favor a convention according to which those sentences (formulae) which we might consider neither true nor false (neither satisfied nor frustrated) are arbitrarily assigned one of these values, usually falsehood (frustration) being preferred. We digress to consider the relationship between this approach and a three-valued semantics.

Given *Pred*, form *Pred*$^+$ [*Pred*$^-$] by replacing each $\zeta \in$ *Pred*(n) with a new n-place predicate-constant ζ^+ [ζ^-]; let *Pred** = *Pred*$^+$ $\cup$ *Pred*$^-$. Where 'z' is replaced by '+' ['−'] ['*'] let L^z be the language generated from *Pred*^{z} and *Funct* using the logical lexicon {'$\perp$', '$\supset$', '$\exists$', '$=^+$', ['$=^-$'] [both '$=^+$' and '$=^-$']}. For a model $\mathscr{A}$ for *Pred*^{z}, *Funct* and $\mathscr{A}$-assignment α (recall these are partial) we define a two-valued satisaction relation $\vDash_2$ with these base clauses added to the usual induction clauses:

$$\mathscr{A} \vDash_2 \zeta^+(\cdots)\,[\alpha] \quad \text{iff} \quad \mathscr{A} \vDash \zeta(\cdots)\,[\alpha];$$

$$\mathscr{A} \vDash_2 \zeta^-(\cdots)\,[\alpha] \quad \text{iff} \quad \mathscr{A} \vDash^{\text{w}} \zeta(\cdots)\,[\alpha];$$

similarly for equations.

Thus $\zeta^+(\cdots)$ [$\zeta^-(\cdots)$] is in effect $\zeta(\cdots)$ according to the familiar Falsehood [unfamiliar Truth] convention: where the three-value approach says 'neither', say 'false' ['true'].

We define translation functions t^+, t^- from $\text{Fml}(L_{\times})$ into $\text{Fml}(L^*)$ so that for all $\mathscr{A}$ and α as above:

$$\mathscr{A} \vDash_2 t^+(\varphi)\,[\alpha] \quad \text{iff} \quad \mathscr{A} \vDash \varphi\,[\alpha];$$

$$\mathscr{A} \vDash_2 t^-(\varphi)\,[\alpha] \quad \text{iff} \quad \mathscr{A} \vDash^{\text{w}} \varphi\,[\alpha].$$

Let $t^+(\zeta(\cdots))$ be $\zeta^+(\cdots)$, $t^-(\zeta(\cdots))$ be $\zeta^-(\cdots)$; similarly for equations.

Also:

t^+('u'), t^+('$\bot$'), t^-('$\bot$') are all '$\bot$';

t^-('u') is '$\neg\bot$';

$t^+(\top\varphi)$ and $t^-(\top\varphi)$ are $t^+(\varphi)$;

$t^+(\varphi \supset \psi)$ is $t^-(\varphi) \supset t^+(\psi)$;

$t^-(\varphi \supset \psi)$ is $t^+(\varphi) \supset t^-(\psi)$;

t^+ and t^- commute with '$\exists$';

$t^+(\varphi \boldsymbol{\supset} \psi)$ is $(t^+(\varphi) \supset t^-(\varphi))\ \&\ (t^-(\varphi) \supset t^+(\varphi))\ \&\ (t^-(\psi) \supset t^+(\psi))$;

$t^-(\varphi \boldsymbol{\supset} \psi)$ is $(t^-(\varphi) \supset t^+(\psi)) \vee \neg(t^-(\varphi) \supset t^+(\varphi)) \vee \neg(t^-(\psi) \supset t^+(\psi))$;

$t^+((\exists v)\varphi)$ is $(\exists v)t^+(\varphi)\ \&\ (\forall v)(t^-(\varphi) \supset t^+(\varphi))$;

$t^-((\exists v)\varphi$ is $(\exists v)t^-(\varphi) \vee \neg(\forall v)(t^-(\varphi) \supset t^+(\varphi))$.

The point of the last four clauses lies here:

$$\mathscr{A} \mid \varphi\,[\alpha] \quad \text{iff} \quad \mathscr{A} \not\models_2 t^-(\varphi) \supset t^+(\varphi)\,[\alpha],$$

by a simultaneous induction.

For the above translation we couldn't get by with only L^+ or L^-. For example, for 'P', 'Q' $\in$ PRED(0) there is no $\psi \in \mathrm{Fml}(L^+)$ so that for all models $\mathscr{A}$ for {'P', 'Q'}:

$$\mathscr{A} \models \text{'P} \supset \text{Q'} \quad \text{iff} \quad \mathscr{A} \models_2 \varphi.$$

Suppose $\mathscr{A}_0 \mid$ 'P', $\mathscr{A}_1 \dashv$ 'P', $\mathscr{A}_0 \dashv$ 'Q', $\mathscr{A}_1 \dashv$ 'Q'; so $\mathscr{A}_0 \mid$ 'P $\supset$ Q' and $\mathscr{A}_1 \models$ 'P $\supset$ Q'. But for any $\psi \in \mathrm{Fml}(L^+)$, $\mathscr{A}_0 \models_2 \psi$ iff $\mathscr{A}_1 \models_2 \psi$. For similar reasons there is no $\psi \in \mathrm{Fml}(L^+)$ so that for all $\mathscr{A}$ and α as above: $\mathscr{A} \models^{w}$ 'P $\supset$ Q' iff $\mathscr{A} \models_2 \psi$.

Using the Falshood (or Truth) convention in our model-theoretic semantics for partial models and assignments is objectionable on two grounds.

(1) It destroys the symmetry of truth and falsehood found in two-valued semantics for total models and assignments, and destroys it in an ad-hoc way.

(2) Use of the Falsehood [Truth] convention amounts to use of L^+ [L^-] under $\models_2$, which as our last example shows, is less expressive then even our weakest language L_0 considered under $\models$ or $\models^{w}$.

4. Natural deduction formalizations

I include the following section on formalization for the sake of the compactness result it yields and because these calculi have not, as far as I know, appeared elsewhere.

Some of our logical lexica can't express '$\supset_w$' or '$\supset_s$'; so axiomatization of the class of valid or weakly valid formulae will not capture all information about the validity and weak validity of sequents. For a uniform approach to formalizing the logics presented in Section 1 we need a direct inductive definition of the classes of valid and/or weakly valid sequents, that is to say a sequent-calculus. Abstractly, we may view a sequent-calculus $\mathscr{K}$ as a class-size function that applies to a language L to yield a simultaneous inductive definition of two sets of sequents in L:

$$\text{th}\mathscr{K}(L) = \text{the set of theorems of } \mathscr{K}(L);$$

$$\text{wkth}\mathscr{K}(L) = \text{the set of weak theorems of } \mathscr{K}(L).$$

The axioms [weak axioms] of $\mathscr{K}(L)$ are those sequents thrown into th$\mathscr{K}(L)$ [wkth$\mathscr{K}(L)$] by the base-clauses of this inductive definition; the rules are the induction-clauses.

For 'x' and 'y' replaced as before: $\mathscr{K}$ is x, y-sound iff for any *Pred, Funct,* letting $L_x = L_x(Pred, Funct)$, all members of th$\mathscr{K}(L_x)$ are y-valid and all members of wkth$\mathscr{K}(L_x)$ are weakly y-valid. $\mathscr{K}$ is x, y-complete iff for any *Pred, Funct,* all y-valid sequents of L_x belong to th$\mathscr{K}(L_x)$ and all weakly y-valid such sequents belong to wkth$\mathscr{K}(L_x)$. We'll use these abbreviations when context fixes $\mathscr{K}$ and L_x:

$$\Gamma, \Delta \vdash \varphi \quad : \quad (\Gamma, \Delta, \varphi) \in \text{th}\mathscr{K}(L_x);$$

$$\Gamma, \Delta \vdash^{w} \varphi \quad : \quad (\Gamma, \Delta, \varphi) \in \text{wkth}\mathscr{K}(L_x).$$

We introduce the calculus $\mathscr{K}_y$. First, for 'y' replaced by a blank. Given x, *Pred, Funct,* the axioms of $\mathscr{K}(L_x)$ are those of the following whose formulae belong to Fml(L_x).

(1) $\{\varphi\}, \{\varphi\} \vdash \varphi$.

(2) $\{\ \}, \{\varphi\} \vdash^{w} \varphi$.

(3) $\{\ \}, \{\bot\} \vdash \bot$.

(4) $\{\mathsf{u}\}, \{\mathsf{u}\} \vdash \bot$ (but unnecessary for $\mathsf{x} = 0, \top, \mathsf{u}$, or $1, \top, \mathsf{u}$).

(5) $\{\neg\mathsf{u}\}, \{\neg\mathsf{u}\} \vdash \bot$.

(6) $\{\ \}, \{\ \} \vdash^{w} \tau = \tau$.

(6_s) (6) with '$=_s$' replacing '$=$'.

(6_b) (6) with '$=_b$' replacing '$=$'.

(7) $\{\mathsf{E}(\tau_0), \mathsf{E}(\tau_1)\}, \{\mathsf{E}(\tau_0), \mathsf{E}(\tau_1), \varphi\} \vdash \varphi$, with φ either $\tau_0 = \tau_1$ or $\tau_0 \neq \tau_1$.

(7_s) $\{\mathsf{E}_s(\tau_i)\}, \{\mathsf{E}_s(\tau_i), \varphi\} \vdash \varphi$, with φ either $\tau_0 =_s \tau_1$ or $\tau_0 \neq_s \tau_1$ and $i \in 2$.

(7_b) $\{\ \}, \{\tau_0 =_b \tau_1\} \vdash \tau_0 =_b \tau_1$.

(8) $\{\varphi\}, \{\varphi\} \vdash \mathsf{E}(\tau_i)$, where one of these holds:

(i) φ is of the form $\zeta(\tau_1, \ldots, \tau_n)$ or $\neg\zeta(\tau_1, \ldots, \tau_n)$ and $i < n$;

(ii) φ and i are as in (7);

(iii) φ is $\mathsf{E}(\tau)$ and τ_i is a subterm of τ.

(8_s) $\{\varphi\}, \{\varphi\} \vdash \mathsf{E}_s(\tau_i)$, where one of these holds:

(i) as in (8.i);

(ii) φ is $\tau_0 =_s \tau_1$ and $i \in 2$;

(iii) φ is $\mathsf{E}_s(\tau_i)$ and τ_i is a subterm of τ.

(8_b.i) As in (8_s) where either (8.i) or (8.iii) holds;

(8_b.ii) $\{\tau_0 =_b \tau_1, \mathsf{E}_s(\tau_i)\}, \{\tau_0 =_b \tau_1, \mathsf{E}_s(\tau_i)\} \vdash \mathsf{E}_s(\tau_{i-1})$, for $i < 2$.

(9_s) $\{\tau_0 \neq_s \tau_1\}, \{\tau_0 \neq_s \tau_1, \neg\exists_s(\tau_i)\} \vdash \mathsf{E}_s(\tau_{i-1})$ for $i < 2$.

(9_b) As in (9_s) with '$=_b$' replacing '$=_s$'.

We need the following structural rules.

(Thinning) If $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ and $\Gamma' \subseteq \Delta'$:

$$\frac{\Gamma,\ \Delta \vdash \varphi}{\Gamma', \Delta' \vdash \varphi} \qquad \frac{\Gamma,\ \Delta \vdash^{w} \varphi}{\Gamma', \Delta' \vdash^{w} \varphi}$$

(Weakening) $$\frac{\Gamma, \Delta \vdash \varphi}{\Gamma, \Delta \vdash^{w} \varphi}$$

(Strengthening) $$\frac{\Gamma, \Delta \vdash^{w} \bot}{\Gamma, \Delta \vdash \bot}$$

Now Introduction and Elimination rules for '$\supset$' and 'T'.

(Strong '$\supset$' Elimination) $$\frac{\Gamma, \Delta \vdash \varphi \supset \psi \qquad \Gamma, \Delta \vdash^{w} \varphi}{\Gamma, \Delta \vdash \psi}$$

(Weak '$\supset$' Elimination) $$\frac{\Gamma, \Delta \vdash^{w} \varphi \supset \psi \qquad \Gamma, \Delta \vdash \varphi}{\Gamma, \Delta \vdash^{w} \psi}$$

(Strong '$\supset$' Introduction) $$\frac{\Gamma, \Delta \cup \{\varphi\} \vdash \psi}{\Gamma, \Delta \vdash \varphi \supset \psi}$$

(Weak '$\supset$' Introduction) $$\frac{\Gamma \cup \{\varphi\}, \Delta \cup \{\varphi\} \vdash^{w} \psi}{\Gamma, \Delta \vdash^{w} \varphi \supset \psi}$$

('T' Elimination) $$\frac{\Gamma, \Delta \vdash^{w} \mathsf{T}\varphi}{\Gamma, \Delta \vdash \varphi}$$

('T' Introduction) $$\frac{\Gamma, \Delta \vdash \varphi}{\Gamma, \Delta \vdash \mathsf{T}\varphi}.$$

The Strong and Weak '$\boldsymbol{\supset}$' Elimination rules and the Weak '$\boldsymbol{\supset}$' Introduction rules are obtained from the corresponding rules for '$\supset$' by replacing '$\supset$' with '$\boldsymbol{\supset}$'; in addition we need the following.

(Strong '$\boldsymbol{\supset}$' Introduction)

$$\frac{\begin{array}{l}\Gamma, \Delta \cup \{\varphi\} \vdash \varphi \\ \Gamma, \Delta \cup \{\varphi\} \vdash \varphi \\ \Gamma, \Delta \cup \{\psi\} \vdash \psi\end{array}}{\Gamma, \Delta \vdash \varphi \boldsymbol{\supset} \psi}$$

('$\boldsymbol{\supset}$' Definedness)

$$\frac{\Gamma, \Delta \vdash \varphi \boldsymbol{\supset} \psi}{\Gamma, \Delta \cup \{\varphi\} \vdash \varphi} \qquad \frac{\Gamma, \Delta \vdash \varphi \boldsymbol{\supset} \psi}{\Gamma, \Delta \cup \{\psi\} \vdash \psi}$$

These rules will make our logic classical.

(Strong RAA)

$$\frac{\Gamma, \Delta \cup \{\neg\varphi\} \vdash \bot}{\Gamma, \Delta \vdash \varphi}$$

(Weak RAA)

$$\frac{\Gamma \cup \{\neg\varphi\}, \Delta \cup \{\neg\varphi\} \vdash \bot}{\Gamma \quad \Delta \vdash^{w} \varphi}$$

For '=' $\in$ lex$_x$, $\mathscr{K}(L_x)$ handles '$\exists$' as follows.

(Strong '$\exists$' Elimination)

$$\frac{\begin{array}{l}\Gamma, \Delta \vdash (\exists v)\varphi \\ \Gamma \cup \{\mathsf{E}(v), \varphi\}, \Delta \cup \{\mathsf{E}(v), \varphi\} \vdash \psi\end{array}}{\Gamma, \Delta \vdash \psi}$$

provided v is not free in ψ or in any member of Δ

(Weak '$\exists$' Elimination)

$$\frac{\begin{array}{l}\Gamma, \Delta \vdash^{w} (\exists v)\varphi \\ \Gamma \cup \{\mathsf{E}(v)\}, \Delta \cup \{\mathsf{E}(v), \varphi\} \vdash^{w} \psi\end{array}}{\Gamma, \Delta \vdash^{w} \psi}$$

provided v is as above.

(Strong '$\exists$' Introduction)

$$\frac{\begin{array}{l}\Gamma, \Delta \vdash \varphi(v/\tau) \\ \Gamma, \Delta \vdash \mathsf{E}(\tau)\end{array}}{\Gamma, \Delta \vdash (\exists v)\varphi}$$

provided that τ is substitutible for v in φ.

(Weak '$\exists$' Introduction)

$$\frac{\begin{array}{l}\Gamma, \Delta \vdash^{w} \varphi(v/\tau) \\ \Gamma, \Delta \vdash \mathsf{E}(\tau)\end{array}}{\Gamma, \Delta \vdash^{w} (\exists v)\varphi}$$

provided that τ is as above.

The Strong and Weak '$\exists$' Elimination rules and the Weak '$\exists$' Introduction rules are obtained from the corresponding rules for '$\exists$' by replacing '$\exists$' by '$\exists$'. We also need these further rules.

(Strong '$\exists$' Introduction)

$$\begin{array}{lll}
\Gamma, & \Delta & \vdash \varphi(v/\tau) \\
\Gamma, & \Delta & \vdash \mathsf{E}(\tau) \\
\Gamma \cup \{\mathsf{E}(v), \}, & \Delta \cup \{\mathsf{E}(v), \varphi\} & \vdash \varphi \\
\hline
\Gamma, & \Delta & \vdash (\exists v)\varphi
\end{array}$$

provided that τ is substitutible for v in φ.

('$\exists$' Definedness)

$$\begin{array}{lll}
\Gamma, & \Delta & \vdash (\exists v)\varphi \\
\hline
\Gamma \cup \{\mathsf{E}(v)\}, & \Delta \cup \{\mathsf{E}(v), \varphi\} & \vdash \varphi
\end{array}$$

We also need these rules.

(Strong Congruence)
$$\begin{array}{l}
\Gamma, \Delta \vdash \varphi(v/\tau_0) \\
\Gamma, \Delta \vdash \tau_0 = \tau_1 \\
\hline
\Gamma, \Delta \vdash \varphi(v/\tau_1)
\end{array}$$

(Weak Congruence)
$$\begin{array}{l}
\Gamma, \Delta \vdash^{\mathrm{w}} \varphi(v/\tau_0) \\
\Gamma, \Delta \vdash \tau_0 = \tau_1 \\
\hline
\Gamma, \Delta \vdash^{\mathrm{w}} \varphi(v/\tau_1)
\end{array}$$

(Positive 't' Elimination)
$$\begin{array}{l}
\Gamma, \Delta \vdash \tau = (\mathrm{t}v)\varphi \\
\hline
\Gamma, \Delta \vdash \varphi(v/\tau)
\end{array}$$

provided τ is substitutible for v in φ.

(Negative 't' Elimination)
$$\begin{array}{l}
\Gamma, \Delta \vdash \tau_0 = (\mathrm{t}v)\varphi \\
\Gamma, \Delta \vdash \varphi(v/\tau_1) \\
\hline
\Gamma, \Delta \vdash \tau_0 = \tau_1
\end{array}$$

provided τ_1 is substitutible for v in φ.

('t' Introduction)

$$\begin{array}{lll}
\Gamma \cup \{\varphi(v/\mu), \mathsf{E}(\mu)\}, & \Delta \cup \{\varphi(v/\mu), \mathsf{E}(\mu)\} & \vdash \mu = \tau \\
\Gamma, & \Delta & \vdash \varphi(v/\tau) \\
\hline
\Gamma, & \Delta & \vdash \tau = (\mathrm{t}v)\varphi
\end{array}$$

provided that μ and τ are substitutible for v in φ.

For '$=_s$' $\in$ lex_x replace '=' and 'E' by '$=_s$' and 'E_s' in the preceding rules; similarly for '$=_b$' $\in$ lex_x. This completes our specification of $\mathscr{H}(L_x)$.

Form $\mathscr{H}_{nn}(L_x)$ by adding the appropriate one of these axioms to $\mathscr{H}(L_x)$:

(nn) $\{\ \}, \{\ \} \vdash (\exists v)\mathsf{E}(v)$; $\{\ \}, \{\ \} \vdash (\exists v)\mathsf{E}(v)$;

or else with 'E_s' in place of 'E'. Form $\mathscr{H}_{et}(L_x)$ by adding these axioms or the result of replacing 'E' in them by 'E_s':

(et) $\{\mathsf{E}(\tau_1), \ldots, \mathsf{E}(\tau_n)\}, \{\mathsf{E}(\tau_1), \ldots, \mathsf{E}(\tau_n), \varphi\} \vdash \varphi$

where φ is either $\zeta(\tau_1, \ldots, \tau_n)$ or $\neg\zeta(\tau_1, \ldots, \tau_n)$.

Form $\mathscr{H}_{nn\&et}(L_x)$ by adding both sorts of axioms to $\mathscr{H}(L_x)$,

It's easy to see that $\mathscr{H}_y$ is x, y-sound.

For $\Gamma \subseteq \Delta \subseteq \text{Fml}(L_x)$ let (Γ, Δ) be x, y-inconsistent iff $\Gamma, \Delta \vdash \bot$; otherwise (Γ, Δ) is y-consistent.

Henkin's Lemma. *If (Γ, Δ) is* x, y*-consistent, then there is a* y*-model $\mathscr{A}$ and an $\mathscr{A}$-assignment α so that $\mathscr{A} \models \Gamma[\alpha]$ and $\mathscr{A} \models^w \Delta[\alpha]$.*

This follows by an easy modification of well-known techniques. It yields the y-completeness of $\mathscr{H}_y$, again by a familiar argument. As usual, compactness follows: if $\Gamma, \Delta \vdash \varphi$ $[\Gamma, \Delta \vdash^w \varphi]$ then there are finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ so that $\Gamma', \Delta' \vdash \varphi$ $[\Gamma', \Delta' \vdash^w \varphi]$.

Notice the interplay between theorems and weak theorems in many of these rules. This is unavoidable; with an eye toward natural deduction (which is to say, with an eye toward logic rather than just algebra) we can't consider validity without weak validity, and vice versa. And therein lines the problem with the traditional use of valuational systems for the semantics of three-valued logics. One such system defines validity and another weak-validity, yielding two distinct logics. (Valuational systems were first introduced as a technical device for independence results concerning axiomatizations of logic. I suspect that an unwarrented preoccuption with valuational systems contributed to some of Michael Dummett's views on truth-value gaps; see Section 11.)

5. Ultraproducts and centers

We'll apply the ultraproduct construction to partial models. Given an index set I and for each $i \in I$ a model $\mathscr{A}_i = (|\mathscr{A}_i|, \mathscr{E}_i, \mathscr{N}_i)$, let:

$$\prod_{i \in I} |\mathscr{A}_i| = \{f : f \text{ is a function on } I \text{ and for each } i \in I, f(i) \in |\mathscr{A}_i|\}.$$

For D an ultrafilter on I and $f, g \in \prod_{i \in I} |\mathscr{A}_i|$, let:

$$f \approx_D g \quad \text{iff} \quad \{i: f(i) \in g(i)\} \in D;$$

$$f_D = \{g: f \approx_D g\};$$

$$\left|\prod_D \mathscr{A}_i\right| = \left\{f_D: f \in \prod_{i \in I} |\mathscr{A}_i|\right\}.$$

For $\zeta \in \text{PRED}(n)$ and $j \in 2$ let:

$$\mathscr{E}'(\zeta)(f_{0,D}, \ldots, f_{n-1,D}) = j$$
$$\text{iff} \quad \{i: \mathscr{E}_i(\zeta)(f_0(i), \ldots, f_{n-1}(i)) = j\} \in D;$$

otherwise $\mathscr{E}'(\zeta)(f_{0,D}, \ldots, f_{n-1,D})\uparrow$. For $\zeta \in \text{FUNCT}(n)$, if $n = 0$, let $\mathscr{N}'(\xi) = f_D$ iff $\{i: \mathscr{N}_i(\xi) = f(i)\} \in D$ for any $f \in \prod_{i \in I} |\mathscr{A}_i|$; if $n \geq 1$, let

$$\mathscr{N}'(\xi)(f_{0,D}, \ldots, f_{n-1,D}) = f_D$$
$$\text{iff} \quad \{i: \mathscr{N}_i(\xi)(f_0(i), \ldots, f_{n-1}(i)) = f(i)\} \in D,$$

again for any $f \in \prod_{i \in I} |\mathscr{A}_i|$. Note that $\mathscr{E}'$ and $\mathscr{N}'$ are well-defined, with $\text{dom}(\mathscr{E}') = \text{PRED}$ and $\bigcup_{1 \leq n < \omega} \text{FUNCT}(n) \subseteq \text{dom}(\mathscr{N}')$. For any $\zeta \in \text{PRED}$ let $\mathscr{E}(\zeta) = \mathscr{E}'(\xi)$ if $\mathscr{E}'(\zeta) \neq \{\ \}$, and otherwise $\mathscr{E}(\zeta)\uparrow$. For any $\xi \in \text{FUNCT}(n)$, if $n = 0$, let $\mathscr{N}(\xi) \simeq \mathscr{N}'(\xi)$; if $n \geq 1$ and $\text{dom}(\mathscr{N}'(\xi)) \neq \{\ \}$, let $\mathscr{N}(\xi) = \mathscr{N}'(\xi)$, and otherwise $\mathscr{N}(\xi)\uparrow$. Thus $\text{dom}(\mathscr{E})$ and $\text{dom}(\mathscr{N})$ are sets. Let $\prod_D \mathscr{A}_i$ be $(|\prod_D \mathscr{A}_i|, \mathscr{E}, \mathscr{N})$. Suppose that for each $i \in I$, α_i is an $\mathscr{A}_i$-assignment. For any variable v and $f \in \prod_{i \in I} |\mathscr{A}_i|$ let $(\prod_D \alpha_i)(v) = f_D$ iff $\{i: \alpha_i(v) = f(i)\} \in D$. So $\prod_D \alpha_i$ is a $\prod_D \mathscr{A}_i$-assignment.

Łos' fundamental theorem on ultraproducts (Theorem 4.1.9 of [1]) easily carries over to partial models as follows. If for each $i \in I$, $\mathscr{A}_i$ is a model for *Pred*, *Funct* and $\varphi \in \text{Fml}(L_\kappa(\mathit{Pred}, \mathit{Funct}))$:

$$\prod_D \mathscr{A}_i \models \varphi\left[\prod_D \alpha_i\right] \quad \text{iff} \quad \{i: \mathscr{A}_i \models \varphi[\alpha_i]\} \in D;$$

$$\prod_D \mathscr{A}_i \dashv \varphi\left[\prod_D \alpha_i\right] \quad \text{iff} \quad \{i: \mathscr{A}_i \dashv \varphi[\alpha_i]\} \in D;$$

The ultraproduct compactness corollary to Los' theorem (Corollary 4.1.11 of [1]) also carries over as follows. For any set X let $S_\omega(X) = \{Y: Y \subseteq X \text{ and } Y \text{ is finite}\}$. Suppose $\Sigma_0 \subseteq \Sigma_1$ are sets of formulae, $I = S_\omega(\Sigma_1)$ and for each $i \in I$ there is a model $\mathscr{A}_i$ with $\mathscr{A}_i \models^w i$ and $\mathscr{A}_i \models i \cap \Sigma_0$. Then there is an ultraproduct D on I so that $\prod_D \mathscr{A}_i \models^w \Sigma_1$ and $\prod_D \mathscr{A}_i \models \Sigma_0$. The usual D such that for each $\sigma \in \Sigma_1$, $\hat{\sigma} = \{i: i \in I$ and $\sigma \in i\} \in D$ does the trick, because

$$\text{if } \sigma \in \Sigma_1, \quad \text{then } \hat{\sigma} \subseteq \{i \in I: \mathscr{A}_i \models^w \sigma\};$$

$$\text{if } \sigma \in \Sigma_0, \quad \text{then } \hat{\sigma} \subseteq \{i \in I: \mathscr{A}_i \models \alpha\}.$$

Consider models $\mathscr{A}_i = (A, \mathscr{E}_i, \mathscr{N}_i)$ for $i \in 2$. For $\zeta \in \text{PRED}(n)$ let $\mathscr{A}_1$ ζ-widen $\mathscr{A}_0$ iff $\mathscr{N}_0 = \mathscr{N}_1$ and:

for any $\zeta' \in \text{PRED}$ if $\zeta' \neq \zeta$, then $\mathscr{E}_0(\zeta') \simeq \mathscr{E}_1(\zeta')$;

$\mathscr{E}_0(\zeta)\uparrow$; $\quad \mathscr{E}_1(\zeta)\downarrow$.

For $\xi \in \text{FUNCT}(n)$ let $\mathscr{A}_1$ ξ-widen $\mathscr{A}_0$ iff $\mathscr{E}_0 = \mathscr{E}_1$ and:

for any $\xi' \in \text{FUNCT}$ if $\xi' \neq \xi$, then $\mathscr{N}_0(\xi') \simeq \mathscr{N}_1(\xi')$;

$\mathscr{N}_0(\xi)\uparrow$; $\quad \mathscr{N}_1(\xi)\downarrow$.

For $K \subseteq \text{MOD}$ we adopt these definitions.

$$Pred(K) = \{\zeta \in \text{PRED}: \text{there are models } \mathscr{A}, \mathscr{B} \text{ so that } \mathscr{A}\ \zeta\text{-widens } \mathscr{B} \text{ and: } \mathscr{A} \in K \text{ iff } \mathscr{B} \notin K\}.$$

Define $Funct(K)$ similarly. $\text{Center}(K) = Pred(K) \cup Funct(K)$; K is bounded iff $\text{Center}(K)$ is a set (i.e. not a proper class); $L_x(K) = L_x(Pred(K), Funct(K))$.

For any class $\mathscr{F} \subseteq \text{PRED} \cup \text{FUNCT}$ and any model $\mathscr{A}$ let $\mathscr{A} \restriction \mathscr{F} = (|\mathscr{A}|, \mathscr{E} \restriction \mathscr{F}, \mathscr{N} \restriction \mathscr{F})$.

Theorem 1. *If* $K \subseteq \text{MOD}$ *is closed under isomorphism, ultraporoducts and* $\text{MOD} - K$ *is closed under ultrapowers, then* $\text{Center}(K)$ *is the minimal* $\mathscr{F} \subseteq \text{PRED} \cup \text{FUNCT}$ *so that for all* $\mathscr{A} \in \text{Mod}$: $\mathscr{A} \in K$ *iff* $\mathscr{A} \restriction \mathscr{F} \in K$.

Proof. Assume that K meets the stated closure conditions. Let $\mathscr{A} = (|\mathscr{A}|, \mathscr{E}, \mathscr{N})$ be a model. For κ a cardinal let $\langle \gamma_\iota \rangle_{\iota<\kappa}$ be a listing of $(\text{dom}(\mathscr{E}) \cup \text{dom}(\mathscr{N})) - \text{Center}(K)$.

Assume that $\mathscr{A} \restriction \text{Center}(K) \in K$. For $\iota \leqslant \kappa$ let $\mathscr{A}_\iota = \mathscr{A} \restriction \{\gamma_{\iota'}: \iota \leqslant \iota' < \kappa\}$; so $\mathscr{A}_{\iota+1}$ γ_ι-widens $\mathscr{A}_\iota$. Claim: for each $\iota < \kappa$, $\mathscr{A}_\iota \in K$. If $\mathscr{A}_\iota \in K$ but $\mathscr{A}_{\iota+1} \notin K$, then $\gamma_\iota \in \text{Center}(K)$ contrary to choice of γ_ι. If $\mathscr{A}$ is finite, we're done. Otherwise suppose $\lambda \leqslant \kappa$ is a limit ordinal and for all $\iota < \lambda$, $\mathscr{A}_\iota \in K$. $E = \{\lambda - \iota: \iota < \lambda\}$ is a filter on λ with the finite intersection property; let D be an ultrafilter on λ with $D \subseteq E$; we have $\prod_D \mathscr{A}_{\iota<\lambda} \in K$. But $\prod_D \mathscr{A}_\lambda$ $(= \prod_D \mathscr{B}_\iota$ taking $\mathscr{B}_\iota = \mathscr{A}_\lambda$ for each $\iota < \lambda)$ is isomorphic to $\prod_D \mathscr{A}_{\iota<\lambda}$. If $\mathscr{A}_\lambda \notin K$, then $\prod_D \mathscr{A}_\lambda \notin K$; so $\mathscr{A}_\lambda \in K$. Thus the claim; in particular $\mathscr{A}_\kappa = \mathscr{A} \in K$.

Assume that $\mathscr{A} \in K$. Let $\mathscr{A}_\iota = \mathscr{A} \restriction \{\iota': \iota' < \iota\}$ for $i < \kappa$. So $\mathscr{A} = \mathscr{A}_0$. As above we show that for all $\iota \leqslant \kappa$, $\mathscr{A}_\iota \in K$. In particular, $\mathscr{A}_\kappa = \mathscr{A} \restriction \text{Center}(K) \in K$.

Finally, Suppose that $\mathscr{F} \subseteq \text{PRED} \cup \text{FUNCT}$ and for all models $\mathscr{A}$: $\mathscr{A} \in K$ iff $\mathscr{A} \restriction \mathscr{F} \in K$. Suppose $\gamma \in \text{Center}(K)$ as witnessed by models $\mathscr{A}$ and $\mathscr{B}$. If $\gamma \notin \mathscr{F}$, $\mathscr{A} \restriction \mathscr{F} = \mathscr{B} \restriction \mathscr{F}$. We have: $\mathscr{A} \in K$ iff $\mathscr{A} \restriction \mathscr{F} \in K$, $\mathscr{B} \restriction \mathscr{F} \in K$ iff $\mathscr{B} \in K$; so $\mathscr{A} \in K$ iff $\mathscr{B} \in K$, contrary to choice of $\mathscr{A}$ and $\mathscr{B}$. So $\gamma \in \mathscr{F}$. So $\text{Center}(K) \subseteq \mathscr{F}$.

6. On x-elementary and weakly x-elementary classes

For the rest of this paper, replace 'x' by any of our subscripts for 'lex' so that '⊤' $\notin \text{lex}_x$. For $\mathscr{A} \in \text{MOD}$ and $K \subseteq \text{MOD}$ let:

$$\text{Th}_x(\mathscr{A}) = \{\varphi: \varphi \in \text{Sent}(\mathbf{L}_x) \text{ and } \mathscr{A} \models \varphi\};$$

$$\text{WkTh}_x(\mathscr{A}) = \{\varphi: \varphi \in \text{Sent}(\mathbf{L}_x) \text{ and } \mathscr{A} \models^w \varphi\};$$

$$\text{Th}_x(K) = \bigcap \{\text{Th}_x(\mathscr{A}): \mathscr{A} \in K\};$$

$$\text{WkTh}_x(K) = \bigcap \{\text{WkTh}_x(\mathscr{A}): \mathscr{A} \in K\}.$$

Note that these classes can be proper.

For $x = i, \top, \ldots,$ $\text{Th}_x(\mathscr{A}) \subseteq \text{Th}_x(\mathscr{B})$ iff $\text{Th}_x(\mathscr{A}) = \text{Th}_x(\mathscr{B})$; for if $\varphi \notin \text{Th}_x(\mathscr{A})$ then $\neg\top\varphi \in \text{Th}_x(\mathscr{A})$, and so if the left-side holds $\neg\top\varphi \in \text{Th}_x(\mathscr{B})$, yielding $\varphi \notin \text{Th}_x(\mathscr{B})$. Furthermore, for $S \subseteq \text{Sent}(\mathbf{L}_x)$ closed under '¬' and any $\mathscr{A}, \mathscr{B} \in$ MOD:

$$\text{Th}_x(\mathscr{A}) \cap S \subseteq \text{Th}_x(\mathscr{B}) \quad \text{iff} \quad \text{WkTh}_x(\mathscr{B}) \cap S \subseteq \text{WkTh}_x(\mathscr{A}).$$

Let K be S-upward$_x$ [S-downward$_x$] closed iff for all $\mathscr{A}$, $\mathscr{B} \in \text{MOD}$ if $\text{Th}_x(\mathscr{A}) \cap S \subseteq \text{Th}_x(\mathscr{B})$ then:

$$\text{if } \mathscr{A} \in K, \text{ then } \mathscr{B} \in K \qquad [\text{if } \mathscr{B} \in K, \text{ then } \mathscr{A} \in K];$$

For $S = \text{Sent}(\mathbf{L}_x)$ we delete mention of S.

For $\Gamma \subseteq \text{Sent}(\mathbf{L}_x)$: Γ defines [weakly defines] K iff: for all $\mathscr{A} \in \text{MOD}$, $\mathscr{A} \in K$ iff $\mathscr{A} \models \Gamma$ [$\mathscr{A} \models^w \Gamma$]. K is x-elementary [weakly x-elementary] iff some $\Gamma \subseteq \text{Sent}(\mathbf{L}_x)$ defines [weakly defines] K; K is basic x-elementary [weakly basic x-elementary] iff for some $\varphi \in \text{Sent}(\mathbf{L}_x)$, $\{\varphi\}$ defines [weakly defines] K, in which case we'll say that φ defines [weakly defines] K.

Lemma 4. *Let Pred and Funct be given. For any* $\varphi \in \text{Sent}(\mathbf{L}_x)$ *there is a* $\varphi^+ \in \text{Sent}(L_x(\textit{Pred}, \textit{Funct}))$ *so that for any* $\mathscr{A} \in \text{MOD}$:

$$\mathscr{A} \restriction (\textit{Pred} \cup \textit{Funct}) \models \varphi \quad \textit{iff} \quad \mathscr{A} \restriction (\textit{Pred} \cup \textit{Funct}) \models \varphi^+.$$

Given φ form φ' by replacing atomic subformulae of φ as follows: replace an atomic formula of the form $\zeta(\cdots)$ or $\tau_0 = \tau_1$ containing a constant not in *Pred* ∪ *Funct* by 'u'; replace $\tau_0 =_s \tau_1$ [$\tau_0 =_b \tau_1$] such that τ_0 and τ_1 each contain a function-constant not in *Pred* ∪ *Funct* by 'u' ['¬⊥']; if τ_i contains a function-constant not in *Pred* ∪ *Funct* and τ_{1-i} does not, replace $\tau_0 =_s \tau_1$ [$\tau_0 =_b \tau_1$] by $\neg E_s(\tau_{1-i}) \vee$ u [$\neg E_s(\tau_{1-i})$]. Easily $\mathscr{A} \restriction (\textit{Pred} \cup \textit{Funct}) \models \varphi$ iff $\mathscr{A} \restriction (\textit{Pred} \cup \textit{Funct}) \models \varphi'$, and all constants in φ' belong to *Pred* ∪ *Funct*; so if 'u' $\in \text{lex}_x$ let φ^+ be φ'. Otherwise we select $\varphi^+ \in \text{Sent}(L_x(\textit{Pred}, \textit{Funct}))$ equivalent$^+$ to φ', using Observation 3(i).

Theorem 2 (weak part). *Suppose* $K \subseteq \mathrm{MOD}$ *is bounded. The following are equivalent*:

(i^{w}) *K is weakly* x-*elementary.*

(ii^{w}) *K is weakly defined by* $\mathrm{WkTh}_{\mathsf{x}}(K) \cap \mathrm{Sent}(L_{\mathsf{x}}(K))$.

($\mathrm{iii}^{\mathrm{w}}$) *K is closed under isomorphism, ultraproducts and is downward*$_{\mathsf{x}}$ *closed and* MOD $-$ *K is closed under ultrapowers.*

Clearly (ii^{w}), implies (i^{w}). It's easy to see that (ii^{w}) implies ($\mathrm{iii}^{\mathrm{w}}$).

To show: ($\mathrm{iii}^{\mathrm{w}}$) implies ($\mathrm{ii}^{\mathrm{w}}$). Assume ($\mathrm{iii}^{\mathrm{w}}$). Suppose $\mathscr{B} \in \mathrm{MOD}$, $\mathscr{B} \vDash^{\mathrm{w}} \mathrm{WkTh}_{\mathsf{x}}(K) \cap \mathrm{Sent}(L_{\mathsf{x}}(K))$. Letting $\mathscr{B}' = \mathscr{B} \restriction \mathrm{Center}(K)$, $\mathscr{B}' \in K$ iff $\mathscr{B} \in K$, by Theorem 1. Let $I = S_{\omega}(\mathrm{Th}_{\mathsf{x}}(\mathscr{B}') \cap \mathrm{Sent}(L_{\mathsf{x}}(K))$; since K is bounded, I is a set. Claim: for each $i \in I$ there is an $\mathscr{A}_i \in K$ so that $\mathscr{A}_i \vDash i$. Otherwise fix an i so that for every $\mathscr{A} \in K$: $\mathscr{A} \nvDash i$, so $\mathscr{A} \vDash^{\mathrm{w}} \neg\&i$; so $\neg\&i \in \mathrm{WkTh}_{\mathsf{x}}(K) \cap \mathrm{Sent}(L_{\mathsf{x}}(K))$; so $\mathscr{B} \vDash^{\mathrm{w}} \neg\&i$; so $\mathscr{B} \vDash i$ by choice of I, a contradiction. Let $\mathscr{A}_i' = \mathscr{A}_i \restriction \mathrm{Center}(K)$; again $\mathscr{A}_i' \in K$. By the corollary to Los' theorem we select an ultrafilter D on I so that $\prod_D \mathscr{A}_i' \vDash \mathrm{Th}_{\mathsf{x}}(\mathscr{B}') \cap \mathrm{Sent}(L_{\mathsf{x}}(K))$. We have $\prod_D \mathscr{A}_i' \in K$. Claim: $\mathrm{Th}_{\mathsf{x}}(\mathscr{B}') \subseteq \mathrm{Th}_{\mathsf{x}}(\prod_D \mathscr{A}_i')$. For $\varphi \in \mathrm{Th}_{\mathsf{x}}(\mathscr{B}')$ form φ^+ as in Lemma 4, taking $Pred = Pred(K)$, $Funct = Funct(K)$. So $\varphi^+ \in \mathrm{Th}_{\mathsf{x}}(\mathscr{B}') \cap \mathrm{Sent}(L_{\mathsf{x}}(K))$; since $(\prod_D \mathscr{A}_i') \restriction \mathrm{Center}(K) = \prod_D \mathscr{A}_i'$, $\prod_D \mathscr{A}_i' \vDash \varphi$ iff $\prod_D \mathscr{A}_i' \vDash \varphi^+$. But D was selected to contain $X = \{i: \varphi^+ \in i\}$; so $\prod_D \mathscr{A}_i' \vDash \varphi^+$; so $\prod_D \mathscr{A}_i' \vDash \varphi$, establishing the claim. So by downward$_{\mathsf{x}}$ closure $\mathscr{B}' \in K$; so $\mathscr{B} \in K$, which suffices for (ii^{w}).

The strong part of Theorem 2 requires further definitions. For $\mathscr{A} \in \mathrm{MOD}$, α an $\mathscr{A}$-assignment and θ a formula, α is total for θ iff for every variable v occurring free in θ $\alpha(v)\downarrow$. θ is $\mathscr{A}$-bivalent iff for every $\mathscr{A}$-assignment α, if α is total for θ then either $\mathscr{A} \vDash \theta[\alpha]$ or $\mathscr{A} \dashv \theta[\alpha]$. For a class Δ of formula, Δ is $\mathscr{A}$-bivalent iff for every $\theta \in \Delta$, θ is $\mathscr{A}$-bivalent. The following definition will be of use only for $\mathsf{x} = 0, \ldots$. Let:

$$\mathrm{Core}_{\mathsf{x}}(K) = \{\theta: \theta \in \mathrm{AtFml}(\mathbb{L}_{\mathsf{x}}) \text{ and for all } \mathscr{A} \in K\ \theta \text{ is } \mathscr{A}\text{-bivalent}\}.$$

$$C_{\mathsf{x}}(K) = \{\varphi \in \mathrm{Fml}(\mathbb{L}_{\mathsf{x}}): \text{every exposed subformula of } \varphi \text{ belongs to } \mathrm{Core}_{\mathsf{x}}(K)\}.$$

Clearly for any $\mathscr{A} \in K$, $\mathrm{Core}_{\mathsf{x}}(K)$ is $\mathscr{A}$-bivalent. Also, if '$\top$' $\notin \mathrm{lex}_{\mathsf{x}}$, then $C_{\mathsf{x}}(K)$ is the class of formula of $\mathbb{L}_{\mathsf{x}}$ generated from $\mathrm{Core}_{\mathsf{x}}(K)$. Also, if $K \neq \{\ \}$, then 'u' $\notin \mathrm{Core}_{\mathsf{x}}(K)$; and so for $\varphi \in C_{\mathsf{x}}(K)$ no occurrence of 'u' in φ is exposed.

For '=' $\in \mathrm{lex}_{\mathsf{x}}$ and $K \neq \{\ \}$, $\mathrm{Core}_{\mathsf{x}}(K) \subseteq \mathrm{AtFml}(L_{\mathsf{x}}(K))$. For suppose $\theta \in \mathrm{Core}_{\mathsf{x}}(K)$. For any function- or predicate-constant γ occurring in θ, pick any $\mathscr{A} \in K$, any θ-total $\mathscr{A}$-assignment α, and any $\mathscr{B} \in \mathrm{MOD}$ so that $\mathscr{A}$ γ-widens $\mathscr{B}$; then $\mathscr{B} \mid \theta[\alpha]$; so $\mathscr{B} \notin K$; so $\gamma \in \mathrm{Center}(K)$; so $\theta \in \mathrm{AtFml}(L_{\mathsf{x}}(K))$. For other x this doesn't hold, e.g. for '$\mathbf{a}$', '$\mathbf{b}$' $\in \mathrm{FUNCT}(0)$ let '$\mathrm{E}_{\mathrm{s}}(\mathbf{a})$' define K; '$\mathbf{a} =_{\mathrm{s}} \mathbf{b}$' $\in \mathrm{Core}_{0,\mathrm{s}}(K)$ though '$\mathbf{b}$' $\notin \mathrm{Center}(K)$.

Lemma 5. *If* $\mathrm{Core}_{\mathsf{x}}(K)$ *is* $\mathscr{A}$*-bivalent and* $\varphi \in C_{\mathsf{x}}(K)$, *then* φ *is* $\mathscr{A}$*-bivalent.* Proof by induction on the construction of φ.

Lemma 6. *For* $\mathsf{x}=0,\ldots$ $\mathrm{Th}_{\mathsf{x}}(K)\subseteq C_{\mathsf{x}}(K)$.

Suppose $\varphi\notin C_{\mathsf{x}}(K)$. There is an exposed occurrence of some $\theta\notin \mathrm{Core}(K)$ in φ. Pick a model $\mathscr{A}\in K$ and an $\mathscr{A}$-assignment total for θ so that $\mathscr{A}\mid\theta\,[\alpha]$. By Lemma 2, $\mathscr{A}\mid\varphi$; so $\varphi\notin\mathrm{Th}_{\mathsf{x}}(K)$.

Note: for $\mathsf{x}=0,\ \top,\ldots$ K is $C_{\mathsf{x}}(K)$-downward$_{\mathsf{x}}$ closed iff K is downward$_{\mathsf{x}}$ closed. Left to right is trivial; from right to left it suffices to note that if $\mathrm{Th}_{\mathsf{x}}(\mathscr{A})\cap C_{\mathsf{x}}(K)\subseteq\mathrm{Th}_{\mathsf{x}}(\mathscr{B})$, then $\mathrm{Th}_{\mathsf{x}}(\mathscr{A})\subseteq\mathrm{Th}_{\mathsf{x}}(\mathscr{B})$: if $\varphi\in\mathrm{Th}_{\mathsf{x}}(\mathscr{A})$ then $\top\varphi\in\mathrm{Th}_{\mathsf{x}}(\mathscr{A})\cap C_{\mathsf{x}}(K)$.

Theorem 2.1 (strong part). *For* $K\subseteq\mathrm{MOD}$ *and bounded, the following are equivalent.*

(i) *K is* x*-elementary.*

(ii) *K is defined by* $\mathrm{Th}_{\mathsf{x}}(K)\cap\mathrm{Sent}(L_{\mathsf{x}}(K))$.

(iii) *K is closed under isomorphism and ultraproducts,* $\mathrm{MOD}-K$ *is closed under ultrapowers and*

(.i) *if* $\mathsf{x}=1,\ldots$ *then K is upward$_{\mathsf{x}}$ closed;*

(.ii) *if* $\mathsf{x}=0,\ldots$ *then K is $C_{\mathsf{x}}(K)$-upward$_{\mathsf{x}}$ closed.*

Clearly (ii) implies (i). First suppose that $\mathsf{x}=1,\ldots$. Clearly (i) implies (iii). Assume (iii). Suppose that $\mathscr{B}\in\mathrm{MOD}$ and $\mathscr{B}\vDash\mathrm{Th}_{\mathsf{x}}(K)\cap\mathrm{Sent}(L_{\mathsf{x}}(K))$. Again let $\mathscr{B}'=\mathscr{B}\restriction$Center (K). Let $I=S_{\omega}(\mathrm{WkTh}_{\mathsf{x}}(\mathscr{B}')\cap\mathrm{Sent}(L_{\mathsf{x}}(K))$; since K is bounded I is a set. Claim: for each $i\in K$ there is an $\mathscr{A}_i\in K$ so that $\mathscr{A}_i\vDash^{\mathrm{w}} i$. Suppose not; fix an i so that for all $\mathscr{A}\in K$, $\mathscr{A}\nvDash^{\mathrm{w}} i$. Then for $\mathscr{A}\in K$, $\mathscr{A}\vDash\neg\&i$; so $\neg\&i\in\mathrm{Th}_{\mathsf{x}}(K)\cap\mathrm{Sent}(L_{\mathsf{x}}(K))$; so $\mathscr{B}\vDash\neg\&i$; since $\mathscr{B}\vDash^{\mathrm{w}} i$, this is a contradiction. Let $\mathscr{A}_i'=\mathscr{A}_i\restriction$Center$(K)$; so $\mathscr{A}_i'\in K$. By the compactness corollary to Los' theorem we find an ultrafilter D on I with $\prod_D\mathscr{A}_i'\vDash^{\mathrm{w}}\mathrm{WkTh}_{\mathsf{x}}(\mathscr{B}')\cap\ \mathrm{Sent}(L_{\mathsf{x}}(K))$. Also $\prod_D\mathscr{A}_i'\in K$. Claim: $\mathrm{WkTh}_{\mathsf{x}}(\mathscr{B}')\subseteq\mathrm{WkTh}_{\mathsf{x}}(\prod_D\mathscr{A}_i')$. For $\varphi\in\mathrm{WkTh}_{\mathsf{x}}(\mathscr{B}')$, let φ^- be $(\neg\varphi)^+$ formed by applying Lemma 4 to $\neg\varphi$. Then $\neg\varphi^-\in\mathrm{WkTh}_{\mathsf{x}}(\prod_D\mathscr{A}_i')$, yielding $\varphi\in\mathrm{WkTh}_{\mathsf{x}}(\prod_D\mathscr{A}_i')$, arguing as in the proof that (iii$^{\mathrm{w}}$) yields (ii$^{\mathrm{w}}$). So $\mathrm{Th}_{\mathsf{x}}(\prod_D\mathscr{A}_i')\subseteq\mathrm{Th}_{\mathsf{x}}(\mathscr{B}')$. By upward$_{\mathsf{x}}$ closure, $\mathscr{B}'\in K$, establishing (ii).

Now suppose that $\mathsf{x}=0,\ldots$. If Γ defines K, then $\Gamma\subseteq\mathrm{Th}_{\mathsf{x}}(K)$, and so by Lemma 6, $\Gamma\subseteq C_{\mathsf{x}}(K)$; with this, (i) implies (iii) as above. Now assume (iii). Consider $\mathscr{B}$ as above. Let $I=S_{\omega}(\mathrm{WkTh}_{\mathsf{x}}(\mathscr{B}')\cap C_{\mathsf{x}}(K)\cap\mathrm{Sent}(L_{\mathsf{x}}(K)))$. Claim: for each $i\in I$ there is an $\mathscr{A}_i\in K$ so that $\mathscr{A}_i\vDash^{\mathrm{w}} i$. Suppose not; fix an i so that for all $\mathscr{A}\in K$, $\mathscr{A}\nvDash^{\mathrm{w}} i$. For any $\mathscr{A}\in K$ and $\sigma\in i$ either $\mathscr{A}\vDash\sigma$ or $\mathscr{A}\ \dashv\ \sigma$, by Lemma 5. Thus $\mathscr{A}\vDash\neg\&i$; so $\neg\&i\in\mathrm{Th}_{\mathsf{x}}(K)\cap\mathrm{Sent}(L_{\mathsf{x}}(K))$; so $\mathscr{B}\vDash\neg\&i$; but $\mathscr{B}\vDash^{\mathrm{w}} i$, for a contradiction. Forming $\mathscr{A}_i'$ as above, we get $\mathrm{WkTh}_{\mathsf{x}}(\mathscr{B}')\cap C_{\mathsf{x}}(K)\subseteq\mathrm{WkTh}_{\mathsf{x}}(\prod_D\mathscr{A}_i')$, and so $\mathrm{Th}_{\mathsf{x}}(\prod_D\mathscr{A}_i')\cap C_{\mathsf{x}}(K)\subseteq\mathrm{Th}_{\mathsf{x}}(\mathscr{B}')$, as above. Since K is $C_{\mathsf{x}}(K)$-upward$_{\mathsf{x}}$ closed and $\prod_D\mathscr{A}_i'\in K$, $\mathscr{B}'\in K$.

Note: K is upward$_{1,\top,\ldots}$ [downward$_{1,\top,\ldots}$] closed iff K is $C_{0,\top,\ldots}(K)$-upward$_{0,\top,\ldots}$ [$C_{0,\top,\ldots}(K)$-downward$_{0,\top,\ldots}$] closed; this by Observations 1(i) and 3(ii).

Corollary. *If K is bounded and* x*-elementary [weakly* x*-elementary], then K is defined [weakly defined] by a set of sentences.*

Theorem 3. *For $K \subseteq$ MOD and bounded, the following are equivalent.*

(i) *K is basic* x*-elementary.*

(ii) *Some $\varphi \in \mathrm{Sent}(L_x(K))$ defines K.*

(iii) *Some finite $\Gamma \subseteq \mathrm{Sent}(L_x(K))$ defines K.*

(iv) *K and* MOD $-$ *K are closed under isomorphism and ultraproducts, and*

(.i) *if $x = 1, \ldots$ then K is upward$_x$-closed;*

(.ii) *if $x = 0, \ldots$ then K is $C_x(K)$-upward$_x$ closed.*

For $x = 1, \ldots$ or $0, \top, \ldots$ the following are also equivalent:

(i^w) *K is basic weakly* x*-elementary.*

(ii^w) *Some $\varphi \in \mathrm{Sent}(L_x(K))$ weakly defines K.*

(iii^w) *Some finite $\Gamma \subseteq \mathrm{Sent}(L_x(K))$ weakly defines K.*

(iv^w) *K and* MOD $-$ *K are closed under isomorphism and ultraproducts, and K is downward$_x$ closed.*

For x *otherwise these are equivalent:* (i^w); (ii^w);

(v^w) *K and* MOD $-$ *K are closed under isomorphism and ultraproducts and K is $C_x(\mathrm{MOD} - K)$-downward$_x$ closed.*

Clearly (ii) implies (i) and (ii^w) implies (i^w). Taking φ to be $\&\Gamma$ or $\&\Gamma$, (ii) is equivalent to (iii); taking φ to be $\&\Gamma$ or $\&\{\neg\mathsf{F}\sigma : \sigma \in \Gamma\}$ for $x = 1, \ldots$ or $0, \top, \ldots$ (ii^w) is equivalent to (iii^w). It's not hard to see that: (i) implies (iv); for $x = 1, \ldots$ or $0, \top, \ldots$ (i^w) implies (iv^w); and for x otherwise (i^w) implies (v^w), using the fact that if φ weakly defines K, then $\neg\varphi$ defines MOD $- K$.

To show: (iv) implies (iii). First note that $\mathrm{Center}(K) = \mathrm{Center}(\mathrm{MOD} - K)$, and so $L_x(K) = L_x(\mathrm{MOD} - K)$. Assume (iv). MOD $- K$ is downward$_x$ closed because K is upward$_x$ closed. By Theorem 2, $\mathrm{Th}_x(K) \cap \mathrm{Sent}(L_x(K))$ defines K and $\mathrm{WkTh}_x(\mathrm{MOD} - K) \cap \mathrm{Sent}(L_x(K))$ weakly defines MOD $- K$. Thus

$$(\mathrm{Th}_x(K) \cap \mathrm{Sent}(L_x(K)), \mathrm{WkTh}_x(\mathrm{MOD} - K) \cap \mathrm{Sent}(L_x(K)))$$

is inconsistent. So by compactness there is a finite $\Gamma \subseteq \mathrm{Th}_x(K) \cap \mathrm{Sent}(L_x(K))$ so that $(\Gamma, \mathrm{WkTh}_x(\mathrm{MOD} - K) \cap \mathrm{Sent}(L_x(K)))$ is inconsistent. If $\mathscr{A} \vDash \Gamma$ then $\mathscr{A} \in K$, for otherwise $\mathscr{A} \vDash^w \mathrm{WkTh}_x(\mathrm{MOD} - K) \cap \mathrm{Sent}(L_x(K))$. So Γ defines K.

To show: for $x = 1, \ldots$ or $0, \top, \ldots$ (iv^w) implies (ii^w). Assume (iv^w). MOD $- K$ is upward$_x$ closed because K is downward$_x$ closed. Then with K and MOD $- K$ switching places, (iv) holds; so by the preceding we obtain a φ defining MOD $- K$; so $\neg\varphi$ weakly defines K.

To show: for x otherwise (v^w) implies (ii^w). Because K is $C_x(\mathrm{MOD} - K)$-downward$_x$ closed, it's downward$_x$ closed; so the previous argument applies.

For 'P', 'Q' $\in$ PRED(0), let {'P', 'Q'} weakly define K; then K is not weakly basic$_0$-elementary; in particular 'P&Q' doesn't weakly define K. Notice that $\mathrm{Core}_0(\mathrm{MOD} - K) = \{`\bot'\} \cup \{v = v\colon v \in \mathit{Var}\}$; so for any $\mathscr{A} \in \mathrm{MOD}$, $\mathrm{Th}_0(\mathscr{A}) \cap C_0(\mathrm{MOD} - K) \subseteq \mathrm{Th}_0(\mathscr{B})$; so K is not $C_0(\mathrm{MOD} - K)$-downward$_0$ closed.

7. On x-elementary class-pairs

For $K_0, K_1 \subseteq \mathrm{MOD}$, the ordered pair (K_0, K_1) will not exist if K_0 or K_1 is proper; since everything we'll say in terms of such ordered pairs can be said without mentioning them, we'll permit ourselves to speak of such pairs in spite of their non-existence. We adopt these definitions.

$$Pred(K_0, K_1) = Pred(K_0) \cup Pred(K_1);$$

$$Funct(K_0, K_1) = Funct(K_0) \cup Funct(K_1);$$

$$\mathrm{Center}(K_0, K_1) = \mathrm{Center}(K_0) \cup \mathrm{Center}(K_1);$$

$$L_x(K_0, K_1) = L_x(Pred(K_0, K_1), Funct(K_0, K_1)).$$

(K_0, K_1) is bounded iff $\mathrm{Center}(K_0, K_1)$ is a set, i.e. iff K_0 and K_1 are bounded.

(K_0, K_1) is center-bivalent [b-center-bivalent] iff $K_1 - K_0$ contains no model total on $\mathrm{Center}(K_0, K_1)$ [$Pred(K_0, K_1)$]. (K_0, K_1) is s-center-bivalent iff $K_1 - K_0$ contains no model $\mathscr{A}$ both total on $Pred(K_0, K_1)$ and so that for every variable-free term τ based on $Funct(K_0, K_1)$, $\tau^{\mathscr{A},\{\}}\downarrow$. (K_0, K_1) is core$_x$-bivalent iff for any model $\mathscr{A}$ with $\mathrm{Core}_x(K_0)$ $\mathscr{A}$-bivalent, $\mathscr{A} \notin K_1 - K_0$.

A class Γ of sentences defines (K_0, K_1) iff Γ defines K_0 and weakly defines K_1. (K_0, K_1) is x-elementary iff for some $\Gamma \subseteq \mathrm{Sent}(\mathbf{L}_x)$ Γ defines (K_0, K_1). (K_0, K_1) is basic x-elementary iff for some $\varphi \in \mathrm{Sent}(\mathbf{L}_x)$, $\{\varphi\}$ defines (K_0, K_1). (K_0, K_1) is u-defined iff {'u'} defines it, i.e. $K_0 = \{\,\}$ and $K_1 = \mathrm{MOD}$.

(K_0, K_1) *is* S-cross$_x$-closed iff for all $\mathscr{A}, \mathscr{B} \in \mathrm{MOD}$ with $\mathrm{Th}_x(\mathscr{A}) \cap S \subseteq \mathrm{WkTh}_x(\mathscr{B})$ if $\mathscr{A} \in K_0$ then $\mathscr{B} \in K_1$; for $S = \mathrm{Sent}(\mathbf{L}_x)$ we omit mention of S. This definition will be convenient: (K_0, K_1) has property 1_x iff

> if $x = 1, \ldots$ then K_0 is upward$_x$ closed, K_1 is downward$_x$ closed and (K_0, K_1) is cross$_x$ closed;
> if $x = 0, \ldots$ then K_1 is $C_x(K_0)$-downward$_x$ closed.

Lemma 7. *Let* $x = 1, \ldots$; *suppose that* (K_0, K_1) *is bounded and* cross$_x$ *closed, both* K_0 *and* K_1 *are closed under isomorphism and ultraproducts and both* $\mathrm{MOD} - K_0$ *and* $\mathrm{MOD} - K_1$ *are closed under ultrapowers. Then for any* $\mathscr{B} \in \mathrm{MOD}$:

$$\textit{if } \mathscr{B} \models^{w} \mathrm{Th}_x(K_0) \cap \mathrm{Sent}(L_x(K_0, K_1)), \quad \textit{then } \mathscr{B} \in K_1.$$

Assume that $\mathscr{B} \models^{w} \mathrm{Th}_x(K_0) \cap \mathrm{Sent}(L_x(K_0, K_1))$. Let $\mathscr{B}' = \mathscr{B} \upharpoonright \mathrm{Center}(K_0, K_1)$. Since $\mathscr{B} \upharpoonright \mathrm{Center}(K_1) = \mathscr{B}' \upharpoonright \mathrm{Center}(K_1)$, with two applications of Theorem 1 if $\mathscr{B}' \in K_1$ then $\mathscr{B} \in K_1$. Let $I = S_\omega(\mathrm{Th}_x(\mathscr{B}') \cap \mathrm{Sent}(L_x(K_0, K_1)))$; by boundedness I is a set. Claim: for each $i \in K$ there is an $\mathscr{A}_i \in K_0$ with $\mathscr{A}_i \models^{w} i$. If this fails for i, $\neg\&i \in \mathrm{Th}_x(K_0) \cap \mathrm{Sent}(L_x(K_0, K_1))$, and so $\mathscr{B} \models^{w} \neg\&i$; since $\mathscr{B}' \models i$, $\mathscr{B} \models i$, a contradiction. Letting $\mathscr{A}_i' = \mathscr{A}_i \upharpoonright \mathrm{Center}(K_0, K_1)$, $\mathscr{A}_i' \upharpoonright \mathrm{Center}(K_0) = \mathscr{A}_i \upharpoonright \mathrm{Center}(K_0)$; so two more uses of Theorem 1 gives $A_i' \in K_0$. We select an ultrafilter D on I so that $\prod_D \mathscr{A}_i' \models^{w} \mathrm{Th}_x(\mathscr{B}') \cap \mathrm{Sent}(L_x(K_0, K_1))$. Using Lemma 4 as we did in Section 6 we get $\mathrm{Th}_x(\prod_D \mathscr{A}_i') \subseteq \mathrm{WkTh}_x(\mathscr{B}')$. Since (K_0, K_1) is cross$_x$-closed and $\prod_D \mathscr{A}_i' \in K_0$, $\mathscr{B}' \in K_1$.

Finally let $\mathrm{Th}_x(K_0, K_1) = \mathrm{Th}_x(K_0) \cap \mathrm{WkTh}_x(K_1)$. For $x = 0, \ldots$ $\mathrm{Th}_x(K_0) \subseteq C_x(K_0)$, and so $\mathrm{Th}_x(K_0, K_1) \subseteq C_x(K_0)$.

Theorem 4. *Let (K_0, K_1) be bounded. The following are equivalent*:

(i) *(K_0, K_1) is x-elementary.*

(ii) *(K_0, K_1) is defined by* $\mathrm{Th}_x(K_0, K_1) \cap \mathrm{Sent}(L_x(K_0, K_1))$.

(iii) *$K_0 \subseteq K_1$, K_0 and K_1 are closed under isomorphism and ultraproducts,* MOD $- K_0$ *and* MOD $- K_1$ *are closed under ultrapowers, (K_0, K_1) has property* 1_x, *and*:

(.i) *if* $x = 1$ *or* $1, \top$ $[= 1, \mathrm{s}]$ $[= 1, \mathrm{b}$ *or* $1, \top, \mathrm{b}]$, *then (K_0, K_1) is center-bivalent* [*s-center-bivalent*] [*b-center-bivalent*];

(.ii) *if* $x = 0$ *or* $0, \top$ *or* $0, \mathrm{s}$ *or* $0, \mathrm{b}$ *or* $0, \top, \mathrm{b}$, *then (K_0, K_1) is core$_x$-bivalent*;

(.iii) *if* $x = 0, \mathrm{u}$ *or* $0, \top, \mathrm{u}$ *or* $0, \mathrm{u}, \mathrm{s}$ *or* $0, \mathrm{u}, \mathrm{b}$, *then (K_0, K_1) is either* u-*defined or core$_x$-bivalent.*

Clearly (ii) entails (i). To show: (i) entails (iii). Assume that $\Gamma \subseteq \mathrm{Sent}(\mathbf{L}_x)$ defines (K_0, K_1). Clearly $K_0 \subseteq K_1$; by Theorem 2, K_1 is closed under isomorphism and ultraproducts, and MOD $- K_i$ is closed under ultrapowers for $i \in 2$. For $x = 1, \ldots$ clearly K_0 is upward$_x$ closed and K_1 is downward$_x$ closed. Suppose $\mathrm{Th}_x(\mathscr{A}) \subseteq \mathrm{WkTh}_x(\mathscr{B})$. If $\mathscr{A} \in K_0$, then $\Gamma \subseteq \mathrm{Th}_x(\mathscr{A})$; so $\Gamma \subseteq \mathrm{WkTh}_x(\mathscr{B})$; so $\mathscr{B} \in K_1$. Thus (K_0, K_1) is cross$_x$-closed. For $x = 0, \ldots$ we have $\Gamma \subseteq C_x(K_0)$, and so K_1 is $C_x(K_0)$-downward$_x$ closed. Thus (K_0, K_1) has property 1_x.

Suppose $x = 1$ or $1, \top$. Suppose that $\mathscr{A} \in K_1 - K_0$ is total on Center(K_0, K_1). Since $\mathscr{A} \notin K_0$ we may select $\varphi \in \Gamma$ so that $\mathscr{A} \not\models \varphi$; since $\mathscr{A} \in K_1$, $\mathscr{A} \mid \varphi$. Fix sets *Pred* and *Funct* so that *Pred*$(K_0, K_1) \subseteq$ *Pred*, *Funct*$(K_0, K_1) \subseteq$ *Funct* and $\varphi \in$ Sent$(L_x(\mathit{Pred}, \mathit{Funct}))$. Let $\mathscr{B}$ be any widening of $\mathscr{A}$ that is total on *Pred* $\cup$ *Funct*. Then either $\mathscr{B} \models \psi\ [\alpha]$ or $\mathscr{B} \mathrel{=\!\!|} \psi\ [\alpha]$ for any $\psi \in \mathrm{Fml}(L_x(\mathit{Pred}, \mathit{Funct}))$ and any $\mathscr{B}$-assignment α, this by induction on the construction of ψ. Thus either $\mathscr{B} \models \varphi$ or $\mathscr{B} \mathrel{=\!\!|} \varphi$. But $\mathscr{A} \restriction \mathrm{Center}(K_0, K_1) = \mathscr{B} \restriction \mathrm{Center}(K_0, K_1)$. So by two uses of Theorem 1, $\mathscr{B} \in K_1 - K_0$, a contradiction. Thus (K_0, K_1) is center-bivalent. A similar argument when $x = \ldots, \mathrm{s}$ $[= \ldots, \mathrm{b}]$ shows that (K_0, K_1) is s-center-bivalent [b-center-bivalent].

Suppose $x = 0$ or $0, \top$ or $0, \mathrm{s}$ or $0, \mathrm{b}$ or $0, \top, \mathrm{b}$. We have $\Gamma \subseteq \mathrm{Th}_x(K_0) \subseteq C_x(K_0)$. For any $\mathscr{A} \in$ MOD, if Core$_x(K_0)$ is $\mathscr{A}$-bivalent, then for each $\varphi \in \Gamma$, $\mathscr{A}$ is φ-bivalent, by Lemma 4; so if $\mathscr{A} \notin K_0$ then $\mathscr{A} \notin K_1$. So (K_0, K_1) is core$_x$-bivalent.

Suppose $x = 0, \mathrm{u}$ or $0, \top, \mathrm{u}$ or $0, \mathrm{u}, \mathrm{s}$ or $0, \mathrm{u}, \mathrm{b}$. By the above paragraph and Lemma 2, either (K_0, K_1) is u-defined or core$_x$-bivalent.

To show: (iii) entails (ii). Assume (iii). Let $\Gamma = \mathrm{Th}_x(K_0, K_1) \cap \mathrm{Sent}(L_x(K_0, K_1))$. So if $\mathscr{A} \in K_0$ then $\mathscr{A} \models \Gamma$, and if $\mathscr{A} \in K_1$ then $\mathscr{A} \models^{w} \Gamma$. We need the converses.

Suppose $x=1,\ldots$. We'll show that if $\mathcal{A}\models^{w}\Gamma$ then $\mathcal{A}\in K_1$. Suppose not. By Lemma 7, $\mathcal{A}\not\models^{w}\mathrm{Th}_x(K_0)\cap\mathrm{Sent}(L_x(K_0,K_1))$; select $\varphi\in\mathrm{Th}_x(K_0)\cap\mathrm{Sent}(L_x(K_0,K_1))$ so that $\mathcal{A}\dashv\varphi$. For any $\psi\in\mathrm{WkTh}_x(K_1)\cap\mathrm{Sent}_x(L(K_1))$, $\varphi\vee\psi\in\Gamma$; so $\mathcal{A}\models^{w}\varphi\vee\psi$; so $\mathcal{A}\models^{w}\psi$. Thus $\mathcal{A}\models^{w}\mathrm{WkTh}_x(K_1)\cap\mathrm{Sent}(L_x(K_1))$. Since K_1 is downward$_x$ closed, by Theorem 2, $\mathcal{A}\in K_1$, a contradiction.

To show: if $\mathcal{A}\models\Gamma$ then $\mathcal{A}\in K_0$. Suppose $\mathcal{A}\models\Gamma$.

If 'u' $\in\mathrm{lex}_x$ then for any $\varphi\in\mathrm{Th}_x(K_0)\cap\mathrm{Sent}(L_x(K_0))$, $\varphi\vee\mathrm{u}\in\Gamma$; so $\mathcal{A}\models\varphi\vee\mathrm{u}$, and so $\mathcal{A}\models\varphi$. Thus $\mathcal{A}\models\mathrm{Th}_x(K_0)\cap\mathrm{Sent}(L_x(K_0))$; since K_0 is upward$_x$ closed, by Theorem 2, $\mathcal{A}\in K_0$.

Suppose 'u' $\notin\mathrm{lex}_x$. Claim: for any $\mathcal{B}\in\mathrm{MOD}$ so that $\mathcal{B}\models\mathrm{WkTh}_x(K_1)\cap\mathrm{Sent}(L_x(K_0,K_1))$: (a) if $x=1$ or $1,\mathrm{T}$ then $\mathcal{B}$ is total on $\mathrm{Center}(K_0,K_1)$; (b) if $x=1,\mathrm{s}$ then $\mathcal{B}$ is total on $\mathit{Pred}(K_0,K_1)$ and for all closed terms τ based on $\mathit{Funct}(K_0,K_1)$, $\mathrm{den}(\mathcal{B},\{\ \},\tau)\downarrow$; (c) if $x=1,\mathrm{b}$ then $\mathcal{B}$ is total on $\mathit{Pred}(K_0,K_1)$.

For $x=1$ or $1,\mathrm{T}$. For any $\zeta\in\mathit{Pred}(K_0,K_1)(n)$ fix n distinct variables $v_0,\ldots,{}_{n-1}$, abbreviating the list $\vec{v}$; $(\forall\vec{v})(\zeta(\vec{v})\vee\neg\zeta(\vec{v}))$ is weakly valid, so belongs to $\mathrm{WkTh}_x(K_1)\cap\mathrm{Sent}(L_x(K_0,K_1))$, and so is true in $\mathcal{B}$. For any $\xi\in\mathit{Funct}(K_0,K_1)(n)$ select $\vec{v}$ as above and a further distinct v; $(\forall\vec{v})(\exists v)v=\xi(\vec{v})$ is weakly valid and so true in $\mathcal{B}$ as above. These facts establish (a). Since $\mathcal{A}\models\Gamma$, $\mathcal{A}\in K_1$. Suppose $\mathcal{A}\notin K_0$; since K_0 is upward$_x$ closed, by Theorem 2, $\mathcal{A}\not\models\mathrm{Th}_x(K_0)\cap\mathrm{Sent}(L_x(K_0))$; select $\varphi\in\mathrm{Th}_x(K_0)\cap\mathrm{Sent}(L_x(K_0))$ so that $\mathcal{A}\not\models\varphi$. For any $\psi\in\mathrm{WkTh}_x(K_1)\cap\mathrm{Sent}(L_x(K_0,K_1))$, $\varphi\vee\psi\in\Gamma$; so $\mathcal{A}\models\varphi\vee\psi$; So $\mathcal{A}\models\psi$. Thus $\mathcal{A}\models\mathrm{WkTh}_x(K_1)\cap\mathrm{Sent}(L_x(K_0,K_1))$. By (a) $\mathcal{A}$ is total on $\mathrm{Center}(K_0,K_1)$. Since (iii.i) has (K_0,K_1) center-bivalent, $\mathcal{A}\in K_0$. For $x=1,\mathrm{s}$: note that for τ a closed term based on $\mathit{Funct}(K_0,K_1)$, $\tau=_{\mathrm{s}}\tau\in\mathrm{WkTh}_x(K_1)\cap\mathrm{Sent}(L_x(K_0,K_1))$; so $\mathcal{B}\models\tau=_{\mathrm{s}}\tau$; so $\tau^{\mathcal{B},\{\}}\downarrow$; thus (b). The argument for $\mathcal{A}\in K_0$ is like the preceding. For $x=1,\mathrm{b}$: the first part of the argument for (a) gives (c). For $\mathcal{A}\in K_0$ the argument is like the preceding.

Suppose $x=0,\ldots$. Assume that $\mathcal{B}\models^{w}\Gamma$. Let $\mathcal{B}'=\mathcal{B}\restriction\mathrm{Center}(K_0,K_1)$; so $\mathcal{B}'\models^{w}\Gamma$. Let $I=S_\omega(\mathrm{Th}_x(\mathcal{B}')\cap C_x(K_0)\cap\mathrm{Sent}(L_x(K_0,K_1))$; as usual I is a set. Claim: for any $i\in I$ there is an $\mathcal{A}_i\in K_1$ so that $\mathcal{A}_i\models i$. Suppose this fails for i; then $\neg\&i\in\mathrm{WkTh}_x(K_1)\cap\mathrm{Sent}(L_x(K_0,K_1))$, since for any $\mathcal{A}$, $\mathcal{A}\models^{w}\neg\&i$ iff $\mathcal{A}\not\models i$. Furthermore, for any $\mathcal{A}\in K_0$, $\mathcal{A}\models\neg\&.$, since $\mathrm{Core}_x(K_0)$ is $\mathcal{A}$-bivalent and $i\subseteq C_x(K_0)$. So $\neg\&i\in\Gamma$, and so $\mathcal{B}'\models^{w}\neg\&i$, contradicting $\mathcal{B}'\models i$, yielding the claim. Let $\mathcal{A}_i'=\mathcal{A}_i\restriction\mathrm{Center}(K_0,K_1)$; so $\mathcal{A}_i'\in K_1$. As usual we select an ultrafilter D on I so that $\prod_D\mathcal{A}_i'\models\mathrm{Th}_x(\mathcal{B}')\cap C_x(K_0)\cap\mathrm{Sent}(L_x(K_0,K_1))$. Since $(\prod_D\mathcal{A}_i')\restriction\mathrm{Center}(K_0,K_1)=\prod_D\mathcal{A}_i'$ with Lemma 6 we may strengthen this to $\prod_D\mathcal{A}_i'\models\mathrm{Th}_x(\mathcal{B}')\cap C_x(K_0)$. But K_1 is $C_x(K_0)$-downward$_x$ closed; so since $\prod_D\mathcal{A}_i'\in K_1$, $\mathcal{B}'\in K_1$, and so $\mathcal{B}\in K_1$. Now assume $\mathcal{B}\models\Gamma$; so by the above $\mathcal{B}\in K_1$. For $\theta\in\mathrm{Core}_x(K_0)$ containing variables in the list $\vec{v}$, $(\forall\vec{v})(\theta\vee\neg\theta)\in\mathrm{Th}_x(K_0)\cap\mathrm{Sent}(L_x(K_0,K_1))$ by the definition of $\mathrm{Core}_x(K_0)$. Since $(\forall\vec{v})(\theta\vee\neg\theta)\in\mathrm{WkTh}_x(K_1)\cap\mathrm{Sent}(L_x(K_0))$, $(\forall\vec{v})(\theta\vee\neg\theta)\in\Gamma$. Thus $\mathrm{Core}_x(K_0)$ is $\mathcal{B}$-bivalent; since (K_0,K_1) is core$_x$-bivalent, $\mathcal{B}\in K_0$. Thus (ii).

Note: suppose $x=0,\ldots$. It's worth noticing that if $K_0\subseteq K_1$ and K_1 is

$C_x(K_0)$-downward$_x$ closed, then (K_0, K_1) is $C_x(K_0)$-cross$_s$ closed. For suppose $\mathrm{Th}_x(\mathcal{A}) \cap C_x(K_0) \subseteq \mathrm{WkTh}_x(\mathcal{B})$; using '¬' $\mathrm{Th}_x(\mathcal{B}) \cap C_x(K_0) \subseteq \mathrm{WkTh}_x(\mathcal{A})$; if $\mathcal{A} \in K_0$ by Lemma 5, $\mathrm{WkTh}_x(\mathcal{A}) \cap C_x(K_0) \subseteq \mathrm{Th}_x(\mathcal{A})$; so $\mathrm{Th}_x(\mathcal{B}) \cap C_x(K_0) \subseteq \mathrm{Th}_x(\mathcal{A})$; since $\mathcal{A} \in K_1$, we get $\mathcal{B} \in K_1$. Also, if '⊤' $\in$ lex$_x$ then K_1 is $C_x(K_0)$-downward$_x$ closed iff K_1 is downward$_x$ closed.

Theorem 4 yields a slick proof of Observation 4(v). Let (K_0, K_1) be defined by '$\mathbf{a} =_s \mathbf{b} \supset \mathbf{c} =_s \mathbf{d}$'. It's not hard to see that $\mathrm{Core}_{0,\top}(K_0) = \{v = v' : v, v' \in Var\}$; so (K_0, K_1) is not core$_{0,\top}$-bivalent, and so not 0, ⊤-elementary.

To analyze basic x-elementary pairs we need two more definitions. (K_0, K_1) is core$'_x$-bivalent iff $K_1 - K_0$ contains no model that is $\mathrm{Core}_x(K_0) \cap \mathrm{Core}_x(\mathrm{MOD} - K_1)$-bivalent. (K_0, K_1) has property 2_x iff:

> if x = 1, ... then K_0 and K_1 are upward$_x$ and downward$_x$ closed and (K_0, K_1) is cross$_x$ closed;
> if x = 0, ... then K_1 is $C_x(K_0) \cap C_x(\mathrm{MOD} - K_1)$-downward$_x$ closed and K_0 is $C_x(K_0) \cap C_x(\mathrm{MOD} - K_1)$-upward$_x$ closed.

Theorem 5. *Let (K_0, K_1) be bounded. For* x = 1, ... *or* 0, ⊤, ... *the following are equivalent*:

(i) (K_0, K_1) *is basic* x*-elementary.*

(ii) (K_0, K_1) *is defined by a* $\varphi \in \mathrm{Sent}(L_x(K_0, K_1))$.

(iii) (K_0, K_1) *is defined by a finite* $\Gamma \subseteq \mathrm{Th}_x(K_0, K_1) \cap \mathrm{Sent}(L_x(K_0, K_1)$.

(iv) $K_0 \subseteq K_1$, K_0 *and* K_1 *are closed under isomorphism,* K_0, K_1, $\mathrm{MOD} - K_0$ *and* $\mathrm{MOD} - K_1$ *are closed under ultrapowers,* (K_0, K_1) *has property* 2_x, *and*:

(.i) *if* x = 1 *or* 1, ⊤ [= 1, s] [= 1, b *or* 1, ⊤, b], *then* (K_0, K_1) *is center-bivalent* [s*-center-bivalent*] [b*-center-bivalent*];

(.ii) *if* x = 0, ... *for* 'u' $\notin$ lex$_x$ *then* (K_0, K_1) *is core$'_x$-bivalent*;

(.iii) *if* x = 0, ... *for* 'u' $\in$ lex$_x$ *then* (K_0, K_1) *is either* u*-defined or core$'_x$-bivalent.*

Furthermore for x = 0, ... *with* '⊤' $\notin$ lex$_x$ *these are equivalent*: (i); (ii); (iv).

Clearly (ii) implies (i); for x = 1, ... or 0, ⊤, ... (iii) is equivalent to (ii). It's easy to see that for x = 1, ... (K_0, K_1) has property 2_x iff (K_0, K_1) and $(\mathrm{MOD} - K_1, \mathrm{MOD} - K_0)$ have property 1_x; so (i) implies (iv). Also for x = 0, ... (i) implies (iv); for if $\varphi \in \mathrm{Sent}(\mathbb{L}_x)$ defines (K_0, K_1), then $\varphi \in C_x(K_0) \cap C_x(\mathrm{MOD} - K_1)$; this suffices to make (K_0, K_1) have property 2_x and be core$'_x$-bivalent.

To show: for x = 1, ... (iv) implies (iii). First notice that $\mathrm{Center}(K_0, K_1) = \mathrm{Center}(\mathrm{MOD} - K_1, \mathrm{MOD} - K_0)$. Applying Theorem 4 to (K_0, K_1) and $(\mathrm{MOD} - K_1, \mathrm{MOD} - K_0)$ we obtain $\Gamma, \Delta \subseteq \mathrm{Sent}(L_x(K_0, K_1))$ so that Γ defines (K_0, K_1) and Δ defines $(\mathrm{MOD} - K_1, \mathrm{MOD} - K_0)$. So $(\Gamma, \Gamma \cup \Delta)$ and $(\Delta, \Gamma \cup \Delta)$ are inconsistent. By compactness there is a finite $\Gamma' \subseteq \Gamma$ so that $(\Gamma', \Gamma' \cup \Delta)$ and $(\Delta, \Gamma' \cup \Delta)$ are inconsistent. Claim: Γ' defines (K_0, K_1). If $\mathcal{A} \models \Gamma'$ but $\mathcal{A} \notin K_0$

then $\mathscr{A} \models^w \Delta$, contrary to the inconsistency of the first pair; so Γ' defines K_0. If $\mathscr{A} \models^w \Gamma'$ but $\mathscr{A} \notin K_1$ then $\mathscr{A} \models \Delta$, contrary to the inconsistency of the second pair; so Γ' weakly defines K_1.

To show: for x = 0, . . . (iv) implies (ii). Assume (iv). Let Γ be as in the proof of Theorem 4, (iii) implies (ii). We show that $\Gamma^* = \Gamma \cap C_x(\mathrm{MOD} - K_1)$ defines (K_0, K_1). Assume that $\mathscr{B} \models^w \Gamma^*$. Let $\mathscr{B}' = \mathscr{B} \upharpoonright \mathrm{Center}(K_0, K_1)$; so $\mathscr{B}' \models^w \Gamma^*$. Let

$$I = S_\omega(\mathrm{Th}_x(\mathscr{B}') \cap C_x(K_0) \cap C_x(\mathrm{MOD} - K_1) \cap \mathrm{Sent}(L_x(K_0, K_1));$$

as usual I is a set. Claim: for any $i \in I$ there is an $\mathscr{A}_i \in K_1$ so that $\mathscr{A}_i \models i$. Suppose this fails for i; then $\neg\&i \in \mathrm{WkTh}_x(K_1) \cap \mathrm{Sent}(L_x(K_0, K_1))$, since for any $\mathscr{A}, \mathscr{A} \models^w \neg\&i$ iff $\mathscr{A} \not\models i$. Furthermore, for any $\mathscr{A} \in K_0$, $\mathscr{A} \models \neg\&i$, since $\mathscr{A}$ is $\mathrm{Core}_x(K_0)$-bivalent and $i \subseteq C_x(K_0)$. So $\neg\&i \in \Gamma$. Since $i \subseteq C_x(\mathrm{MOD} - K_1)$ we have $\mathscr{B}' \models^w \neg\&i$, contradicting $\mathscr{B}' \models i$, yielding the claim. Let $\mathscr{A}_i' = \mathscr{A}_i \upharpoonright \mathrm{Center}(K_0, K_1)$; so $\mathscr{A}_i' \in K_1$. As usual we select an ultrafilter D on I so that

$$\prod_D \mathscr{A}_i' \models \mathrm{Th}_x(\mathscr{B}') \cap C_x(K_0) \cap C_x(\mathrm{MOD} - K_1) \cap \mathrm{Sent}(L_x(K_0, K_1)).$$

Since $(\prod_D \mathscr{A}_i') \upharpoonright \mathrm{Center}(K_0, K_1) = \prod_D \mathscr{A}_i'$ with Lemma 6 we may strengthen this to $\prod_D \mathscr{A}_i' \models \mathrm{Th}_x(\mathscr{B}') \cap C_x(K_0) \cap C_x(\mathrm{MOD} - K_1)$. But K_1 is $C_x(K_0) \cap C_x(\mathrm{MOD} - K_1)$-downward$_x$ closed; so since $\prod_D \mathscr{A}_i' \in K_1$, $\mathscr{B}' \in K_1$, and so $\mathscr{B} \in K_1$. So Γ^* weakly defines K_1. For any $\theta \in \mathrm{Core}_x(K_0) \cap \mathrm{Core}_x(\mathrm{MOD} - K_1)$, $(\forall v)(\theta \vee \neg\theta) \in \Gamma^*$. So if $\mathscr{B} \models \Gamma$ then $\mathrm{Core}_x(K_0) \cap \mathrm{Core}_x(\mathrm{MOD} - K_1)$ is $\mathscr{B}$-bivalent; since (K_0, K_1) is core_x'-bivalent $\mathscr{B} \in K_0$. So Γ^* defines (K_0, K_1). As in the case of x = 1, . . . we obtain a finite $\Gamma' \subseteq \Gamma^*$ defining (K_0, K_1). To see that $\&\Gamma'$ defines (K_0, K_1) we need only show that if $\mathscr{A} \notin K_1$ then $\mathscr{A} \dashv \&\Gamma'$; for this we need that for every $\varphi \in \Gamma'$ either $\mathscr{A} \models \varphi$ or $\mathscr{A} \dashv \varphi$. Since $\Gamma' \subseteq C_x(\mathrm{MOD} - K_1)$ this is the case.

The following deserves mention. For x = 0 or 0, b and 'T' $\notin \mathrm{lex}_x$ and (K_0, K_1) basic x-elementary: if $K_0 \neq \{\}$, then $\mathrm{Core}_x(K_0) \subseteq \mathrm{Core}_x(\mathrm{MOD} - K_1)$; so if $K_0 \neq \{\}$ and $K_1 \neq \mathrm{MOD}$, then $\mathrm{Core}_x(K_0) = \mathrm{Core}_x(\mathrm{MOD} - K_1)$. For suppose $\varphi \in L_0$ defines (K_0, K_1) and $\mathscr{A} \notin K_1$. Suppose θ isn't $\mathscr{A}$-bivalent. Then θ contains some γ, either a predicate- or a function-constant, so that for some $\vec{a} \in |\mathscr{A}|^n$, $\gamma^{\mathscr{A}}(\vec{a})\uparrow$. Then γ doesn't occur in φ, since $\mathscr{A} \dashv \varphi$, using Lemma 4. So in fact $\gamma \notin \mathrm{Center}(K_0, K_1)$. Now pick $\mathscr{B} \in K_0$; then $\mathscr{B}' = \mathscr{B} \upharpoonright \mathrm{Center}(K_0, K_1) \in K_0$; but θ is not $\mathscr{B}'$-bivalent; so $\theta \in \mathrm{Core}_0(K_0)$. This argument also works for x = 0, b, except that then γ must be a predicate-constant.

For x = 0, s this fact doesn't hold. For '$\mathbf{a}$', '$\mathbf{b}$' $\in \mathrm{FUNCT}(0)$ let '$\mathbf{a} =_s \mathbf{b}$' define (K_0, K_1); then '$\mathbf{a} =_s \mathbf{a}$' $\in \mathrm{Core}_{0,s}(K_0) - \mathrm{Core}_{0,s}(\mathrm{MOD} - K_1)$.

8. An algebraic classification of the x-elementary class-pairs

For $K \subseteq \mathrm{MOD}$ let K be upward monotonic [s-monotonic] iff for any $\mathscr{A}, \mathscr{B} \in \mathrm{MOD}$ so that $\mathscr{A} \sqsubseteq \mathscr{B}$ [$\mathscr{A} \sqsubseteq_s \mathscr{B}$]: if $\mathscr{A} \in K$ then $\mathscr{B} \in K$. Let K be downward monotonic [s-monotonic] iff for any such $\mathscr{A}, \mathscr{B}$: if $\mathscr{B} \in K$ then $\mathscr{A} \in K$.

For $K_0 \subseteq K_1 \subseteq \text{MOD}$, let (K_0, K_1) be monotonic [s-monotonic] iff K_0 is upward monotonic [s-monotonic] and K_1 is downward monotonic [s-monotonic]. Let (K_0, K_1) be crosstonic [s-crosstonic] iff for any $\mathcal{A}, \mathcal{B} \in \text{MOD}$ so that $\mathcal{A} * \mathcal{B}$ $[\mathcal{A} *_s \mathcal{B}]$, if $\mathcal{A} \in K_0$ then $\mathcal{B} \in K_1$.

Let Λ be a class of terms closed under subterms (i.e. for any $\tau \in \Lambda$ and any subterm σ of τ, $\sigma \in \Lambda$). Given a model $\mathcal{A} = (|\mathcal{A}|, \mathscr{E}, \mathcal{N})$ we construct $\mathcal{N}_\Lambda$ to be the $\sqsubseteq$-least $\mathcal{N}'$ so that $\mathcal{N}' \sqsubseteq \mathcal{N}$ and for any term $\tau \in \Lambda$ and any $\mathcal{A}$-assignment α, $\tau^{(|\mathcal{A}|, \mathscr{E}, \mathcal{N}'), \alpha} \simeq \tau^{\mathcal{A}, \alpha}$. Set $\mathcal{N}_0 = \{\}$. Given $\mathcal{N}_m$ let $\mathcal{A}_m = (|\mathcal{A}|, \mathscr{E}, \mathcal{N}_m)$; for $\xi \in \text{FUNCT}(n)$ and $\vec{a} \in |\mathcal{A}|^n$ let $\mathcal{N}_{m+1}(\xi)(\vec{a}) = a$ iff either $\mathcal{N}_m(\xi)(\vec{n}) = a$ or there are terms $\tau_0, \ldots, \tau_{n-1}$ and an $\mathcal{A}$-assignment α so that $\xi(\tau_0, \ldots, \tau_{n-1}) \in \Lambda$ has depth $\leqslant m+1$, for all $i < n$, $\tau_i^{\mathcal{A}_m, \alpha} = a_i$ and $\mathcal{N}(\xi)(\vec{a}) = a$; otherwise $\mathcal{N}_{m+1}(\xi)(\vec{a})\uparrow$. Let $\mathcal{N}_\Lambda = \bigcup_n \mathcal{N}_n$; it is as desired.

For Δ a class of atomic formula let $\bar{\Delta}$ be the class of terms occurring in members of Δ; so $\bar{\Delta}$ is closed under subterms. For $\zeta \in \text{PRED}(n)$ and $\vec{a} \in |\mathcal{A}|^n$ let $\mathscr{E}_\Delta(\zeta)(\vec{a}) \simeq \mathscr{E}(\zeta)(\vec{a})$ if there are terms $\tau_0, \ldots, \tau_{n-1}$ so that $\zeta(\tau_0, \ldots, \tau_{n-1}) \in \Delta$ and $\text{den}((|\mathcal{A}|, \mathscr{E}, \mathcal{N}_{\bar{\Delta}}), \alpha, \tau_i) = a_i$ for all $i < n$; otherwise $\mathscr{E}_\Delta(\zeta)(\vec{a})\uparrow$. Let $\mathcal{A} \restriction \Delta = (|\mathcal{A}|, \mathscr{E}_\Delta, \mathcal{N}_{\bar{\Delta}})$.

It's easy to see that for '=' $\in \text{lex}_x$ and $\Delta \subseteq \text{Fml}(\mathbb{L}_x)$, $\mathcal{A} \restriction \Delta$ is the $\sqsubseteq$-least model $\mathcal{B}$ so that $\mathcal{B} \sqsubseteq \mathcal{A}$ and for any $\theta \in \Delta$ and $\mathcal{A}$-assignment α:

$$\mathcal{A} \vDash \theta[\alpha] \quad \text{iff} \quad \mathcal{B} \vDash \theta[\alpha];$$

$$\mathcal{A} \dashv \theta[\alpha] \quad \text{iff} \quad \mathcal{B} \dashv \theta[\alpha].$$

Note that for $\mathcal{A} \vDash \{$'$\mathbf{a} \neq_s \mathbf{b}$', '$E_s(\mathbf{a})$', '$E_s(\mathbf{b})$'$\}$ there is no $\sqsubseteq$-least $\mathcal{B} \sqsubseteq \mathcal{A}$ so that $\mathcal{B} \vDash$ '$\mathbf{a} \neq_s \mathbf{b}$'.

Let (K_0, K_1) be core_x-closed iff for all $\mathcal{A} \in \text{MOD}$ if $\mathcal{A} \restriction \text{Core}_x(K_0) \in K_1$ then $\mathcal{A} \in K_1$.

What follows is the three-valued version of Corollary 6.1.16 of [1].

Theorem 6. *Let (K_0, K_1) be bounded. (K_0, K_1) is* x*-elementary iff $K_0 \subseteq K_1$, K_0 and K_1 are closed under isomorphism and ultraproducts,* $\text{MOD} - K_0$ *and* $\text{MOD} - K_1$ *are closed under ultrapowers, and these conditions are met*:

(i) *If* x = 0 [= 0, s *or* 0, b], *then K_1 is downward monotonic* [s-*monotonic*] *and (K_0, K_1) is* core_x*-closed and* core_x*-bivalent.*

(ii) *If* x = 0, u [= 0, u, s *or* 0, u, b], *then K_1 is downward monotonic* [s-*monotonic*] *and (K_0, K_1) is* core_x*-closed and either* core_x*-bivalent or* u*-defined.*

(iii) *If* x = 0, ⊤ *or* 0, ⊤, b, *then (K_0, K_1) is* core_x*-bivalent.*

(iv) *If* x = 0, ⊤, u, *then either (K_0, K_1) is either* core_x*-bivalent or* u*-defined.*

(v) *If* x = 1 [= 1, s] [1, b], *then (K_0, K_1) is monotonic* [s-*monotonic*] [s-*monotonic*], *crosstonic* [s-*crosstonic*] [s-*crosstonic*] *and center-bivalent* [s-*center-bivalent*] [b-*center-bivalent*].

(vi) *If* x = 1, u [= 1, u, s], *then (K_0, K_1) is monotonic* [s-*monotonic*] *and crosstonic* [s-*crosstonic*].

(vii) *If* x = 1, ⊤ [= 1, ⊤, b], *then (K_0, K_1) is center-bivalent* [b-*center-bivalent*].

Note: in cases (i) and (ii) (K_0, K_1) is crosstonic [s-crosstonic] and monotonic [s-monotonic]. Suppose $\mathscr{A} \in K_0$. For '=' $\in$ lex$_x$, if $\mathscr{A} * \mathscr{B}$, since $\mathrm{Core}_x(K_0)$ is $\mathscr{A}$-bivalent $\mathscr{B} \restriction \mathrm{Core}_x(K_0) \sqsubseteq \mathscr{A}$; we have $\mathscr{A} \in K_1$; since K_1 is downward monotonic $\mathscr{B} \restriction \mathrm{Core}_x(K_0) \in K_1$; by core$_x$-closure $\mathscr{B} \in K_1$. If $\mathscr{A} \sqsubseteq \mathscr{B}$, $\mathscr{B}$ is also $\mathrm{Core}(K_0)$-bivalent and $\mathscr{A} * \mathscr{B}$; so if $\mathscr{A} \in K_1$ then $\mathscr{B} \in K_1$, and so $\mathscr{B} \in K_0$. Similarly for $*_s$ and $\sqsubseteq_s$.

Proof from left to right. Assume the left-hand side; by Theorem 4 we may suppose that $\Gamma \subseteq L_x(K_0, K_1)$ defines (K_0, K_1). For '$\top$' $\notin$ lex$_x$ and '=' $\in$ lex$_x$ ['=' $\notin$ lex$_x$] it's easy to see that (K_0, K_1) is monotonic [s-monotonic] and crosstonic [s-crosstonic], using Lemma 1. If x $\neq$ 1, ... and $\neq$ 0, $\top$, ... we may suppose that $\Gamma \subseteq C_x(K_0)$; thus for any $\mathfrak{A} \in$ MOD, $\mathscr{A} \models \Gamma$ iff $\mathscr{A} \restriction \mathrm{Core}_x(K_0) \models \Gamma$, and similarly for $\models^w$; so (K_0, K_1) is core$_x$-closed. For the remaining conditions, use Theorem 4.

The next two lemmas will get us from right to left.

Lemma 8. *Let* $\mathscr{A}, \mathscr{B} \in$ MOD.

(i) *For* x = 1, ... *or* 0, $\top$, ... *supppose that* $\mathrm{Th}_x(\mathscr{A}) \subseteq \mathrm{Th}_x(\mathscr{B})$:

(i.i) *if* x = 0, $\top$ *or* 0, $\top$, u *or* 1, $\top$ *or* 1, $\top$, u: *there is an ultrafilter D so that* $\prod_D \mathscr{A} \cong \prod_D \mathscr{B}$;

(i.ii) *if* x = 1 *or* 1, u *there is an ultrafilter D so that* $\prod_D \mathscr{A} \sqsubseteq \prod_D \mathscr{B}$;

(i.iii) *if* x = 1, s *or* 1, u, s *or* 1, b *there is an ultrafilter D so that* $\prod_D \mathscr{A} \sqsubseteq_s \prod_D \mathscr{B}$.

(ii) *For* x *otherwise suppose that* $\mathrm{Th}_x(\mathscr{A}) \cap C_x(K_0) \subseteq \mathrm{Th}_x(\mathscr{B})$:

(ii.i) *if* x = 0 *or* 0, u *there is an ultrafilter D so that* $(\prod_D \mathscr{A}) \restriction \mathrm{Core}_x(K_0) \sqsubseteq \prod_D \mathscr{B}$;

(ii.ii) *if* x = 0, s *or* 0, u, s *or* 0, b *or* 0, u, b *there is an ultrafilter D so that* $(\prod_D \mathscr{A}) \restriction \mathrm{Core}_x(K_0) \sqsubseteq_s \prod_D \mathscr{B}$.

Lemma 9. *For* x = 1 *or* 1, u [1, s *or* 1, u, s *or* 1, b] *and* $\mathscr{A}, \mathscr{B} \in$ MOD: *if* $\mathrm{Th}_x(\mathscr{A}) \subseteq \mathrm{WkTh}_x(\mathscr{B})$, *then there is an ultrafilter D so that* $\prod_D \mathscr{A} \stackrel{*}{\sqsubseteq} \prod_D \mathscr{B}$ [$\prod_D \mathscr{A} \stackrel{*}{\sqsubseteq}_s \prod_D \mathscr{B}$].

Assume the right-hand side of Theorem 5. We'll use these lemmas to prove the left-hand side. By Theorem 4 it suffices to show that (K_0, K_1) has property 1_x.

Suppose that $\mathrm{Th}_x(\mathscr{A}) \subseteq \mathrm{Th}_x(\mathscr{B})$. If x = 1 or 1, u [1, s or 1, u, s or 1, b], Lemma 8 gives an ultrafilter D with $\prod_D \mathscr{A} \sqsubseteq \prod_D \mathscr{B}$ [$\prod_D \mathscr{A} \sqsubseteq_s \prod_D \mathscr{B}$]. If $\mathscr{B} \in K_1$ then $\prod_D \mathscr{B} \in K_1$; since K_1 is downward monotonic [s-monotonic] $\prod_D \mathscr{A} \in K_1$; thus $\mathscr{A} \in K_1$. So K_1 is downward$_x$ closed. Reversing direction, K_0 being upward monotonic [s-monotonic] yields that K_0 is upward$_x$ closed. If x = 1, $\top$, ... Lemma 8 yields a D so that $\prod_D \mathscr{A}$ is isomorphic to $\prod_D \mathscr{B}$; so $\mathscr{A} \in K_i$ iff $\mathscr{B} \in K_i$, making K_1 downward$_x$ closed and K_0 upward$_x$ closed.

For x = 1, $\top$, ..., if $\mathrm{Th}_x(\mathscr{A}) \cap C_x(K_0) \subseteq \mathrm{Th}_x(\mathscr{B})$ then $\mathrm{Th}_x(\mathscr{A}) \subseteq \mathrm{Th}_x(\mathscr{B})$; so the argument used for x = 1, $\top$, ... shows that K_1 is $C_x(K_0)$-downward$_x$ closed. For x = 0 or 0, u [0, s or 0, u, s or 0, b or 0, u, b] suppose that $Th_x(\mathscr{A}) \cap C_x(K_0) \subseteq Th_x(\mathscr{B})$. Lemma 8 yields a D so that $(\prod_D \mathscr{A}) \upharpoonright Core_x(K_0) \sqsubseteq \prod_D \mathscr{B}$ $[(\prod_D \mathscr{A}) \upharpoonright Core_x(K_0) \sqsubseteq_s \prod_D \mathscr{B}]$. If $\mathscr{B} \in K_1$, $\prod_D \mathscr{B} \in K_1$; since K_1 is downward monotonic [s-monotonic] $(\prod_D \mathscr{A}) \upharpoonright Core_x(K_0) \in K_1$; by core$_x$-closure $\prod_D \mathscr{A} \in K_1$, and so $\mathscr{A} \in K_1$. So K_1 is $C_x(K_0)$-downward$_x$ closed.

For x = 1 or 1, u [1, s or 1, u, s or 1, b], suppose that $Th_x(\mathscr{A}) \subseteq WkTh_x(\mathscr{B})$. By Lemma 9 there is an ultrafilter D so that $\prod_D \mathscr{A} \stackrel{*}{\rightharpoonup} \prod_D \mathscr{B}$ $[\prod_D \mathscr{A} \stackrel{*}{\rightharpoonup}_s \prod_D \mathscr{B}]$. If $\mathscr{A} \in K_0$, $\prod_D \mathscr{A} \in K_0$; by crosstonicity [s-crosstonicity] $\prod_D \mathscr{B} \in K_1$, and so $\mathscr{B} \in K_1$. So (K_0, K_1) is cross$_x$ closed. For x = 1, T, . . . (K_0, K_1) is cross$_x$-closed because K_0 is upward$_x$ closed. The point is that if $Th_x(\mathscr{A}) \subseteq WkTh_x(\mathscr{B})$ then $Th_x(\mathscr{A}) \subseteq Th_x(\mathscr{B})$: if $\mathscr{A} \models \varphi$ then $\mathscr{A} \models \top\varphi$, so $\mathscr{B} \models^{w} \top\varphi$, so $\mathscr{B} \models \varphi$.

Assuming GCH the technology of saturation and good ultrafilters as presented in [1] can be modified to prove Lemmas 8 and 9. The need to deal with both $\models$ and $\models^{w}$ leads to some revisions in the classical apparatus: we need two notions of an n-type, and thus two notions of saturation. To avoid assuming GCH we'll modify Shelah's technology as presented in [1] (avoiding some minor errors found there).

For λ and κ infinite cardinals let μ be the least cardinal so that $\lambda < \lambda^{\mu}$; thus $\mu \leqslant \lambda$ and μ is regular. Suppose that F is a set of functions $f\colon \to \mu$, and G is a set of functions $g\colon \lambda \to \beta(g)$ for $\beta(g)$ a cardinal less than μ. Suppose D is a filter over λ. (F, G, D) is κ-consistent iff:

(i) D is generated by some $E \subseteq D$ with card$(E) \leqslant \kappa$ (i.e. E is closed under finite intersection and for every $X \in D$ there is a $Y \in E$ with $Y \subseteq X$);

(ii) for any cardinal $\beta < \mu$ and sequences $\langle f_\rho \rangle_{\rho<\beta}$ in F, $\langle \sigma_\rho \rangle_{\rho<\beta}$ in μ, the first without repetitions,

(.i) $\{\{i < \lambda\colon f_\rho(i) = \sigma_\rho \text{ for all } \rho < \beta\}\} \cup D = D'$ generates a non-trivial filter over λ;

(.ii) for any $f \in F$ and $g \in G$, $\{\{i\colon f(i) = g(i)\}\} \cup D'$ generates a non-trivial filter over λ.

(Note: in [1, p. 315], the authors try to collapse clauses (ii.i) and (ii.ii) into a definition that is not as intended. Their definition makes their Lemma 6.1.10 vacuously true and their proof of Lemma 6.1.12 incorrect.)

The following lemmas are quoted directly from [1, p. 315–7].

Lemma [1, 6.1.10]. *There is an F with* card$(F) = 2^{\lambda}$ *and* $(F, \{\ \}, \{\lambda\})$ *μ-consistent.*

Lemma [1, 6.1.13(ii)]. *Suppose that* $(F, \{\ \}, D)$ *is κ-consistent,* $\mu \leqslant \kappa$ *and for* $\iota < \kappa$, $A_\iota \subseteq \lambda$. *There are* $F' \subseteq F$ *and* D' *with* $D \subseteq D'$ *so that* card$(F - F') \leqslant \kappa$, $(F', \{\ \}, D')$ *is κ-consistent, and for each* $\iota < \kappa$ *either* $A_\iota \in \Delta'$ *or* $\lambda - A_\iota \in D'$. For proofs see [1].

The following lemma replaces Lemma 6.1.14 of [1].

Lemma 10. *Suppose μ and λ are as above, $\lambda \leq \kappa < 2^\lambda$, $\mathscr{A}$ is a model of cardinality $< \mu$ for L_x and $(F, \{\,\}, D)$ is κ-consistent. Let $\langle \varphi_\iota \rangle_{\iota<\kappa}$ be a sequence of formulae of L_x which is closed under conjunction, $\varphi_\iota = \varphi_\iota(v, v_0, \ldots, v_{n_\iota-1})$. For each $\iota < \kappa$ and $j < n_\iota$ suppose $a_{\iota,j}: \lambda \to |\mathscr{A}|$. Let*

$$C_\iota^+ = \{i < \lambda: \mathscr{A} \models (\exists v)\varphi_\iota\,[a_{\iota,0}(i), \ldots, a_{\iota,n-1}(i)]\};$$
$$C_\iota^- = \{i < \lambda: \mathscr{A} \models^w (\exists v)\varphi_\iota\,[a_{\iota,0}(i), \ldots, a_{\iota,n-1}(i)]\}.$$

For $a: \lambda \to |\mathscr{A}|$ let:

$$A_{a,\iota}^+ = \{i < \lambda: \mathscr{A} \models \varphi_\iota\,[a(i), a_{\iota,0}(i), \ldots, a_{\iota,n-1}(i)]\};$$
$$A_{a,\iota}^- = \{i < \lambda: \mathscr{A} \models^w \varphi_\iota\,[a(i), a_{\iota,0}(i), \ldots, a_{\iota,n-1}(i)]\}.$$

If for each $\iota < \kappa$, $C_\iota^+ \in D$ [$C_\iota^- \in D$], then there are $a: \lambda \to |\mathscr{A}|$, $F' \subseteq F$ and D' with $D \subseteq D'$ so that $(F', \{\,\}, D')$ is κ-consistent and for each $\iota < \lambda$, $A_{a,\iota}^+ \in D'$ [$A_{a,\iota}^- \in D'$].

This follows by straightforward modifications of the proof of Lemma 6.1.14 in [1]. (As formulated in [1], Lemma 6.1.14 is too weak, and its proof involves a fallacy on p. 319 line 15; the "notational difficulties" the authors tried to avoid can't be avoided; when it is formulated sufficiently strongly, as in the observation on the lower half of p. 319, the fallacy disappears.)

Suppose that $\mathscr{A}$ and $\mathscr{B}$ are given models for L_x. Fix cardinals λ and μ so that card($\mathscr{A}$), card($\mathscr{B}$) $< \mu$, μ is the least so that $\lambda < \lambda^\mu$, and card(Fml(L_x)) $\leq \lambda$.

We now prove Lemma 8 for $x = 1, \ldots$. Assume that $\mathrm{Th}_x(\mathscr{A}) \subseteq \mathrm{Th}_x(\mathscr{B})$. We'll construct sequences $\langle F_\rho \rangle_{\rho<2^\lambda}$, $\langle D_\rho \rangle_{\rho<2^\lambda}$, $\langle a_\rho \rangle_{\rho<2^\lambda}$, $\langle b_\rho \rangle_{\rho<2^\lambda}$ so that:

(0) $a_\rho: \lambda \to |\mathscr{A}|$, $b_\rho: \lambda \to |\mathscr{B}|$; for every $a: \lambda \to |\mathscr{A}|$ there is a ρ so that $a = a_\rho$; similarly for every $b: \lambda \to |\mathscr{B}|$.

(1) If $\rho \leq \rho'$ then $F_{\rho'} \subseteq F_\rho$ and $D_\rho \subseteq D_{\rho'}$.

(2) card(F_ρ) $= 2^\lambda$; $(F_\rho, \{\,\}, D_\rho)$ is λ + card(ρ)-consistent; if η is a limit ordinal $F_\eta = \bigcap \{F_\rho: \rho < \eta\}$, $D_\eta = \bigcup \{D_\rho: \rho < \eta\}$.

(3) For every $B \subseteq \lambda$ there is a $\rho < 2^\lambda$ so that either $B \in D_\rho$ or $\lambda - B \in D_\rho$.

(4) For every $\varphi(v_0, \ldots, v_{n-1}) \in \mathrm{Fml}(L_x)$ and $\rho_0, \ldots, \rho_{n-1} < \rho$ either $\{i < \lambda: \mathscr{A} \models \varphi[a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)]\} \in D_\rho$ or $\{i < \lambda: \mathscr{A} \not\models \varphi[a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)]\} \in D_\rho$.

(5) For every $\varphi(v_0, \ldots, v_{n-1}) \in \mathrm{Fml}(L_x)$ and $\rho_0, \ldots, \rho_{n-1} < \rho$ if $\{i < \lambda: \mathscr{A} \models \varphi[a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)]\} \in D_\rho$ then $\{i < \lambda: \mathscr{B} \models \varphi[b_{\rho_0}(i), \ldots, b_{\rho_{n-1}}(i)]\} \in D_\rho$.

We construct these sequences by a back-and-forth induction. Let $D_0 = \{\lambda\}$ and select F_0 as in Lemma 6.1.10. Notice that for each $\varphi \in \mathrm{Sent}(L_x)$ either $\{i < \lambda: \mathscr{A} \models \varphi\}$ or $\{i < \lambda: \mathscr{A} \not\models \varphi\} = \lambda$; similarly with $\mathscr{B}$ in place of $\mathscr{A}$. Thus for $\rho = 0$ (4) and (5) hold, relying on our hypothesis. Furthermore, if all these conditions hold for all $\rho < \eta$ and η is a limit, then they also hold for η, where F_η and D_η are defined as required by (2).

Suppose that $\rho = \lambda_0 + 2m$, λ_0 a limit and $m < \omega$. Let a_ρ be the first member of ${}^\lambda|\mathscr{A}| - \{a_{\rho'}: \rho' < \rho\}$; let B be the first subset of λ so that B, $\lambda - B \notin D$; we'll define $F_{\rho+1}$, $D_{\rho+1}$ and $b_{\rho+1}$. For each $\varphi(v, v_0, \ldots, v_{n-1}) \in \mathrm{Fml}(L_x)$ and

$\rho_0, \ldots, \rho_{n-1} < \rho$ let:

$$X^+ = X^+(\varphi, \rho_0, \ldots, \rho_{n-1})$$
$$= \{i < \lambda : \mathscr{A} \models [a_\rho(i), a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)]\}.$$

There are $\lambda + \text{card}(\rho)$ such sets. Since $(F_\rho, \{\}, D_\rho)$ is $\lambda + \text{card}(\rho)$-consistent, by Lemma 6.1.13(ii) there are $F' \subseteq F$ and D' with $D_\rho \subseteq D'$ so that:

$\text{card}(F_\rho - F') \leq \lambda + \text{card}(\rho)$;

either $B \in D'$ or $\lambda - B \in D'$;

for each $\varphi, \rho_0, \ldots, \rho_{n-1}$ as above either $X^+ \in D'$ or $\lambda - X^+ \in D'$;

$(F', \{\}, D')$ is $\lambda + \text{card}(\rho)$-consistent.

Let:

$$\Gamma^+ = \{(\varphi, \rho_0, \ldots, \rho_{n-1}) : X^+(\varphi, \rho_0, \ldots, \rho_{n-1}) \in D'\};$$
$$Y^+ = Y^+(\varphi, \rho_0, \ldots, \rho_{n-1})$$
$$= \{i < \lambda : \mathscr{A} \models (\exists v)\varphi\, [a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)]\};$$
$$Z^+ = Z^+(\varphi, \rho_0, \ldots, \rho_{n-1})$$
$$= \{i < \lambda : \mathscr{B} \models (\exists v)\varphi\, [b_{\rho_0}(i), \ldots, b_{\rho_{n-1}}(i)]\}.$$

If $(\varphi, \rho_0, \ldots, \rho_{n-1}) \in \Gamma^+$ then $Y^+ \in D'$, since $X^+ \subseteq Y^+$ (using our choice of x). Then $Z^+ \in D_\rho$, for if otherwise then by (5), $Y^+ \notin D_\rho$; so by (4), $\lambda - Y^+ \in D_\rho$, giving $\lambda - Y^+ \in D'$, contrary to D' being non-trivial. Applying Lemma 10 to $\mathscr{B}$ we get $b_\rho : \lambda \to |\mathscr{B}|$, $F_{\rho+1} \subseteq F'$, and $D_{\rho+1}$ with $D' \subseteq D_{\rho+1}$ so that $(F_{\rho+1}, \{\}, D_{\rho+1})$ is $\lambda + \text{card}(\rho)$-consistent and:

for each $(\varphi, \rho_0, \ldots, \rho_{n-1}) \in \Gamma^+$,

$\{i < \lambda : \mathscr{B} \models \varphi\, [b_\rho(i), b_{\rho_0}(i), \ldots, b_{\rho_{n-1}}(i)]\} \in D_{\rho+1}$.

(0) through (5) are now satisfied for $\rho + 1$ in place of ρ.

Now let $b_{\rho+1}$ be the first member of ${}^\lambda|B| - \{b_{\rho'} : \rho' < \rho\}$. We define $F_{\rho+2}$, $D_{\rho+2}$ and $a_{\rho+2}$. For φ and $\rho_0, \ldots, \rho_{n-1}$ as above let:

$$X^- = X^-(\varphi, \rho_0, \ldots, \rho_{n-1})$$
$$= \{i < \lambda : \mathscr{B} \models^{\text{w}} \varphi\, [b_\rho(i), b_{\rho_0}(i), \ldots, b_{\rho_{n-1}}(i)]\}.$$

Form $F'' \subseteq F_{\rho+1}$ and D'' with $D_{\rho+1} \subseteq D''$ so that:

$\text{card}(F_{\rho+1} - F'') \leq \lambda + \text{card}(\rho)$;

for each $\varphi, \rho_0, \ldots, \rho_{n-1}$ as above either X^- or $\lambda - X^- \in D''$;

$(F'', \{\}, D'')$ is $\lambda + \text{card}(\rho)$-consistent.

Let:

$$\Gamma^- = \{(\varphi, \rho_0, \ldots, \rho_{n-1}) : X^-(\varphi, \rho_0, \ldots, \rho_{n-1}) \in D''\},$$
$$Y^- = Y^-(\varphi, \rho_0, \ldots, \rho_{n-1}) = \{i < \lambda : \mathscr{A} \models^w (\exists v)\varphi\, [a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)]\},$$
$$Z^- = Z^-(\varphi, \rho_0, \ldots, \rho_{n-1}) = \{i < \lambda : \mathscr{B} \models^w (\exists v)\varphi\, [a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)]\}.$$

If $(\varphi, \rho_0, \ldots, \rho_{n-1}) \in \Gamma^-$ then $Z^- \in D''$, since $X^- \subseteq Z^-$. Then $Y^- \in D_{\rho+1}$, for otherwise

$$\{i < \lambda : \mathscr{A} \models \neg(\exists v)\varphi\, [a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)]\} \in D_{\rho+1},$$

since $\rho + 1$ satisfies (4); since $\rho + 1$ also satisfies (5):

$$\{i < \lambda : \mathscr{B} \models \neg(\exists v)\varphi\, [a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)]\} \in D_{\rho+1}.$$

Since $Z^- \in D''$, this contradicts the non-triviality of D''. Applying Lemma 10 to $\mathscr{A}$ we get $a_{\rho+1} : \lambda \to |\mathscr{A}|$, $F_{\rho+2} \subseteq F''$, and $D_{\rho+1}$ with $D'' \subseteq D_{\rho+1}$ so that:

$\mathrm{card}(F'' - F_{\rho+2}) \leq \lambda + \mathrm{card}(\rho)$;

$(F_{\rho+2}, \{\ \}, D_{\rho+2})$ is $\lambda + \mathrm{card}(\rho)$-consistent;

for each $(\varphi, \rho_0, \ldots, \rho_{n-1}) \in \Gamma^-$

$\{i < \lambda : \mathscr{A} \models^w \varphi\, [a_{\rho+1}(i), a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)]\} \in D_{\rho+2}$.

Now (0) through (5) are satisfied with $\rho + 2$ for ρ.

Finally, we let $D = \bigcup \{D_\rho : \rho < 2^\lambda\}$ and $\pi(a_\rho) = b_\rho$ for all $\rho < 2^\lambda$. The construction insures that D is an ultrafilter on λ and π maps $|\mathscr{A}|$ one–one onto $|\mathscr{B}|$. For $\vec{a} \in |\prod_D \mathscr{A}|^n$ and $\varphi \in \mathrm{Fml}(L_x)$ with free variables among $v_0, \ldots, v_{n-1}$, letting $\pi\vec{a} = (\pi(a_0), \ldots, \pi(a_{n-1}))$, we have:

(*) if $\prod_D \mathscr{A} \models \varphi\, [\vec{a}]$, then $\prod_D \mathscr{B} \models \varphi\, [\pi\vec{a}]$;

if $\prod_D \mathscr{A} \dashv \varphi\, [\vec{a}]$, then $\prod_D \mathscr{B} \dashv \varphi\, [\pi\vec{a}]$.

Say $\prod_D \mathscr{A} = (A, \mathscr{E}_0, \mathscr{N}_0)$, $\prod_D \mathscr{B} = (B, \mathscr{E}_1, \mathscr{N}_1)$. (*) insures that $\pi : \mathscr{E}_0 \sqsubseteq \mathscr{E}_1$. If '=' $\in \mathrm{lex}_x$ then (*) insures that $\pi : \mathscr{N}_0 \sqsubseteq \mathscr{N}_1$; thus $\pi : \prod_D \mathscr{A} \sqsubseteq \prod_D \mathscr{B}$. If '$=_s$' $\in \mathrm{lex}_x$ then (*) insures that $\pi : \mathscr{N}_0 \cong \mathscr{N}_1$; thus $\pi : \prod_D \mathscr{A} \sqsubseteq_s \prod_D \mathscr{B}$. Finally, if '⊤' $\in \mathrm{lex}_x$ we can strengthen (*) to:

(**) $\prod_D \mathscr{A} \models \varphi\, [\vec{a}]$ iff $\prod_D \mathscr{B} \models \varphi\, [\pi\vec{a}]$;

$\prod_D \mathscr{A} \dashv \varphi\, [\vec{a}]$ iff $\prod_D \mathscr{B} \dashv \varphi\, [\pi\vec{a}]$;

so $\pi : \prod_D \mathscr{A} \cong \prod_D \mathscr{B}$. This completes the proof for $x = 1, \ldots$.

For $x = 0, \ldots$ we modify the previous construction as follows. Restrict conditions (4) and (5) to $\varphi(v_0, \ldots, v_{n-1}) \in C_x(K_0) \cap \mathrm{Fml}(L_x)$; for

$\rho_0, \ldots, \rho_{n-1} < \rho < 2^\lambda$ and $\varphi(v, v_0, \ldots, v_{n-1}) \in C_x(K_0) \cap \mathrm{Fml}(L_x)$ define $X^+(\varphi, \rho_0, \ldots, \rho_{n-1})$ to be:

$$\{i < \lambda: \mathscr{A} \models \varphi\,[a_\rho(i), a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)] \text{ and }$$
$$\mathscr{A} \models (\forall v)(\varphi \vee \neg\varphi)\,[a_{\rho_0}(i), \ldots, a_{\rho_{n-1}}(i)]\}.$$

(*) then holds for $\varphi \in C_x(K_0)$. Using our inductive characterization of $N_{i,\mathrm{Core}_x(K_0)}$ we have:

$$\pi: \Big(\prod_D \mathscr{A}\Big) \upharpoonright \mathrm{Core}_x(K_0) \sqsubseteq \prod_D \mathscr{B};$$

and if '=' $\notin \mathrm{lex}_x$ we can strengthen this by replacing '$\sqsubseteq$' with '$\sqsubseteq_s$'. This gives us the lemma for '$\top$' $\notin \mathrm{lex}_x$. For $x = 0, \top, \ldots$ we actually have (**) for all φ, since $\top\varphi \in C_x(K_0)$; so as above $\pi: \prod_D \mathscr{A} \cong \prod_D \mathscr{B}$.

The proof of Lemma 9 is a straightforward modification of the previous construction and is left to the reader.

Let (K_0, K_1) be core$'_x$-closed iff for any $\mathscr{A} \in \mathrm{MOD}$ if $\mathscr{A} \upharpoonright \mathrm{Core}_x(K_0) \cap \mathrm{Core}_x(\mathrm{MOD} - K_1) \in K_1$ then $\mathscr{A} \in K_1$.

Theorem 7. *Let (K_0, K_1) be bounded. (K_0, K_1) is basic x-elementary iff $K_0 \subseteq K_1$, K_0, K_1, $\mathrm{MOD} - K_0$ and $\mathrm{MOD} - K_1$ are closed under isomorphism and ultraproducts, and these conditions are met*:

(i) *If $x = 0$ [$= 0, s$ or $0, b$] then K_1 is downward monotonic* [*s-monotonic*] *and (K_0, K_1) is core$'_x$-closed and core$'_x$-bivalent.*

(ii) *If $x = 0, u$ [$= 0, u, s$ or $0, u, b$] then K_1 is downward monotonic* [*s-monotonic*] *and (K_0, K_1) is core$'_x$-closed and either core$'_x$-bivalent or u-defined.*

(iii) *If $x = 0, \top$ or $0, \top, b$ then (K_0, K_1) is core$'_x$-bivalent.*

(iv) *If $x = 0, \top, u$ then either (K_0, K_1) is either core$'_x$-bivalent or u-defined.*

(v) *If $x = 1$ [$= 1, s$] [$= 1, b$] then (K_0, K_1) is monotonic* [*s-monotonic*] [*s-monotonic*], *cross-closed* [*s-crosstonic*] [*s-crosstonic*] *and center-bivalent* [*s-center-bivalent*] [*b-center-bivalent*];

(vi) *If $x = 1, u$ [$= 1, u, s$] then (K_0, K_1) is monotonic* [*s-monotonic*] *and crosstonic* [*s-crosstonic*].

(vii) *If $x = 1, \top$ [$= 1, \top, b$] then (K_0, K_1) is center-bivalent* [*b-center-bivalent*].

Only one new element is involved in this proof. For $x = 0$ we need to show: if $\mathrm{Th}_x(\mathscr{A}) \cap C_x(K_0) \cap C_x(\mathrm{MOD} - K_1) \subseteq \mathrm{Th}_x(\mathscr{B})$ then there is an ultrafilter D so that $(\prod_D \mathscr{A}) \upharpoonright C_x(K_0) \cap C_x(\mathrm{MOD} - K_1) \sqsubseteq \prod_D \mathscr{B}$; analogously for $x = 0, s$ and $\sqsubseteq_s$, etc. The proof is a straightforward modification of that of Lemma 8.

9. Back to x-elementary and weakly x-elementary classes

For '$\top$' $\in \mathrm{lex}_x$ and any $K \subseteq \mathrm{MOD}$ the following are equivalent: (i); (i^w); (ii).

(i) K is x-elementary.

(i^{w}) K is weakly x-elementary.

(ii) (K, K) is x-elementary.

If $\Gamma \subseteq \mathrm{Sent}(\mathbb{L}_x)$ defines K, then $\{\top\varphi: \varphi \in \Gamma\}$ weakly defines K; so (i) implies (i^{w}). If Γ weakly defines K then $\{\neg F\varphi: \varphi \in \Gamma\}$ defines (K, K); so (i^{w}) implies (ii). Clearly (ii) implies (i).

Let K be closed under compatibility [s-compatibility] iff (K, K) is crosstonic [s-crosstonic]. Then for '$=$' $\in \mathrm{lex}_x$ ['$=$' $\notin \mathrm{lex}_x$] by Theorem 5, if K is bounded, then (i) is equivalent to:

(iii) K is closed under isomorphism, ultraproducts and compatibility [s-compatibility] and $\mathrm{MOD} - K$ is closed under ultrapowers.

For $x = 1, u$ [$1, u, s$ or $1, u, b$] and K bounded, the following are equivalent: (i); (iv); (v).

(iv) (K, MOD) is x-elementary.

(v) K is upward monotonic [s-monotonic] and closed under isomorphism, ultraproducts, and $\mathrm{MOD} - K$ is closed under ultrapowers.

Furthermore these are equivalent: (i^{w}); (iv^{w}); (v^{w}).

(iv^{w}) $(\{\,\}, K)$ is x-elementary.

(v^{w}) K is downward monotonic [s-monotonic] and closed under isomorphism, ultraproducts, and $\mathrm{MOD} - K$ is closed under ultrapowers.

Clearly (iv) implies (i) and (i) implies (v); by Theorem 6 (v) implies (iv). Similarly for the weak versions of these.

For $x = 1$ [$1, s$ or $1, b$], by Observation 3(ii), (i) and (v) are equivalent, as are (i^{w}) and (v^{w}). For such x we can say more. For any $K \subseteq \mathrm{MOD}$ let K be closed under core_x-restriction iff for any $\mathscr{A} \in \mathrm{K}$, $\mathscr{A} \restriction \mathrm{Core}_x(K) \in K$. Let

$$K^* = \{\mathscr{B} \in \mathrm{MOD}\colon \text{ for some } \mathscr{A} \in K, \mathscr{A} * \mathscr{B}\};$$
$$K^{*s} = \{\mathscr{B} \in \mathrm{MOD}\colon \text{ for some } \mathscr{A} \in K, \mathscr{A} *_s \mathscr{B}\}.$$

Let K^x be the intersection af all $K' \subseteq \mathrm{MOD}$ such that $K^* \subseteq K'$ [$K^{*s} \subseteq K'$], K' is downward monotonic [s-monotonic] and closed under isomorphism and ultraproducts and $\mathrm{MOD} - K'$ is closed under ultrapowers. Since MOD is such a class this intersection is non-vacuous. Since $K^* \subseteq K^1$ [$K^{*s} \subseteq K^{1,s} = K^{1,b}$], (K_0, K_1) is crosstonic [s-crosstonic]; furthermore K^1 [$K^{1,s}$] is downward monotonic [s-monotonic] and closed under isomorphism and ultraproducts and $\mathrm{MOD} - K^1$ [$\mathrm{MOD} - K^{1,s}$] is closed under ultrapowers. For $x = 0$ [$0, s$ or $0, b$] form K^x by adding to the conditions on K' that for any $\mathscr{A} \in \mathrm{MOD}$ if $\mathscr{A} \restriction \mathrm{Core}_x(K) \in K'$ then $\mathscr{A} \in K'$. Then K^x meets all of the above conditions and furthermore (K, K^x) is core_x-closed.

For $x = i$ [i, s or i, b], $i \in 2$, and any bounded $K \subseteq \mathrm{MOD}$, these are equivalent: (i); (vi); (v′).

(vi) (K, K^x) is x-elementary.

(v′) (v) holds and if $x = 0$ [$0, s$ or $0, b$] then K is closed under core_x-restriction.

Clearly (vi) implies (i) and (i) implies (v′). Assume (v′). For $x = 1$ [$1, s$ or $1, b$] we show that (K, K^x) is center-bivalent. Consider $x = 1$. For any $\gamma \in \mathrm{Center}(K)$

let

$$K_\gamma = K^* \cup \{\mathscr{A} \in \text{MOD}: \mathscr{A} \text{ is not total on } \{\gamma\}\}.$$

Suppose $\mathscr{A} \in \text{MOD} - K$ is total on Center(K). Then $\mathscr{A} \notin K^*$; for otherwise fix $\mathscr{B} \in K$ so that $\mathscr{B} * \mathscr{A}$; then $\mathscr{B} \restriction \text{Center}(K) \sqsubseteq \mathscr{A}$; since $\mathscr{B} \restriction \text{Center}(K) \in K$, $\mathscr{A} \in K$. Thus $\mathscr{A} \notin K_\gamma$; since K_γ meets the conditions on the K' in the definition of K^1, $\mathscr{A} \notin K^1$. A similar argument applies for $\mathsf{x} = 1, \mathsf{s}$ or $1, \mathsf{b}$.

For $\mathsf{x} = 0$ [$0, \mathsf{s}$ or $0, \mathsf{b}$] we show that (K, K^x) is core$_\mathsf{x}$-bivalent. Consider $\mathsf{x} = 0$. For $\theta \in \text{Core}_0(K)$ let:

$$K_\theta = K^* \cup \{\mathscr{A} \in \text{MOD}: \theta \text{ is not } \mathscr{A}\text{-bivalent}\}.$$

Consider $\mathscr{A} \in \text{MOD} - K$ so that Core$_0$(K) is $\mathscr{A}$-bivalent. Since K is closed under core$_0$-restriction we can argue as above to get $\mathscr{A} \notin K_\theta$; so as above (K, K^0) is core$_0$-bivalent. A similar argument applies for $\mathsf{x} = 0, \mathsf{s}$ or $0, \mathsf{b}$. So the right hand-side of Theorem 6 is satisfied; by Theorem 6, (vi) follows.

Note: for x as above, if K is x-elementary, then K^x is the minimum K' so that (K, K') is x-elementary.

Problems: Is there a more constructive way to describe K^x? For K satisfying (v$^\mathsf{w}$) is there a maximum $K' \subseteq K$ so that (K', K) is x-elementary?

10. Partial continuous monadic quantifiers

In this section we'll prove a quantificational analog of the fact that {'⊃', 'u', 'T'} is truth-functionally complete for three-valued logic. This involves extending the notion of a continuous quantifier, as presented in [5], to partial models.

A signature z is a finite sequence $\langle \zeta_0, \ldots, \zeta_{n-1} \rangle$ of predicates, $\zeta_i \in \text{PRED}(n_i)$ for $i < n$. Set $Pred = \{\zeta_0, \ldots, \zeta_{n-1}\}$. For $K \subseteq \text{MOD}$ let K be closed under *Pred*-restricted isomorphism iff for any $\mathscr{A}, \mathscr{B} \in \text{MOD}$ with $\mathscr{A} \restriction Pred \cong \mathscr{B} \restriction Pred$, if $\mathscr{A} \in K$ then $\mathscr{B} \in K$.

Let (K_0, K_1) be a partial quantifier with signature z iff $K_0 \subseteq K_1 \subseteq \text{MOD}$, Center$(K_0, K_1) \subseteq \{\zeta_0, \ldots, \zeta_{n-1}\}$, and K_0 and K_1 are closed under *Pred*-restricted isomorphisms. (K_0, K_1) is monadic iff for all $i < n$, $n_i = 1$.

A quantifier-expression χ with signature z has the following formation-rule in $L_{\mathsf{x},\chi}$:

> if for each $i < n$, $\vec{v}_i$ is a sequence of n_i distinct variables and $\varphi_0, \ldots, \varphi_{n-1}$ are formulae, then $(\chi: \vec{v}; \ldots; \vec{v}_{n-1})(\varphi_0, \ldots, \varphi_{n-1})$ is a formula.

For $\vec{v} = (v_0, \ldots, v_{n-1})$ a sequence of distinct variables, φ a formula, $\mathscr{A} \in$ MOD and α an $\mathscr{A}$-assignment let:

$$\text{ext}^+_{\vec{v}} = \{\vec{a} \in |\mathscr{A}|^n: \mathscr{A} \vDash \varphi[\alpha^{\vec{v}}_{\vec{a}}]\};$$

$$\text{ext}^-_{\vec{v}} = \{\vec{a} \in |\mathscr{A}|^n: \mathscr{A} \dashv \varphi[\alpha^{\vec{v}}_{\vec{a}}]\}.$$

Let $\mathscr{A}(z, \varphi_0, \ldots, \varphi_{n-1})$ be the model $(|\mathscr{A}|, \mathscr{E}, \{\ \})$ with $\mathrm{dom}(\mathscr{E}) = Pred$ and for all $i < n$:

$$\mathscr{E}(\zeta_i) = \begin{cases} 1 & \text{if } \vec{a} \in \mathrm{ext}^{+}_{\vec{v}_i}(\mathscr{A}, \alpha, \varphi_i), \\ 0 & \text{if } \vec{a} \in \mathrm{ext}^{-}_{\vec{v}_i}(\mathscr{A}, \alpha, \varphi_i). \end{cases}$$

To let χ represent the partial quantifier (K_0, K_1) is to add these clauses to our inductive definition of $\models$ and $\dashv$:

$$\mathscr{A} \models (\chi\colon \vec{v}_0; \ldots; \vec{v}_{n-1})(\varphi_0, \ldots, \varphi_{n-1})\,[\alpha]$$
$$\text{iff} \quad \mathscr{A}(z, \varphi_0, \ldots, \varphi_{n-1}) \in K_0;$$
$$\mathscr{A} \dashv (\chi\colon \vec{v}_0; \ldots; \vec{v}_{n-1})(\varphi_0, \ldots, \varphi_{n-1})\,[\alpha]$$
$$\text{iff} \quad \mathscr{A}(z, \varphi_0, \ldots, \varphi_{n-1}) \notin K_1.$$

Suppose $\mathscr{A} = (|\mathscr{A}|, \mathscr{E}, \mathscr{N})$ and $B \subseteq |\mathscr{A}|$. For $n > 0$ let

$$\mathscr{E} \restriction B(\zeta) = \mathscr{E}(\zeta) \cap B^n \times 2 \quad \text{for } \zeta \in \mathrm{PRED}(n);$$
$$\mathscr{N} \restriction B(\xi) = \mathscr{N}(\xi) \cap B^{n+1} \quad \text{for all } \zeta \in \mathrm{PRED}(n);$$

for $n = 0$, $\mathscr{E} \restriction B(\zeta) = \mathscr{E}(\zeta)$ and $\mathscr{N} \restriction B(\xi) = \mathscr{N}(\xi)$ for $\mathscr{N}(\xi) \in B$, $\mathscr{N} \restriction B(\xi)\uparrow$ otherwise. Let $\mathscr{A} \restriction B = (\mathscr{B}, \mathscr{E} \restriction B, \mathscr{N} \restriction B)$. Finally let $\mathscr{B} \subseteq \mathscr{A}$ iff for some B, $\mathscr{B} = \mathscr{A} \restriction B$.

For $K \subseteq \mathrm{MOD}$ let $\mathscr{B}$ secure $\mathscr{A}$ into [out of] K iff $\mathscr{B} \subseteq \mathscr{A}$ and for every $\mathscr{A}'$, if $\mathscr{B} \subseteq \mathscr{A}' \subseteq \mathscr{A}$, then $\mathscr{A}' \in K$ [$\mathscr{A}' \notin K$]. Let $\mathscr{B}$ secure $\mathscr{A}$ for (K_0, K_1) iff for each $i \in 2$ either $\mathscr{B}$ secures $\mathscr{A}$ into K_i or secures $\mathscr{A}$ out of K_i. A partial quantifier (K_0, K_1) is continuous [uniformly continuous with bound $q \in \omega$] iff for every $\mathscr{A} \in \mathrm{MOD}$ there is a $\mathscr{B} \in \mathrm{MOD}$, $\mathscr{B}$ finite [$\mathrm{card}(\mathscr{B}) \leq q$] and $\mathscr{B}$ secures $\mathscr{A}$ for (K_0, K_1).

Theorem 9. *For (K_0, K_1) a monadic partial quantifier with signature $s = (\zeta_0, \ldots, \zeta_{n-1})$ the following are equivalent:*
(i) *(K_0, K_1) is continuous.*
(ii) *(K_0, K_1) is uniformly continuous.*
(iii) *A sentence of $L = L_{1,\mathsf{T},\mathsf{u}}(Pred, \{\ \})$ defines (K_0, K_1).*

To show: (iii) implies (ii). For $f\colon n \to 3 = \{0, 1, 2\}$, let $\theta_f(v)$ be:

$$\&\{\mathsf{T}\zeta_i(v)\colon f(i) = 0\}\ \&\ \&\{\mathsf{F}\zeta_i(v)\colon f(i) = 1\}$$
$$\&\ \&\{\mathsf{U}\zeta_i(v)\colon f(i) = 2\}.$$

Let a basic sentence for f be one of the form

$$(\exists v_0) \cdots (\exists v_r)(\&\{\theta_f(v_i)\colon i \leq q\}\ \&\ \&_{i<j\leq r} v_i \neq v_j)$$

for some $r < \omega$. Let a pre-normal sentence be a conjunction of sentences that may be basic, negated basic or ‘u’; let a normal sentence be a disjunction of pre-normal sentences. Familiar transformations involving ‘$\exists$’, together with some obvious new ones for ‘T’, yield the following: if $\varphi \in \mathrm{Sent}(L)$, then there is a normal sentence φ' equivalent to φ. But then φ may be prenexed into each of these forms:

$$(\exists v_0)\cdots(\exists v_{q-1})(\forall \mu_0)\cdots(\forall \mu_{t-1})\bar{\varphi};$$
$$(\forall \mu_0)\cdots(\forall \mu_{t-1})(\exists v_0)\cdots(\exists v_{q-1})\bar{\varphi}.$$

Suppose φ defines (K_0, K_1) and $\mathscr{A}$ is given. If $\mathscr{A} \in K_0$ then $\mathscr{A} \models (\exists \vec{v})(\forall \vec{\mu})\bar{\varphi}$; select witnesses $a_0, \ldots, a_q \in |\mathscr{A}|$ so that $\mathscr{A} \models (\forall \vec{\mu})\bar{\varphi}\,[\vec{a}]$; then $\mathscr{A} \restriction \{a_0, \ldots, a_{q-1}\}$ secures $\mathscr{A}$ into K_0. If $\mathscr{A} \notin K_0$ then $\mathscr{A} \models^{w} (\exists \vec{\mu})(\forall \vec{v})\neg\bar{\varphi}$; a similar argument yields $b_0, \ldots, b_{t-1} \in |\mathscr{A}|$ so that $\mathscr{A} \restriction \{b_0, \ldots, b_{t-1}\}$ secures $\mathscr{A}$ out of K_0. An analogous argument applies to K_1. Thus (K_0, K_1) is uniformly continuous with bound $\max(q, t)$.

To show: (i) implies (iii). Assume (i). If $\mathscr{A}$ is a model for *Pred* and $f : n \to 3$, let

$$s(\mathscr{A})(f) = \operatorname{card}\{a : \mathscr{A} \models \theta_f(v)[a]\};$$

think of $s(\mathscr{A})$ as a 3^n-tuple with components indexed by the $f \in 3^n$ listed in lexicographic order. For $i \in 2$ let $s(K_i) = \{s(\mathscr{A}) : \mathscr{A} \in K_i\}$. We then have $\mathscr{A} \in K_i$ iff $s(\mathscr{A}) \in s(K_i)$, using the fact that K_i is closed under *Pred*-restricted isomorphisms and that if $s(\mathscr{A}) = s(\mathscr{B})$ then $\mathscr{A} \restriction \text{Pred} \cong \mathscr{B} \restriction \text{Pred}$.

For $\vec{\kappa} = (\kappa_0, \ldots, \kappa_{m-1})$ and $\vec{\kappa}' = (\kappa'_0, \ldots, K'_{m-1})$ m-tuples of cardinals let $\vec{\kappa} \subseteq \vec{\kappa}'$ iff for all $i < m$, $\kappa_i \leq \kappa'_i$; let $\operatorname{card}(\vec{\kappa}) = \sum_{i<m} \kappa_i$. For C_0 and C_1 classes of such m-tuples, our definitions of continuity and uniform continuity with bound q may be extended to (C_0, C_1), following the analogous definition in [5]. As in [5] we may prove: if (C_0, C_1) is continuous, then (C_0, C_1) is uniformly continuous. This involves an induction on m; see the proof of Theorem 5 of [5] for details. Finally, as in [5] we have:

(K_0, K_1) is [uniformly] continuous
iff $(s(K_0), s(K_1))$ is [uniformly] continuous.

Assuming (K_0, K_1) to be continuous, we have $(s(K_0), s(K_1))$ uniformly continuous. From that fact it's not hard to produce a sentence of $L_{1,\mathrm{T},\mathrm{u}}(\mathit{Pred}, \{\,\})$ defining (K_0, K_1).

Alternatively, we could simply take the above to show that (i) implies (ii) and get from (ii) to (iii) using Theorem 7. For suppose that (K_0, K_1) is uniformly continuous with bound q. It suffices to show that for $j \in 2$, K_j and $\text{MOD} - K_j$ are closed under ultraproducts.

Suppose that for each $i \in I$, $\mathscr{A}_i \in K_j$. Without loss of generality, suppose each $\mathscr{A}_i$ is a model for *Pred*. For each $i \in I$ fix $\mathscr{B}_i \subseteq \mathscr{A}_i$ so that $\operatorname{card}(\mathscr{B}_i) \leq q$ and $\mathscr{B}_i$ secures $\mathscr{A}_i$ into K_j. Now let $\mathscr{B}$ secure $\prod_D \mathscr{A}_i$ for (K_0, K_1), $\operatorname{card}(\mathscr{B}) \leq q$. Setting $B_i = \{f(i) : f \in |\mathscr{B}|\}$, $\operatorname{card}(B_i) \leq q$; set $\mathscr{C}'_i = \mathscr{A}_i \restriction (B_i \cup |\mathscr{B}_i|)$; so $\operatorname{card}(\mathscr{C}_i) \leq 2q$. Claim: for some $i_0 \in K$, $\prod_D \mathscr{C}_i \cong \mathscr{C}_{i_0}$. If I is finite, D is principal, so this is trivial. Otherwise it suffices to note that there are finitely many isomorphism-types for models for *Pred* with cardinality $\leq 2q$; so for some $X \in D$ for all $i, i' \in X$: $\mathscr{C}_i \cong \mathscr{C}_{i'}$; any $i_0 \in X$ is as desired. Since $\mathscr{B}_{i_0}$ secures $\mathscr{A}_{i_0}$ into K_j and $\mathscr{B}_{i_0} \subseteq \mathscr{C}_{i_0}$, $\mathscr{C}_{i_0} \in K_j$; so

$\prod_D \mathscr{C}_i \in K_j$. Since $\mathscr{B} \subseteq \prod_D \mathscr{C}_i \subseteq \prod_D \mathscr{A}_i$ and $\mathscr{B}$ either secures $\prod_D \mathscr{A}_i$ into or out of K_j, it must secure $\prod_D \mathscr{A}_i$ into K_j, as required. A similar argument shows that $\mathrm{MOD} - K_j$ is closed under ultraproducts. Since Center$(K_0, K_1) \subseteq Pred$, (iii) follows by Theorem 7.

11. A truth-value gap or a third truth-value?

According to Michael Dummett, the semantic component of a theory of meaning might utilize more than two truth-values, but it would have no use for a truth-value gap:

> Given that, e.g., "King Arthur did not defeat the Saxons" is construed as the negation of "King Arthur defeated the Saxons", we need a distinction between ... being false and being neither true nor false; but nothing has emerged to give any ground for regarding this latter state as one of having no truth-value at all, rather than as one of having a second undesignated truth-value, which we may call 'the value X'. ([2], p. 425)

Of course, as Dummett acknowledges, "It might be thought that ... the difference between, saying that it has not truth-value and that it has the value X is a mere indifferent matter of terminology." In this section I'll try to make some sense of this distinction.

The objection Dummett acknowledges is right to this extent: on philosophically neutral semantic grounds, there is no distinction allowing for lack of a truth-value and allowing for a third truth-value. But some philosophical positions provide a background against which one logical lexicon may be said to merely open a truth-value gap, while another introduces a third truth-value.

In presenting the truth-tables in Section 1 it was convenient to use '|' in addition to '⊨' and '⫤'; and this might suggest that we are using three truth-values. But such tables are merely an alternative presentation of the inductive clauses in a simultaneous inductive definition of ⊨ and ⫤; '|' was introduced afterwards as a convenient abbreviation for "the waste case", one obviously parasitic on the fundamental inductive definition. So at least in our order of exposition, truth and falsity were fundamental in a way in which neither-true-nor-false was not. But we can't conclude that *our semantics* only involves two truth-values, and a truth-value gap. Other orders of exposition were possible.

What we make of the question "A third truth-value or a truth-value gap?" depends on what we make of talk about truth-values. After his quoted remark about "the value X", Dummett goes on to point out that Frege's philosophical apparatus, rather than facts about linguistic practice, can make this question into a genuine issue.

Frege took talk of truth-values at face value: according to him they are genuine objects, and sentences are really singular terms "designed" to stand for them. This background lends some substance to the issue. A sense-bearing language embodying a three-valued semantics would add a new object to the ontological package carried by other two-valued languages. For Frege, 'if ... then ...' stands for a function from $\{\text{True, False}\}^2$ into {True, False}. If φ and ψ are sentences that fail to stand for anything, then the concatenation of 'if', φ, 'then' and ψ (hereafter 'if φ then ψ') also doesn't stand for anything; it's on all fours with '𝐟(𝐚)' for '𝐟' a function-constant when '𝐚' fails to designate. Thus the Fregean "if ... then ..." is modelled by our '$\supset$'. If our semantic theory takes our logic to be modelled by that of lex_0, or $\text{lex}_{0,\text{s}}$, we have not introduced a third truth-value, but only recognized a truth-value gap. But any step beyond $\text{lex}_{0,\text{u}}$, e.g. to lex_1 or $\text{lex}_{0,\text{T}}$, would introduce a third truth-value. And that Frege would not want us to do: lex_0, or perhaps $\text{lex}_{0,\text{u}}$, is the Fregean lexicon.

If we reject Frege's assimilation of sentences to singular terms, and thus reject the doctrine that truth-values are genuine objects, our question seems to loose it's content. But we may reconstrue it as asking whether truth and falsehood, or better the status of being true or being false, differ significantly from the status of being neither true nor false. And the answer depends on the logical lexicon that we take to be in place in our language. (Note: all our lexica express negation; so all of them are symmetric with respect to truth and falsehood.)

Since {'$\supset$', 'u', 'T'} is a definitional base for all three-valued functions of finitely many arguments, for it, truth, falsehood and the third status are all on a par, with no asymmetries: the image of any expressible truth-function under any permutation of three truth-values is itself expressible. Similarly it plausible to label the first-order logic based on $\text{lex}_{1,\text{T},\text{u}}$ 'full three-valued elementary logic'; and it too yields no asymmetries between truth and falsehood on one hand and the third status on the other. More precisely, let (K_0, K_1) be any partial quantifier expressible using $\text{lex}_{1,\text{T},\text{u}}$; set $Z_0 = K_0$, $Z_1 = K_1 - K_0$ and Z_2 be $\text{MOD} - K_1$; for any permutation π of $\{1,2,3\}$, $(Z_{\pi 0}, Z_{\pi 0} \cup Z_{\pi 1})$ is a monadic partial quantifier expressible using $\text{lex}_{1,\text{T},\text{u}}$.

When we restrict ourselves to more narrow lexica, asymmetries emerge. For example, using $\text{lex}_{1,\text{T}}$, 'u' is not expressible; this may be viewed as a significant difference between ⊨ and ⫤ on one hand and | on the other. Do these differences in expressive power provide reasons for saying that one of these sub-lexica of $\text{lex}_{1,\text{T},\text{u}}$ introduce no third truth-value? Only a philosophical background could make such a metaphor apt. We'll now consider two such backgrounds.

What picture of language could lead us to say that $\text{lex}_{1,\text{u}}$ doesn't introduce a third truth-value? Suppose we replace 'true' and 'false' by 'verified' and 'falsified', and think of inquiry as involving a computational process that either terminates in verification or falsification (after a finite time), or which diverges, yielding no answer at any time. So with '𝐏' representing a 1-piece total decidable predicate and '𝐚' representing a name, '𝐏(𝐚)' is associated with a computation that first looks for a designatum for 'a'; if one is found, the decision-procedure associated with 'P' is then applied to that object; if none is found, the computation diverges.

Initially we might want to follow Frege and represent 'if ... then ...' as '⊃'. Given "If φ then ψ", suppose our computation for φ [ψ] yields verification [falsification]; we now know that if the computation for φ [ψ] converges, "If φ then ψ" will have been verified. We might then jump the gun and declare it already verified, without waiting for the second convergence. In doing this, we replace '⊃' by '⊃'. Then $\text{lex}_{1,u}$ would be the richest lexicon this picture could accommodate. Use of 'T' would be impossible, since at no time is it established that a computation diverges.

Though this picture does select a specific sub-lexicon of $\text{lex}_{1,T,u}$, it's not what we want, since it replaced truth-conditions by knowledge-conditions. But there is a picture of the sort we want that also selects $\text{lex}_{1,u}$. Consider the neo-Fregean who rejects Frege's assimilation of sentences to singular terms, but who accepts the more basic Fregean thesis underlying that assimilation: there are exactly two semantic roles available for sentences, namely being true and being false. ('Semantic role' is Dummett's term for the way in which an expression contributes to determining the truth-values of the sentences in which it occurs.) Thus a sentence that is neither true nor false plays no semantic role; it makes no contribution to determining whether a sentence of which it is a constituent is true or false. So if φ is a constituent of φ' and φ is neither true nor false, but φ' is true [false], φ makes no contribution to the latter fact; thus if we were to change the status of φ, leaving as much else as possible the same (e.g. by assigning a name a referent, or enlarging the domain of a function-expression or extension or antiextension of a predicate) this would not effect the truth [falsity] of φ'. This amounts to imposing monotonicity on our semantics; thus the neo-Fregean can use $\text{lex}_{1,u}$. But use of 'T' would not be allowed: it would require a third sort of semantic role for sentences; and that may be aptly described as introducing a third truth-value. (By constraining the allowable changes envisioned above to changes in the extensions of predicates, we'd impose s-monotonicity, thus enriching the permissible lexicon to $\text{lex}_{1,u,s}$ (equivalently $\text{lex}_{1,u,b}$). This of course is for neo-Fregeans who think that '=' and 'E' don't adequately model 'is identical to' and 'exists'.)

Up to now we've only considered sentences as bearers of truth-value. If propositions are the fundamental bearers of truth-value, I take it that these principles are axiomatic:

(1) If σ expresses p then: σ is true [false] iff p is true [false].

(2) If σ is true [false] then there is a proposition expressed by σ.

All this is compatible with orthodox Fregean doctrine and the neo-Fregean position just sketched. Suppose we go on to assume propositional bivalence:

(3) Each proposition is either true or false.

Using (1), (3) implies that if σ is neither true nor false, then σ fails to express a proposition. (Frege did say that some sentences express propositions (in his terminology, thoughts) without being true or false. But Evans has shown [3] that this involves a tension in the Fregean notion of sense; he also finds some textual suggestions that Frege was worried about allowing the existence of such sentences. In [4] McDowell finds this half-way position rather Russellian: had Russell abandoned his epistemological view that logically proper names couldn't fail to designate, he would have held this view.)

Now suppose we reject the neo-Fregean doctrine presented above, and allow that sentences which are neither true nor false play a third semantic role. We may still claim not to have, in effect, introduced a third truth-value by citing our adherence to (3), provided we also accept the following principle:

(4) Whether a sentence expresses a proposition can't depend on whether a constituent sentence is true or false.

This principle then rules out use of '$\supset$'. For suppose σ is $\varphi \supset \psi$ and φ is neither true nor false. If ψ is true, then so is σ; so by (2), σ expresses a proposition. If ψ is false, then σ is neither true nor false, and so fails to express a proposition. So whether or not σ expresses a proposition is sensitive to the truth-value of ψ, violating (4). These principles don't rule out use of '$\top$'; so again the permissible lexicon goes beyond that of the orthodox Fregean, this time to $\text{lex}_{0,\top}$, or perhaps $\text{lex}_{0,\top,u}$.

Note: (4) should not be confused with the content of 2.0211–2 of Wittgenstein's *Tractatus*; Wittgenstein's claim is that whether one sentence expresses a proposition can't depend on whether another sentence is true or false; that principle would rule out use of '$\top$', '$=_s$' or '$=_b$', since whether '$\mathbf{P}(\mathbf{a})$' expresses a proposition would depend on whether '$E_s(\mathbf{a})$' is true or false.

None of these attempts to clarify the difference between allowing for truth-valuelessness and introducing a third truth-value help vindicate Dummett's main claim. If the question of which of these holds for a given language is merely one of which of our lexica model the logic of that language, I see no reason to be sure that the favored lexica would not be of the former sort. In that case our theory of meaning for that language has use for a truth-value gap.

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