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UPPER BOUNDS ON LOCALLY COUNTABLE ADMISSIBLE INITIAL SEGMENTS OF A TURING DEGREE HIERARCHY[1]

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Abstract. Where AR is the set of arithmetic Turing degrees, $\mathbf{0}^{(\omega)}$ is the least member of $\{\boldsymbol{a}^{(2)} \mid \boldsymbol{a}$ is an upper bound on AR$\}$. This situation is quite different if we examine HYP, the set of hyperarithmetic degrees. We shall prove (Corollary 1) that there is an $\boldsymbol{a}$, an upper bound on HYP, whose hyperjump is the degree of Kleene's $\mathcal{O}$. This paper generalizes this example, using an iteration of the jump operation into the transfinite which is based on results of Jensen and is detailed in [3] and [4]. In § 1 we review the basic definitions from [3] which are needed to state the general results.

§1. Introduction. Where $A \subseteq \omega$, $\boldsymbol{a}$ is a Turing degree, and $A \in \boldsymbol{a}$, we may define a hierarchy of Turing degrees $\lambda\xi.\ \boldsymbol{a}^{(\xi)}$ on $\aleph_1^{L[A]}$. This hierarchy is studied in [2]. We shall review the basic definitions. Where X is any set, let

$L_0[X] = M_0[X] = \langle HF; \varepsilon \upharpoonright HF, X \cap HF; HF\rangle$;

$L_{\xi+1}[X] = \langle Y; \varepsilon \upharpoonright Y, X \cap Y; Y\rangle$ where Y is the collection of all sets first-order definable over $L_\xi[X]$;

$L_\lambda[X] = \bigcup_{\xi<\lambda} L_\xi[X]$, where λ is a limit ordinal;

$M_{\omega\xi}[X] = L_\xi[X]$;

$M_{\omega\cdot\xi+n}[X] = \langle Y; \varepsilon \upharpoonright Y; X \cap Y; Y\rangle$ where $Y = \Delta_n(L_\xi[X])$ and $1 \leq n < \omega$.

Both $L_\xi[X]$ and $M_\xi[X]$ are, by definition, structures; we shall abuse notation by letting '$L_\xi[X]$' and '$M_\xi[X]$' also stand for the universes of these structures. Note that if $A \equiv_T B$ then $M_\xi[A] = M_\xi[B]$ for any ordinal ξ. For $B \subseteq \omega$, B is a master code for ξ relative to $A \subseteq \omega$ iff:

$$\{F \in \omega^\omega \mid F \leq_T B\} = M_\xi[A] \cap \omega^\omega.$$

Master codes for ξ relative to A are unique up to Turing degree. $\lambda\xi\cdot\boldsymbol{a}^{(\xi)}$ is the sequence of the Turing degrees of the master codes relative to A, taken in increasing order, where $\boldsymbol{a} = \deg(A)$. More explicitly, let ξ be an $M[A]$-index iff $M_{\xi+1}[A] - M_\xi[A]$ contains a real. The fundamental theorem on master codes, due to Jensen, tells us:

ξ is an $M[A]$-index iff there is a master code for ξ relative to A.

The proof of this theorem provides a "normal form" for master codes. A structure $\langle X; E, F; X\rangle$, where $E \subseteq \omega \times \omega$, $F \subseteq \omega$ and $X = \text{Field}(E)$, is an $E_\alpha[Z]$ iff

$$\langle X; E, F; X\rangle \simeq \langle L_\alpha[Z]; \varepsilon \upharpoonright L_\alpha[Z], Z; L_\alpha[Z]\rangle.$$

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$\mathrm{Th}_n(E_\alpha[Z])$ is the $(\Sigma_n \cup \Pi_n)$ theory of $E_\alpha[Z]$. Then the master code for ξ relative to A is the least degree of the form $\deg(\mathrm{Th}_n(E_\alpha[A]))$, where $\xi = \omega \cdot \alpha + n, n < \omega$.

Let $\mathrm{Ind}^{\boldsymbol{a}}$ enumerate the $M[A]$-index ordinals in increasing order. $\boldsymbol{a}^{(\xi)}$ is the Turing degree of master codes for $\mathrm{Ind}^{\boldsymbol{a}}(\xi)$ relative to A. By the normal form theorem, $\boldsymbol{a}^{(\xi)} = \deg(\mathrm{Th}_n(E_\alpha[A]))$ for some $E_\alpha[A]$, where $\omega \cdot \alpha + n = \mathrm{Ind}^{\boldsymbol{a}}(\xi)$.

What is the relation between $\boldsymbol{a}^{(\lambda)}$ and $\{\boldsymbol{a}^{(\xi)} \mid \xi < \lambda\}$? We shall only consider this question for $\boldsymbol{a} = \mathbf{0}$; by standard relativization arguments, results for $\mathbf{0}$ extend easily to arbitrary $\boldsymbol{a}$. Let I_λ be the ideal of Turing degrees generated by $\{\mathbf{0}^{(\xi)} \mid \xi < \lambda\}$. In [3] and [4], the above question was answered in terms of exact pairs for I_λ; in this paper we approach the question in terms of upper bounds on I_λ. To make clear the differences, we restate the central results of [3].

Let $J^{\boldsymbol{a}}(\xi)$ be the least strict upper bound on $\{\mathrm{Ind}^{\boldsymbol{a}}(\eta) \mid \eta < \xi\}$, and $F^{\boldsymbol{a}}(\alpha)$ be the length of the $M[A]$-gap started at α; in other words, where $\boldsymbol{a} = \deg(A)$:

$$F^{\boldsymbol{a}}(\alpha) = \text{the maximum } \beta \text{ such that } (M_{\alpha+\beta}[A] - M_\alpha[A]) \cap \omega^\omega = \varnothing.$$

Thus $\mathrm{Ind}^{\boldsymbol{a}}(\alpha) = J^{\boldsymbol{a}}(\alpha) + F^{\boldsymbol{a}}(J^{\boldsymbol{a}}(\alpha))$.

If $\boldsymbol{a} = \mathbf{0}$, we may take $A = \varnothing$ and omit explicit relativization. A degree $\boldsymbol{a}$ is I-exact, where I is an ideal of Turing degrees, iff $\boldsymbol{a} = \boldsymbol{b} \vee \boldsymbol{c}$ and $I = \{\boldsymbol{d} \mid \boldsymbol{d} \le \boldsymbol{b}$ and $\boldsymbol{d} \le \boldsymbol{c}\}$. In [3] it is proved that $\mathbf{0}^{(\lambda)}$ is the least member of $\{\boldsymbol{a}^{(\mu_\lambda)} \mid \boldsymbol{a} \text{ is } I_\lambda\text{-exact}\}$, where

$$\mu_\lambda = \begin{cases} 2 + F(J(\lambda)) & \text{if } J(\lambda) \text{ is not a limit of } M\text{-gaps;} \\ 3 + F(J(\lambda)) & \text{otherwise.} \end{cases}$$

(α is an M-gap iff $F(\alpha) > 0$.)

Furthermore, for $\xi < \mu_\lambda$, $\{\boldsymbol{a}^{(\xi)} \mid \boldsymbol{a} \text{ is } I_\lambda\text{-exact}\}$ has no least member. Thus the "distance" between I_λ and $\mathbf{0}^{(\lambda)}$, measured in terms of I_λ-exact degrees, is determined by the "distance' between' $J(\lambda)$ and the next index ordinal.

In this paper we prove that if $J(\lambda)$ is admissible, then the "distance" between I_λ and $\mathbf{0}^{(\lambda)}$, measured in terms of upper bounds on I_λ, is as great as possible, namely $\mathrm{Ind}(\lambda)$! Notice that $J(\lambda)$ is admissible iff λ is admissible and locally countable. Furthermore, if $J(\lambda)$ is admissible, $\lambda = J(\lambda)$ and $\mathrm{Ind}(\lambda) = \lambda + F(\lambda)$. Hereafter assume that $\lambda < (\aleph_1)^L$ is admissible and locally countable.

THEOREM 1. *$\mathbf{0}^{(\lambda)}$ is the least member of $\{\boldsymbol{a}^{(\mathrm{Ind}(\lambda))} \mid \boldsymbol{a}$ is an upper bound on $I_\lambda\}$.*

If we require the upper bounds in question to have low hyper-degree, and if $F(\lambda) < \omega$, then the situation is slightly less pathological.

THEOREM 2. *If $F(\lambda) < \omega$ then $\mathbf{0}^{(\lambda)}$ is the least member of*

$$\{\boldsymbol{a}^{(\mathrm{Ind}(\lambda)-1)} \mid \boldsymbol{a} \text{ is an upper bound on } I_\lambda \text{ and } \omega_1^{\boldsymbol{a}} = \lambda\}.$$

However, if $F(\lambda) \ge \omega$, even this small comfort must be abandoned.

THEOREM 3. *If $F(\lambda) \ge \omega$, then $\mathbf{0}^{(\lambda)}$ is the least member of*

$$\{\boldsymbol{a}^{(\mathrm{Ind}(\lambda))} \mid \boldsymbol{a} \text{ is an upper bound on } I_\lambda \text{ and } \omega_1^{\boldsymbol{a}} = \lambda\}.$$

Theorem 1 is a generalization of the main negative result of [5].

§2. The basic construction. One direction of Theorems 1, 2 and 3 is trivial. For any $\boldsymbol{a}$, $\mathbf{0}^{(\lambda)} \le \boldsymbol{a}^{(\mathrm{Ind}(\lambda))}$. Suppose λ is admissible, $F(\lambda) < \omega$, and $\omega_1^{\boldsymbol{a}} = \lambda$. Then

$\boldsymbol{a}^{(\lambda)} = \deg(\mathrm{Th}_1(E_\lambda[A]))$ for some $E_\lambda[A]$, since $F^{\boldsymbol{a}}(\lambda) = 1$. There is an E_λ such that $\mathrm{Th}_1(E_\lambda) \leq_T \mathrm{Th}_1(E_\lambda[A])$, implying that $\mathrm{Th}_{F(\lambda)}(E_\lambda) \leq_T \mathrm{Th}_{F(\lambda)}(E_\lambda[A])$. So $\boldsymbol{a}^{(\lambda+F(\lambda)-1)} = \deg(\mathrm{Th}_{F(\lambda)}(E_\lambda[A]))$ and $\mathbf{0}^{(\lambda)} \leq \deg(\mathrm{Th}_{F(\lambda)}(E_\lambda))$, implying that $\mathbf{0}^{(\lambda)} \leq \boldsymbol{a}^{(\mathrm{Ind}(\lambda)-1)}$.

The nontrivial content of Theorem 1 is built into this lemma.

LEMMA. *Suppose* $F(\lambda) = \omega \cdot \beta + n$. *There is an* $A \subseteq \omega$ *such that*

(i) $\omega_1^A = \lambda$;

(ii) A *is a Turing upper bound on* $L_\lambda \cap \omega^\omega$;

(iii) *for some* $E_{\lambda+\beta}[A]$, $\mathrm{Th}_n(E_{\lambda+\beta}[A]) \in \Delta_{n+1}(L_{\lambda+\beta})$.

Note that $\bigcup I_\lambda = L_\lambda \cap \omega^\omega$ and that for any real f, $\deg(f) \leq \mathbf{0}^{(\lambda)}$ iff $f \in \Delta_{n+1}(L_{\lambda+\beta})$.

PROOF OF LEMMA. Our strategy is to combine a Henkin construction in an infinitary language with a forcing argument in a ramified finitary language. The Henkin construction will "produce" $\mathrm{Th}_0(E_\lambda[A])$ and the forcing construction will produce the rest of $\mathrm{Th}_n(E_{\lambda+\beta}[A])$. Let $\mathscr{L}$ be the L_λ-fragment of $\mathscr{L}_{\aleph_1,\aleph_0}$ with a binary predicate letter 'ε', a constant '$\boldsymbol{t}$' for each $t \in L_\lambda$, and a new constant '$\boldsymbol{A}$'. (As usual, a formula has only finitely many variables.) Let $\boldsymbol{L}_\lambda[\boldsymbol{A}]$ and $\boldsymbol{L}_{\lambda+\beta}[\boldsymbol{A}]$ be the ramified languages for set-theory with the one-place predicate '$\boldsymbol{A}$' of heights λ and $\lambda + \beta$ respectively, containing ranked abstraction terms as usual. (See for example [3].) If $\beta > 0$, an unranked formulae of $\boldsymbol{L}_\lambda[\boldsymbol{A}]$ shall be identified with a formula in $\boldsymbol{L}_{\lambda+\beta}[\boldsymbol{A}]$ of rank λ by replacing unranked variables by suitable new variables of rank λ. If φ is a formula of $\boldsymbol{L}_\lambda[\boldsymbol{A}]$, it may be translated to a finite formulae φ^* of $\mathscr{L}$ as follows: replace variables of rank ξ by ordinary variables restricted to $L_\xi[A]$, where '$x \in L_\xi[A]$' is the obvious Σ_1 formula with only x free and constants $\boldsymbol{\xi}$ and $\boldsymbol{A}$ of $\mathscr{L}$; eliminate abstraction terms; replace any new ranked variables as before; eliminate abstraction terms, etc.

We introduce two sequences of new constants to $\mathscr{L}$: $\langle \boldsymbol{k}_n \rangle_{n\in\omega}$, designed to denote nonstandard ordinals, and $\langle \boldsymbol{h}_n \rangle_{n\in\omega}$, the Henkin constants. Let $\mathscr{L}^+$ be the resulting extension of $\mathscr{L}$, where any formula contains only finitely many $\boldsymbol{k}_n$'s and $\boldsymbol{h}_n$'s. As usual, $\mathscr{L}$, $\mathscr{L}^+$ and $\boldsymbol{L}_\lambda[\boldsymbol{A}]$ are identified with subsets of L_λ, $\boldsymbol{L}_{\lambda+\beta}[\boldsymbol{A}]$ with a subset of $L_{\lambda+\beta}$.

Let T be the following $\Delta_1(L_\lambda)$ theory in $\mathscr{L}$:

$$\{\text{Extensionality}, \boldsymbol{A} \subseteq \boldsymbol{\omega}, (\forall\xi)(\exists x)(x = L_\xi[\boldsymbol{A}])\}$$
$$\cup \{\boldsymbol{t} \leq_T \boldsymbol{A} \mid t \in L_\lambda \cap \omega^\omega\} \cup \text{Diagram}(L_\lambda)$$
$$\cup \{(\forall x)(x \in \boldsymbol{t} \equiv \bigvee_{s\in t} x = \boldsymbol{s}) \mid t \in L_\lambda\}.$$

Let T' be the following $\Delta_1(L_\lambda)$ theory in $\mathscr{L}^+$:

$$T' = T \cup \{\boldsymbol{k}_n \text{ is an ordinal} \mid n \in \omega\} \cup \{\boldsymbol{k}_{n+1} < \boldsymbol{k}_n \mid n \in \omega\}.$$

T' is consistent by an easy Henkin argument. Call a set p of sentences of $\boldsymbol{L}_\lambda[\boldsymbol{A}]$ essentially Π_1 iff each member of p is ranked or Π_1. For such p, let p^* be the set of sentences φ^* where either φ is ranked and $\varphi \in p$ or for some ranked ψ, $(\forall x)\psi \in p$ and φ is $\psi(x/\boldsymbol{c})$, where $\boldsymbol{c}$ is an abstraction term of $\boldsymbol{L}_\lambda[\boldsymbol{A}]$. A condition is a pair $\langle p, s \rangle$ where p is finite and essentially Π_1, s is a finite set of sentences in $\mathscr{L}^+$ and

(i) $T' \cup p^* \cup s$ is consistent.

(ii) If $T' \cup p^* \cup s \vdash$ '$\boldsymbol{h}_n$ is an ordinal' then either for some $\xi < \lambda$ '$\boldsymbol{h}_n = \boldsymbol{\xi}$' $\in s$ or for some m, '$\boldsymbol{k}_m \leq \boldsymbol{h}_n$' $\in s$.

Notice that if $T' \cup p^* \cup s$ is consistent and $T' \cup p^* \cup s \vdash$ '$\boldsymbol{h}_n$ is an ordinal' then $\boldsymbol{h}_n$ occurs in s; otherwise $T' \cup p^* \cup s \vdash (\forall x)$ (x is an ordinal), contradicting Diagram(L_λ). $\langle p, s\rangle$ extends $\langle p', s'\rangle$ iff $p' \subseteq p$ and $s' \subseteq s$. Let P be the set of conditions.

Let $S = \{p^* \cup s \mid \langle p, s\rangle \in P\}$. Although not quite a consistency property, since (c0) of [1, p. 85] fails, S is almost one:

SUBLEMMA. *S satisfies* (c1)–(c7) *of* [1, p. 85].

As usual, the only nontrivial clauses concern '$\exists$' and '$\bigvee$'. We prove that (c6) is satisfied: *if* '$(\exists x)\theta$' $\in p^* \cup s \in S$ *then for some* $\boldsymbol{h}_n$ *and* s', $\langle p, s'\rangle \in P$, $s \subseteq s'$ *and* '$\theta(x/\boldsymbol{h}_n)$' $\in s'$.

Let $\boldsymbol{h}_n$ be the least Henkin constant not occurring in s. Then $U = T' \cup p^* \cup s \cup \{\theta(x/\boldsymbol{h}_n)\}$ is consistent. If $U \nvdash$ '$\boldsymbol{h}_n$ is an ordinal', let $s' = s \cup \{\theta(x/\boldsymbol{h}_n)\}$ and we are done. Suppose that $U \vdash$ '$\boldsymbol{h}_n$ is an ordinal'. Let $\boldsymbol{k}_m$ be the least such constant such that no $\boldsymbol{k}_q$, for $q \geq m$, occurs in $s \cup \{\theta(x/\boldsymbol{h}_n)\}$. If $U \cup \{\boldsymbol{k}_m \leq \boldsymbol{h}_n\}$ is consistent, let $s' = s \cup \{\theta(x/\boldsymbol{h}_n), \boldsymbol{k}_m \leq \boldsymbol{h}_n\}$, and we are done. Suppose that $U \cup \{\boldsymbol{k}_m \leq \boldsymbol{h}_n\}$ is inconsistent. In any model $\mathfrak{M}$ of U, $h = \boldsymbol{h}_n^{\mathfrak{M}}$ is a standard ordinal. If h were nonstandard, we could select a descending sequence $\langle d_i\rangle_{i\in\omega}$ of ordinals in $\mathfrak{M}$ such that $d_0 = h$ and reinterpret '$\boldsymbol{k}_{m+i}$' to denote d_i; where $\mathfrak{M}'$ is the model produced by this revision, $\mathfrak{M}' \models U$, since the only occurrences of '$\boldsymbol{k}_{m+i}$' in U are in $T' - T$, and these sentences remain true in $\mathfrak{M}'$. But $\mathfrak{M}' \models \boldsymbol{k}_m \leq \boldsymbol{h}_n$. Suppose that for every $\mathfrak{M}$, if $\mathfrak{M} \models U$ then $\{\langle x, y\rangle \mid \mathfrak{M} \models x < y \leq \boldsymbol{h}_m\}$ has order type $\geq \lambda$. Then using the formula '$x < \boldsymbol{h}_m$', U pins down all ordinals below λ. By Theorem 7.4 of [1, p. 107] this is impossible. So for some $\mathfrak{M} \models U \cup \{\boldsymbol{h}_n = \boldsymbol{\xi}\}$, for some $\xi < \lambda$. Thus $U \cup \{\boldsymbol{h}_n = \boldsymbol{\xi}\}$ is consistent. Let $s' = s \cup \{\theta(x/\boldsymbol{h}_n), \boldsymbol{h}_n = \boldsymbol{\xi}\}$. We have ensured that $\langle p, s'\rangle$ is a condition; so $p^* \cup s' \in S$, as claimed.

The proof that if '$\bigvee \Phi$' $\in p^* \cup s \in S$ then for some $\theta \in \Phi$, and some s', $\langle p, s'\rangle \in P$, $s \subseteq s'$ and 'θ' $\in s'$ is similar. We omit details.

Suppose that $G = \langle\langle p_i, s_i\rangle\rangle_{i\in\omega}$ is a sequence of conditions such that for every $i \in \omega$, $\langle p_{i+1}, s_{i+1}\rangle$ extends $\langle p_i, s_i\rangle$. We say that G has the Henkin property iff:

(1) For any $i \in \omega$, if '$(\exists x)\theta$' $\in p_i^* \cup s_i$ then for some Henkin constant $\boldsymbol{h}_n$ and some $j \in \omega$, '$\theta(x/\boldsymbol{h}_n)$' $\in s_j$.

(2) For any $i \in \omega$, if '$\bigvee\Phi$' $\in p_i^* \cup s_i$ then for some $\theta \in \Phi$, and some $j \in \omega$, 'θ' $\in s_j$.

(3) For any θ in $\mathscr{L}^+$ there is a j such that either 'θ' $\in s_j$ or '$\neg\, \theta$' $\in s_j$.

A sequence G with the Henkin property determines a path through S, which in turn determines a canonical term model $\mathfrak{M} = \mathfrak{M}(G)$ of $\bigcup_{i\in\omega}(p_i^* \cup s_i)$. Let $A = A(G) = \{n \mid \mathfrak{M} \models \boldsymbol{n} \in \boldsymbol{A}\}$. A is a Turing upper bound on $L_\lambda \cap \omega^\omega$, since for any real $t \in L_\lambda$, $\mathfrak{M} \models \boldsymbol{t} \leq_T \boldsymbol{A}$ and $t(n) = m$ iff $\boldsymbol{t}^{\mathfrak{M}}(\boldsymbol{n}^{\mathfrak{M}}) = m^{\mathfrak{M}}$. Furthermore λ is the supremum of the order-types of the standard ordinals of $\mathfrak{M}$. This is because P was defined to ensure that the type of λ was omitted. Thus $\omega_1^A = \lambda$. Letting $\widetilde{\mathfrak{M}} = \bigcup_{\xi<\lambda}(L_\xi[A])^{\mathfrak{M}}$, we have $\widetilde{\mathfrak{M}} \simeq L_\lambda[A]$. This is obvious, since $\mathfrak{M} \models (\forall\xi)(\exists x)(x = L_\xi[A])$ and $\boldsymbol{\xi}^{\mathfrak{M}}$ is standard. Finally, $\widetilde{\mathfrak{M}} \models \bigcup_{i\in\omega} p_i$. Suppose $\varphi \in p_i$. If φ is ranked in $\boldsymbol{L}_\lambda[\boldsymbol{A}]$, $\mathfrak{M} \models \varphi^*$; so $\widetilde{\mathfrak{M}} \models \varphi^*$; so $\widetilde{\mathfrak{M}} \models \varphi$. If φ is $(\forall x)\,\psi$ where ψ is ranked in $\boldsymbol{L}_\lambda[\boldsymbol{A}]$, then for any abstraction term $\boldsymbol{c}$ of $\boldsymbol{L}_\lambda[\boldsymbol{A}]$, $\mathfrak{M} \models \psi(x/\boldsymbol{c})^*$; so $\widetilde{\mathfrak{M}} \models \psi(x/\boldsymbol{c})^*$; so $\widetilde{\mathfrak{M}} \models \psi(x/\boldsymbol{c})$. Since every element of $\widetilde{\mathfrak{M}}$ is denoted by some such abstraction term, $\widetilde{\mathfrak{M}} \models \varphi$.

We now define forcing and consider sequences of conditions which have the Henkin property and are generic. Where φ is a sentence of $\boldsymbol{L}_{\lambda+\beta}[\boldsymbol{A}]$, let:

$\langle p, s\rangle \Vdash \varphi$ iff $\varphi \in p$ where $\rho(\varphi) < \lambda$ or $\rho(\varphi) = \lambda$ and φ is a Π_1 sentence of $\boldsymbol{L}_\lambda[\boldsymbol{A}]$;

$\langle p, s\rangle \Vdash \neg\varphi$ iff for every condition $\langle p', s'\rangle$ extending $\langle p, s\rangle$, $\langle p', s'\rangle \nVdash \varphi$, where $\rho(\varphi) \geq \lambda$ and if $\rho(\varphi) = \lambda$ then φ is not a Σ_1 sentence of $\boldsymbol{L}_\lambda[\boldsymbol{A}]$;

$\langle p, s\rangle \Vdash (\varphi_1 \& \varphi_2)$ iff $\langle p, s\rangle \Vdash \varphi_1$ and $\langle p, s\rangle \Vdash \varphi_2$ where $\rho(\varphi_1), \rho(\varphi_2) \geq \lambda$;

$\langle p, s\rangle \Vdash (\exists x^\gamma)\psi$ iff for some $\boldsymbol{c}$, an abstraction term in $\boldsymbol{L}_{\lambda+\beta}[\boldsymbol{A}]$ of rank γ, $\langle p, s\rangle \Vdash \psi(x^\gamma/\boldsymbol{c})$, where $\rho((\exists x^\gamma)\psi) \geq \lambda$;

$\langle p, s\rangle \Vdash (\exists x)\psi$ iff for some abstraction term $\boldsymbol{c}$, $\langle p, s\rangle \Vdash \psi(x/\boldsymbol{c})$.

(ρ is the rank function.)

Suppose $G = \langle\langle p_i, s_i\rangle\rangle_{i\in w}$ is a sequence of conditions which is generic with respect to the $\Sigma_n \cup \Pi_n$ sentences of $\boldsymbol{L}_{\lambda+\beta}[\boldsymbol{A}]$ and which has the Henkin property. Let $\mathfrak{M} = \mathfrak{M}(G)$. The set of sentences of $\boldsymbol{L}_{\lambda+\beta}[\boldsymbol{A}]$ forced by conditions in this sequence also determines a term model $\mathfrak{N} = \mathfrak{N}(G)$. Where $\mathfrak{N}_\lambda$ is $\mathfrak{N}$ restricted to denotations of terms of rank $< \lambda$, we clearly have $\tilde{\mathfrak{M}} \simeq \mathfrak{N}_\lambda$. Thus $\{n|\mathfrak{N} \models A(\boldsymbol{n})\} = \{n\,|\mathfrak{M} \models \boldsymbol{n} \in \boldsymbol{A}\} = A(G)$. The Henkin component of the construction "built" $\tilde{\mathfrak{M}}$; the forcing component was designed to ensure agreement with the Henkin component, so $\tilde{\mathfrak{M}} \simeq \mathfrak{N}_\lambda$, and to control the construction of the rest of $\mathfrak{N}$ in the usual way. This is why the definition of forcing required that the sentences φ such that $\rho(\varphi) < \lambda$, or $\rho(\varphi) = \lambda$ and φ is Π_1 in $\boldsymbol{L}_\lambda[\boldsymbol{A}]$, be handled differently from other sentences.

We now examine the definitional complexity of forcing. $P \in \Pi_1(L_\lambda)$. Thus forcing restricted to the $\Sigma_1 \cup \Pi_1$ sentences of $\boldsymbol{L}_\lambda[\boldsymbol{A}]$ is Δ_2 over L_λ. Forcing restricted to $\Sigma_n \cup \Pi_n$ sentences of $\boldsymbol{L}_{\lambda+\beta}[\boldsymbol{A}]$ is Δ_{n+1} over $L_{\lambda+\beta}$.

Fix countings of the $\Sigma_n \cup \Pi_n$ sentences of $\boldsymbol{L}_{\lambda+\beta}[\boldsymbol{A}]$, the abstraction terms of $\boldsymbol{L}_{\lambda+\beta}[A]$, and all sentences of $\mathscr{L}^+$, which are Δ_{n+1} over $L_{\lambda+\beta}$—say $\langle\varphi_i\rangle_{i\in\omega}$, $\langle \boldsymbol{c}_i\rangle_{i\in\omega}$ and $\langle\theta_i\rangle_{i\in\omega}$ respectively. Define a sequence G as follows:

$\langle p_0, s_0\rangle = \langle\varnothing, \varnothing\rangle$;

$\langle p_{2i+1}, s_{2i+1}\rangle$ = the $<_L$-least condition $\langle p, s\rangle$ extending $\langle p_{2i}, s_{2i}\rangle$ such that $\langle p, s\rangle$ decides φ_i;

$\langle p_{2i+2}, s_{2i+2}\rangle$ = the $<_L$-least condition $\langle p, s\rangle$ extending $\langle p_{2i+1}, s_{2i+1}\rangle$ such that

(1) either θ_i or $\neg\theta_i \in s$;

(2) for any $\theta(x)$, if $(\exists x)\theta \in s_{2i+1}$ then for some $\boldsymbol{h}_n$, $\theta(x/\boldsymbol{h}_n) \in s_{2i+2}$;

(3) for any Φ, if $\bigvee\Phi \in s_{2i+1}$ then for some $\theta \in \Phi$, $\theta \in s_{2i+2}$.

G is Δ_{n+1} over $L_{\lambda+\beta}$, is generic, and has the Henkin property. Letting $A = A(G)$, $\mathfrak{N} = \mathfrak{N}(G) \simeq L_{\lambda+\beta}[A]$. By the usual forcing = truth lemma, $\mathfrak{N} \models \varphi_i$ iff $\langle p_{2i+1}, s_{2i+1}\rangle \Vdash \varphi_i$. Thus $\mathrm{Th}_n(\mathfrak{N}) \in \Delta_{n+1}(L_{\lambda+\beta})$. "Pulling back to ω" by the counting $\langle c_i\rangle_{i\in\omega}$, $\mathfrak{N}$ becomes an $E_{\lambda+\beta}[A]$ for which (iii) is satisfied. (i) and (ii) are true by remarks on the Henkin property. QED

COROLLARY. *There is an* $A \subseteq \omega$, *a Turing upper bound on* HYP, *the set of hyperarithmetic reals, whose hyperjump has the Turing degree of Kleene's* $\mathcal{O}$.

PROOF. Consider the A constructed in this proof where $\lambda = \omega_1^{\mathrm{ck}}$. For any $E_\lambda[A]$, $\omega_1^A = \lambda$ implies that $\mathcal{O}^A \leq_T \mathrm{Th}_1(E_\lambda[A])$. For the $E_\lambda[A]$ constructed in the lemma, $\mathrm{Th}_1(E_\lambda[A]) \in \Delta_2(L_\lambda)$, and so $\mathrm{Th}_1(E_\lambda[A]) \leq_T \mathcal{O} \leq_T \mathcal{O}^A$. Thus $\mathcal{O} \equiv_T \mathcal{O}^A$.

Theorem 2 is also an immediate consequence of the lemma. Suppose $F(\lambda) < \omega$; so $\beta = 0$ and $F(\lambda) = n$. Taking $\boldsymbol{a} = \deg(A)$, A and $E_\lambda[A]$ as in the lemma, we have $\boldsymbol{a}^{(\lambda+n-1)} \leq \deg(\mathrm{Th}_n(E_\lambda[A])) \leq \boldsymbol{0}^{(\lambda)}$.

§3. Theorems 1 and 3. Again, we assume that λ is admissible and locally countable. If $F(\lambda) < \omega$, to prove Theorem 1 we have to construct a $C \subseteq \omega$ so that C is a Turing upper bound on $L_\lambda \cap \omega^\omega$ and, for an appropriate $E_\lambda[C]$, $\boldsymbol{c}^{(\lambda+F(\lambda))} = \deg(\mathrm{Th}_{F(\lambda)}(E_\lambda[C]))$, where $\boldsymbol{c} = \deg(C)$. For the latter condition it will suffice to ensure that $\omega_1^C > \lambda$. But if $F(\lambda)$ is big we face a further worry. Suppose we can construct a C as desired and so that $\mathrm{Th}_n(E_{\lambda+\beta}[C]) \in \Delta_{n+1}(L_{\lambda+\beta})$ for some $E_{\lambda+\beta}[C]$, where $F(\lambda) = \omega \cdot \beta + n$. If $\mathrm{Ind}^{\boldsymbol{c}}(\mathrm{Ind}(\lambda)) > \mathrm{Ind}(\lambda)$ this will not ensure that $\boldsymbol{c}^{(\mathrm{Ind}(\lambda))} \leq \boldsymbol{0}^{(\lambda)}$. To avoid this problem we will ensure that $\omega_1^C > \lambda + \beta$; then $\omega_1^C > \lambda + \omega \cdot \beta + n = \mathrm{Ind}(\lambda)$, and so $\mathrm{Ind}^{\boldsymbol{c}}(\mathrm{Ind}(\lambda)) = \mathrm{Ind}(\lambda)$. We construct a $B \subseteq \omega$ suitably generic over $L_{\lambda+\beta}[A]$, where A is as in the lemma of §2, so that $\omega_1^B > \lambda + \beta$ and for an appropriate $E_{\lambda+\beta}[A \oplus B]$, $\mathrm{Th}_n(E_{\lambda+\beta}[A \oplus B]) \in \Delta_{n+1}(L_{\lambda+\beta}[A])$. Then $C = A \oplus B$ will be as desired. The trick is to take B generic in the sense of forcing with Steel's tagged trees of height $< \delta$, where δ is the maximum admissible or limit of admissibles $\leq \lambda + \beta$. The details are routine. Basic lemmas concerning Steel forcing are presented in [3].

Theorem 3 is immediate from the lemma of §2, if $\mathrm{Ind}^{\boldsymbol{a}}(\mathrm{Ind}(\lambda)) = \mathrm{Ind}(\lambda)$, where $\boldsymbol{a} = \deg(A)$, A as in the lemma. But in general this is not the case. Where $F(\lambda) = \omega \cdot \beta + n$ and $\beta \geq 1$, our strategy is to produce a $B \subseteq \omega$, suitably generic over $L_{\lambda+\beta}[A]$, and an $E_{\lambda+\beta}[A \oplus B]$ so that if $A \oplus B = C$, then:

(1) $\omega_1^C = \lambda$;

(2) $\omega_2^C > \lambda + \beta$;

(3) $\mathrm{Th}_n(E_{\lambda+\beta}[C]) \in \Delta_{n+1}(L_{\lambda+\beta}[A])$.

By (1), $\boldsymbol{c}^{(\lambda+\omega)} \leq \deg(\mathrm{Th}_0(E_{\lambda+1}[C]))$, where $\boldsymbol{c} = \deg(C)$. (2) implies that $\omega_2^C \geq \lambda + \omega \cdot \beta + n = \mathrm{Ind}(\lambda)$, and so $\mathrm{Ind}^{\boldsymbol{c}}(\mathrm{Ind}(\lambda)) = \mathrm{Ind}(\lambda)$. Thus $\boldsymbol{c}^{(\mathrm{Ind}(\lambda))} \leq \deg(\mathrm{Th}_n(E_{\lambda+\beta}[C]))$. Because there is an $E_{\lambda+\beta}[A]$ so that $\mathrm{Th}_n(E_{\lambda+\beta}[A]) \in \Delta_{n+1}(L_\lambda)$, we obtain an $E_{\lambda+\beta}[C]$ so that $\mathrm{Th}_n(E_{\lambda+\beta}[C]) \in \Delta_{n+1}(L_\lambda)$.

To construct B, we take several generic extensions of $L_{\lambda+\beta}[A]$. Let δ be the maximum admissible or limit of admissibles $\leq \lambda + \beta$. Take B_1 to be a well-founded tree of height δ which is generic in the sense of Steel forcing and such that $\mathrm{Th}_n(E_{\lambda+\beta}[A \oplus B_1]) \in \Delta_{n+1}(L_{\lambda+\beta}[A])$ for some $E_{\lambda+\beta}[A \oplus B_1]$.

Working within $L_\lambda[A, B_1]$, we shall construct an appropriately generic extension $L_\lambda[A, X]$ of $L_\lambda[A]$, where $X \subseteq \lambda$, $L_\lambda[A, X]$ is admissible relative to A and X, and so that X encodes B_1. Let a condition r be a function from some $\gamma < \lambda$ into 2 so that $r \in L_\lambda[A]$. Let Q be the set of these conditions. Define forcing for sentences of $\boldsymbol{L}_\lambda[\boldsymbol{A}, \boldsymbol{X}]$ by the following base clauses:

$$r \Vdash \boldsymbol{X}(\boldsymbol{\xi}) \quad \text{iff } r(\xi) = 1,$$

$$r \Vdash \boldsymbol{A}(\boldsymbol{n}) \quad \text{iff } n \in A.$$

The other clauses are as usual. Forcing Σ_1 sentences of $\boldsymbol{L}_\lambda[\boldsymbol{A}, \boldsymbol{X}]$ is Σ_1 over $L_\lambda[A, B_1]$. A sentence of the form $(\forall x^\gamma)(\exists y)\varphi(x^\gamma, y)$ $((\exists x^\gamma)(\forall y)\varphi(x^\gamma, y))$, where $\varphi(x^\gamma, y)$ is ranked in $\boldsymbol{L}_\lambda[\boldsymbol{A}, \boldsymbol{X}]$ and $\gamma < \lambda$, shall be called extended Σ_1 (extended Π_1). Suppose $r \Vdash (\forall x^\gamma)(\exists y)\varphi(x^\gamma, y)$, where φ is ranked. This is a Π_2 statement over L_λ:

$$(*) \quad (\forall r' \supseteq r)(\forall c)(c \text{ an abstraction term of rank } \leq \gamma \supset (\exists r'' \supseteq r')(\exists c')$$
$$(c' \text{ an abstraction term } \& \ r'' \Vdash \varphi(c, c'))).$$

L_λ is admissible, and so reflects Π_2 statements, so (*) is true in L_η for some η such that $\gamma \leq \eta < \lambda$. Let $\langle c_i \rangle_{i\in\omega}$ be a counting in $L_\eta[A, B_1]$ of the abstraction terms of rank $\leq \gamma$. Let $r_0 = r$; r_{i+1} extends r_i and for some c in L_η, $r_{i+1} \Vdash \varphi(c_i, c')$. Let $\hat{r} = \lim_{i<\omega} r_i$. Because λ is admissible relative to A, $\hat{r} \in L_\lambda[A]$; clearly $\hat{r} \Vdash (\forall x^\gamma)(\exists y^\eta)\varphi(x^\gamma, y)$.

Let $\langle \varphi_i \rangle_{i\in\omega}$ and $\langle c_i \rangle_{i\in\omega}$ be $\Delta_2(L_\lambda[A])$ countings of the sentences of $\boldsymbol{L}_\lambda[\boldsymbol{A}, \boldsymbol{X}]$ which are extended $\Sigma_1 \cup$ extended Π_1 and the abstraction terms of $\boldsymbol{L}_\lambda[\boldsymbol{A}, \boldsymbol{X}]$. We construct a generic sequence as follows:

$r_0 = \langle\ \rangle$;

$r_{3i+1} =$ the $<_{L[A]}$-least condition extending r_{3i} and deciding φ_i;

$$r_{3i+2} = \begin{cases} r_{3i+1} & \text{if } r_{3i+1} \not\Vdash \varphi_i \text{ or } \varphi_i \text{ is not extended } \Sigma_1; \\ \text{that } r \text{ such that } \langle r, \eta\rangle \text{ is } <_{L[A]}\text{-least such that} \\ r \text{ extends } r_{3i+1} \text{ and } r \Vdash (\forall x^\gamma)(\exists y^\eta)\varphi(x^\gamma, y^\eta) \\ \qquad \text{if } r_{3i+1} \Vdash \varphi_i \text{ and } \varphi_i \text{ is } (\forall x^\gamma)(\exists y)\varphi(x^\gamma, y); \\ \qquad \text{and } \varphi \text{ is ranked.} \end{cases}$$

$r_{3i+3} = r_{3i+2} {}^\frown \langle B_1(i)\rangle$.

(Here B_1 is identified with its characteristic function.) Clearly $\langle r_i \rangle_{i\in\omega} \in \Delta_2(L_\lambda[A, B_1])$. Let $X = \{\xi | (\exists i) r_i(\xi) = 1\}$. "Pulling back to ω" by the counting of the abstraction terms of $\boldsymbol{L}_\lambda[\boldsymbol{A}, \boldsymbol{X}]$, we obtain an $E_\lambda[A, X]$ so that $\mathrm{Th}_1(E_\lambda[A, X]) \in \Delta_2(L_\lambda[A])$. This easily extends to an $E_{\lambda+\beta}[A, X]$ so that $\mathrm{Th}_n(E_{\lambda+\beta}[A, X]) \in \Delta_{n+1}(L_{\lambda+\beta}[A])$. Stages of the form $3i + 2$ ensure that $L_\lambda[A, X]$ is admissible relative to A and X. Finally, $B_1 \in \Delta_2(L_\lambda[A, X])$. To see this, we construct $\langle r_i \rangle_{i\in\omega}$ in a Δ_2 way over $L_\lambda[A, X]$. At each stage of the form $3i + 3$ we consult X and r_{3i+2} to determine $B_1(i)$ and r_{3i+3}. Thus λ is the only ordinal below $\lambda + \beta$ admissible relative to A and X.

Working within $L_{\lambda+\beta}[A, X]$, we use almost disjoint coding to code X into a real B. Because λ must remain admissible relative to $A \oplus B$, we carry out the construction of B using the machinery from §2. Let $\langle f_\xi \rangle_{\xi<\lambda}$ be a listing of $L_\lambda[A] \cap \omega^\omega$ in the order imposed by $<_{L[A]}$. Let

$$S(f_\xi) = \{\sigma | \sigma \text{ is a sequence number } \& \ \sigma \text{ represents an initial segment of } f_\xi\}.$$

Thus $S(f_\xi) \cap S(f_\eta)$ is infinite iff $\xi = \eta$. Let $\mathscr{L}$ be the fragment of $\mathscr{L}_{\aleph_1,\aleph_0}$ with predicate 'ε', for each member t of $L_\lambda[A]$ the constant '$\boldsymbol{t}$', ('$\boldsymbol{A}$' for A), and a new constant '$\boldsymbol{B}$'. Let T be the following theory in $\Delta_1(L_\lambda[A, X])$:

$$\begin{aligned} &\{\text{Extensionality}, (\forall\xi)(\exists x)(x = L_\xi[\boldsymbol{A}, \boldsymbol{B}]), \boldsymbol{B} \subseteq \boldsymbol{\omega}\} \\ &\quad \cup \{\boldsymbol{B} \cap S(f_\xi) \text{ is finite } | \xi \in X\} \\ &\quad \cup \{\boldsymbol{B} \cap S(f_\xi) \text{ is infinite } | \xi \notin X\} \\ &\quad \cup \mathrm{Diag}(L_\lambda[A]) \cup \{(\forall x)(x \in \boldsymbol{t} \equiv \bigcup_{s\in t} x = \boldsymbol{s}) \mid t \in L_\lambda[A]\}. \end{aligned}$$

Form T' from T as in §2. Define P, S and forcing as in §2, where the ramified language in question is $\boldsymbol{L}_{\lambda+\beta}[\boldsymbol{A}, \boldsymbol{B}]$. Fix $\langle \varphi_i \rangle_{i\in\omega}$, a $\Delta_{n+1}(L_{\lambda+\beta}[A, X])$ counting of the $\Sigma_n \cup \Pi_n$ sentences of $\boldsymbol{L}_{\lambda+\beta}[\boldsymbol{A}, \boldsymbol{B}]$. As in §2, we construct a sequence of conditions in P,

$\langle\langle p_i, s_i\rangle\rangle_{i\in\omega}$, which is Δ_{n+1} over $L_{\lambda+\beta}[A, X]$, has the Henkin property, and is such that $\langle p_{2i+1}, s_{2i+1}\rangle$ decides φ_i for each $i \in \omega$. Once again, we obtain a real $B \subseteq \omega$ such that $\omega_1^{A\oplus B} = \lambda$, $L_\lambda[A, B] \simeq \tilde{\mathfrak{M}}$, where $\mathfrak{M} = \mathfrak{M}(\langle\langle p_i, s_i\rangle\rangle_{i\in\omega})$, $\tilde{\mathfrak{M}} \models T$ and $L_{\lambda+\beta}[A, B] \simeq \mathfrak{N} = \mathfrak{N}(\langle\langle p_i, s_i\rangle\rangle_{i\in\omega})$. As usual, $\mathfrak{N}$ may be turned into an $E_{\lambda+\beta}[A, B] = E_{\lambda+\beta}[A \oplus B]$ so that $\mathrm{Th}_n(E_{\lambda+\beta}[A \oplus B]) \in \Delta_{n+1}(L_{\lambda+\beta}[A, X])$. Recall that for appropriate $E_{\lambda+\beta}[A, X]$, $E_{\lambda+\beta}[A, B_1]$ and $E_{\lambda+\beta}[A]$ we had:

$\mathrm{Th}_n(E_{\lambda+\beta}[A]) \in \Delta_{n+1}(L_{\lambda+\beta})$,

$\mathrm{Th}_n(E_{\lambda+\beta}[A, B_1]) \in \Delta_{n+1}(L_\lambda[A])$,

$\mathrm{Th}_n(E_{\lambda+\beta}[A, X]) \in \Delta_{n+1}(L_{\lambda+\beta}[A, B_1])$.

Putting this all together we obtain an $E_{\lambda+\beta}[A \oplus B]$ so that $\mathrm{Th}_n(E_{\lambda+\beta}[A \oplus B]) \in \Delta_{n+1}(L_{\lambda+\beta})$. Letting $C = A \oplus B$, $\boldsymbol{c} = \deg(C)$, we have

$$\boldsymbol{c}^{(\lambda+\omega\cdot\beta+n)} = \boldsymbol{c}^{(\mathrm{Ind}(\lambda))} \leq \deg(\mathrm{Th}_n(E_{\lambda+\beta}[C])),$$

using the fact that $\beta \geq 1$ and $\omega_2^c \geq \lambda + \omega \cdot \beta + n$, and so $\mathrm{Ind}^c(\mathrm{Ind}(\lambda)) = \mathrm{Ind}(\lambda)$. Thus $\boldsymbol{c}^{(\mathrm{Ind}(\lambda))} \leq \mathbf{0}^{(\lambda)}$ as desired. QED

How "far" are $\mathbf{0}^{(\lambda)}$ and I_λ, in terms of upper bounds, where λ is a locally countable nonadmissible limit of admissibles? In [2] the following is proved:

THEOREM. *If $\{\alpha < \lambda \mid \alpha$ is not Σ_1 projectible to $\omega\}$ is bounded in λ, and λ is a locally countable nonadmissible limit of admissibles, then $\mathbf{0}^{(\lambda)}$ is the least member of $\{\boldsymbol{a}^{(3)} \mid \boldsymbol{a}$ is an upper bound on $I_\lambda\}$.*

If, however, $L_\lambda \models (\Pi_3^0 \cup \Sigma_3^0)$ determinacy, there is an $\boldsymbol{a}$, an upper bound on I_λ, such that $\boldsymbol{a}^{(4)} \leq \mathbf{0}^{(\lambda)}$.

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