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WHERE DO SETS COME FROM?

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Many philosophers take set-theoretic discourse to be about objects[1] of a special sort, namely sets; correlatively, they regard truth in such discourse as quite like truth in discourse about nonmathematical objects. There is a thin "disquotational" way of construing this construal;[2] but that may candy-coat a philosophically substantive semantic theory: the Mathematical-Object theory[3] of the basis for the distribution of truth and falsehood to sentences containing set-theoretic expressions. This theory asserts that truth and falsity for sentences containing set-theoretic expressions are grounded in semantic facts (about the relation between language and the world) of the sort modelled by the usual model-theoretic semantics for an uninterpreted formal first-order language. For example, it would maintain that '$\{\ \} \in \{\{\ \}\}$' is true in virtue of the set-theoretic fact that the empty set is a member of its singleton, and the semantic facts that '$\{\ \}$' designates the empty set, '$\{\{\ \}\}$' designates its singleton, and '$\in$' applies to an ordered pair of objects iff that pair's first component is a member of its second component.

Now this theory may come so naturally as to seem trivial. My purpose here is to loosen its grip by "modelling" an alternative account of the alethic underpinnings of set-theoretic discourse. According to the Alternative theory,[4] the point of having set-theoretic expressions ('set' and '$\in$' will do) in a language is *not* to permit its

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[1] I use 'object' as Frege did: to be an object is to be capable of being designated by a closed singular term (what Frege called a proper name).

[2] This is explained in detail in [H4, §8].

[3] I am reluctant to call this view 'Platonism': that label should be reserved for a view held by Plato at least at some point in his career. Whether Plato ever accepted the Mathematical-Object theory is a matter of controversy.

[4] In [H1] I called the Alternative theory "Coding-Fictionalism", in part because of this analogy: there are statements with definite truth-values containing mathematical [fictional] singular terms, even though these terms do not designate. But I have found that this label led to confusion, largely because of an apparent disanalogy, one not sufficiently emphasized in [H1]: as naturally construed, 'Hamlet existed' is literally false; but 'The null set exists' is literally true. So I have replaced 'Coding-Fictionalism' by a less informative, but less misleading, label.

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speakers to talk about some special objects under a special relation; rather it is to clothe a higher-order language in lower-order garments. A first-order language containing expressions for set-theoretic notions is an encoding of a sort of second-order language, one that is not about *objects* of a distinctively mathematical sort; furthermore, fundamental set-theoretic principles are encodings of validities in an appropriate second-order logic. (Unfortunately, this encoding can be philosophically misleading, creating the illusion of a distinctively philosophical subject-matter.)

I suspect that those who accept the Mathematical-Object theory begin by sliding from attributions of existence to claims about designation, e.g. from the uncontroversial mathematical claim that the empty set exists to the semantic claim that 'the empty set' designates something. This slide involves a hasty generalization from a syntactic uniformity (between mathematical singular terms and our paradigmatic singular terms that designate people, places, events, etc.) to a semantic uniformity.

This paper reports research in model-theory; its philosophical force is oblique. In §2 I shall present a model-theoretic semantics that reflects the Mathematical-Object theory; in §3, §5 and Appendix 3 I shall present several other such semantics, all reflecting the Alternative theory. I offer these semantics in part because they are intrinsically interesting, and in part because they may serve as instruments of philosophical therapy: once one understands them the above-mentioned slide will, I hope, seem more preventable than it did before.

This paper is intended to be self-contained. I have made four sections into appendices because they are more technical than the rest, and of less philosophical interest. The rest overlaps with [H4], though the latter is somewhat less technical and more philosophical. [H3] presents a parallel discussion of arithmetic; for more motivational background, see also [H1].

Let me emphasize that my purpose is in no way to reform classical mathematical practice. Indeed the Alternative theory is intended (1) to show the metaphysical harmlessness of the "Mathematical-Object picture" which is embedded in the syntax of actual mathematical practice, and (2) to explicate the sense in which the ontological commitments of mathematical theories are, using a word from [H4], "thin".

The exposition will be within familiar set theory with proper classes. Models are, of course, sets with a certain structure. The Axiom of Choice will not be assumed without warning. Cardinals are Scott-cardinals. For any set x, $\operatorname{card}(x) =$ the cardinality of x. An aleph is the cardinality of an infinite well-orderable set; so the Axiom of Choice is equivalent to: all infinite cardinals are alephs. You may assume AC if you wish, and identify cardinals with initial ordinals.

Where κ and μ are cardinals, let $\kappa^{<\mu} = \sum_{\xi<\mu}\kappa^{\xi}$. We adopt these definitions: μ is *κ-acceptable* iff $\mu = \mu^{<\kappa}$, and κ is *acceptable* iff κ is κ-acceptable.

Note. κ^{+} is acceptable iff the continuum hypothesis holds for κ; thus GCH implies that all infinite cardinals are acceptable.

As I shall understand it, a model-theoretic semantics is: (i) a logical vocabulary and a set of types for variables, which determine a class of languages (one for each choice as to nonlogical vocabulary) involving those logical expressions, and (ii) a notion of a model for an arbitrary nonlogical vocabulary, together with (iii) a definition of model-theoretic truth and falsity, these being two-place relations that may hold between a model for a given nonlogical vocabulary and sentences in the language based on that vocabulary. A model-theoretic semantics determines a logic, that is, logical notions of consequence, validity, equivalence, bivalence, etc. In this paper we shall consider several logics, of which I regard A_{1s} as the most important.

§1. Background logics. The semantics with which we shall be most concerned with are three-valued: relative to a model for a language, a sentence in that language may be true, false, or neither. This makes it reasonable, though by no means necessary, to begin with a simple three-valued semantics.

There are several logical lexica for a three-valued semantics. We shall use what in [H2] I called $\text{lex}_{1,s}$. (Of course all the basic lexical classes to be introduced are disjoint from one another.) Fix the 0-place connective $\perp$, the 2-place connective $\supset$, the quantifier $\exists$, and the 2-place predicate $=$. Fix a countable set Var(0) of variables of type 0. Given any vocabulary-set Vcb of predicate-constants and function-constants (taking individual-constants to be 0-place function-constants) we define the language $L^0 = L^0(\text{Vcb})$, i.e. we define the set of terms based of L^0, the set of formulae of L^0, and the set $\text{Sent}(L^0)$ of sentences of L^0, as usual.

We define what it is for $\mathscr{A}$ to be a model for Vcb as usual, with one change: where $\zeta \in \text{Vcb}$ is an n-place function-constant:

if $n = 0$, we allow that $\zeta^{\mathscr{A}}\uparrow$;

if $n \geq 1$, we allow that $\zeta^{\mathscr{A}}$ be a function into $|\mathscr{A}|$ with $\text{dom}(\zeta^{\mathscr{A}}) \subseteq |\mathscr{A}|^n$.

Let $\mathscr{A}$ be *total* iff, for every n-place function-constant $\zeta \in \text{Vcb}$, if $n = 0$ then $\zeta^{\mathscr{A}}\downarrow$, and if $n \geq 1$ then $\text{dom}(\zeta^{\mathscr{A}}) = |\mathscr{A}|^n$. Of course $|\mathscr{A}|$ is the universe of $\mathscr{A}$, which we require to be nonempty.[5] As usual, $\text{card}(\mathscr{A}) = \text{card}(|\mathscr{A}|)$ and $\text{Power}(\mathscr{A}) = \text{Power}(|\mathscr{A}|)$.

For a model $\mathscr{A}$ for Vcb we define $\text{des}^{\mathscr{A}}$ to be a perhaps partial function on the set of closed terms of L^0 into $|\mathscr{A}|$; this definition runs as usual; as usual we shall write $\text{des}^{\mathscr{A}}(\tau)$ (the designatum of τ relative to $\mathscr{A}$) as $\tau^{\mathscr{A}}$. Expand L^0 to $L^0_{\mathscr{A}}$ by introducing a new individual-constant $\underline{a}$ for each $a \in |\mathscr{A}|$; expand $\mathscr{A}$ to a model $\mathscr{B}$ by taking $\underline{a}^{\mathscr{B}} = a$ for each a as above. We define truth and falsity ($\models$ and $\dashv$) relative to $\mathscr{B}$ for sentences of $L^0_{\mathscr{A}}$ by a simultaneous induction. Here are the clauses for atomic sentences, where ξ is any n-place predicate-constant:

$\mathscr{B} \dashv \perp$;

$\mathscr{B} \models \xi(\tau_1, \ldots, \tau_n)$ iff, for all $1 \leq i \leq n$, $\tau_i^{\mathscr{B}}\downarrow$ and $\langle \tau_1^{\mathscr{B}}, \ldots, \tau_n^{\mathscr{B}} \rangle \in \xi^{\mathscr{B}}$;

$\mathscr{B} \dashv \xi(\tau_1, \ldots, \tau_n)$ iff, for all $1 \leq i \leq n$, $\tau_i^{\mathscr{B}}\downarrow$ and $\langle \tau_1^{\mathscr{B}}, \ldots, \tau_n^{\mathscr{B}} \rangle \notin \xi^{\mathscr{B}}$;

$\mathscr{B} \models \tau_0 = \tau_1$ iff $\tau_0^{\mathscr{B}}\downarrow$, $\tau_1^{\mathscr{B}}\downarrow$ and $\tau_0^{\mathscr{B}} = \tau_1^{\mathscr{B}}$;

$\mathscr{B} \dashv \tau_0 = \tau_1$ iff either $\tau_0^{\mathscr{B}}\downarrow$, $\tau_1^{\mathscr{B}}\downarrow$ and $\tau_0^{\mathscr{B}} \neq \tau_1^{\mathscr{B}}$, or, for some $i < 2$, $\tau_i^{\mathscr{B}}\downarrow$ and $\tau_{1-i}^{\mathscr{B}}\uparrow$.

Note. A nondesignating term in $\xi(\tau_1, \ldots, \tau_n)$ makes that sentence neither true nor false. $=$ has a strong, though not quite bivalent, semantics; that accorded to $=_s$ in

[5] Without much more effort we could have allowed a model to be partial in its action on predicate-constants: in the terminology of [H2], all the models we are considering here are extensionwise-total. We could also have allowed that $|\mathscr{A}|$ be empty; see [H2].

[H2]. We then use the following induction-clauses:

$$\mathscr{B} \models \varphi \supset \psi \text{ iff either } \mathscr{B} \mathrel{=\!\!|} \varphi \text{ or } \mathscr{B} \models \psi;$$
$$\mathscr{B} \mathrel{=\!\!|} \varphi \supset \psi \text{ iff } \mathscr{B} \models \varphi \text{ and } \mathscr{B} \mathrel{=\!\!|} \psi;$$
$$\mathscr{B} \models (\exists \upsilon)\varphi \text{ iff, for some } a \in |\mathscr{B}|, \mathscr{B} \models \varphi(\underline{a}/\upsilon);$$
$$\mathscr{B} \mathrel{=\!\!|} (\exists \upsilon)\varphi \text{ iff, for each } a \in |\mathscr{B}|, \mathscr{B} \mathrel{=\!\!|} \varphi(\underline{a}/\upsilon).$$

Note. $\supset$ and $\exists$ have the strong Kleene semantics. Finally, for $\varphi \in \mathrm{Sent}(L^0)$, let $\mathscr{A} \models \varphi$ iff $\mathscr{B} \models \varphi$, and $\mathscr{A} \mathrel{=\!\!|} \varphi$ iff $\mathscr{B} \mathrel{=\!\!|} \varphi$.

Let $\mathscr{A} \mid \varphi$ abbreviate $\mathscr{A} \not\models \varphi$ and $\mathscr{A} \not\mathrel{=\!\!|} \varphi$.

We adopt the usual abbreviations, e.g. $\neg\varphi$ for $\varphi \supset \bot$, $(\forall \upsilon)\varphi$ for $\neg(\exists \upsilon)\neg\varphi$, etc. When υ is a type-0 variable not occurring in a term τ of L^0, $E(\tau)$ abbreviates $(\exists \upsilon)\tau = \upsilon$; E expresses the bivalent notion of existence, which in [H2] was expressed as E_s. For τ variable-free, Wright's principle (see [H4]) in model-theoretic form holds, as does its negative counterpart:

$$\mathscr{A} \models E(\tau) \quad \text{iff} \quad \tau^{\mathscr{A}}\!\downarrow; \qquad \mathscr{A} \mathrel{=\!\!|} E(\tau) \quad \text{iff} \quad \tau^{\mathscr{A}}\!\uparrow.$$

Fact 1.1. *We can contextually define the truth-function T governed by the clauses: $\mathscr{A} \models T\varphi$ if $\mathscr{A} \models \varphi$; otherwise $\mathscr{A} \mathrel{=\!\!|} T\varphi$.*

This is a consequence of our restriction to models that are extensionwise-total and our strong semantics for $=$; see Observation 2 of [H2] for a proof.

We enrich L^0 to $L^{0,\underline{u}}$ by introducing the 0-place connective $\underline{u}$ into the logical lexicon; syntactically it is a formula; it is to be governed by the semantic rules $\mathscr{A} \not\models \underline{u}$ and $\mathscr{A} \not\mathrel{=\!\!|} \underline{u}$. Since T is already expressible in L^0, in $L^{0,\underline{u}}$ all three-valued truth-functions are expressible.

Digression. For our ultimate purposes in §3 and beyond, we could at this stage have used the more familiar two-valued semantics, handling atomic sentences containing nondenoting terms with the falsehood convention, as is done in [B]. But the falsehood convention introduces an *ad hoc* asymmetry between truth and falsehood, and we face truth-value gaps down the road anyway; so I have allowed them here.[6]

We now form an ω-order language L^ω by enriching L^0 as follows. For each $1 \leq j < \omega$ introduce a countable set $\mathrm{Var}(j)$ of new variables of type j. We define being a formula of L^ω with these familiar formation rules:

for any $\gamma \in \mathrm{Var}(1)$ and any term τ, $\gamma\tau$ is a formula;

for any $\gamma \in \mathrm{Var}(j+1)$ and $\delta \in \mathrm{Var}(j)$, $\gamma\delta$ is a formula; and

for any $\gamma \in \mathrm{Var}(j)$ and φ a formula, $(\exists\gamma)\varphi$ is a formula.

Given a model $\mathscr{A}$ for Vcb, a *restriction-sequence* for $\mathscr{A}$ is a function S on ω with $S(0) = |\mathscr{A}|$ and $S(j+1) \subseteq \mathrm{Power}(S(j))$ for $j < \omega$. We define truth and falsity relative to $(\mathscr{A}, S)$ by letting variables of type j range over $S(j)$. In other words, after expanding L^ω to $L^\omega_{\mathscr{A}}$ as above, we expand to $L^\omega_{\mathscr{A},S}$ by introducing a new constant $\underline{A}$ of type j for each $A \in S(j)$ and $1 \leq j < \omega$. Relative to $\mathscr{B}$ formed as before and S, we

[6] We could have adopted a weaker semantics for $=$ and E; similarly for $\supset$ and $\exists$. Alternatively, we also could have adopted a stronger bivalent semantics for $=$, counting $\tau_0 = \tau_1$ as true when $\tau_i^{\mathscr{A}}\!\uparrow$ for both $i < 2$. See [H2] for details on all these alternatives.

define truth and falsity by adding the following clauses to the definition of truth relative to $\mathscr{B}$ for sentences of $L^0_{\mathscr{A}}$:

for $A \in S(1)$:

$$\mathscr{B}, S \models \underline{A}\tau \quad \text{iff} \quad \tau^{\mathscr{B}}\downarrow \text{ and } \tau^{\mathscr{B}} \in A;$$
$$\mathscr{B}, S \mathrel{=\!\!|} \underline{A}\tau \quad \text{iff} \quad \tau^{\mathscr{B}}\downarrow \text{ and } \tau^{\mathscr{B}} \notin A;$$

for $A \in S(j+1)$ and $B \in S(j)$:

$$\mathscr{B}, S \models \underline{A}\underline{B} \quad \text{iff} \quad B \in A;$$
$$\mathscr{B}, S \mathrel{=\!\!|} \underline{A}\underline{B} \quad \text{iff} \quad B \notin A;$$

for $1 \leq j < \omega$ and $\gamma \in \mathrm{Var}(j)$:

$$\mathscr{B}, S \models (\exists\gamma)\varphi \quad \text{iff} \quad \text{for some } A \in S(j), \mathscr{B}, S \models \varphi(\underline{A}/\gamma);$$
$$\mathscr{B}, S \mathrel{=\!\!|} (\exists\gamma)\varphi \quad \text{iff} \quad \text{for each } A \in S(j), \mathscr{B}, S \mathrel{=\!\!|} \varphi(\underline{A}/\gamma).$$

Again, for $\varphi \in \mathrm{Sent}(L^\omega)$ let

$$\mathscr{A}, S \models \varphi \quad \text{iff} \quad \mathscr{B}, S \models \varphi; \qquad \mathscr{A}, S \mathrel{=\!\!|} \varphi \quad \text{iff} \quad \mathscr{B}, S \mathrel{=\!\!|} \varphi.$$

For $1 \leq j < \omega$ let L^j be the fragment of L^ω formed by eliminating variables of type $> j$. According to standard misuse of Russell's word 'order', L^j is a $(j+1)$st-order language. Abbreviations introduced for L^0 carry over straightforwardly for L^ω and L^j. Everything said for L^ω carries over to $L^{\omega,\underline{u}}$ in the obvious ways.

Two ways of choosing S deserve special mention. For a set x and a cardinal κ, let

$$\mathrm{Power}^{<\kappa}(x) = \{y\colon y \subseteq x \text{ and } \mathrm{card}(y) < \kappa\}.$$

Let S be the weak restriction sequence for a model $\mathscr{A}$ iff, for all $j < \omega$, $S(j+1) = \mathrm{Power}^{<\mathrm{card}(\mathscr{A})}(S(j))$. For such S we shall write '$\overset{\mathrm{w}}{\models}$' and '$\overset{\mathrm{w}}{=\!\!|}$' in place of '$S \models$' and '$S \mathrel{=\!\!|}$'. The resulting semantics and logic for L^1 will be called "weak monadic second-order", and for L^ω, "weak monadic ω-order". ('Weak monadic second-order' has been used to indicate that the type-1 variables range over all finite subsets of $|\mathscr{A}|$; I have seen this in contexts in which all models might as well have been countable, so perhaps my usage will not cause confusion.) Let S be the null restriction sequence for $\mathscr{A}$ iff, for $j < \omega$, $S(j+1) = \mathrm{Power}(S(j))$. For such an S we shall simply drop the reference to S to the left of '$\models$' and '$\mathrel{=\!\!|}$'; this yields the model-theoretic semantics for full monadic ω-order, or full monadic jth order, logic.

Fact 1.2. *Over models of regular cardinality, in weak-monadic logic, L^ω collapses to L^3. In other words, for each $\varphi \in \mathrm{Sent}(L^\omega)$ there is a $\psi \in \mathrm{Sent}(L^3)$ so that, for any infinite model $\mathscr{A}$ of regular cardinality,*

$$\mathscr{A} \overset{\mathrm{w}}{\models} \varphi \quad \textit{iff} \quad \mathscr{A} \overset{\mathrm{w}}{\models} \psi; \qquad \mathscr{A} \overset{\mathrm{w}}{=\!\!|} \varphi \quad \textit{iff} \quad \mathscr{A} \overset{\mathrm{w}}{=\!\!|} \psi.$$

The proof of this is implicit in the proof of Fact A2 below.

Weak monadic second-order logic is, I contend, the encoded logic behind the limitation-of-size conception of sethood; Fact 3.2 will make this contention precise.

Our languages of the form $L^j(\mathrm{Vcb})$ are, of course, uninterpreted. In fact, it is quite misleading to call models "interpretations": they do not assign senses to the non-logical constants, and so do not assign thoughts to sentences of their languages. A model is merely a set. Model-theoretic truth and falsity are two-place set-theoretic relations, and should not be confused with real, live truth and falsity, genuinely semantic properties that apply only to sentences or statements of interpreted, or, as I prefer to call them, sense-bearing, languages (which, of course, may be "artificial" and "formalized"). A model-theoretic semantics is philosophically interesting in so far as: (1) the formation-rules for an uninterpreted language L can represent or model (in the engineer's sense, and at some level of abstraction) those for a sense-bearing language (or fragment thereof) $\mathscr{L}$; (2) models for L (i.e. for the nonlogical vocabulary of L) can model the semantic facts underlying the distribution of truth and falsity to sentences or statements in $\mathscr{L}$; (3) the model-theory's assignment of truth and falsehood relative to models for L to sentences of L models the possible distributions of real truth and falsity to sentences or statements of $\mathscr{L}$; and (4) the logical notions defined model-theoretically coincide, or at least mesh plausibly, with the use, especially the deductive practices, animating $\mathscr{L}$.

I construe higher-order quantification in sense-bearing languages as Frege did: as quantification over "unsaturated" entities;[7] so, for example, type-1 variables would range over Fregean concepts of level one. This is a point of potential confusion. Over any model $\mathscr{A}$, the values of the type-1 variables are subsets of $|\mathscr{A}|$, and thus (speaking "within the Mathematical-Object picture") are objects. But under the modelling, these objects *represent* Fregean concepts; so second-order quantification in an uninterpreted language relative to a model can represent second-order quantification in a sense-bearing language. Similarly for the values of variables of yet higher type. The syntax of natural languages permits, and sometimes requires, misleading constructions that "try" to make singular terms do the work of predicates; witness 'the concept horse' in Frege's well-known discussion. Thus, in much of the literature, type-1 variables in sense-bearing languages are said to range over classes. In this usage (or misusage) a class is what Russell in ***Principles of Mathematics***, Chapter 6, called "a class as many"; I suggest that we construe this as an attempt to say that type-1 variables range over Fregean concepts.[8]

More controversially, I think that quantification over proper classes is best taken as a misleading way of expressing second-order quantification, with '$\in$' followed by a type-1 variable expressing predication. A formal theory like GNB uses the first-order fragment of second-order logic: from a proof-theoretic point of view, the type-1 variables may just as well be a second sort of first-order variables; this is what encourages the benighted to take proper classes to be peculiar objects.[9] Suffice to say: the Alternative theory takes higher-order logic to be ontologically more fundamental than set theory; in this respect it is in keeping with the tradition of Russell and Whitehead.

Suppose that L^0 represents a sense-bearing language that lacks expressions for set-theoretic notions; that is, it lacks predicates for sethood and membership, and an operator for set abstraction. In §2 we shall model the Mathematical-Object theory's

[7] See [H4], where I refer the reader to [Fu].

[8] I take a Russellian "class as one" to be a set.

[9] Peculiar in that they, unlike nonclasses and nonproper classes, belong to no classes. Why should they be so peculiar? To say "They are too big to be elements of classes" explains nothing. This explanatory vacuum should embarrass those who do not regard quantification over proper classes as a notation for quantification over Fregean concepts.

account of how such a language might be enriched with such vocabulary; in §3 we shall model the Alternative theory's account.

§2. Mathematical-Object semantics. We now enrich L^0 by introducing new expressions Set, $\in$, and $\hat{}$, forming the language $L^{0,\wedge}$. We define the sets of terms and formulae of $L^{0,\wedge}$ by simultaneous induction, with Set and $\in$ working like 1-place and 2-place predicates respectively, and with this novel rule:

if υ is a variable of type 1 and φ is a formula, then $\hat{\upsilon}\varphi$ is a term.

For the moment, we leave open whether Set, $\in$ and $\hat{}$ are new nonlogical expressions being added to Vcb or new logical expressions added to the logical lexicon.

For a model $\mathscr{A}$ we adopt these definitions:

$\mathscr{e}$ is an *extensor* for $\mathscr{A}$ iff $\mathscr{e}$ is a one-to-one function into $|\mathscr{A}|$ with $\mathrm{dom}(\mathscr{e}) \subseteq \mathrm{Power}(\mathscr{A})$.

$\mathscr{e}$ is an *ls-extensor* for $\mathscr{A}$ iff $\mathscr{e}$ is an extensor for $\mathscr{A}$ with $\mathrm{dom}(\mathscr{e}) = \mathrm{Power}^{<\mathrm{card}(\mathscr{A})}(\mathscr{A})$.

Note: 'ls' abbreviates 'limitation-of-size'. A model $\mathscr{A}$ is *acceptable* iff $\mathrm{card}(\mathscr{A}) > 1$ and $\mathrm{card}(\mathscr{A})$ is acceptable. Obviously all acceptable models are infinite.

Fact 2.1. *If* $\mathrm{card}(\mathscr{A}) > 1$, *then* $\mathscr{A}$ *has an ls-extensor iff* $\mathscr{A}$ *is acceptable.*

Let an MO-model be a pair $\langle \mathscr{A}, \mathscr{e} \rangle$, $\mathscr{A}$ a model and $\mathscr{e}$ an extensor for $\mathscr{A}$. We define designation, truth and falsity relative to an MO-model as follows (τ, σ and $\hat{v}\varphi$ contain no free-variables):

$\mathscr{A}, \mathscr{e} \models \mathrm{Set}(\tau)$ iff $\tau^{\mathscr{A},\mathscr{e}}\downarrow$ and $\tau^{\mathscr{A},\mathscr{e}} \in \mathrm{rng}(\mathscr{e})$;

$\mathscr{A}, \mathscr{e} \dashv \mathrm{Set}(\tau)$ iff $\tau^{\mathscr{A},\mathscr{e}}\downarrow$ and $\tau^{\mathscr{A},\mathscr{e}} \notin \mathrm{rng}(\mathscr{e})$;

$\mathscr{A}, \mathscr{e} \models \tau \in \sigma$ iff $\tau^{\mathscr{A},\mathscr{e}}\downarrow$, $\sigma^{\mathscr{A},\mathscr{e}}\downarrow$ and, for some $A \in \mathrm{dom}(\mathscr{e})$, $\sigma^{\mathscr{A},\mathscr{e}} = \mathscr{e}(A)$ and $\tau^{\mathscr{A},\mathscr{e}} \in A$;

$\mathscr{A}, \mathscr{e} \dashv \tau \in \sigma$ iff $\tau^{\mathscr{A},\mathscr{e}}\downarrow$, $\sigma^{\mathscr{A},\mathscr{e}}\downarrow$ and for no $A \in \mathrm{dom}(\mathscr{e})$ do we have $\sigma^{\mathscr{A},\mathscr{e}} = \mathscr{e}(A)$ and $\tau^{\mathscr{A},\mathscr{e}} \in A$;

$\hat{\upsilon}\varphi^{\mathscr{A},\mathscr{e}} = a$ if $\mathscr{e}(\{b: \mathscr{A}, \mathscr{e} \models \varphi(\underline{b}/\upsilon)\}) = a$;

$\hat{\upsilon}\varphi^{\mathscr{A},\mathscr{e}}\uparrow$ if $\{b: \mathscr{A}, \mathscr{e} \models \varphi(\underline{b}/\upsilon)\} \notin \mathrm{dom}(\mathscr{e})$.

Note. $\tau^{\mathscr{A},\mathscr{e}}$ abbreviates $\mathrm{des}^{\mathscr{A},\mathscr{e}}(\tau)$, etc. This semantics is mildly unusual in that designation is defined simultaneously with truth and falsity.

Let $L^{0,\in}$ be the language obtained by eliminating $\hat{}$ from $L^{0,\wedge}$. Form $L^{0,\in,\underline{u}}$ by adding $\underline{u}$ to the lexicon of $L^{0,\in}$. All of the other definitions above extend to these languages in the obvious ways.

According to Frege's conception of sethood, to each concept of level one there corresponds a set that is its extension—that is to say, an object to which all and only those objects falling under that concept bear the relation of membership. Suppose that $\mathscr{A}$ models the referential basis of non-set-theoretic discourse, with the members of $|\mathscr{A}|$ representing everything that exists (including, of course, all sets). If there were such a correspondence, it would be represented relative to $\mathscr{A}$ by a minimal extensor for $\mathscr{A}$ with domain equal to Power($\mathscr{A}$). But by Russell's paradox, there is no such correspondence; the model-theoretic counterpart of this is the fact that no model has an extensor whose domain is its power-set, which is really just Cantor's theorem. (As Anil Nerode once put it, somewhat unfairly, "Russell's paradox is Cantor's theorem; the difference was that Cantor knew what to do with it.")

Frege saw Russell's paradox as undoing not only his foundational project in the ***Grundgesetze***, but also any sort of set-theory. (See [Fr, p. 269].) Fortunately others were not so hasty. Though we cannot allow that all Fregean concepts have extensions, we can allow that some do. The Mathematical-Object theory is committed to the existence of a "standard extensor" assigning extensions to those Fregean concepts fortunate enough to have them.[10] And which are so fortunate?

Cantor had a suggestion here, based on distinguishing absolutely infinite collections from those collections which may be infinite but are not absolutely infinite. Following Cantor's lead, we can preserve some of Frege's approach in the face of Russell's paradox by accepting the limitation-of-size comprehension principle: any level-one concept under which there does not fall an absolute infinity of objects has an extension. We could then construe the paradoxes as merely showing that an absolute infinity of objects fall under the "paradoxical" concepts like non-self-membership.

We could also take a further Cantorian step by accepting the limitation-of-size restriction principle: any level-one concept under which there falls an absolute infinity of objects lacks an extension. These two principles constitute what I shall call "the limitation of size conception of sethood" (using a phrase from [R]).

If we accepted the Mathematical-Object theory about the semantics of set-theoretic discourse, given a model $\mathscr{A}$ and an extensor e for $\mathscr{A}$, we could think of e as representing whatever is left of Frege's standard extensor after Russell's paradox: the members of dom(e) represent those Fregean concepts that have extensions, and their values under e represent those extensions. Thus for the Mathematical-Object theorist, Set, $\in$ and $\wedge$ are nonlogical expressions, and appropriate models for the nonlogical vocabulary formed by adding Set, $\in$ and $\wedge$ to Vcb are MO-models: they, rather than mere models for Vcb, can model the basic semantic facts underlying the distribution of truth-values to sentences in languages containing set-theoretic expressions. With $\langle \mathscr{A}, e \rangle$ modelling these facts, $\mathrm{des}^{\mathscr{A}, e}$ models designation of closed terms, while $\mathscr{A}, e \models$ and $\mathscr{A}, e \dashv$ model truth and falsity for sentences.

For C a class of models and K a class of extensors, let an MO_K^C-model be an MO-model belonging to $C \times K$. These determine the Mathematical-Object logic MO_K^C as follows. For $\Gamma \subseteq \mathrm{Sent}(L^{0,\wedge})$ and $\varphi, \psi \in \mathrm{Sent}(L^{0,\wedge})$:

Γ MO_K^C-*implies* φ iff, for every MO_K^C-model $\langle \mathscr{A}, e \rangle$, if $\langle \mathscr{A}, e \rangle \models \Gamma$ then $\langle \mathscr{A}, e \rangle \models \varphi$;

φ is MO_K^C-*valid* iff, for every MO_K^C-model $\langle \mathscr{A}, e \rangle$, $\mathscr{A}, e \models \varphi$;

φ and ψ are *positively* MO_K^C-*equivalent* iff, for every MO_K^C-model $\langle \mathscr{A}, e \rangle$, $\mathscr{A}, e \models \varphi$ iff $\mathscr{A}, e \models \psi$;

φ and ψ are MO_K^C-*equivalent* iff φ and ψ are positively MO_K^C-equivalent, and so are $\neg\varphi$ and $\neg\psi$.

We shall construe absolute infinity as the "cardinality" of the universe, modelled relative to $\mathscr{A}$ by card($\mathscr{A}$). Then according to the Mathematical-Object theory, the limitation-of-size conception is reflected by logics of the form $\mathrm{MO}_{\mathrm{ls}}^C$, for ls = the

[10] Any set is the extension of the Fregean concept: is a member of that set.

class of ls-extensors. For $C =$ the class of all acceptable models, we omit the superscript; $\mathrm{MO}_{\mathrm{ls}}$ is the Mathematical-Object limitation-of-size logic. Similarly for other conceptions of sethood that might be reflected by other choices of K. For $K =$ the class of all extensors, we omit the subscript.

Fact 2.2. *For any model $\mathscr{A}$ and any extensor e for $\mathscr{A}$,* $\mathrm{des}^{\mathscr{A},e}$ *is not total on the closed terms of* $L^{0,\wedge}$; *e.g.* $\hat{x}(x \notin x)^{\mathscr{A},e}\uparrow$.

This is why it was convenient to allow partial designation from the start.

Proof. Let $r = \{b: \mathscr{A}, e \models \underline{b} \notin \underline{b}\}$, and suppose that $r \in \mathrm{dom}(e)$; let $a = e(r)$. Thus:

$$\begin{array}{ll} \mathscr{A}, e \models \underline{a} \in \underline{a} \text{ iff } a \in r, & \text{by the definition of } \models; \\ a \in r \text{ iff } \mathscr{A}, e \models \underline{a} \notin \underline{a}, & \text{by choice of } r; \\ \mathscr{A}, e \models \underline{a} \notin \underline{a} \text{ iff } \mathscr{A}, e \not\models \underline{a} \in \underline{a} & \text{because } \underline{a}^{\mathscr{A},e}\downarrow. \end{array}$$

This is a contradiction.

Thus $\mathscr{A}, e \,|\, \hat{x}(x \notin x) \in \hat{x}(x \notin x)$.

§3. The Alternative limitation-of-size semantics. If we reject the Mathematical-Object theory, we cannot regard MO-models as modelling the basic semantic facts underlying the distribution of truth values to sense-bearing statements containing set-theoretic expressions. Instead we may model the semantic facts underlying our set-theoretic talk as follows.

Let K be a class of extensors; a K-extensor is a member of K. For $\varphi \in \mathrm{Sent}(L^{0,\wedge})$ and $\mathscr{A}$ an acceptable model, let:

$\mathscr{A} \models_K \varphi$ iff, for each K-extensor e for $\mathscr{A}$, $\mathscr{A}, e \models \varphi$;

$\mathscr{A} \dashv_K \varphi$ iff, for each K-extensor e for $\mathscr{A}$, $\mathscr{A}, e \dashv \varphi$;

$\mathrm{des}^{\mathscr{A}}_K(\tau) = a$ iff, for each K-extensor e for $\mathscr{A}$, $\tau^{\mathscr{A},e} = a$;

$\mathrm{des}^{\mathscr{A}}_K(\tau)\uparrow$ iff there is no a so that $\mathrm{des}^{\mathscr{A}}_K(\tau) = a$;

$\mathscr{A} \,|_K\, \varphi$ iff $\mathscr{A} \not\models_K \varphi$ and $\mathscr{A} \not\dashv_K \varphi$.

Fact 3.1. *These definitions extend those given in* §1: *for* $\varphi \in \mathrm{Sent}(L^0)$, $\mathscr{A} \models \varphi$ *iff* $\mathscr{A} \models_K \varphi$; *similarly for* $\mathscr{A} \dashv_K$; *and, for any term* τ *of* L^0, *if* $\tau^{\mathscr{A}} = a$ *then* $\mathrm{des}^{\mathscr{A}}_K(\tau) = a$.

Notice that even if our definitions of $\mathscr{A}, e \models$ and $\mathscr{A}, e \dashv$ had been two-valued, $\mathscr{A} \models_K$ and $\mathscr{A} \dashv_K$ would permit truth-value gaps.

Where C is a class of acceptable models we now introduce the Alternative logic A^C_K. For $\Gamma \subseteq \mathrm{Sent}(L^{0,\wedge})$ and $\varphi, \psi \in \mathrm{Sent}(L^{0,\wedge})$:

Γ A^C_K-*entails* φ iff, for each $\mathscr{A} \in C$, if $\mathscr{A} \models_K \Gamma$ then $\mathscr{A} \models_K \varphi$;

φ is A^C_K-*valid* iff, for each $\mathscr{A} \in C$, $\mathscr{A} \models_K \varphi$;

φ is A^C_K-*anti-valid* iff, for each $\mathscr{A} \in C$, $\mathscr{A} \dashv_K \varphi$;

φ is A^C_K-*bivalent* iff, for each $\mathscr{A} \in C$, either $\mathscr{A} \models_K \varphi$ or $\mathscr{A} \dashv_K \varphi$;

φ is A^C_K-*truth-valueless* iff, for each $\mathscr{A} \in C$, $\mathscr{A} \,|_K\, \varphi$;

φ and ψ are *positively* A^C_K-*equivalent* iff, for each $\mathscr{A} \in C$, $\mathscr{A} \models_K \varphi$ iff $\mathscr{A} \models_K \psi$;

φ and ψ are A^C_K-*equivalent* iff φ and ψ are positively A^C_K-equivalent and so are $\neg\varphi$ and $\neg\psi$.

Let A^C_K-$\mathrm{Biv}(L^{0,\wedge})$ be the set of A^C_K-bivalent sentences of $L^{0,\wedge}$; similarly for A^C_K-$\mathrm{Biv}(L^{0,\in})$. If $C =$ the class of acceptable models, we shall drop the superscript on all these notions. Let tot $=$ the class of total acceptable models.

We shall devote most attention to the case $K = \mathrm{ls}$; so we hereafter omit the subscripted 'ls' on '$\models_{\mathrm{ls}}$', etc., and abbreviate $\mathrm{des}^{\mathscr{A}}_{\mathrm{ls}}(\tau)$ as $\tau^{\mathscr{A}}$. But do not confuse the relations between MO-models and sentences represented by '$\models$' and '$\dashv$' with the relations between acceptable models and sentences also represented (now homonymously) by those symbols!

Let A_{ls} be the Alternative limitation-of-size logic; its semantics reflects the Alternative theory of the alethic underpinnings for set-theoretic discourse when that discourse employs the limitation-of-size conception; so logics of the form $\mathrm{A}_{\mathrm{ls}}^{C}$ are appropriate for Alternative theorists who accept the limitation-of-size conception of sethood. According to the Alternative theory, there is no standard extensor. A mere model, rather than an MO-model, can model all the basic semantic facts underlying the distribution of truth values within a sense-bearing language which is parsed by $L^{0,\wedge}$ or $L^{0,\in}$: taking $\mathscr{A}$ to model these facts, $\mathrm{des}^{\mathscr{A}}$ models genuine designation, and $\mathscr{A} \models$ and $\mathscr{A} \dashv$ model truth and falsehood. Unlike an MO-model, an acceptable model does not "interpret" (i.e. assign values to) Set, $\in$ or $^\wedge$; so for the Alternative limitation-of-size semantics these are to be counted as logical expressions. Indeed, according to the Alternative theory, an MO-model $\langle \mathscr{A}, \mathscr{e} \rangle$, $\mathrm{des}^{\mathscr{A},\mathscr{e}}$, $\mathscr{A}, \mathscr{e} \models$ and $\mathscr{A}, \mathscr{e} \dashv$ all do no modelling; they are only "supervaluational" stepping-stones to the semantically significant definitions of designation, truth, and falsity relative to $\mathscr{A}$.

The model-theoretic versions of Wright's principle and its negative counterpart still hold for terms of L^0, but they fail for some abstraction-terms: for any such τ, $\tau^{\mathscr{A}}\uparrow$; but, for example, $E(\hat{x}\perp)$ is A_{ls}-valid.

We can now give an example of how set-theoretic expressions may permit the encoding of a higher-order logic into a notationally and conceptually more tractable first-order syntax. For the limitation-of-size conception, weak ω-order logic encodes into $L^{0,\in}$ as follows.

Fact 3.2. *There is a translation* t: $\mathrm{Sent}(L^{\omega}) \to \mathrm{A}_{\mathrm{ls}}^{\mathrm{tot}}\text{-}\mathrm{Biv}(L^{0,\in})$ *such that, for any* $\varphi \in \mathrm{Sent}(L^{\omega})$ *and any acceptable model* $\mathscr{A}$,

$$\mathscr{A} \overset{\mathrm{w}}{\models} \varphi \quad \textit{iff} \quad \mathscr{A} \models t(\varphi); \qquad \mathscr{A} \overset{\mathrm{w}}{\dashv} \varphi \quad \textit{iff} \quad \mathscr{A} \dashv t(\varphi).$$

Similarly for $L^{\omega,\underline{u}}$ *and* $L^{0,\wedge}$.

Proof. Suppose that $\mathscr{A}$ is a model, $\mathscr{e}$ is an extensor for $\mathscr{A}$, and S is the weak restriction-sequence for $\mathscr{A}$. Assume without loss of generality that $\bigcup_{0<j<\omega} S(j)$ and $|\mathscr{A}|$ are disjoint. Define S' and $\mathscr{e}'$ as follows:

$$\begin{aligned} S'(0) &= |\mathscr{A}|; \\ S'(j+1) &= \{\mathscr{e}(X) \colon X \subseteq S'(j) \text{ and } X \in \mathrm{dom}(\mathscr{e})\}; \\ \mathscr{e}'(x) &= \mathscr{e}(x) \quad \text{for } x \in S(1) = \mathrm{dom}(\mathscr{e}); \\ \mathscr{e}'(x) &= \mathscr{e}(\{\mathscr{e}'(y) \colon y \in x\}) \quad \text{for } x \in S(j+1),\, 0 < j < \omega. \end{aligned}$$

Though $\mathscr{e}'$ is not one-to-one, it maps $S(j)$ one-to-one onto $S'(j)$ for $1 \leq j < \omega$. For such a j we shall represent an $x \in S(j)$ by $e'(x)$, and the typing of a variable of type j by restricting a corresponding type-0 variable to $S'(j)$.

Let $\mathrm{Set}_1(\upsilon)$ be $\mathrm{Set}(\upsilon)$. For $0 < j < \omega$ let $\mathrm{Set}_{j+1}(\upsilon)$ be

$$\mathrm{Set}(\upsilon) \mathbin{\&} (\forall \upsilon')(\upsilon' \in \upsilon \supset \mathrm{Set}_j(\upsilon')),$$

where υ' is a type-0 variable distinct from υ. Note that, for $1 \leq j < \omega$ and any $a \in |\mathscr{A}|$

and $x, y \in \bigcup_{j<\omega} S(j)$:

$$\begin{aligned}&\text{if } a \in S'_j \text{ then } \mathscr{A}, e \models \mathrm{Set}_j(\underline{a});\\ &\quad\text{otherwise } \mathscr{A}, e \dashv \mathrm{Set}_j(\underline{a});\\ &\text{if } x \in y \text{ then } \mathscr{A}, e \models \underline{e'(x)} \in \underline{e'(y)};\\ &\quad\text{otherwise } \mathscr{A}, e \dashv \underline{e'(x)} \in \underline{e'(y)}.\end{aligned}$$

Given $\varphi \in \mathrm{Sent}(L^\omega)$, form $t(\varphi)$ as follows: replace each variable γ of type > 0 by a distinct new type-0 variable υ_γ, replacing subformulas of the form $\gamma\tau$ by $\tau \in \upsilon_\gamma$ and subformulas of the form $(\exists\gamma)\theta$ in which γ is a type-j variable for $j > 0$ by $(\exists\upsilon_\gamma)(\mathrm{Set}_j(\upsilon_\gamma)\ \&\ \theta')$, where θ' is the result of doing all this within θ. By induction on the construction of φ we have

$$\mathscr{A} \overset{\mathrm{w}}{\models} \varphi \quad \text{iff} \quad \mathscr{A}, e \models t(\varphi); \qquad \mathscr{A} \overset{\mathrm{w}}{\dashv} \varphi \quad \text{iff} \quad \mathscr{A}, e \dashv t(\varphi).$$ [11]

The desired biconditionals follow.

If $\mathscr{A}$ is total, then either $\mathscr{A} \overset{\mathrm{w}}{\models} \varphi$ or $\mathscr{A} \overset{\mathrm{w}}{\dashv} \varphi$. Thus $t(\varphi)$ is $\mathrm{A}^{\mathrm{tot}}_{\mathrm{ls}}$-bivalent. For $\varphi \in \mathrm{Sent}(L^{\omega,u})$, proceed as above, but remember that in $t(\varphi)$ $\underline{u}$ abbreviates '$\hat{x}(x \notin x) \in \hat{x}(x \notin x)$'.

In Appendix 2 we prove a further encoding result.

Because the point of set-theoretic notions is to encode discourse that could be parsed in a fragment of L^ω, our mathematical practice can be parsed within $\mathrm{A}^{\mathrm{tot}}_{\mathrm{ls}}$-$\mathrm{Biv}(L^{0,\wedge})$; so in effect this practice is based on two-valued logic. But beyond the fringe of actual mathematical practice lie statements, of interest only to philosophers, that are plausibly truth-valueless, e.g. 'Julius Caesar is a set' or 'Julius Caesar = the empty set'. Of course the Mathematical-Object theory requires that these statements have a truth value (provided if 'Julius Caesar' designates). The Alternative theory makes no such demand; and this is reflected in A_{ls}: for any term τ of L^0, $\mathrm{Set}(\tau)$ is not A_{ls}-bivalent, and so has no mathematical content; similarly for $\tau = \hat{x}\bot$. In fact both are A_{ls}-truth-valueless. This reflects a natural reaction to such statements: they sound peculiar precisely because they have no mathematical content, and thus no role in mathematical practice; I suspect that most people, unless they have already swallowed a philosophical theory, would be reluctant to even consider them false. But nonetheless they are not ill-formed or meaningless: a logically truth-valueless sentence is a sentence!

Things may seem rather different with a statement like 'Everything is a set'. Though it too would play no role in actual mathematical practice, one might insist that it is straightforwardly false, as would be permitted by the Mathematical-Object theory. But $(\forall x)\mathrm{Set}(x)$ is A_{ls}-truth-valueless. Still, such a statement has at most philosophical, rather than properly mathematical, content; its falsity is not built into our mathematical practice; so a theory of the alethic underpinnings of set-theoretic discourse is not obliged to reconstruct it as false. The Alternative theory maintains

[11] As this suggests, the Mathematical-Object theory also takes set-theoretic statements to encode statements in weak ω-order logic; but it does not take this to be the "raison d'être" of set-theoretic vocabulary.

that this statement lacks content because it lacks mathematical content: the doctrine that there are nonsets should be dismissed as an intrusion of the Mathematical-Object theory into philosophical thinking.

We now compare $\mathrm{A}_{\mathrm{ls}}^{C}$ to $\mathrm{MO}_{\mathrm{ls}}^{C}$ for C any class of acceptable models.

Fact 3.3. *For any* $\Gamma \subseteq \mathrm{Sent}(L^{0,\wedge})$ and $\varphi \in \mathrm{Sent}(L^{0,\wedge})$:

φ *is* $\mathrm{MO}_{\mathrm{ls}}^{C}$*-valid iff* φ *is* $\mathrm{A}_{\mathrm{ls}}^{C}$*-valid; and*

if Γ $\mathrm{MO}_{\mathrm{ls}}^{C}$*-implies* φ, *then* Γ $\mathrm{A}_{\mathrm{ls}}^{C}$*-implies* φ.

Fact 3.4. *For* $\Gamma \subseteq \mathrm{A}_{\mathrm{ls}}^{C}\text{-Biv}(L^{0,\wedge})$ *and* $\varphi \in \mathrm{A}_{\mathrm{ls}}^{C}\text{-Biv}(L^{0,\wedge})$: Γ $\mathrm{MO}_{\mathrm{ls}}^{C}$*-implies* φ *iff* Γ $\mathrm{A}_{\mathrm{ls}}^{C}$*-implies* φ.

The Alternative theory construes mathematical principles as validities in appropriate logics. Applying that theory to set-theoretic discourse based on the limitation-of-size conception of sethood, the appropriate logic would be A_{ls}, or else a logic obtained from it by restriction to some special class of acceptable models. Fact 3.3 suggests that the Alternative theory and the Mathematical-Object theory do not disagree about any properly mathematical principles; and Fact 3.4 suggests that the actual mathematical practice of a Mathematical-Object theorist will be virtually indiscernible from that of an Alternative theorist. This is as it should be: these are semantic (or even philosophical), not mathematical, theories.

Is there a translation s: $\mathrm{A}_{\mathrm{ls}}\text{-Biv}(L^{0,\in}) \to \mathrm{Sent}(L^{\omega})$ so that for any acceptable model $\mathscr{A}$ and any $\varphi \in \mathrm{dom}(s)$:

(1) $$\mathscr{A} \overset{\mathrm{w}}{\models} s(\varphi) \quad \text{iff} \quad \mathscr{A} \models \varphi; \qquad \mathscr{A} \overset{\mathrm{w}}{\dashv} s(\varphi) \quad \text{iff} \quad \mathscr{A} \dashv \varphi?$$

Conjecture. No. But A_{ls} is a fragment of full second-order logic, in the following sense.

Enrich L^1 to $L^{(0,0)}$ by introducing a countable set Var((0,0)) of new type-(0,0) variables, with the following formation rules: if γ is a type-(0,0) variable and τ and σ are terms of L^0 then $\gamma\tau\sigma$ is a formula; and of course quantification of such variables. Define truth and falsity in $\mathscr{A}$ by letting type-1 variables range over Power($\mathscr{A}$) and type-(0,0) variables range over Power($|\mathscr{A}|^2$). Enrich $L^{(0,0)}$ to $L^{(0,0),\underline{u}}$ by adding $\underline{u}$ as a primitive.

Fact 3.5. *There is a translation*

$$s\colon \mathrm{A}_{\mathrm{ls}}^{\mathrm{tot}}\text{-Biv}(L^{0,\in}) \to \mathrm{Sent}(L^{(0,0)})$$
$$[s\colon \mathrm{Sent}(L^{0,\wedge}) \to \mathrm{Sent}(L^{(0,0),\underline{u}})]$$

so that, for any acceptable model $\mathscr{A}$ *and any* $\varphi \in \mathrm{dom}(s)$, (1) *holds.*

Proof. Any extensor e codes as

$$\{\langle a,b\rangle\text{: for some } A \in \mathrm{dom}(e) \text{ we have } a \in A \text{ and } e(A) = b\}.$$

It is not hard to construct formulae $\mathrm{Ext}(\delta)$ and $\varphi'(\delta)$ of $L^{(0,0)}$ in which δ is the only free variable and is of type (0,0), so that for any $R \subseteq |\mathscr{A}|^2$ and $\mathscr{A}'$ the expansion of $\mathscr{A}$ formed by setting $\underline{R}^{\mathscr{A}'} = R$:

$$\mathscr{A}' \models \text{ls-Ext}(\underline{R}) \text{ iff } R \text{ codes an ls-extensor for } \mathscr{A};$$

and if R codes the ls-extensor e then

$$\mathscr{A}, e \models \varphi \quad \text{iff} \quad \mathscr{A}' \models \varphi'(\underline{R}).$$

Then, for any acceptable model $\mathscr{A}$,

$$\mathscr{A} \models \varphi \quad \text{iff} \quad \mathscr{A} \models (\forall\delta)(\text{ls-Ext}(\delta) \supset \varphi'(\delta)),$$

and, if $\varphi \in \mathrm{A}_{\mathrm{ls}}^{\mathrm{tot}}\text{-Biv}(L^{0,\in})$ and $\mathscr{A}$ is total,

$$\mathscr{A} \mathrel{=\!\!|} \varphi \quad \text{iff} \quad \mathscr{A} \mathrel{=\!\!|} (\forall\delta)(\text{ls-Ext}(\delta) \supset \varphi'(\delta)).$$

More generally, for $\varphi \in \text{Sent}(L^{0,\wedge})$, first translate into $L^{0,\underline{u},\in}$ as in Fact A1.2; then take $s(\varphi)$ to be

$$[(\forall\delta)(\text{ls-Ext}(\delta) \supset \varphi'(\delta))] \vee [(\exists\delta)(\text{ls-Ext}(\delta) \;\&\; \neg T \neg \varphi'(\delta)) \;\&\; \underline{u}].$$

§4. Axioms and the cardinality of the universe. We shall now consider the status of familiar axioms within $\mathrm{A}_{\mathrm{ls}}^{C}$ for $C =$ the class of acceptable models meeting these conditions:

$C = \text{reg}$	:	regular cardinality;
$C = \infty$	:	uncountable cardinality;
$C = \infty, \text{reg}$	:	uncountable regular cardinality;
$C = \text{s-lim}$	:	strong-limit cardinality;
$C = \text{inacc}$	:	strongly inaccessible cardinality.

Clearly all instances of the Axiom of Separation are A_{ls}-valid. Similarly for the Axiom of Replacement.

The Union Axiom (also known as Sum),

$$(\forall x)(\text{Set}(x) \supset (\exists y)[\text{Set}(y) \;\&\; (\forall z)(z \in y \equiv (\exists u)(z \in u \;\&\; u \in x))]),$$

is not A_{ls}-valid. For suppose $\text{card}(\mathscr{A}) = \aleph_\omega$, e is an ls-extensor for $\mathscr{A}$, and for $j < \omega$ we take $A_j \subseteq |\mathscr{A}|$ of cardinality $\aleph_j$; then $e(\{e(A_j): j \in \omega\})\downarrow$, and $\langle \mathscr{A}, e\rangle \mathrel{=\!\!|}$ Union. In fact for any acceptable model $\mathscr{A}$, $\mathscr{A} \models$ Union iff $\text{card}(\mathscr{A})$ is regular; so Union is $\mathrm{A}_{\mathrm{ls}}^{\mathrm{reg}}$-valid.

The Axiom of Infinity is not A_{ls}-valid, since it fails in all acceptable models of cardinality $\aleph_0$. But for any $\mathscr{A}$ as above, $\mathscr{A} \models$ Infinity iff $\text{card}(\mathscr{A})$ is uncountable. So Infinity is $\mathrm{A}_{\mathrm{ls}}^{\infty}$-valid.

The Power-set Axiom is not A_{ls}-valid; indeed, for any $\mathscr{A}$ as above, $\mathscr{A} \models$ Power-set iff $\text{card}(\mathscr{A})$ is a strong limit. So Power-set is $\mathrm{A}_{\mathrm{ls}}^{\text{s-lim}}$-valid.

Recall that a strongly inaccessible cardinal is an uncountable regular strong limit cardinal. The above remarks show the axioms of ZF, excluding Regularity, to be $\mathrm{A}_{\mathrm{ls}}^{\mathrm{inacc}}$-valid.

Unlike the Sum, Infinity, and Power-set Axioms, the validity of the Axiom of Choice is not sensitive to relativization to a class of acceptable models; similarly for CH and $V = L$: appropriate sentences expressing these three propositions are valid iff they are true. In this sense their status is mathematically substantive, while that of the Sum, Infinity and Power-set Axioms merely reflects a view about "the size of the universe". This leads us to an important issue.

Russell and Whitehead, in trying to substantiate their hunch that the truths of pure mathematics were logical truths, ran into a significant embarassment: they had to assume that there are infinitely many urelements.

I regard the Alternative theory as partly in the spirit of logicism. So it is not surprising that the Alternative theory, applied to arithmetic discourse, dictates that such discourse presuppose the existence of infinitely many objects; this is borne out by the need, in [H3], for a restriction to infinite models in specifying the Alternative logic for arithmetic.[12]

Straightforward logicism is compromised, but by very little: pure arithmetic truths are logical truths modulo the assumption "There are infinitely many objects." And that assumption is not intrinsically mathematical: it can be expressed by an infinite set of pure first-order sentences (using =, of course).

The Alternative theory applied to set-theoretic discourse requires a similar, but stronger, caveat. Taking A_{ls} as the Alternative theorist's logic, that theory construes set-theoretic principles as logical truths modulo at least the assumption that may be loosely put as: "There are acceptably many objects." This is loose because acceptability is defined for cardinals, and the real universe is, at least on the limitation-of-size conception, absolutely infinite, and so literally without a cardinality. (An analogy: anyone who accepts second-order ZF thinks that, loosely speaking, there are inaccessibly many objects; but literally inaccessibility is defined only for cardinals.)

The needed assumption can be precisely expressed without use of set-theoretic locutions. For example, using a weak second-order version of Henkin's branching quantifier, the "acceptability" of the size of the universe can be expressed as:

$$\begin{matrix}(\forall X)(\exists x)\\(\forall Y)(\exists y)\end{matrix}[x = y \equiv (\forall z)(X(z) \equiv Y(z))].$$

Thus even on the Alternative theory, the limitation-of-size comprehension principle has "factual content", though of a very narrow sort: concerning the "cardinality" of the universe.

Our previous discussion shows that familiar axioms have similar content. The Power-set Axiom requires that the universe have "strong limit size". This can be expressed without set-theoretic vocabulary by:

$$(\forall X)(\exists Y)\begin{matrix}(\forall U)(\exists u)\\(\forall V)(\exists v)\end{matrix}((\forall x)(Ux \supset Xx) \supset [Yu \;\&\; ((\forall x)(Ux \equiv Vx) \supset u = v)]),$$

again with type-1 variables understood weakly. The Infinity Axiom says that the universe is uncountable, and can be expressed in weak monadic second-order logic without branching quantifiers and without set-theoretic vocabulary. The "regularity" of the universe can be expressed in full second-order logic (that is, $L^{(0,0)}$), so without use of set-theoretic vocabulary. It seems unlikely that it can be expressed with branching quantifiers in weak monadic logic.

[12] In [H1] I said that the modal gambit there sketched could dodge the conclusion that arithmetic presupposes the existence of infinitely many objects. I have since come to my senses; see [H3].

§5. The iterative conception of sethood, and others. We now consider the Axiom of Regularity and its motivation: the so-called iterative conception of sethood.

Express "$\upsilon = \omega$" as usual; let $E(\omega)$ be $(\exists x)x = \omega$; so $\mathscr{A} \models E(\omega)$ iff $\aleph_0 <$ card($\mathscr{A}$). Let "f is a descending $\in$-chain" mean "f is a function with $\mathrm{dom}(f) \leq \omega$ and $f(n + 1) \in f(n)$ for each $n + 1 \in \mathrm{dom}(f)$". For any model $\mathscr{A}$, extensor e for $\mathscr{A}$, and $a \in \mathrm{rng}(e)$, let a be well-founded for e iff there is no $f\colon \omega \to |\mathscr{A}|$ such that, for each $n < \omega$, $\mathscr{A}, e \models \underline{f(n+1)} \in \underline{f(n)}$. Let e be well-founded iff every $a \in \mathrm{rng}(e)$ is well-founded for e.

The Axiom of Regularity is not A_{ls}-valid; in fact, it is A_{ls}-truth-valueless: clearly every acceptable model has a well-founded ls-extensor, relative to which Regularity is true; the compactness of first-order logic permits us to construct a non-well-founded ls-extensor, relative to which it is false.

Under the Alternative limitation-of-size semantics we can express the well-foundedness of membership in $L^{0,\in}$, taking advantage of that semantics' second-order nature.

For any type-0 variable υ, express "$\upsilon = \{\ \}$" as usual. Let $\mathrm{Ord}(\upsilon)$ be the obvious formula saying

$$(\upsilon \text{ is a transitive set well-ordered by } \in) \,\&\, (\upsilon = \{\ \} \vee \{\ \} \in \upsilon).$$

As usual, for ordinals υ and υ' we write $\upsilon < \upsilon'$ for $\upsilon \in \upsilon'$, etc. This is an adequate definition of being an ordinal; easily $(\forall \upsilon)(\mathrm{Ord}(\upsilon) \supset \neg\, \upsilon < \upsilon)$ is A_{ls}-valid, since a well-ordering is irreflexive. The second conjunct is needed to insure comparability of any two ordinals, since $(\forall x)\mathrm{Set}(x)$ is not valid; comparability follows by the argument in [L] for Theorem 3.12. Let $\mathrm{FinOrd}(\upsilon)$ be

$$\mathrm{Ord}(\upsilon) \,\&\, \neg(\exists \upsilon')(\upsilon' \leq \upsilon \,\&\, \mathrm{LimOrd}(\upsilon')),$$

where $\mathrm{LimOrd}(\upsilon')$ says "υ' is a limit ordinal" in the obvious way.

For a cardinal κ let $\mu^* = \sum_{\kappa < \mu} \kappa^+$. (Recall that $\kappa^+ = \sup\{\xi\colon (\exists x)(\mathrm{card}(x) \leq \kappa \,\&\, x$ has a well-ordering of order-type $\xi)\}$.) Clearly, for any model $\mathscr{A}$ and ls-extensor e for $\mathscr{A}$, $\{a \in |\mathscr{A}|\colon \mathscr{A}, e \models \mathrm{Ord}(\underline{a})\}$ is well-ordered by $\{\langle a, b\rangle \in |\mathscr{A}|^2\colon \mathscr{A}, e \models \underline{a} \in \underline{b}\}$ with order-type $\mathrm{card}(\mathscr{A})^*$. In particular, all finite ordinals are "available" over any acceptable model. Let $\mathrm{Wfd}(x)$ be

$$\begin{aligned} &[\neg E(\omega) \supset (\exists y)(\mathrm{Ord}(y) \,\&\, (\forall z)((z \text{ is a descending } \in\text{-chain} \\ &\qquad\qquad\qquad\qquad \&\, z(\{\ \}) = x) \supset \mathrm{dom}(z) < y))] \\ &\&\, [E(\omega) \supset (\forall z)((z \text{ is a descending } \in\text{-chain } \&\, z(\{\ \}) = x) \\ &\qquad\qquad\qquad\qquad \supset \mathrm{FiinOrd}(\mathrm{dom}(z))]. \end{aligned}$$

Assume this much Choice: that for any cardinal κ, if κ is infinite then $\kappa \geq \aleph_0$ (i.e. finite sets are Dedekind-finite).

Fact 5.1. *For any model $\mathscr{A}$, ls-extensor e for $\mathscr{A}$, and $a \in \mathrm{rng}(e)$, $\mathscr{A}, e \models \mathrm{Wfd}(\underline{a})$ iff a is well-founded for e.*

Proof. If $\mathrm{card}(\mathscr{A}) > \aleph_0$ this should be clear. If $\mathrm{card}(\mathscr{A}) = \aleph_0$, left to right should be clear; going from right to left requires care, since any f in virtue of which a was not well-founded for e would not be represented in $\mathrm{dom}(e)$.

Let a finite sequence be a function whose domain is a finite ordinal. Let Z be a tree in a set X iff Z is a set of nonempty finite sequences of members of X, Z is closed under initial segments and, for any $\alpha, \beta \in Z$, $\alpha(0) = \beta(0)$. An infinite branch of Z is a function on ω every proper initial segment of which belongs to Z. We associate each $a \in \mathrm{rng}(e)$ with a tree Z_a by induction as follows. Put $\{\langle 0, a\rangle\}$ into Z_a. Suppose that α with length $n + 1$ has been put into Z_a; for each b such that $\mathscr{A}, e \models \underline{b} \in \underline{\alpha(n)}$ put $\alpha \cup \{\langle n + 1, b\rangle\}$ into Z_a. Clearly a is well-founded for e iff Z_a has no infinite branch. If e is an ls-extensor for $\mathscr{A}$ then Z_a is finite-branching; by König's lemma either Z_a has an infinite branch or there is an $n \in \omega$ so that $\mathrm{dom}(\alpha) < n$ for each $\alpha \in Z_a$. So if a is well-founded for e and $\mathrm{card}(\mathscr{A}) = \aleph_0$, then $\mathscr{A}, e \models \mathrm{Wfd}(\underline{a})$.

Fact 5.2. *For $\mathscr{A}$ and e as above, $\mathscr{A}, e \models (\forall x)(\mathrm{Set}(x) \supset \mathrm{Wfd}(x))$ iff e is well-founded.*

This follows from Fact 5.1.

Suppose we accept the limitation-of-size comprehension and restriction principles. According to the Mathematical-Object theory, whether or not all sets are well-founded is a question with a yes or no answer. On the Alternative theory the issue has no mathematical content; correspondingly $(\forall x)(\mathrm{Set}(x) \supset \mathrm{Wfd}(x))$ is A_{ls}-truth-valueless. Indeed, from the viewpoint of a Mathematical-Object theorist who employs the limitation-of-size conception, adopting the more restrictive iterative conception of sethood may seem merely to be putting "blinders" on quantification in $L^{0,\in}$: replacing $\mathrm{Set}(v)$ by $(\mathrm{Set}(v)\ \&\ \mathrm{Wfd}(v))$ in all sentences we accept. Doing this to Regularity yields a trivial validity:

$$(\forall x)((\mathrm{Wfd}(x)\ \&\ \mathrm{Set}(x)) \supset (x \neq \{\ \} \supset x \text{ has a } \in\text{-minimal member})).$$

An analogy: set-theorists frequently restrict their attention to pure sets. This notion can also be expressed under the Alternative limitation-of-size semantics. Let $\mathrm{Pure}(x)$ be:

$$\neg(\exists y)(y \text{ is a descending } \in\text{-chain}\ \&\ y(\{\ \}) = x\ \&\ (\exists z)\neg\mathrm{Set}(y(z))).$$

No one thinks that all sets are pure. Nonetheless set-theorists frequently restrict their attention to pure sets, in effect replacing $\mathrm{Set}(v)$ by $(\mathrm{Set}(v)\ \&\ \mathrm{Pure}(v))$. Of course the Alternative theory does not tell us that some sets are impure: $(\forall x)(\mathrm{Set}(x) \supset \mathrm{Pure}(x))$ is A_{ls}-truth-valueless.

This analogy seems to count against the iterative concept of sethood. As I understand it, a conception of sethood is at least a view about which level-one Fregean concepts have extensions, and which do not. (The previous discussion shows, I hope, that this statement involves no commitment to the Mathematical-Object theory.) Model-theoretically, a conception of sethood is represented by a class K of extensors, which determines both the Mathematical-Object logics $\mathrm{MO}_K^{\mathrm{C}}$ and the Alternative logics $\mathrm{A}_K^{\mathrm{C}}$ appropriate to that conception. The iterative conception is a strengthening of the limitation-of-size conception: it yields the limitation-of-size comprehension and restriction principles, but it takes the latter not to be fundamental, but rather to be a consquence of a stronger restriction: that membership be well-founded.[13] This conception is represented by $i =$ the class of well-founded extensors. So the logics appropriate for the Alternative theorist who adopts the iterative conception are those of the form A_i^C.

Clearly all A_{ls}-valid sentences of $L^{0,\wedge}$ are also A_i-valid; similarly for equivalences. In this sense A_i is a strengthening of A_{ls}. The Axiom of Regularity is A_i-valid; so all the axioms of ZF are A_i-valid with respect to models of inaccessible cardinality. Since $A_{ls}\text{-Biv}(L^{0,\wedge}) \subseteq A_i\text{-Biv}(L^{0,\wedge})$, for any acceptable model $\mathscr{A}$ and $\varphi \in A_{ls}\text{-Biv}(L^{0,\wedge})$ we have

$$\mathscr{A} \models \varphi \quad \text{iff} \quad \mathscr{A} \models_i \varphi; \qquad \mathscr{A} \mathrel{=\!\!|} \varphi \quad \text{iff} \quad \mathscr{A} \mathrel{=\!\!|}_i \varphi;$$

so the Alternative i-semantics is at least as expressive as the Alternative ls-semantics.

PROBLEM. For each $\varphi \in A_{ls}\text{-Biv}(L^{0,\wedge})$ is there a $\varphi' \in \text{Sent}(L^{0,\wedge})$ so that, for any acceptable model $\mathscr{A}$,

$$\mathscr{A} \models \varphi' \quad \text{iff} \quad \mathscr{A} \models_i \varphi; \qquad \mathscr{A} \mathrel{=\!\!|} \varphi' \quad \text{iff} \quad \mathscr{A} \mathrel{=\!\!|}_i \varphi?$$

If not, the A_i-semantics is more expressive than the A_{ls}-semantics. For the Alternative theorist, this would show that there is really a good reason for adopting the iterative conception, that doing so is not merely a matter of wearing blinders that hide non-well-founded sets from our quantifiers.

It has been claimed that, unlike ZF and its cousins, set theories that posit a universal set are mere formalisms, that they cannot purport to express a body of set-theoretic truths because they are not backed up by any notion of sethood. This "cannot" strikes me as too harsh.

For example, we may adopt what I shall call the Boolean limitation-of-size notion of sethood, based on these principles. Add to the limitation-of-size comprehension principle its coprinciple: any level-one concept under whose complement there does not fall an absolute infinity of objects has an extension; weaken the limitation-of-size restriction principle to: any set either has fewer than absolutely infinitely many members or has fewer than absolutely infinitely many nonmembers. This conception is reflected in the following Alternative logic.

Let e be a *Bls-extensor* for a model $\mathscr{A}$ iff e is an extensor for $\mathscr{A}$ and

$$\text{dom}(e) = \{A \subseteq |\mathscr{A}|\text{: either card}(A) < \text{card}(\mathscr{A}) \\ \text{or card}(|\mathscr{A}| - A) < \text{card}(\mathscr{A})\}.$$

Let $\mathscr{A} \models_B \varphi$ iff, for every Bls-extensor e for $\mathscr{A}$, $\mathscr{A}, e \models \varphi$; define $\mathrel{=\!\!|}_B$ analogously. With A_B the resulting logic, $(\exists x)(\forall y)y \in x$ is A_B-valid. Of course the Axiom of Separation is not A_B-valid. (Here 'B' is for 'Boolean'.)

[13] My slight acquaintance with Cantor's writings left me with the impression that Cantor held the limitation-of-size conception, but not the stronger iterative conception. Wang seems to disagree, e.g. "One feels vaguely that the iterative concept corresponds pretty well to Cantor's 1895 'genetic' definition of set" [W, p. 188; see also p. 187]. Is there a textual basis for Wang's vague feeling? Wang also thinks that the iterative conception is "quite different" from "the dichotomy concept which regards each set as obtained by dividing the totality of all things into two categories" [W, p. 187]; of course I also disagree with this: each set is in a sense "so obtained"—see note 10 above; conceptions of sethood differ as to which such divisions yield sets.

PROBLEM. Is there a notion of sethood, i.e. a reasonable constraint on extensors, whose alternative logic incorporates NF or any of its relatives? Does the λ-calculus offer a conception of sethood in the above sense?

Appendix 1. On abstraction-terms.

Fact A1.1. *For any* $\varphi \in \mathrm{Sent}(L^{0,\in})$, *any total model* $\mathscr{A}$ *and any extensor* e *for* $\mathscr{A}$, *either* $\mathscr{A}, e \models \varphi$ *or* $\mathscr{A}, e \mathrel{=\!\!|} \varphi$.

The point here is that a term τ of $L^{0,\in}$ is a term of L^0, with $\tau^{\mathscr{A},e} = \tau^{\mathscr{A}}\downarrow$.

Thus under the MO semantics, $\underline{u}$ is not expressible in $L^{0,\in}$. Form $L^{0,\underline{u},\in}$ from $L^{0,\in}$ by adding $\underline{u}$ as a new primitive.

Fact A1.2. *For any* $\varphi \in \mathrm{Sent}(L^{0,\wedge})$ *there is a* $\varphi' \in \mathrm{Sent}(L^{0,\underline{u},\in})$ MO-*equivalent (and so* $\mathrm{MO}_{\mathrm{1s}}$-*equivalent) to* φ.

Proof (by example). If $\underline{P}$ is a 1-place predicate-constant, then $\mathscr{A}, e \models \underline{P}(\hat{x}\varphi)$ iff

$$\mathscr{A}, e \models (\exists y)(\underline{P}(y) \;\&\; \mathrm{Set}(y) \;\&\; (\forall x)(x \in y \equiv T\varphi)) \\ \vee (\neg(\exists y)[\mathrm{Set}(y) \;\&\; (\forall x)(y \in x \equiv T\varphi)] \;\&\; \underline{u});$$

and similarly with '$\mathrel{=\!\!|}$' replacing '$\models$'. Details (an induction on the construction of sentences of $L^{0,\wedge}_{\mathscr{A}}$) are left to the reader.

Facts 2.2 and A1.2 show that under the MO-semantics $L^{0,\wedge}$ and $L^{0,u,\in}$ have the same expressive power.[14]

Fact A1.3. *For every sentence of* $L^{0,\wedge}$ $[L^{0,\underline{u},\in}]$ *there is an* A_{1s}-*equivalent sentence of* $L^{0,\underline{u},\in}$ $[L^{0,\wedge}]$.

This is a trivial consequence of Facts 2.2 and A2.2.

By Fact 2.2, $\hat{x}(x \notin x) \in \hat{x}(x \notin x)$ is A_{1s}-truth-valueless. But we do not need $\wedge$ to form such a sentence; consider $(\forall x)\mathrm{Set}(x)$. This suggests the following conjecture: For every sentence of $L^{0,\underline{u},\in}$ there is an A_{1s}-equivalent sentence of $L^{0,\in}$; i.e. $L^{0,\in}$ has the expressive strength of $L^{0,\wedge}$. One might try to prove this by replacing all occurrences of 'u' by $(\forall x)\mathrm{Set}\,(x)$ in a sentence of $L^{0,\underline{u},\in}$. But this will not preserve A_{1s}-equivalence; keep in mind that $\mathscr{A} \models$ and $\mathscr{A} \mathrel{=\!\!|}$ are not defined inductively; e.g. consider $\underline{u} \supset \underline{u}$. Lacking a proof of this conjecture, I shall prove something weaker.

Fact A1.4. *For every* $\varphi \in \mathrm{Sent}(L^{0,\underline{u},\in})$ *there are* $\varphi^+, \varphi^- \in \mathrm{Sent}(L^{0,\in})$ *so that* φ *is positively* A_{1s}-*equivalent to* φ^+ *and* $\neg\varphi$ *is positively* A_{1s}-*equivalent to* $\neg\varphi^-$.

Given φ, form φ^- $[\varphi^+]$ by replacing all positive occurrences of $\underline{u}$ in φ by $\neg\bot$ $[\bot]$ and all negative occurrences by $\bot$ $[\neg\bot]$. For any model $\mathscr{A}$ and extensor e for $\mathscr{A}$:

1) if $\mathscr{A}, e \models \varphi$ then $\mathscr{A}, e \models \varphi^+$ and $\mathscr{A}, e \models \varphi^-$;

2) if $\mathscr{A}, e \mathrel{=\!\!|} \varphi$ then $\mathscr{A}, e \mathrel{=\!\!|} \varphi^+$ and $\mathscr{A}, e \mathrel{=\!\!|} \varphi^-$;

3) if $\mathscr{A}, e \mid \varphi$ then $\mathscr{A}, e \not\mathrel{=\!\!|} \varphi^-$ and $\mathscr{A}, e \not\models \varphi^+$.

All this follows by induction on the construction of φ. If $\mathscr{A} \models \varphi$, clearly $\mathscr{A} \models \varphi^+$. If $\mathscr{A} \not\models \varphi$, fix an e so that $\mathscr{A}, e \not\models \varphi$; then $\mathscr{A}, e \not\models \varphi^+$; so $\mathscr{A} \not\models \varphi^+$. Similarly for $\neg\varphi$ and $\neg\varphi^-$.

[14]If we had handled nondesignating terms with the falsehood convention, rather than with truth-value gaps, we could eliminate $\wedge$ within $L^{0,\in}$. If we preserved the truth-value gap approach but changed things so as to render T unexpressible (by weakening semantics for $=$ or allowing non-extensionwise-total models), $\wedge$ would not be eliminible in $L^{0,u,\in}$.

Appendix 2. Further encoding. Set theory may seem "richer" than higher-order logic in that set-formation is cumulative, and so extends into the transfinite, while the abstraction of Fregean unsaturated entities is not cumulative, and so does not. However, over models of regular cardinality, the encoding described in Fact 3.2 can be pushed into the transfinite.

Given a set A with $\mathrm{card}(A) = \kappa$, let

$$V_0^A = A; \qquad V_{\xi+1}^A = \mathrm{Power}^{<\kappa}(V_\xi^A) \cup V_\xi^A; \qquad V_\lambda^A = \bigcup_{\xi<\lambda} V_\xi^A,$$

where ξ is any ordinal and λ is any limit ordinal. Then κ^* is the least ordinal ξ so that $V_\xi^A = V_{\xi+1}^A$. Let $V^A = V_{\kappa^*}^A$.

Since according to the Alternative theory Set and $\in$ are logical constants, and we shall want to consider models in the usual sense for set-theoretic vocabulary, it will be convenient to have available different nonlogical predicates: $\underline{\mathrm{Set}}$ and $\underline{\in}$. Let $L^* = L^0(\mathrm{Vcb} \cup \{\underline{\mathrm{Set}}, \underline{\in}\})$. Given a model $\mathscr{A}$, assume without loss of generality that $|\mathscr{A}|$ and $\bigcup_{0<\xi<\kappa^*} V_\xi^{|\mathscr{A}|}$ are disjoint. Form the model $\mathscr{A}^*$ for $\mathrm{Vcb} \cup \{\underline{\mathrm{Set}}, \underline{\in}\}$ by taking $|\mathscr{A}^*| = V^{|\mathscr{A}|}$, $\underline{\mathrm{Set}}^{\mathscr{A}^*} = V^{|\mathscr{A}|} - |\mathscr{A}|$, and $\underline{\in}^{\mathscr{A}^*} = \in \restriction V^{|\mathscr{A}|}$.

Fact A2. *There is a translation* t: $\mathrm{Sent}(L^*) \to \mathrm{Sent}(L^{0,\in})$ *such that, for* $\psi \in \mathrm{Sent}(L^*)$ *and any model* $\mathscr{A}$ *of regular cardinality,*

$$\mathscr{A}^* \models \psi \quad \textit{iff} \quad \mathscr{A} \models t(\psi); \qquad \mathscr{A}^* \dashv \psi \quad \textit{iff} \quad \mathscr{A} \dashv t(\psi).$$

Proof. Given an ls-extensor e for $\mathscr{A}$, define $\bar{e}$: $V^{|\mathscr{A}|} \to |\mathscr{A}|$ as follows: $\bar{e}(a) = a$ for $a \in |\mathscr{A}|$, and $\bar{e}(x) = e(\{\bar{e}(y) : y \in x\})$ for $x \in V^{\mathscr{A}} - |\mathscr{A}|$. Let a tree on a set U be a set of finite sequences of members of U closed under initial segments. For a tree T on U, $\alpha \in T$, and $\mathrm{length}(\alpha) = n + 1$, let $\mathrm{label}(\alpha) = \alpha(n)$; let α be a leaf of T iff $\alpha \in T$ and α is not a proper initial segment of any element of T. We associate with $x \in V^{\mathscr{A}}$ a tree T^x in $V^{\mathscr{A}}$ representing x, as follows. Put $\{\langle 0, x\rangle\}$ into T^x; if $\alpha \in T^x$ and $\mathrm{dom}(\alpha) = n + 1$, for each $y \in \alpha(n)$ put $\alpha \cup \{\langle n+1, y\rangle\}$ into T^x. So the label of a leaf of T^x is $\{\ \}$ or an "urelement", i.e. a member of $|\mathscr{A}|$. We would like to code x by a tree Z in $|\mathscr{A}|$ isomorphic to T^x so that a leaf of T^x and one of Z that are matched have the same label. But there is a problem when $\alpha \in T^x$ has label $\{\ \}$: since $\{\ \} \in V_1^{\mathscr{A}}$, we have assumed that $\{\ \} \notin |\mathscr{A}|$. Letting $\mathrm{TrCl}(x) =$ the transitive closure of x, for $c \in |\mathscr{A}| - \mathrm{TrCl}(x)$ and Z any tree on $|\mathscr{A}|$, let $\langle Z, c\rangle$ code x iff there is an isomorphism π: $T^x \to Z$ such that, for any leaf α of T^x, if $\mathrm{label}(\alpha) \in |\mathscr{A}|$ then $\mathrm{label}(\pi(\alpha)) = \mathrm{label}(\alpha)$, and if $\mathrm{label}(\alpha) = \{\ \}$ then $\mathrm{label}(\pi(\alpha)) = c$. For any well-founded tree Z with $\mathrm{card}(Z) < \mathrm{card}(\mathscr{A})$ and any $c \in |\mathscr{A}|$, $\langle Z, c\rangle$ codes a unique $x \in V^{\mathscr{A}}$. Assume that $\mathrm{card}(\mathscr{A})$ is regular. Then for any x as above $\mathrm{card}(\mathrm{TrCl}(x)) < \mathrm{card}(\mathscr{A})$, and so there is a $\langle Z, c\rangle$ coding x with $\mathrm{card}(Z) < \mathrm{card}(\mathscr{A})$.

Handling finite ordinals as in §5, we may construct a formula $\mathrm{Code}(v, v')$ in $L^{0,\in}$ so that, for any a, $c \in |\mathscr{A}|$, $\mathscr{A}, e \models \mathrm{Code}(\underline{a}, \underline{c})$ if, for some code $\langle Z, c\rangle$, $a = \bar{e}(Z)$; and $\mathscr{A}, e \dashv \mathrm{Code}(\underline{a}, \underline{c})$ otherwise. With $\mathrm{Code}(v, v')$ we construct formulae $\mathrm{Equal}(v_0, v_0', v_1, v_1')$ and $\mathrm{Element}(v_0, v_0', v_1, v_1')$ so that for a, c, a', $c' \in |\mathscr{A}|$:

$$\begin{aligned} &\mathscr{A}, e \models \mathrm{Equal}(\underline{a}, \underline{c}, \underline{a}', \underline{c}') \text{ if for some } Z, Z' \text{ and } x \in V^{\mathscr{A}}, \\ &\qquad\qquad a = \bar{e}(Z),\ a' = \bar{e}(Z'),\ \langle Z, c\rangle \text{ and } \langle Z', c'\rangle \text{ code } x; \\ &\mathscr{A}, e \dashv \mathrm{Equal}(\underline{a}, \underline{c}, \underline{a}', \underline{c}') \quad \text{otherwise;} \end{aligned}$$

$\mathscr{A}, \mathscr{e} \models \text{Element}(\underline{a}, \underline{c}, \underline{a}', \underline{c}')$ if for some $Z, Z', x, y \in V^{\mathscr{A}}$,

$$a = \bar{\mathscr{e}}(Z), \text{ and } a' = \bar{\mathscr{e}}(Z'), \langle Z, c\rangle \text{ codes } x, \langle Z', c'\rangle \text{ codes } y, \text{ and } x \in y;$$

$\mathscr{A}, \mathscr{e} \dashv \text{Element}(\underline{a}, \underline{c}, \underline{a}', \underline{c}')$ otherwise.

Use Equal to construct Element. To form $t(\psi)$ from ψ, introduce a new variable υ' for each υ in ψ; replace prefixes $(\exists\upsilon)(\cdots)$ by $(\exists\upsilon)(\exists\upsilon')(\text{Code}(\upsilon, \upsilon') \;\&\; \cdots)$, and replace $\upsilon_0 \in \upsilon_1$ by $\text{Element}(\upsilon_0, \upsilon_0', \upsilon_1, \upsilon_1')$. Verify that this works by induction on the construction of ψ.

Appendix 3. Weak alternative logics. We now consider some Alternative semantics significantly less expressive than the Alternative limitation-of-size semantics. These semantics are stronger in a sense to be explained, and they yield logics stronger than A_{ls} in that all of their validities (and thus their equivalences) are validities (equivalences) of A_{ls}, but not conversely. But, taking expressive power as the criterion for strength, I shall call them *weak Alternative logics.*

For any model $\mathscr{A}$ we adopt these definitions:

$\mathscr{e}$ is a *weak κ-extensor* [*κ-extensor*] for $\mathscr{A}$ iff $\mathscr{e}$ is an extensor for $\mathscr{A}$ with

$$\text{Power}^{<\kappa}(\mathscr{A}) \subseteq \text{dom}(\mathscr{e}) \qquad [\text{Power}^{<\kappa}(\mathscr{A}) = \text{dom}(\mathscr{e})].$$

$\mathscr{e}$ is a *weak ls-extensor* for $\mathscr{A}$ iff $\mathscr{e}$ is a weak card($\mathscr{A}$)-extensor for $\mathscr{A}$.

Note. For any model $\mathscr{A}$, $\mathscr{A}$ has a weak ls-extensor iff $\mathscr{A}$ is acceptable.

Let wls [κ-wls] = the class of weak extensors [weak κ-extensors]. We shall take $\models^*$ [$\models^*_\kappa$] to be $\models_{\text{wls}}$ [$\models_{\kappa\text{-wls}}$], and similarly for $\dashv^*$ [$\dashv^*_\kappa$]. Let $\mathscr{A} \mid^* \varphi$ iff $\mathscr{A} \not\models^* \varphi$ and $\mathscr{A} \not\dashv^* \varphi$.

The semantics for A_{wls} reflects the limitation-of-size comprehension principle without a corresponding restriction principle. $\models^*$ and $\dashv^*$ are strong notions of truth and falsity in that the following holds.

Fact A3.1. *For any model $\mathscr{A}$ and $\varphi \in \text{Sent}(L^{0,\wedge})$: if $\mathscr{A} \models^* \varphi$ then $\mathscr{A} \models \varphi$; if $\mathscr{A} \dashv^* \varphi$ then $\mathscr{A} \dashv \varphi$; but if $\mathscr{A} \mid \varphi$ then $\mathscr{A} \mid^* \varphi$.*

Similarly for $\models^*_\kappa$ and $\models_\kappa$, etc. It is easy to find counterexamples for the converses.

Let A_{wls} be the weak Alternative ls-logic.

Fact A3.2. A_{ls}*-validity entails* A_{wls}*-validity*; *similarly for implication, equivalence and bivalence*; *but* A_{ls}*-truth-valuelessness entails* A_{wls}*-truth-valuelessness.*

Fact A3.3. *If φ is* A_{ls}*-valid and* A_{wls}*-bivalent, then φ is* A_{wls}*-valid*; *similarly with relativization to κ.*

Here are some examples to illustrate the relationship between A_{ls} and A_{wls}.

$(\exists x)(\text{Set}(x) \;\&\; (\forall y) y \notin x)$ is A_{wls}-valid.

$(\forall x)\underline{P}x$ is A_{wls}-bivalent.

$(\forall x)((\forall y) y \in x \supset \underline{P}x)$ is A_{ls}-valid, not A_{wls}-bivalent, and not A_{wls}-truth-valueless.

$(\forall x)(\underline{P}x \supset (\forall y) y \in x)$ is A_{ls}-bivalent, not A_{wls}-bivalent, neither A_{ls}-valid nor A_{ls}-anti-valid, and not A_{wls}-truth-valueless.

$\neg(\exists x)(\forall y) y \in x$ is A_{ls}-valid and A_{wls}-truth-valueless.

$(\forall x)(\underline{P}x \equiv (\forall y) y \in x)$ is A_{ls}-bivalent, A_{wls}-truth-valueless, and neither A_{ls}-valid nor A_{ls}-anti-valid.

$(\exists x)(\underline{P}(x) \;\&\; \text{Set}(x))$ is not A_{ls}-bivalent and not A_{wls}-truth-valueless.

$(\exists x)(\forall y) y \in x \equiv (\exists x)(\underline{P}(x) \;\&\; \text{Set}(x))$ is A_{wls}-truth-valueless and neither A_{ls}-bivalent nor A_{ls}-truth-valueless.

Fact A3.4. *For any infinite cardinals* $\kappa < \kappa'$ *and any* $\varphi \in \mathrm{Sent}(L^{0,\wedge})$:

(i) *if* φ *is* $A_{\kappa\text{-wls}}$*-valid, then* φ *is* $A_{\kappa'\text{-wls}}$*-valid; and*

(ii) *if* φ *is* $A_{\kappa\text{-wls}}$*-bivalent, then* φ *is* $A_{\kappa'\text{-wls}}$*-bivalent.*

The point here is just that for any model $\mathscr{A}$, any weak κ'-extensor for $\mathscr{A}$ is also a weak κ-extensor.

A_{ls} and its strengthenings are much stronger than first-order logic (as is shown by Fact 3.2); what follows shows that A_{wls} is essentially first-order. In brief, this is because a model only slightly constrains the domains for its weak extensors, whereas a model determines the domain of its extensors.

Let SETHOOD be the sentence $(\forall x)(\forall y)(y \in x \supset \mathrm{Set}(x))$. Let $\mathrm{EXT}(x, y)$ be $(\forall z)(z \in x \equiv z \in y) \supset x = y$, and let EXT be

$$(\forall x)(\forall y)((\mathrm{Set}(x)\ \&\ \mathrm{Set}(y)) \supset \mathrm{EXT}(x, y)).$$

For $n \in \omega$ let $\mathrm{CA}_n \in \mathrm{Sent}(L^{0,\in})$ say "Given any n objects, there is a set whose members are precisely those objects." Let minimal set theory, hereafter MST, be the first-order theory in the vocabulary Set and $\in$ axiomatized by SETHOOD, EXT, and CA_n for all $n < \omega$.

Note. In calling MST 'first-order', we are treating Set and $\in$ as nonlogical predicates. The arguments that follow create the danger of confusion between this treatment and their treatment as logical predicates in Alternative logics. When such confusion threatens, Set and $\in$ remain logical predicates, but we let $\underline{\mathrm{Set}}$ and $\underline{\in}$ be nonlogical predicates as in Appendix 2. For $\varphi \in \mathrm{Sent}(L^{0,\in})$, form $\underline{\varphi}$ from φ by replacing all occurrences of Set and $\in$ by $\underline{\mathrm{Set}}$ and $\underline{\in}$ respectively; similarly for sets of sentences.

Fact A3.5. *For any* $\varphi \in \mathrm{Sent}(L^{0,\in})$ *the following are equivalent*:

(i) φ *is* A_{wls}*-valid.*

(ii) *For every infinite cardinal* κ, φ *is* $A_{\kappa\text{-wls}}$*-valid.*

(iii) $\underline{MST} \vdash \underline{\varphi}$ *(this in first-order logic).*

Furthermore, if we assume that, for every infinite cardinal κ, $\kappa \geq \aleph_0$ *(i.e. that all Dedekind-finite sets are finite), these are equivalent to*:

(iv) φ *is* $A_{\aleph_0\text{-wls}}$*-valid.*

By Fact A3.4, (iv) $\Rightarrow$ (ii), under our assumption about κ. Trivially (ii) $\Rightarrow$ (i) and (iv). It is easy to see that the axioms of MST are A_{wls}-valid, giving (iii) $\Rightarrow$ (ii). We must show that (i) $\Rightarrow$ (iii).

Suppose $\varphi \in \mathrm{Sent}(L^{0,\in})$ and $\underline{\varphi}$ is not a theorem of MST. Thus $\underline{\mathrm{MST}} \cup \{\neg \underline{\varphi}\}$ is consistent (in two-valued first-order logic). Fix a countable total model $\mathscr{A}$ for $\mathrm{Vcb} \cup \{\underline{\mathrm{Set}}, \underline{\in}\}$ so that $\mathscr{A} \models \underline{\mathrm{MST}} + \neg \underline{\varphi}$. Define e by setting

$$e(\{b \in |\mathscr{A}|: \mathscr{A} \models \underline{b} \mathrel{\underline{\in}} \underline{a}\}) = a, \quad \text{for each } a \in \underline{\mathrm{Set}}^{\mathscr{A}}.$$

Let $\mathscr{A}'$ be the reduct of $\mathscr{A}$ to Vcb. The axioms of MST insure that e is a weak $\aleph_0$-extensor, and thus a weak ls-extensor, for $\mathscr{A}'$. Furthermore $\mathscr{A}', e \models \neg \varphi$; thus φ is not A_{wls}-valid.

Notice that (i) $\Rightarrow$ (iii) for $\varphi \in \mathrm{Sent}(L^{0,\in})$ is a completeness result. Clearly MST is a subtheory of ZF. Perhaps the axioms of MST are analytic to any notion of sethood, i.e. one who did not accept MST was not employing the notion of a set. Thus my use of 'minimal'.[15]

Fact A3.6. *For any* $\varphi \in \mathrm{Sent}(L^{0,\in})$ *the following are equivalent*:

(i) φ *is* $\mathrm{A}_{\mathrm{wls}}$-*bivalent.*

(ii) *For each infinite cardinal* κ, φ *is* $\mathrm{A}_{\kappa\text{-wls}}$-*bivalent.*

(iii) *There is a* $\varphi' \in \mathrm{Sent}(L^0)$ *so that, for any acceptable model* $\mathscr{A}$, $\mathscr{A} \models^* \varphi$ *iff* $\mathscr{A} \models \varphi'$.

Furthermore, assuming that, for every infinite κ, $\kappa \geq \aleph_0$, *these are equivalent to*:

(iv) φ *is* $\mathrm{A}_{\aleph_0\text{-wls}}$-*bivalent.*

By Fact A3.4, (iv) $\Rightarrow$ (ii). Trivially (ii) $\Rightarrow$ (i) and (iv). Trivially (iii) $\Rightarrow$ (ii). Assume (i), to get (iii). Fix new 1-place function-constants $\underline{e}$ and e', new 1-place predicate-constants $\underline{\mathrm{Set}}$ and $\underline{\mathrm{Set}}'$, and new 2-place predicate-constants $\underline{\in}$ and $\underline{\in}'$. Form $\varphi(\underline{e})$ from φ by replacing occurrences of $\mathrm{Set}(\tau)$ in φ by $(\exists \upsilon)(\underline{\mathrm{Set}}(\upsilon) \;\&\; \tau = \underline{e}(\upsilon))$ for a variable υ not free in τ, and occurrences of $\sigma \in \tau$ in φ by $(\exists \upsilon)(\underline{\mathrm{Set}}(\upsilon) \;\&\; \tau = \underline{e}(\upsilon) \;\&\; \sigma \underline{\in} \upsilon))$ for υ not free in τ or σ. Let $\theta(\underline{e})$ say "$\underline{e}$ is a one-to-one function from $\neg\,\underline{\mathrm{Set}}$ onto $\underline{\mathrm{Set}}$". Let T be the theory consisting of $\underline{\mathrm{CA}_0}$, $\underline{\mathrm{EXT}}$ and

$$(\forall x)(\forall y)((x \underline{\in} y) \supset (\underline{\mathrm{Set}}(y) \;\&\; \neg\,\underline{\mathrm{Set}}(x))),$$
$$(\forall \vec{x})((\neg\,\underline{\mathrm{Set}}(x_1) \;\&\; \cdots \;\&\; \neg\,\underline{\mathrm{Set}}(x_n)) \supset (\exists y) y = \{\vec{x}\}),$$

for $n \geq 1$. Form $\varphi(\underline{e}')$, $\theta(e')$ and T' similarly using the primed expressions. For any mode $\mathscr{A}$ and weak ls-extensors e and e', form a model $\mathscr{B} = \mathscr{A}_{e,e'}$ for

$$\mathrm{Vcb} \cup \{\underline{\mathrm{Set}}, \underline{\mathrm{Set}}', \underline{\in}, \underline{\in}', \underline{e}, \underline{e}'\}$$

by taking $|\mathscr{B}| = |\mathscr{A}| \cup \mathrm{dom}(e) \cup \mathrm{dom}(e')$, assuming without loss of generality that $|\mathscr{A}|$ and $\mathrm{Power}(\mathscr{A})$ are disjoint, $\underline{\mathrm{Set}}^{\mathscr{B}} = \mathrm{dom}(e)$, $\underline{\in}^{\mathscr{B}} = \{\langle a, A\rangle : a \in A \in \mathrm{dom}(e)\}$, $\underline{e}^{\mathscr{B}} = e$, and similarly for the primed vocabulary. Thus

$$\mathscr{A}_{e,e'} \models T \cup T' \cup \{\theta(\underline{e}), \theta(\underline{e}')\}.$$

Also if $\mathscr{B} \models T \cup T' \cup \{\theta(\underline{e}), \theta(\underline{e}')\}$ and $\mathscr{B}$ is countable, then $\mathscr{B} = \mathscr{A}_{e,e'}$ for some $\mathscr{A}$, e, and e'; the countability of $\mathscr{B}$ is needed to be sure that e and e' are weak ls-extensors for $\mathscr{A}$. Clearly:

$$\mathscr{A}, e \models \varphi \text{ iff } \mathscr{A}_{e,e'} \models \varphi(\underline{e});$$
$$\mathscr{A}, e' \models \varphi \text{ iff } \mathscr{A}_{e,e'} \models \varphi(\underline{e}').$$

For any total countable model $\mathscr{B}$, if

$$\mathscr{B} \models T + \theta(\underline{e}) + \varphi(\underline{e}) + T' + \theta(\underline{e}'),$$

then $\mathscr{B} = \mathscr{A}_{e,e'}$; thus $\mathscr{A}, e \models \varphi$; by bivalence $\mathscr{A}, e' \models \varphi$; thus $\mathscr{B} \models \varphi(\underline{e}')$. Thus $T + \theta(\underline{e}) + \varphi(\underline{e}) + T' + \theta(\underline{e}')$ entails $\varphi(\underline{e}')$ in two-valued first-order logic. By

[15] MST is not finitely axiomatizable. There are models $\mathscr{A}_i$ for $i \in \omega$ so that $\mathscr{A}_i \not\models \underline{\mathrm{MST}}$ but $\Pi_U \mathscr{A}_i \models \underline{\mathrm{MST}}$, for an ultrafilter U on ω. If σ is a conjunction of finite axiomatization of $\underline{\mathrm{MST}}$, then by Łoś' theorem $\Pi_U \mathscr{A}_i \models \neg\sigma$, a contradiction.

compactness there are finite conjunctions θ and θ' of elements of T and T' respectively so that $\underline{\theta}$ & $\theta(\underline{e})$ & $\varphi(\underline{e})$ entails $(\underline{\theta}'$ & $\theta(\underline{e}')) \supset \varphi(\underline{e}')$ in first-order logic. By Craig's interpolation theorem there is a sentence $\varphi' \in L^0(\mathrm{Vcb})$, i.e. not containing the six new expressions, so that $\underline{\theta}$ & $\theta(\underline{e})$ & $\varphi(\underline{e})$ entails $\underline{\varphi}'$, and $\underline{\varphi}'$ entails $(\underline{\theta}'$ & $\theta(\underline{e}')) \supset \varphi(\underline{e}')$, again in first-order logic. Thus $\mathscr{A}, e \models \varphi$ iff $\mathscr{A}_{e,e'} \models \underline{\varphi}'$; since φ' is based on Vcb, $\mathscr{A}_{e,e'} \models \varphi'$ iff $\mathscr{A} \models \varphi'$; so φ' is as desired.

Fact A3.7. *For any infinite cardinal* κ *and any* $\varphi \in \mathrm{Sent}(L^{0,\in})$:

(i) *If* φ *is* $\mathrm{A}_{\kappa\text{-wls}}$*-valid then* φ *is* $\mathrm{A}^{\geq\kappa}_{\mathrm{wls}}$*-valid, for* $\geq\kappa$ = *the class of acceptable models of cardinality* $\geq \kappa$.

(ii) *Assuming the Axiom of Choice, the converse of* (i) *holds.*

(i) follows by the argument for Fact A3.4. Suppose that φ is not $\mathrm{A}_{\kappa\text{-wls}}$-valid; fix a total κ-acceptable model $\mathscr{A}$ and a weak κ-extensor so that $\mathscr{A}, e \not\models \varphi$; then $\mathrm{card}(\mathscr{A}) \geq \kappa$ (since by Choice, for any κ-acceptable μ, $\kappa \leq \mu$). Form the model $\mathscr{A}_e$ for $\mathrm{Vcb} \cup \{\underline{\mathrm{Set}}, \underline{\in}, \underline{e}\}$ with $|\mathscr{A}_e| = |\mathscr{A}| \cup \mathrm{dom}(e)$, $\underline{\mathrm{Set}}^{\mathscr{A}_e} = \mathrm{rng}(e)$, $\underline{\in}^{\mathscr{A}_e} = \{\langle a, A\rangle : a \in A \in \mathrm{dom}(e)\}$, and $e^{\mathscr{A}_e} = e$; without loss of generality we assume $|\mathscr{A}|$ and $\mathrm{dom}(e)$ to be disjoint. So $\mathscr{A}_e \not\models \underline{\varphi}$. Using Choice, we may form an elementary submodel of $\mathscr{A}_e$ of the form $\mathscr{A}'_{e'}$ for e' a weak κ-extensor for $\mathscr{A}'$ with $\mathrm{card}(\mathscr{A}') = \kappa$; the sole novelty here is that we must throw in every subset of $|\mathscr{A}'|$ of cardinality less than κ. Then $\mathscr{A}'_{e'} \not\models \underline{\varphi}$; so $\mathscr{A}', e' \not\models \varphi$; so φ is not $\mathrm{A}^*_{\mathrm{ls}}$-valid with respect to models of cardinality $\geq \kappa$.

One other weak Alternative logic deserves mention. For a model $\mathscr{A}$ let e be a weak Boolean (wB-) extensor for $\mathscr{A}$ iff e is a weak ls-extensor for $\mathscr{A}$ with $\mathrm{dom}(e)$ a Boolean algebra under $\subseteq$ and $|\mathscr{A}| \in \mathrm{dom}(e)$. Setting wB = the class of wB-extensors we obtain the logic A_{wB}, which is also essentially first-order. The A_{wB}-validities are axiomatized by SETHOOD + EXT + CA_0 + CA_1 + $(\exists x)(\forall y)x \in y$ + sentences asserting the existence of a union, intersection and relative complement of any two sets; this follows by slight changes in the argument for Fact A3.5. The resulting Boolean set theory (BST)[16] is a finitely axiomatized strengthening of MST. Clearly A_{wB} is to A_{B} as $\mathrm{A}_{\mathrm{wls}}$ is to A_{ls}. BST is not a subtheory of ZF, though it is a subtheory of NF:

$$(\forall u)(\mathrm{Set}(u) \supset (\exists y)(\forall z)(z \in y \equiv (z \in u \,\&\, z \notin z)))$$

is A_{wB}-anti-valid.

Appendix 4. Second-order and infinitary axiomatizations of A_{ls}. Some of the results in Appendix 3 have analogs for A_{ls}. We shall let κ-ls be the class of κ-extensors, and write $\models_\kappa$ for $\models_{\kappa\text{-ls}}$, etc.

Fact A4.1. *For any* $\varphi \in \mathrm{Sent}(L^{0,\in})$:

(i) *If* φ *is* $\mathrm{A}_{\kappa\text{-ls}}$*-valid for each acceptable* κ, *then* φ *is* A_{ls}*-valid.*

(ii) *If* φ *is* $\mathrm{A}_{\kappa\text{-ls}}$*-bivalent for each acceptable* κ, *then* φ *is* A_{ls}*-bivalent.*

(iii) *Assuming Choice, the converses of* (i) *and* (ii) *also hold.*

Proof. (i) and (ii) are obvious. Suppose that κ is acceptable, $\mathscr{A}$ is a κ-acceptable model, e is a κ-extensor for $\mathscr{A}$, and $\mathscr{A}, e \not\models \varphi$. Using Choice, $\kappa \leq \mathrm{card}(\mathscr{A})$. Form $\mathscr{A}_e$

[16] [Fo] informed me that Boolean set theory has been discussed under the name 'NF_2'.

and, using Choice, take an elementary submodel of it of the form $\mathscr{A}'_{\mathscr{e}'}$ with card($\mathscr{A}'$) $= \kappa$, as was done above. Since all members of dom($\mathscr{e}$) are of cardinality less than κ, so are all members of dom($\mathscr{e}'$); so $\mathscr{e}'$ is an ls-extensor for $\mathscr{A}'$ and $\mathscr{A}', \mathscr{e}' \not\models \varphi$; so φ is not A_{ls}-valid, yielding the converse of (i). A similar argument, using $\mathscr{A}_{\mathscr{e},\mathscr{e}'}$ for κ-extensors $\mathscr{e}$ and $\mathscr{e}'$, delivers the converse of (ii).

Let 2CA (that is, the Second-Order Comprehension Axiom) and 2LA (the Second-Order Limitation Axiom) be the following sentences, respectively:

$$(\forall X)(\exists y)(\mathrm{Set}(y)\ \&\ (\forall z)(z \in y \equiv Xy));$$
$$(\forall y)(\mathrm{Set}(y) \supset (\exists X)(\forall z)(z \in y \equiv Xy)).$$

Let W2ST (= Weak-Monadic Second-Order Set Theory) be the axioms SETHOOD, EXTENSIONALITY, 2CA, and 2LA, taken under weak-monadic second-order logic.

Fact A4.2. *For any* $\varphi \in \mathrm{Sent}(L^{0,\in})$: φ *is* A_{ls}*-valid iff* $\underline{\mathrm{W2ST}}$ *entails* $\underline{\varphi}$ *in the sense of weak-monadic second-order logic.*

For any acceptable model $\mathscr{A}$ and any ls-extensor $\mathscr{e}$ for $\mathscr{A}$ we have $\mathscr{A}, \mathscr{e} \models$ W2ST, yielding the fact from left to right. Suppose that φ is not a weak-monadic second-order consequence of $\underline{\mathrm{W2ST}}$. There is a total model $\mathscr{A}$ for Vcb $\cup$ $\{\underline{\mathrm{Set}}, \underline{\in}\}$, $\mathscr{A} \overset{\mathrm{w}}{\models} \underline{\mathrm{W2ST}} + \neg\varphi$. Let $\mathscr{A}'$ be the reduct of $\mathscr{A}$ to Vcb. Let $\mathscr{e}(\{b\colon \mathscr{A} \models \underline{b} \mathrel{\underline{\in}} \underline{a}\}) = a$ for each $a \in \underline{\mathrm{Set}}^{\mathscr{A}}$. 2CA and 2LA insure that $\mathscr{e}$ is an ls-extensor for $\mathscr{A}'$; clearly $\mathscr{A}, \mathscr{e} \models \neg\varphi$. Thus φ is not A_{ls}-valid.

This "soundness and completeness" result is too second-order to be particularly satisfying; we shall now consider a similar result for an infinitary language. We consider first-order infinitary languages $L_{\mu,\mu} = L^0_{\mu,\mu}(\mathrm{Vcb} \cup \{\underline{\mathrm{Set}}, \underline{\in}\})$ formed from μ-many type-0 variables, allowing '$\exists$'-prefexes that bind κ variables for any $\kappa < \mu$, and allowing conjunctions and disjunctions of length $< \mu$. Let entailment$_{\mu,\mu}$ be entailment in the sense of the usual logic for $L_{\mu,\mu}$.

For any $\kappa < \mu$, let CA_κ be the sentence of $L_{\mu,\mu}$ saying "given κ objects, there is a set whose members are precisely those objects". Express "There are at least κ many objects" and "there are at least κ many z so that $z \in y$" in the obvious ways. Form ST_μ by adding to the axioms of MST the following infinitary axioms for each $\kappa < \mu$:

a) There are at least κ^+ many objects $\supset \mathrm{CA}_\kappa$.

b) Some set has at least κ many objects $\supset$ there are at least κ^+ many objects.

If μ is a limit cardinal, then all axioms of ST_μ are sentences of $L_{\mu,\mu}$. If μ is singular, fix an increasing sequence $\langle \mu_\xi \rangle_{\xi < \mathrm{cf}(\mu)}$ and let Limit_μ be the axiom:

$$(\forall y)\left(\mathrm{Set}(y) \supset \bigvee_{\xi < \mathrm{cf}(\mu)} \neg(\text{there are at least } \mu_\xi\text{-many } z \text{ such that } z \in y)\right).$$

If μ is singular, then Limit_μ is also a sentence of $L_{\mu,\mu}$.

Fact A4.3. *For any singular limit cardinal* μ *and any* $\varphi \in \mathrm{Sent}(L^{0,\in})$, *if* φ *is* A_{ls}*-valid then* $\mathrm{ST}_\mu + \underline{\mathrm{Limit}}_\mu$ *entails*$_{\mu,\mu}$ $\underline{\varphi}$.

Suppose that the consequent fails. Fix a total model $\mathscr{A}$ for Vcb $\cup$ $\{\mathrm{Set}, \underline{\in}\}$ so that $\mathscr{A} \models \underline{\mathrm{ST}}_\mu + \underline{\mathrm{Limit}}_\mu$ and $\mathscr{A} \not\models \underline{\varphi}$. $\mathrm{ST}_\mu + \mathrm{Limit}_\mu$ insure that card($\mathscr{A}$) $< \mu$. Define $\mathscr{e}$ by

$$\mathscr{e}(\{b \in |\mathscr{A}|\colon \mathscr{A} \models \underline{b} \mathrel{\underline{\in}} \underline{a}\}) = a, \quad \text{for } a \in \mathrm{Set}^{\mathscr{A}};$$

the axioms of ST_μ insure that $\mathrm{dom}(e) = \mathrm{Power}^{<\mathrm{card}(\mathscr{A})}(\mathscr{A})$; so e is an ls-extensor for $\mathscr{A}$. Clearly $\mathscr{A}, e \not\models \varphi$.

We now assume the Axiom of Choice. For each $\varphi \in \mathrm{Sent}(L^{0,\in})$ let:

$$\mu(\varphi) = \begin{cases} \text{the least } \kappa \text{ such that, for some acceptable} \\ \text{model } \mathscr{A} \text{ of cardinality } \kappa,\ \mathscr{A} \models \varphi, \\ \qquad \text{if there is such a } \kappa; \\ 0 \qquad \text{otherwise.} \end{cases}$$

The Skolem-number of $L^{0,\in} = \sup\{\mu(\varphi): \varphi \in \mathrm{Sent}(L^{0,\in})\}$.

By Choice the Skolem-number for $L^{0,\in}$ exists; let μ be it. Since $\mathrm{Sent}(L^{0,\in})$ has $\aleph_0$-many members, μ has cofinality $\aleph_0$. We show that μ is a limit cardinal, in fact a strong limit cardinal.

Fact A4.4. *For each* $\varphi \in \mathrm{Sent}(L^{0,\in})$, *there is a* $\varphi' \in \mathrm{Sent}(L^{0,\in})$ *so that* $\mu(\varphi') = 2^{\mu(\varphi)}$.

Without loss of generality suppose that $\mu(\varphi) > 0$. Take φ' to be "$(\exists x)$ (x is a model such that $x \models \varphi$)", spelled out in terms of $\in$, taking $\models$ to represent the notion of truth for sentences of $L^{0,\in}$ defined in §2. Clearly there is a model $\mathscr{A}$ so that $\mathscr{A} \models \varphi'$; fix one. Fix any extensor e for $\mathscr{A}$; fix $\mathscr{A}, e \models$ "$\underline{a}$ is a model and $\underline{a} \models \varphi$"; then there is a model $\mathscr{B}$ so that $\bar{e}(\mathscr{B}) = a$; for any extensor e' for $\mathscr{B}$, $e' \in V^{\mathscr{A}}$; let $e = \bar{e}(e')$; so

$$\mathscr{A}, e \models \text{“}\underline{e} \text{ is an extensor for } \underline{a}\ \&\ \underline{a}, \underline{e} \models \varphi\text{”}.$$

Thus, in fact, $\mathscr{B}, e' \models \varphi$. So $\mathscr{B} \models \varphi$; so $\mu(\varphi) \leq \mathrm{card}(\mathscr{B}) < \mathrm{card}(\mathscr{A})$. But then since $\mathrm{card}(\mathscr{A})$ is acceptable, $2^{\mu(\varphi)} \leq 2^{\mathrm{card}(\mathscr{B})} < \mathrm{card}(\mathscr{A})$. So φ' is as required.

Fact A4.5. *For any* $\varphi \in \mathrm{Sent}(L^{0,\in})$, *if* $\underline{\mathrm{ST}}_\mu + \underline{\mathrm{Limit}}_\mu$ *entails*$_{\mu,\mu}$ $\underline{\varphi}$ *then* φ *is* A_{ls}-*valid.*

Proof. Suppose φ is not A_{ls}-valid; then there is an $\mathrm{MO}_{\mathrm{ls}}$-model $\langle \mathscr{A}, e \rangle$ so that $\mathscr{A}, e \not\models \varphi$; so $\mathscr{A}, e \models \neg T\varphi$; so we may suppose that $\mathrm{card}(\mathscr{A}) = \mu(\neg\varphi) < \mu$. Clearly $\mathscr{A}, e \models \mathrm{ST}_\mu$. Form the model $\mathscr{B}$ for $\mathrm{Vcb} \cup \{\underline{\mathrm{Set}}, \underline{\in}\}$ with $|\mathscr{B}| = |\mathscr{A}|$, $\underline{\mathrm{Set}}^{\mathscr{B}} = \mathrm{rng}(e)$, and $\underline{\in}^{\mathscr{B}} = \{\langle a, b \rangle : \mathscr{A}, e \models a \in b\}$. So $\mathscr{B} \models \underline{\mathrm{ST}}_\mu + \underline{\mathrm{Limit}}_\mu + \neg\underline{\varphi}$. Thus $\underline{\mathrm{ST}}_\mu + \underline{\mathrm{Limit}}_\mu$ does not entail$_{\mu,\mu}$ $\underline{\varphi}$.

Facts A4.3 and A4.5 give a "sound and complete" infinitary first-order characterization of A_{ls}-validity. But the "soundness" result is quite peculiar: Limit_μ is not itself A_{ls}-valid (true in every $\mathrm{MO}_{\mathrm{ls}}$-model)!

Is there an infinitary but first-order theory, perhaps using the Chang quantifier, complete with respect to A_{ls}-validity, but with A_{ls}-valid axioms? This would be nice, but seems unlikely.

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