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WELL-BEHAVED MODAL LOGICS

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§1. Introduction. Much of the literature on the model theory of modal logics suffers from two weaknesses. Firstly, there is a lack of generality; theorems are proved piecemeal about this or that modal logic, or at best small classes of logics. Much of the literature, e.g. [1], [2], and [3], confines attention to structures with the expanding domain property (i.e., if wRu then $\bar{A}(w) \subseteq \bar{A}(u)$); the syntactic counterpart of this restriction is assumption of the converse Barcan scheme, a move which offers (in Russell's phrase) "all the advantages of theft over honest toil". Secondly, I think there has been a failure to hit on the best ways of extending classical model theoretic notions to modal contexts. This weakness makes the literature boring, since a large part of the interest of modal model theory resides in the way in which classical model theoretic notions extend, and in some cases divide, in the modal setting. (The relation between α-recursion theory and classical recursion theory is analogous to that between modal model theory and classical model theory. Much of the work in α-recursion theory involved finding the right definitions (e.g., of recursive-in) and separating concepts which collapse in the classical case (e.g. of finiteness and boundedness).)

The notion of a well-behaved modal logic is introduced in §3 to make possible rather general results; of course our attention will not be restricted to structures with the expanding domain property. Rather than prove piecemeal that familiar modal logics are well-behaved, in §4 we shall consider a class of "special" modal logics, which obviously includes many familiar logics and which is included in the class of well-behaved modal logics. The notion of specialness, though at first glance ad-hoc, does seem to isolate one reason for the peculiar tractability of the most familiar modal logics. The second above-mentioned weakness will be addressed elsewhere.

Fix the logical lexion "$\perp$", "$\approx$", "$\supset$", "$\forall$", "$\Box$", and a countable set **Var** of variables. A modal language $\mathscr{L} = \mathscr{L}(\mathbf{C}, \mathbf{Pred})$ is determined by a set **C** of individual constants and a set **Pred** of predicate-constants, each of a definite finite number of places. (If all members of **Pred** are 0-place, we may drop "$\forall$", "$\approx$" and **Var**, and we have a propositional modal language.) Standard abbreviations are in force (for "$\exists$", "&", "$\Diamond$", etc.) "T", "$\neg\phi$" and "$E(\sigma)$" *abbreviate*: $(\perp \supset \perp)$, $(\phi \supset \perp)$ and $(\exists v)(v \approx \delta)$ respectively, where $v \in \mathbf{Var}$, $v \neq \sigma$, $\sigma \in \mathbf{Var} \cup \mathbf{C}$. We shall also have reason to

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consider $\mathscr{L}^{\dot\forall}$, formed by enriching $\mathscr{L}$ with "$\dot\forall$", whose syntactic role is like "$\forall$". The sets $\mathrm{Fml}(\mathscr{L})$, $\mathrm{Sent}(\mathscr{L})$, $\mathrm{Fml}(\mathscr{L}^{\dot\forall})$ and $\mathrm{Sent}(\mathscr{L}^{\dot\forall})$ of formulae and sentences of $\mathscr{L}$ and $\mathscr{L}^{\dot\forall}$ are defined as usual. An extended formula of $\mathscr{L}$ is a formula of $\mathscr{L}$ on the result of prefixing to such a formula a string of the form "$(\dot\forall v_1)\cdots(\dot\forall v_n)$"; an extended sentence of $\mathscr{L}$ is an extended formula with no free variables, $\mathrm{ExFml}(\mathscr{L})$ and $\mathrm{ExSent}(\mathscr{L})$ are the sets of extended formulae and sentences of $\mathscr{L}$ respectively. Let $\dot\forall\phi$ and $\dot\exists\phi$ be the closures of ϕ under "$\dot\forall$" and "$\dot\exists$" respectively. Let $\Box^*(\phi_0) = \Diamond^*(\phi_0) = \phi_0$ and

$$\Box^*(\phi_0,\ldots,\phi_{k+1}) = (\phi_0 \supset \Box^*(\phi_0,\ldots,\phi_k)),$$
$$\Diamond^*(\phi_0,\ldots,\phi_{k+1}) = (\phi_0 \,\&\, \Diamond^*(\phi_0,\ldots,\phi_k)).$$

When Γ is a set of formulae, $\Box\Gamma = \{\Box\phi : \phi \in \Gamma\}$ and $\bigwedge\Gamma =$ the conjunction of all members of Γ if Γ is finite; otherwise $\bigwedge\Gamma = \{\bigwedge\Gamma_0 : \Gamma_0 \subseteq \Gamma, \Gamma_0 \text{ finite}\}$.

A frame is a binary structure (W, R), W a nonempty set, $R \subseteq W^2$. Let wR^nu iff either $n = 0$ and $w = u$ or $u > 0$ and there are $w_1 = w, w_2, \ldots, w_n = u$ so that for all i, if $0 < i < n$ then w_iRw_{i+1}. For $w \in W$, let $(W,R)_w = (W', R')$, $W' = \{u \in W: \text{for some } n < \omega, wR^nu\}$, $R' = R \restriction W'$, For our purposes, a modal logic L is a class of frames which is closed under order-isomorphism, and such that, if $(W,R) \in L$ and $w \in W$, then $(W,R)_w \in L$.

An extended frame has the form $(W, R, A, \bar{A})$, where (W, R) is a frame, A is a nonempty set and $\bar{A}$ maps W into $\mathrm{Power}(A)$. (We could introduce a finer notion of a modal logic, taking one to be a class of extended frames meeting certain conditions; but in this paper we shall have no need for such refinements.)

A structure for $\mathscr{L}$ has the form $(W, R, A, \bar{A}, V)$, where $\mathfrak{A} = (W, R, A, \bar{A})$ is an extended frame and V is a function on $\mathbf{C} \cup \mathbf{Pred}$:

$$V(\mathbf{c}) \in A \quad \text{for } \mathbf{c} \in \mathbf{C};$$
$$V(\underline{P}) \subseteq W \times A^n \quad \text{for } \underline{P} \text{ an } n\text{-place member of } \mathbf{Pred}.$$

A model for $\mathscr{L}$ has the form $\mathscr{M} = (\mathfrak{A}, w)$, for $\mathfrak{A}$ as above, $w \in W$. $\mathrm{Frame}(\mathfrak{A}) = \mathrm{Frame}(\mathscr{M}) = (W, R)$, $|\mathfrak{A}| = |\mathscr{M}| = A$. (Where $\mathscr{L}$ is propositional, A and $\bar{A}$ may be dropped.) An $\mathfrak{A}$-assignment (or an $\mathscr{M}$-assignment) is a function from $\mathbf{Var}$ into $|\mathfrak{A}| = |\mathscr{M}|$. When $\sigma \in \mathbf{Var} \cup \mathbf{C}$ and α is an $\mathfrak{A}$-assignment, we put

$$\mathrm{den}(\mathfrak{A}, \alpha, \sigma) = \begin{cases} V(\sigma) & \text{if } \sigma \in \mathbf{C}, \\ \alpha(\sigma) & \text{if } \sigma \in \mathbf{Var}. \end{cases}$$

The notion of satisfaction (of a formula of $\mathscr{L}^{\dot\forall}$ in a model by an assignment) is defined by the familiar recursion. Here are most of the clauses.

$(\mathfrak{A}, w) \not\models \bot[\alpha]$;

$(\mathfrak{A}, w) \models (\sigma \approx \tau)[\alpha]$ iff $\mathrm{den}(\mathfrak{A}, \alpha, \sigma) = \mathrm{den}(\mathfrak{A}, \alpha, \tau)$;

$(\mathfrak{A}, w) \models \underline{P}(\sigma_1, \ldots, \sigma_n)[\alpha]$ iff $(w, \mathrm{den}(\mathfrak{A}, \alpha, \sigma_1), \ldots, \mathrm{den}(\mathfrak{A}, \alpha, \sigma_n)) \in V(\underline{P})$;

$(\mathfrak{A}, w) \models (\forall v)\phi[\alpha]$ iff $(\mathfrak{A}, w) \models \phi[\alpha^v_\xi]$ for every $\xi \in \bar{A}(\mathrm{w})$; notice the restriction on ξ which makes "$\forall$" actualistic;

$(\mathfrak{A}, w) \models (\dot\forall v)\phi[\alpha]$ iff $(\mathfrak{A}, w) \models \phi[\alpha^v_\xi]$ for every $\xi \in A$; notice the lack of such a restriction making "$\dot\forall$" possibilitistic;

$(\mathfrak{A}, w) = \Box\phi[\alpha]$ iff for every $u \in W$, if wRu then $(\mathfrak{A}, u) = \phi[\alpha]$.

When ϕ has free variables among $v_1, \ldots, v_n$ and $\vec{a} = (a_1, \ldots, a_n) \in |\mathfrak{A}|^n$, by $(\mathfrak{A}, w) \models \phi[\vec{a}]$ we mean $(\mathfrak{A}, w) \models \phi[\alpha]$ for any α such that $\alpha(v_i) = a_i$.

When Γ is a set of formulae, $(\mathfrak{A}, w) \models \Gamma[\alpha]$ iff for all $\phi \in \Gamma$, $(\mathfrak{A}, w) \models \phi[\alpha]$. Let $\mathfrak{A} \models \phi[\alpha]$ iff, for all $w \in W$, $(\mathfrak{A}, w) \models \phi[\alpha]$; $(\mathfrak{A}, w) \models \phi$ iff for all $\mathfrak{A}$-assignments α, $(\mathfrak{A}, w) \models \phi[\alpha]$; similarly for $\mathfrak{A} \models \phi$. Let $(W, R) \models \phi$ iff for all structures $\mathfrak{A}$, if $(W, R) = \mathrm{Frame}(\mathfrak{A})$ then $\mathfrak{A} \models \phi$; notice the second-order aspect of this last definition. All these notions apply with Γ in place of ϕ in the obvious way.

$\mathfrak{A}$ is an L-structure iff $\mathrm{Frame}(\mathfrak{A}) \in L$; similarly $\mathscr{M}$ is an L-model iff $\mathrm{Frame}(\mathscr{M}) \in L$. When $\Gamma \cup \{\phi\}$ is a set of formulae of $\mathscr{L}^{\dot\forall}$, let Γ L-imply ϕ iff for all L-models $\mathscr{M}$ and $\mathscr{M}$-assignments α:

$$\text{if } \mathscr{M} \models \Gamma[\alpha] \quad \text{then } \mathscr{M} \models \phi[\alpha].$$

L-validity is L-implication by the empty set; L-equivalence is mutual L-implication. Let K be the "universal" logic, i.e., the class of all frames. By "implies", etc. we shall understand "K-implies", etc.

When $w \in W$ and $(W, R)_w = (W', R')$ and $\mathfrak{A} = (W, R, A, \bar{A}, V)$, let $\mathfrak{A}_w = (W', R', A, \bar{A}, V')$, where $V' \restriction \mathbf{C} = V \restriction \mathbf{C}$ and $V'(\underline{P}) = V(\underline{P}) \cap (W' \times A^n)$ (here $\underline{P}$ is n-place). If $\mathfrak{A} = \mathfrak{A}_w$, $\mathfrak{A}$ is centered at w; if $\mathscr{M} = (\mathfrak{A}_w, w)$, $\mathscr{M}$ is centered at w. Observation: $(\mathfrak{A}, w) \models \phi[\alpha]$ iff $(\mathfrak{A}_w, w) \models \phi[\alpha]$. In defining an extended frame we required only that $\bigcup \mathrm{Range}(\bar{A}) \subseteq A$; this is not because of a taste for the impossible, but rather to ensure that $\mathfrak{A}_w$ is always a structure.

§2. A sequent calculus for K. A sequent in $\mathscr{L}_0^{\dot\forall}$ is an $(n+1)$-tuple $(n \geq 1)$ of the form $(\Gamma_0, \ldots, \Gamma_{n-1}, \phi)$, hereafter written $\Gamma_0 | \cdots | \Gamma_{n-1} \vdash \phi$, where $\Gamma_0 \cup \cdots \cup \Gamma_{n-1} \cup \{\phi\} \subseteq \mathrm{Fml}(\mathscr{L}_0^{\dot\forall})$. $\Gamma_0 | \cdots | \Gamma_{n-1} \vdash \phi$ is valid iff for any structure $\mathfrak{A}$, R-chain $w_0, \ldots, w_{n-1}$ and $\mathfrak{A}$-assignment α, if $(\mathfrak{A}, w_i) \models \Gamma_i[\alpha]$ for all $i < n$ then $(\mathfrak{A}, w_{n-1}) \models \phi[\alpha]$. The class of theorems is defined inductively.

All sequents of these forms are axioms:

$$\Gamma \cup \{\phi\} \vdash \phi; \qquad \Gamma_0 \cup \{\Box\phi\} \,|\, \Gamma_1 \vdash \phi; \qquad \Gamma \vdash \sigma \approx \sigma;$$
$$\{\sigma \approx \tau\} \vdash \Box(\sigma \approx \tau); \qquad \{(\neg\sigma \approx \tau)\} \vdash \Box(\neg\sigma \approx \tau).$$

The following rules are structural: For $n \leq m$

$$\text{(Compounding)} \quad \frac{\Gamma_n | \cdots | \Gamma_{m-1} \vdash \phi}{\Gamma_0 | \cdots | \Gamma_{m-1} \vdash \phi};$$

$$\text{(Cut)} \quad \frac{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash \phi \qquad \Gamma_{n-1} \cup \{\phi\} | \cdots | \Gamma_{m-1} \vdash \psi}{\Gamma_0 | \cdots | \Gamma_{m-1} \vdash \psi};$$

$$\text{(Thinning)} \quad \frac{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash \phi}{\Gamma'_0 | \cdots | \Gamma'_{n-1} \vdash \phi}, \quad \text{where } \Gamma_i \subseteq \Gamma'_1 \text{ for } i < n.$$

Then we have the usual sorts of rules for "$\bot$", "$\supset$", "$\forall$", and "$\dot\forall$":

$$\frac{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash \bot}{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash \phi}; \qquad \frac{\Gamma_0 | \cdots | \Gamma_{n-1} \cup \{\neg\phi\} \vdash \bot}{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash \neg\phi};$$

$$\frac{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash (\phi \supset \psi) \qquad \Gamma_0 | \cdots | \Gamma_{n-1} \vdash \phi}{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash \psi}; \qquad \frac{\Gamma_0 | \cdots | \Gamma_{n-1} \cup \{\phi\} \vdash \psi}{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash (\phi \supset \psi)};$$

$$\frac{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash (\forall v)\phi \qquad \Gamma_0 | \cdots | \Gamma_{n-1} \vdash E(\sigma)}{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash \phi(v/\sigma)}, \qquad \frac{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash (\dot\forall v)\phi}{\Gamma_0 | \cdots | \Gamma_{n-1} \vdash \phi(v/\sigma)},$$

where in the last two rules σ is substitutable for v in ϕ;

$$\frac{\Gamma_0|\cdots|\Gamma_{n-1} \vdash \phi \,\&\, E(v)}{\Gamma_0|\cdots|\Gamma_{n-1} \vdash (\forall v)\phi}, \qquad \frac{\Gamma_0|\cdots|\Gamma_{n-1} \vdash \phi}{\Gamma_0|\cdots|\Gamma_{n-1} \vdash (\dot{\forall} v)\phi},$$

where in the last two rules v is not free in $\Gamma_0, \ldots, \Gamma_{n-1}$. Furthermore we need one rule governing "$\approx$":

$$\text{(Congruence)} \quad \frac{\Gamma_0|\cdots|\Gamma_{n-1} \vdash \phi(v/\sigma_0) \qquad \Gamma_0|\cdots|\Gamma_{n-1} \vdash \sigma_0 \approx \sigma_1}{\Gamma_0|\cdots|\Gamma_{n-1} \vdash \phi(v/\sigma_1)}.$$

Finally these novel rules:

$$\frac{\Gamma_0|\cdots \Gamma_i|\Gamma_{i+1} \cup \{\phi\} \cdots \Gamma_{n-1} \vdash \psi}{\Gamma_0|\cdots \Gamma_i \cup \{\Box\phi\}|\Gamma_{i+1} \cdots \Gamma_{n-1} \vdash \psi}; \qquad \frac{\Gamma_0|\cdots|\Gamma_{n-2}|\{\ \} \vdash \phi}{\Gamma_0|\cdots|\Gamma_{n-2} \vdash \Box\phi}, \quad \text{when } n \geq 2.$$

A sequent is a theorem iff it is an axiom or derivable from axioms by these rules. Clearly all theorems are valid. Let $(\Gamma_0, \ldots, \Gamma_{n-1})$ be inconsistent iff $\Gamma_0|\cdots|\Gamma_{n-1} \vdash \bot$ is a theorem. So in particular, Γ is consistent iff $\Gamma \vdash \bot$ is not a theorem. To prove completeness, it suffices to show that if $(\Gamma_0, \ldots, \Gamma_{n-1})$ is consistent then there exist a structure $\mathfrak{A}$ for $\mathscr{L}_0$, and R-chain $w_0, \ldots, w_{n-1}$ in $(W, R) = \mathrm{Frame}(\mathfrak{A})$ and an $\mathfrak{A}$-assignment α such that $(\mathfrak{A}, w_i) \models \Gamma_i[\alpha]$ for all $i < n$. (For if $\Gamma_0|\cdots|\Gamma_{n-1} \vdash \phi$ is not a theorem then $(\Gamma_0, \ldots, \Gamma_{n-1} \cup \{\neg\phi\})$ is consistent; but by the above, $\Gamma_0|\cdots|\Gamma_{n-1} \vdash \phi$ is not valid.)

For $R \subseteq W^2$ let R be a relational tree with root w_0 iff:

(i) R is anti-symmetric (i.e., for all $w, u \in W$ if wRu then not uRw);

(ii) R is anti-transitive (i.e., for all $w, u, v \in W$, if wRu and uRv then not wRv);

(iii) for any $w \in \mathrm{Fld}(R)$ there is a unique n so that w_0R^nw;

(iv) for any $w \in W$, not wRw_0; and

(v) for any $w \in \mathrm{Fld}(R)$, if $w \neq w_0$ then there is a unique u so that uRw.

When $w_0 \in W$, $\mathscr{D}$ is a diagram for $(\mathscr{L}, W)$ centered at w_0 iff $\mathscr{D} = (D, R)$, where R is a relational tree with root w_0 and $D \subseteq W' \times \mathrm{Sent}(\mathscr{L}^{\dot{\forall}})$, for $W' = \mathrm{Fld}(R) \cup \{w_0\}$.

Let $D(w) = \{\phi : (w, \phi) \in D\}$; when $\mathscr{D}'$ is also a diagram for $\mathscr{L}$, W centered at w_0, let $\mathscr{D}' \subseteq \mathscr{D}$ iff $D' \subseteq D$ and $R' \subseteq R$.

$\mathscr{D}$ is finite iff D and R are finite; when $\mathscr{D}$ is finite, let

$$\mathscr{D}^*(w) = \bigwedge D(w) \,\&\, \bigwedge\{\Diamond\mathscr{D}^*(u) : wRu\}.$$

Since $\mathrm{Fld}(R)$ is finite and $\bigwedge\{\ \} = \top$, this is well-defined. $\mathscr{D}$ is consistent iff for each finite $\mathscr{D}_0 \subseteq \mathscr{D}$ and centered at w_0, $\mathscr{D}_0^*(w_0)$ is consistent. This is equivalent to: For all such $\mathscr{D}_0$ and all $w \in W$, $\mathscr{D}_0^*(w)$ is consistent. For $D' \subseteq W \times \mathrm{Sent}(\mathscr{L}^{\dot{\forall}})$, let $\mathscr{D} \cup D' = (D \cup D', R)$. In what follows $\mathscr{L}^{\dot{\forall}}$ could be replaced by $\mathscr{L}$.

LEMMA 1. *Let $\mathscr{D}^i = \mathscr{D} \cup \{(w, \phi^i)\}$, where $\phi^0 = \phi$ and $\phi^1 = \neg\phi$, $i \in 2$ and $w \in W'$. If $\mathscr{D}$ is consistent then either $\mathscr{D}^0$ or $\mathscr{D}^1$ is consistent.*

For suppose not. Suppose $\mathscr{D}_0^0 \subseteq \mathscr{D}^0$ and $\mathscr{D}_0^1 \subseteq \mathscr{D}^1$ are finite, while $\mathscr{D}_0^{0*}(w_0)$ and $\mathscr{D}_0^{1*}(w_0)$ are inconsistent. Without loss of generality, suppose that for $u \neq w$ we have $D_0^0(u) = D_0^1(u)$, and $D_0^0(w) - \{\phi\} = D_0^1(w) - \{\neg\phi\}$; let $w_0, \ldots, w_n = w$ be the R

chain from w_0 to w. Then $\mathscr{D}_0^{0}{}^*(w_0)$ and $\mathscr{D}_0^{1}{}^*(w_0)$ have the forms $\Diamond^*(\phi_0, \ldots, \phi_n \,\&\, \phi)$ and $\Diamond^*(\phi_0, \ldots, \phi_n \,\&\, \neg\phi)$. Thus

$$\{\ \} \vdash \Box^*(\phi_0, \ldots, \phi_n \supset \neg\phi\} \quad \text{and} \quad \{\ \} \vdash \Box^*(\phi_0, \ldots, \phi_n \supset \phi)$$

are theorems. It is not hard to show then that $\Diamond^*(\phi_0, \ldots, \phi_n) \vdash \bot$. So $\mathscr{D}$ is inconsistent.

LEMMA 2. *Let* $(w, (\exists v)\phi) \in D$ *and* $\mathscr{D}' = \mathscr{D} \cup \{(w, \phi(v/\mathbf{c})), (w, E(\mathbf{c}))\}$ *where* $\mathbf{c}$ *does not occur in* $\mathscr{D}$. *If* $\mathscr{D}$ *is consistent then so is* $\mathscr{D}'$. *Similarly for* $(w, (\dot{\exists} v)\phi) \in D$ *and* $\mathscr{D}' = \mathscr{D} \cup \{(w, \phi(v/\mathbf{c})\}$.

Suppose $\mathscr{D}'$ is inconsistent. Let $w_0, \ldots, w_n = w$ be as above. There is a finite $\mathscr{D}_0 \subseteq \mathscr{D}$ so that $\mathscr{D}'_0 = \mathscr{D}_0 \cup \{(w, \phi(v/\mathbf{c})), (w, E(\mathbf{c}))\}$ is inconsistent, $\mathscr{D}'_0 \subseteq \mathscr{D}'$. Without loss of generality, $\mathscr{D}'_0{}^*(w_0)$ has the form $\Diamond^*(\phi_0, \ldots, \phi_n \,\&\, (\exists v)\phi \,\&\, \phi(v/\mathbf{c}) \,\&\, E(\mathbf{c}))$. Then the following sequents are theorems:

$$\begin{gathered}
\{\ \} \vdash \Box^*(\phi_0, \ldots, (\phi_n \,\&\, (\exists v)\phi) \supset (E(\mathbf{c}) \supset \neg\phi(v/\mathbf{c}))),\\
\{\ \} \vdash \Box^*(\phi_0, \ldots, (\phi_n \,\&\, (\exists v)\phi) \supset (E(v) \supset \neg\phi))),\\
\phi_0 | \cdots \phi_{n-1} | \phi_n \,\&\, (\exists v)\phi \vdash E(v) \supset \neg\phi,\\
\phi_0 | \cdots \phi_{n-1} | \phi_n \,\&\, (\exists v)\phi \vdash (\forall v)\neg\phi,\\
\phi_0 | \cdots \phi_{n-1} | \phi_n \,\&\, (\exists v)\phi \vdash \bot,\\
\Diamond^*(\phi_0, \ldots, \phi_n \,\&\, (\exists v)\phi) \vdash \bot.
\end{gathered}$$

Thus $\mathscr{D}_0$ is inconsistent, and so is $\mathscr{D}$.

LEMMA 3. *Let* $(w, \Diamond\phi) \in D$ *and* $D' = D \cup \{(u, \phi)\}$, $R' = R \cup \{(w, u)\}$, *where* $u \in W$ *does not occur in* D; *let* $D' = (D', R')$. *If* $\mathscr{D}$ *is consistent then so is* $\mathscr{D}'$.

Proof of this is routine. $\mathscr{D}$ is $\neg$-complete (for $\mathscr{L}$, W) iff for each $w \in W'$ and $\phi \in \mathrm{Sent}(\mathscr{L})$ either ϕ or $\neg\phi \in D(w)$. $\mathscr{D}$ is $\exists$-complete (for $\mathscr{L}, W$) iff for each $(w, (\exists v)\phi) \in D$ there is a $\mathbf{c} \in \mathbf{C}$ so that $\phi(v/\mathbf{c})$, $E(\mathbf{c}) \in D(w)$. $\mathscr{D}$ is $\dot{\exists}$-complete (for $\mathscr{L}, W$) iff for each $(w, (\dot{\exists} v)\phi) \in D$ there is a $\mathbf{c} \in \mathbf{C}$ so that $\phi(v/\mathbf{c}) \in D(w)$. $\mathscr{D}$ is $\Diamond$-complete (for $\mathscr{L}, W$) iff for each $(w, \Diamond\phi) \in D$ there is a u so that wRu and $(u, \phi) \in D$. $\mathscr{D}$ is complete iff $\mathscr{D}$ is $\neg$-complete, $\exists$-complete, $\dot{\exists}$-complete, and $\Diamond$-complete. Such a $\mathscr{D}$ determines a unique model for $\mathscr{L}$ centered at w_0. The construction is standard: for $\mathbf{c}, \mathbf{d} \in \mathbf{C}$ let $\mathbf{c} \sim \mathbf{d}$ iff $\mathbf{c} \approx \mathbf{d} \in D(w)$ for some w; $\sim$ is an equivalence relation; let $A = \mathbf{C}/\!\sim$; let $A(w) = \{[\mathbf{c}] : E(\mathbf{c}) \in D(w)\}$, where $[\mathbf{c}]$ is the $\sim$-class of $\mathbf{c}$; $V(\mathbf{c}) = [\mathbf{c}]$; $V(\underline{P}) = \{(w, [\mathbf{c}_1], \ldots, [\mathbf{c}_n] : \underline{P}(\mathbf{c}_1, \ldots, \mathbf{c}_n) \in D(w)\}$, and $\mathfrak{A} = (W', R, A, \bar{A}, V)$. By a familiar argument, for all $\phi \in \mathrm{Sent}(\mathscr{L}^{\dot{\forall}})$:

$$(\mathfrak{A}, w_0) \models \phi \quad \text{iff} \quad \phi \in D(w_0).$$

Given $\mathscr{L}_0 = \mathscr{L}(\mathbf{C}_0, \mathrm{Pred})$ and W_0, let $\kappa = \mathrm{card}(\omega \cup \mathbf{C}_0 \cup W_0)$, select W_1, $\mathbf{C}_1$ of cardinality κ disjoint from W_0 and $\mathbf{C}_0$, and let $\mathbf{C} = \mathbf{C}_0 \cup \mathbf{C}_1$, $W = W_0 \cup W_1$.

THE DIAGRAM-COMPLETION THEOREM. *If* $\mathscr{D}_0$ *is a consistent diagram for* $(\mathscr{L}_0, W_0)$ *then there is a complete consistent diagram* $\mathscr{D}$ *for* $(\mathscr{L}, W)$ *so that* $\mathscr{D}_0 \subseteq \mathscr{D}$.

PROOF OF THE DIAGRAM-COMPLETION THEOREM. Fix a κ-ordering (κ is here an initial ordinal) of $W \times \mathrm{Sent}(\mathscr{L}^{\dot{\forall}})$. We construct a sequence of diagrams $\mathscr{D}_\xi$ for $(\mathscr{L}, W)$ centered at w_0. Suppose we have $\mathscr{D}_\xi$; let $W'_\xi = \mathrm{Fld}(R_\xi) \cup \{w_0\}$. Let (w_ξ, ϕ_ξ) be the least unused (w, ϕ) in our listing with $w \in W'_\xi$. Let ϕ'_ξ be ϕ_ξ if $\mathscr{D}_\xi \cup \{(w_\xi, \phi_\xi)\}$ is consistent; otherwise ϕ'_ξ is $\neg\phi_\xi$; let $\mathscr{D}'_\xi = \mathscr{D}_\xi \cup \{(w_\xi, \phi'_\xi)\}$. *Fact*: $\mathscr{D}'_\xi$ *is consistent.*

This is easy to prove. If ϕ'_ξ is not of the form $(\exists v)\psi$, $(\dot{\exists} v)\psi$ or $\diamond\psi$, let $\mathscr{D}_{\xi+1} = \mathscr{D}'_\xi$. If ϕ'_ξ is $(\exists v)\psi$ or $(\dot{\exists} v)\psi$, let $\mathbf{c}$ be the least member of $\mathbf{C}$ not occurring in $\mathscr{D}'_\xi$; in the first case let $\mathscr{D}_{\xi+1} = \mathscr{D}'_\xi \cup \{(w_\xi, \psi(v/\mathbf{c})), (w_\xi, E(\mathbf{c}))\}$, and in the second case let $\mathscr{D}_{\xi+1} = \mathscr{D}'_\xi \cup \{(w_\xi, \psi(v/\mathbf{c}))\}$. *Fact*: $\mathscr{D}_{\xi+1}$ *is then consistent*. Finally, if ϕ'_ξ is ψ, let w' be the least member of W not occurring in $\mathscr{D}'_\xi$. Let $D_{\xi+1} = D'_\xi \cup \{(w', \psi)\}$ and $R_{\xi+1} = R_\xi \cup \{(w_\xi, w')\}$. *Fact*: $\mathscr{D}_{\xi+1}$ *is then consistent*. For λ a limit ordinal let $\mathscr{D}_\lambda = \bigcup_{\xi<\lambda}\mathscr{D}_\xi$. Then $\mathscr{D}_\kappa = \mathscr{D}$ is as desired. Further details are left to the reader.

The model existence theorem immediately follows: *For $\Gamma \subseteq \mathrm{Fml}(\mathscr{L}_0^{\dot\forall})$, if Γ is consistent then for some centered model $\mathscr{M}$ for $\mathscr{L}_0$ and some $\mathscr{M}$-assignment α, $\mathscr{M} \models \Gamma[\alpha]$.*

§3. Well-behaved modal logics.

When $\mathfrak{A}$ and $\mathfrak{B}$ are structures for $\mathscr{L}$ let $\mathfrak{A}$ and $\mathfrak{B}$ be *almost identical* (in symbols $\mathfrak{A} =^* \mathfrak{B}$) iff, when $\mathfrak{A} = (W, R, A, \bar{A}, V)$, $\mathfrak{B} = (W', S, B, \bar{B}, V')$, $W = W'$, $A = B$ and for all $\phi \in \mathrm{Fml}(\mathscr{L})$, $w \in W$ and α an $\mathfrak{A}$-assignment, we have

$$(\mathfrak{A}, w) \models \phi[\alpha] \quad \text{iff} \quad (\mathfrak{B}, w) \models \phi[\alpha].$$

(Notice that if $\mathfrak{A} =^* \mathfrak{B}$ then $\bar{A} = \bar{B}$ and $V = V'$; but $W = W'$, $\bar{A} = \bar{B}$ and $V = V'$ does not imply that $\mathfrak{A} =^* \mathfrak{B}$.) Let $(\mathfrak{A}, w) =^* (\mathfrak{B}, w)$ iff $\mathfrak{A} =^* \mathfrak{B}$.

Let $\mathscr{L}'$ be a propositional language. For $\phi \in \mathrm{Fml}(\mathscr{L}')$ and $\psi \in \mathrm{Fml}(\mathscr{L})$, ψ is an $\mathscr{L}$-instance of ϕ if ψ is obtained from ϕ by uniformly replacing 0-place predicates in ϕ by formulae of $\mathscr{L}$; when $\Phi \subseteq \mathrm{Fml}(\mathscr{L}')$, let $\Phi(\mathscr{L})$ be the set of $\dot\forall$-closures of $\mathscr{L}$-instances of members of Φ.

DEFINITION. LET L be a modal logic, and Φ be as above. Φ guarantees L iff for any $\mathscr{L}$ and any centered model $\mathscr{M}$ for $\mathscr{L}$, if $\mathscr{M} \models \Phi(\mathscr{L})$ then there is an L-model $\mathscr{N}$ for $\mathscr{L}$ such that $\mathscr{M} =^* \mathscr{N}$.

Φ weakly guarantees L iff for any $\mathscr{L}$ and any structure $\mathfrak{A}$ for L, if $\mathfrak{A} \models \Phi(\mathscr{L})$ then there is an L-structure $\mathfrak{B}$ for $\mathscr{L}$ such that $\mathfrak{A} =^* \mathfrak{B}$.

Φ makes L well-behaved (weakly well-behaved) iff Φ is a set of L-validities which guarantees (weakly guarantees) L.

Note. If Φ is a set of L-validities, then so is $\Phi(\mathscr{L})$. Φ is $\square$-closed iff for all $n < \omega$ and $\phi \in \Phi$, Φ implies $\square^n\phi$.

OBSERVATIONS. (i) If Φ guarantees L then Φ weakly guarantees L.

(ii) Φ makes L well-behaved iff for all $\mathscr{L}$ and all centered models $\mathscr{M}$ for $\mathscr{L}$:

$$\mathscr{M} \models \Phi(\mathscr{L}) \quad \text{iff} \quad \mathscr{M} =^* \mathscr{N} \text{ for some } L\text{-model } \mathscr{N} \text{ for } \mathscr{L}.$$

To see this, suppose Φ makes L well-behaved, $\mathscr{M}$ is a centered model for $\mathscr{L}$, $\mathscr{N}$ is an L-model for $\mathscr{L}$ and $\mathscr{M} =^* \mathscr{N}$; since Φ is a set of L-validities, $\mathscr{N} \models \Phi(\mathscr{L})$, so $\mathscr{M} \models \Phi(\mathscr{L})$.

(iii) Φ makes L weakly well-behaved iff for all $\mathscr{L}$ and all structures $\mathfrak{A}$ for $\mathscr{L}$,

$$\mathfrak{A} \models \Phi(\mathscr{L}) \quad \text{iff} \quad \mathfrak{A} =^* \mathfrak{B} \text{ for some } L\text{-structure } \mathfrak{B} \text{ for } \mathscr{L}.$$

The reason is as above.

(iv) If Φ weakly guarantees L and is $\square$-closed then Φ guarantees L.

Suppose $\mathscr{M}$ is a centered model for $\mathscr{L}$, $\mathscr{M} = (\mathfrak{A}, w)$; if $\mathscr{M} \models \Phi(\mathscr{L})$, then $\mathscr{M} \models \Phi'(\mathscr{L})$ where $\Phi' = \{\square^n\phi : \phi \in \Phi \,\&\, n < \omega\}$. Thus $\mathfrak{A} \models \Phi(\mathscr{L})$, since $\mathscr{M}$ was centered.

(v) If Φ makes L weakly well-behaved then:

$$\Phi \text{ is } \Box\text{-closed} \quad \text{iff} \quad \Phi \text{ makes } L \text{ well-behaved.}$$

Suppose that Φ is not $\Box$-closed but makes L well-behaved. Suppose $\phi \in \Phi$ but Φ does not imply $\Box^n\phi$. Let $(\mathfrak{C}, w)$ be a model for $\mathscr{L}'$ so that $(\mathfrak{C}, w) \models \Phi \cup \{\neg \Box^n \phi\}$. $(\mathfrak{C}, w)$ may be converted into a model $(\mathfrak{A}, w)$ for $\mathscr{L}$ so that $(\mathfrak{A}, w) \models \Phi(\mathscr{L}) \cup \{\neg \Box^n \phi(\mathscr{L})\}$. Let $(\mathfrak{B}, w)$ be an L-model so that $(\mathfrak{A}, w) = {}^*(\mathfrak{B}, w)$. Since $(\mathfrak{B}, w) \models \neg \Box^n \phi(\mathscr{L})$, for some u, $wR^n u$ and $(\mathfrak{B}, u) \models \neg \phi(\mathscr{L})$, where R is the accessibility relation for $\mathfrak{B}$; but $\phi(\mathscr{L})$ is L-valid and $(\mathfrak{B}, u)$ is an L-model—a contradiction.

Observation. When Γ is a set of sentences of $\mathscr{L}$, $\Gamma \cup \Phi(\mathscr{L})$ has a model iff Γ has an L-model.

From right to left holds because $\Phi(\mathscr{L})$ is a set of L-validities. From left to right holds because Φ guarantees L.

This observation is the reason for the notion of well-behavedness: $\Phi(\mathscr{L})$ is an "instant axiomatization" of L, given the underlying formalization of K; furthermore, we have a reduction of model-existence questions from L to K; so, for example, this "instant axiomatization" is complete (given the completeness of our underlying formalization of K). (The argument is easy: if $\Phi(\mathscr{L}) \cup \Gamma \nvdash \phi$ then $\Phi(\mathscr{L}) \cup \Gamma \cup \{\neg \phi\}$ has a model, so $\Gamma \cup \{\neg \phi\}$ has an L-model; so Γ does not L-imply ϕ.) Let L be well-behaved iff for some Φ, Φ makes L well-behaved. For most purposes, study of the model theory of L reduces to study of the model theory of K. Consider this property of a model logic L: for any frame (W, R) there is a frame $(W, R') \in L$, where $R \subseteq R' \subseteq W^2$.

If L lacks this property, L is not well-behaved. So, for example, G(= the class of transitive well-capped (i.e., converse well-founded) frames) is not well-behaved. Let $(W, R) \in L$ iff for any $w, u \in W$ if wRu then there are $w_1, w_2 \in W$ so that wRw_1, w_1Rw_2 and w_2Ru. Let ϕ_0 be: $\Diamond \underline{P} \supset \Diamond^3 \underline{P}$. Claim: $\{\phi_0\}$ defines L. Clearly if $(W, R) \in L$, $(W, R) \models \phi_0$. Suppose $(W, R) \models \phi_0$ and $w, u \in W$, wRu; let $V(\underline{P}) = \{u\}$ and $\mathfrak{A} = (W, R, V)$; then $(\mathfrak{A}, w) \models \phi_0$ & $\Diamond \underline{P}$; so $(\mathfrak{A}, w) \models \Diamond^3 \underline{P}$; so for some $w_1, w_2, w_3 \in W$, wRw_1, w_1Rw_2, w_2Rw_3, and $(\mathfrak{A}, w_3) \models \underline{P}$; so $w_3 = u$. Clearly for any (W, R) there is an R' so that $R \subseteq R' \subseteq W^2$ and $(W, R) \in L$. Nonetheless L is not well-behaved.

To see this, let $W = \omega$, $R = \{(i, i+1) : i \in \omega\}$, $V(\underline{P}) = \{2i : i \in \omega\}$ and $\mathfrak{A} = (W, R, V)$. Then for all $i \in W$ and all sentences ϕ of $\mathscr{L}'$,

$$(\mathfrak{A}, i) \models \phi \quad \text{iff} \quad (\mathfrak{A}, i+2) \models \phi.$$

This follows by induction on the construction of ϕ. Thus $\mathfrak{A} \models \{\phi_0\}(\mathscr{L}')$; when Φ is the set of all L-validities, ϕ_0 implies each $\phi \in \Phi$; so $\mathfrak{U} \models \Phi(\mathscr{L}')$. Suppose $R \subseteq R' \subseteq W$ and $(W, R') \in L$. Since $(0,1) \in R$, either $(0,0) \in R'$ or $(1,1) \in R'$. But $(\mathfrak{A}, 0) \models \underline{P} \,\&\, \Box \neg \underline{P}$ and $(\mathfrak{A}, 1) \models \neg \underline{P} \,\&\, \Box \underline{P}$; so for $\mathfrak{A}' = (W, R', V)$, $\mathfrak{A} \neq {}^*\mathfrak{A}'$.

§4. Special logics. We now turn to the question: which familiar modal logics are well-behaved? Rather than answer this question piecemeal, we shall introduce a class of special logics and show them all to be well-behaved. Familiar logics like T, $K4$, B, $S4$ and $S5$ are readily seen to be special.

Let $\mathscr{L}'$ be the propositional modal language based on the sole 0-place predicate

"$\underline{P}$"; for $\phi \in \mathrm{Fml}(\mathscr{L}')$, ϕ is pre-special iff ϕ is built up from "$\underline{P}$" and "$\top$" using "&" and "$\Diamond$". ϕ is a special formula of $\mathscr{L}'$ iff ϕ is of the form

$$\Box^*(\phi_k,\ldots,\phi_1,\phi_0 \supset \Diamond \underline{P}),$$

where $\phi_0,\ldots,\phi_k$ are pre-special and "$\underline{P}$" occurs exactly once in only one ϕ_i for $i \le k$. Notice that the characteristic axioms for T, $K4$, B, $S4$ and $S5$ in any $\mathscr{L}$ are all instances of the following special formulae:

$$\underline{P} \supset \Diamond \underline{P}; \qquad \Diamond^2 \underline{P} \supset \Diamond \underline{P};$$
$$\underline{P} \supset \Box(\top \supset \Diamond \underline{P}) \quad \text{(equivalent to } \underline{P} \supset \Box\Diamond \underline{P});$$
$$\Diamond \underline{P} \supset \Box(\top \supset \Diamond \underline{P}) \quad \text{(equivalent to } \Diamond \underline{P} \supset \Box\Diamond \underline{P}).$$

A set of formulae Φ of $\mathscr{L}'$ defines a modal logic L iff for all frames (W,R), $(W,R) \in L$ iff $(W,R) \models \Phi$. L is special iff some set Φ of special formulae defines L.

We consider another representation of special formulae. Let $\mathscr{K}$ be the quantificational (nonmodal) language based on the 2-place predicate-constant "$\mathbf{R}$" (without identity). Let "$\models *$" represent satisfaction for formulae of $\mathscr{K}$. We define a "translation" between special formulae of $\mathscr{L}'$ and formulae of $\mathscr{K}$, which we shall also call special. Let T be a finite tree on ω (i.e., $T \subseteq \omega^{<\omega}$, T closed under initial segments). For each $t \in T$ introduce a distinct variable v_t. For any $r, s \in T$ we define a formula $\psi(T,r,s)$ of κ. When $t \in T$ and $|t| > 0$, let t^- be $\langle t_0,\ldots,t_i\rangle$, where $t = \langle t_0,\ldots,t_{i+1}\rangle$ (here $i = |t| - 2$). For $0 < i \le |r|$ let:

$$\theta_i = \bigwedge\{\mathbf{R}(v_{t^-}, v_t): t \restriction i \ne r \restriction i\};$$
$$\theta = \bigwedge\{\mathbf{R}(v_{r\restriction i}, v_{r\restriction(i+1)}): i < |r|\};$$
$$\psi(T,r,s) = ((\theta_{|r|} \mathbin{\&} \theta) \supset \mathbf{R}(v_r, v_s));$$
$$\hat{\psi}(T,r,s) = \text{the universal closure of } \psi(T,r,s).$$

A special formula of κ is one of the form $\hat{\psi}(T,r,s)$. We shall now transform a special $\hat{\psi}(T,r,s) = \psi$ into $f(\psi)$, a special formula of $\mathscr{L}'$.

For $t \ne s$, let:

$$\phi_t = \begin{cases} \bigwedge\{\Diamond \phi_{t'}: t' \in T, t'^- = t\} & \text{if } t \not\subseteq r^-; \\ \bigwedge\{\Diamond \phi_{t'}: t' \in T, t'^- = t, t'(|t|) \ne r(|t|)\} & \text{if } t \subseteq r^-. \end{cases}$$

Let:

$$\phi_s = \begin{cases} \underline{P} \mathbin{\&} \bigwedge\{\Diamond \phi_{t'}: t' \in T, t'^- = t\} & \text{if } s \not\subseteq r^-; \\ \underline{P} \mathbin{\&} \bigwedge\{\Diamond \phi_{t'}: t' \in T, t'^- = t, t'(|t|) \ne r(|t|)\} & \text{if } s \subseteq r^-. \end{cases}$$

Then $\phi_r, \phi_{r\,(|r|-1)}, \ldots, \phi_{r\restriction 1}, \phi_{r\restriction 0}$ is a sequence of pre-special formulae; furthermore, "$\underline{P}$" occurs exactly once in at most one element of this sequence. For suppose s_0 is maximal such that $s_0 \subseteq r$ and $s_0 \subseteq s$; for t so that $s_0 \subseteq t^-$, $t \subseteq r$, "$\underline{P}$" does not occur in ϕ_t; "$\underline{P}$" occurs once in ϕ_{s_0}; for $t' \subseteq s_0$ and $t = t'^-$ (where $|s_0| < 0$), "$\underline{P}$" does not occur in ϕ_t, because $t'(|t|) = r(|t|)$. Let

$$\phi^{r\restriction i} = \Box^*(\phi_{r\restriction i},\ldots,\phi_{r^-},\phi_r \supset P);$$

let $f(\psi) = \phi^{\langle\,\rangle}$. Thus $f(\psi)$ is a special formula of $\mathscr{L}'$. Conversely, given a special formula ϕ of $\mathscr{L}'$ it is easy to construct T, r and s so that $f(\hat{\psi}(T,r,s)) = \phi$; this construction is left to the reader.[1]

LEMMA 1. *For any special formula ψ of $\mathscr{K}$ and any frame (W, R), $(W, R) \models^* \psi$ iff $(W, R) \models f(\psi)$.*

PROOF. Assume $(W, R) \models^* \psi$, where $\psi = \hat{\psi}(T, r, s)$. Fix $\mathfrak{A} = (W, R, V)$ for $\mathscr{L}'$, $w_0 \in W$. *Claim*: $(\mathfrak{A}, w_0) \models f(\psi)$. If $(\mathfrak{A}, w_0) \not\models \phi_{r \upharpoonright 0}$, we are done; so assume $(\mathfrak{A}, w_0) \models \phi_{r\ 0}$. Then there is a β_0 mapping $\{v_{\langle\,\rangle}\} \cup \{v_t : t \upharpoonright 1 \neq r \upharpoonright 1\}$ into W so that $(W, R) \models {}^* \theta_1[\beta_0]$, $\beta_0(v_{\langle\,\rangle}) = w_0$, and if $s = \langle\ \rangle$ or $s \upharpoonright 1 = r \upharpoonright 1$ then $\beta_0(v_s) \in V(\underline{P})$. If $|r| = 0$, θ is $\top$; so $(W, R) \models {}^*(\theta_0 \,\&\, \theta)[\beta_0]$; furthermore $s = \langle\ \rangle$ or $s \upharpoonright 1 \neq \langle\ \rangle = r \upharpoonright 1$; so $\beta_0(v_s)$ is defined; so $(W, R) \models {}^*\mathbf{R}(v_{\langle\,\rangle}, v_s)[\beta_0]$, i.e., $w_0 R \beta_0(v_s)$; so $(\mathfrak{A}, w_0) \models \Diamond \underline{P}$; in this case $f(\psi)$ is $\phi_r \supset \Diamond \underline{P}$, $(\mathfrak{A}, w_0) \models f(\psi)$. Suppose that $|r| \geq 1$. Consider a w_1 so that $w_0 R w_1$. If $(\mathfrak{A}, w_1) \models \phi_{r \upharpoonright 1}$, $(\mathfrak{A}, w_1) \models \phi^{t \upharpoonright 1}$. So then β_0 extends to a β_1 mapping $\{v_{\langle\,\rangle}\} \cup \{v_t : t \upharpoonright 2 \neq r \upharpoonright 2\}$ into W, where $(W, R) \models^* \theta_2[\beta_1]$, $\beta_1(v_{r \upharpoonright 1}) = w_1$, and if $s = \langle\ \rangle$ or $s \upharpoonright 2 \neq r \upharpoonright 2$ then $\beta_1(v_s) \in V(\underline{P})$. If $|r| = 1$, as before we have $(\mathfrak{A}, w_1) \models \Diamond P$; and thus $(\mathfrak{A}, w_1) \models \phi^{r \upharpoonright 1}$. Since in this case $f(\psi)$ is $\phi_{r \upharpoonright 0} \supset \Box(\phi_r \supset \Diamond \underline{P})$, and w was arbitrary, $(\mathfrak{A}, w_0) \models f(\psi)$. Suppose that $|r| \geq 2$. Proceeding inductively we show that, for any R-chain $w_0, \ldots, w_{|r|}$ such that $(\mathfrak{A}, w_i) \models \phi_{r \upharpoonright i}$ for $i \leq |r|$, we have $(\mathfrak{A}, w_{|r|}) \models \Diamond \underline{P}$; so $(\mathfrak{A}, w_i) \models \phi^{r \upharpoonright i}$; in particular, $(\mathfrak{A}, w_0) \models f(\psi)$.

Now suppose that $(W, R) \models f(\psi)$. Suppose β maps $\{v_t : t \in T\}$ into W and $(W, R) \models^* (\theta_{|r|} \,\&\, \theta)[\beta]$. *Claim*: $(W, R) \models^* \mathbf{R}(v_r, v_s)[\beta]$. Let $V(\underline{P}) = \{\beta(v_s)\}$, $\mathfrak{A} = (W, R, V)$, $w_i = \beta(v_{r \upharpoonright i})$ for $i \leq |r|$; so $w_0, \ldots, w_{|r|}$ is an R-chain. For $i \leq |r|$, $\theta_{|r|}$ implies θ_i; so by construction of $\theta_{r \upharpoonright i}$, $(\mathfrak{A}, w_i) \models \phi_{r \upharpoonright i}$. Since $(\mathfrak{A}, w_0) \models f(\psi)$, $(\mathfrak{A}, w_i) \models \phi^{r \upharpoonright i}$; thus $(\mathfrak{A}, w_{|r|}) \models \Diamond \underline{P}$; by choice of $V(\underline{P})$, $w_{|r|} R \beta(v_s)$, as claimed. Q.E.D.

In what follows, Φ is a set of special formulae of $\mathscr{L}'$ defining L.

LEMMA 2. *For any frame (W, R) there is an R, $R \subseteq R^* \subseteq W^2$, so that $(W, R^*) \in L$.*

PROOF. Suppose $f(\psi) \in \Phi$, $\psi = \bar{\psi}(T, r, s)$. Given (W, R), let

$$R_\psi = \{(\beta(v_r), \beta(v_s)) : (W, R) \models^* (\theta_{|r|} \,\&\, \theta)[\beta]\}.$$

Let $\hat{R} = R \cup \bigcup\{R_\psi : f(\psi) \in \Phi\}$; let $R_0 = R$ and $R_{n+1} = \hat{R}_n$ for $n < \omega$, and let $R^* = \bigcup\{R_n : n < \omega\}$. Considering the form of $\hat{\psi}(T, r, s)$ we have $(W, R^*) \models^* \psi$. So $(W, R^*) \models \Phi$, using Lemma 1. Q.E.D.

For $\mathfrak{A} = (W, R, A, \bar{A}, V)$, let $\hat{\mathfrak{A}}$, $\mathfrak{A}^n$ and $\mathfrak{A}^*$ be the result of replacing "R" by $\hat{R}$, R_n and R^* respectively.

LEMMA 3. *Let $\mathfrak{A}$ be a structure for $\mathscr{L}'$, $\mathfrak{A} \models \Phi$. If $w \hat{R} u$ and $u \in V(\underline{P})$ then $(\mathfrak{A}, w) \models \Diamond \underline{P}$.*

PROOF. If wRu this is trivial. Otherwise for some ψ with $f(\psi) \in \Phi$, $w R_\psi u$. Let $\psi = \bar{\psi}(T, r, s)$; for some β mapping $\{v_t : t \in T\}$ into W, $(W, R) \models^* (\theta_{|r|} \,\&\, \theta)[\beta]$ and $\beta(v_r) = w$, $\beta(v_s) = u$. Let $w_i = \beta(v_{r \upharpoonright i})$ for $i \leq |r|$; so $w_0, \ldots, w_{|r|} = w$ is an R-chain. Trivially $(W, B) \models^* \theta_i[\beta]$ for $1 \leq |r|$; so by the construction of $\phi_{r \upharpoonright i}$, and since $\beta(v_s) \in V(\underline{P})$, we have $(\mathfrak{A}, w_i) \models \phi_{r \upharpoonright i}$. But $(\mathfrak{A}, w_0) \models f(\psi)$; so $(\mathfrak{A}, w_i) \models \phi^{r \upharpoonright i}$. Thus by induction $(\mathfrak{A}, w_{|r|}) \models \Diamond \underline{P}$, as claimed. Q.E.D.

[1] *Added in proof.* A few examples may help the reader understand the relation of ψ to $f(\psi)$; on the left we have a choice of T, r, s, and on the right $f(\psi(T, r, s))$:

$\{\langle\ \rangle = r = s\}$	$\underline{P} \supset \Diamond \underline{P}$
$\{\langle\ \rangle = r, \langle 0 \rangle, \langle 0, 0 \rangle = s\}$	$\Diamond^2 \underline{P} \supset \Diamond \underline{P}$
$\{\langle\ \rangle = s, \langle 0 \rangle = r\}$	$\underline{P} \supset \Box \Diamond \underline{P}$
$\{\langle\ \rangle, \langle 0 \rangle = s, \langle 1 \rangle = r\}$	$\Diamond \underline{P} \supset \Box \Diamond \underline{P}$

LEMMA 4. *Let $\mathfrak{A}$ be a structure for any modal language $\mathscr{L}$* where $\mathfrak{A} \models \Phi(\mathscr{L})$. *Let ϕ be any formula of $\mathscr{L}$, α any $\mathfrak{A}$-assignment. If wRu and $(\mathfrak{A}, u) \models \phi[\alpha]$ then $(\mathfrak{A}, w) \models \Diamond\phi[\alpha]$.*

PROOF. Let $\hat{V}(\underline{P}) = \{v:(\mathfrak{A}, v) \models \phi[\alpha]\}$; then $\mathfrak{B} = (W, R, \hat{V})$ is a structure for $\mathscr{L}'$ (where $(W, R) = \text{Frame}(\mathfrak{A})$). Furthermore $\mathfrak{B} \models \Phi$, since $\mathfrak{A} \models \Phi(\underline{P}/\phi)$. Because $u \in \hat{V}(\underline{P})$, by Lemma 3 $(\mathfrak{B}, w) \models \Diamond\underline{P}$. Suppose $(\mathfrak{B}, v) \models \underline{P}$ and wRv; then $(\mathfrak{A}, v) \models \phi[\alpha]$; so $(\mathfrak{A}, w) \models \Diamond\phi[\alpha]$, as claimed. Q.E.D.

LEMMA 5. *Let $\mathfrak{A}$ be a structure for $\mathscr{L}$. If $\mathfrak{A} \models \Phi(\mathscr{L})$ then $\hat{\mathfrak{A}} \models \Phi(\mathscr{L})$.*

PROOF. Let θ be a formula of $\mathscr{L}$, $\phi \in \Phi$, and α an $\mathfrak{A}$-assignment. Suppose $f(\bar{\psi}(T, r, s)) = \phi$. We shall show that $\hat{\mathfrak{A}} \models \phi(\underline{P}/\theta)[\alpha]$. Suppose that $w = w_0, \ldots, w_k$ is an $\hat{R}$-chain and $(\hat{\mathfrak{A}}, w_i) \models \phi^{r\restriction i}(\underline{P}/\theta)[\alpha]$ for all $i \leq k$; thus we associate with each $t \in T$ a $w_t \in W$ such that for $t \in T$, $|t| > 0$, we have $w_{t^-} R w_t$, where $w_i = w_{r\restriction i}$, and such that $(\hat{\mathfrak{A}}, w_t) \models \phi_t(\underline{P}/\theta)[\alpha]$, for all $t \in T$. This relies on the construction of ϕ_t and $\phi^{r\restriction i}$. But by Lemma 4, $(\mathfrak{A}, w_t) \models \phi_t(\underline{P}/\theta)[\alpha]$ for all $t \in T$. It suffices to show that $(\hat{\mathfrak{A}}, w_r) \models \Diamond\theta[\alpha]$. If not, $(\mathfrak{A}, w_r) \not\models \Diamond\theta[\alpha]$, since $R \subseteq \hat{R}$. Thus $(\mathfrak{A}, w_r) \not\models \phi^r(\underline{P}/\theta)[\alpha]$, since $\phi^r = (\phi_r \supset \Diamond\underline{P})$. But by Lemma 4, $(\mathfrak{A}, w_{r^-}) \not\models \phi^{r^-}(\underline{P}/\theta)[\alpha]$. Iterating this argument, we get

$$(\mathfrak{A}, w_0) \not\models \phi^{\langle\rangle}(\underline{P}/\theta)[\alpha];$$

since $\phi^{\langle\rangle} = \phi$, this is a contradiction. Q.E.D.

COROLLARY. *For all $n < \omega$, if $\mathfrak{A} \models \Phi(\mathscr{L})$ then $\mathfrak{A}^n \models \Phi(\mathscr{L})$.*

THEOREM. *If L is special then L is well-behaved.*

PROOF. Let Φ define L; $\Phi' = \{\Box^n\phi: \phi \in \Phi\}$ is a set of L-validities. *Claim*: Φ' makes L well-behaved. Let $\mathscr{L}$ be a modal language and $\mathfrak{A}$ a structure for $\mathscr{L}$ centered at $w \in W$; let $(\mathfrak{A}, w) \models \Phi'(\mathscr{L})$.

Since $\mathfrak{A}$ is centered at w, $\mathfrak{A} \models \Phi(\mathscr{L})$. For $\phi \in \text{Fml}(\mathscr{L})$, $u \in W$, $n \in \omega$ and α an $\mathfrak{A}$-assignment, we have

$$(\mathfrak{A}, u) \models \phi[\alpha] \quad \text{iff} \quad (\mathfrak{A}^n, u) \models \phi[\alpha] \quad \text{iff} \quad (\mathfrak{A}^*, u) \models \phi[\alpha].$$

Proof of this will suffice to show that $\mathfrak{A}^* =^* \mathfrak{A}$. We induce on the construction of ϕ. The only nontrivial case is where $\phi = \Box\psi$. It suffices to show that if $(\mathfrak{A}, u) \models \phi[\alpha]$ then $(\mathfrak{A}^n, u) \models \phi[\alpha]$ for all n. (If for all n, $(\mathfrak{A}^n, u) \models \phi[\alpha]$, then by the construction of R^*, $(\mathfrak{A}^*, u) \models \phi[\alpha]$.) Suppose $(\phi, u) \models \phi[\alpha]$. We argue by induction on n. For $n = 0$ there is nothing to prove. Suppose that $(\mathfrak{A}^{n+1}, u) \not\models \phi[\alpha]$. Then for some $u \in W$, $uR_{n+1}v$ and $(\mathfrak{A}^{n+1}, v) \not\models \psi[\alpha]$. By the induction hypothesis for ψ, $(\mathfrak{A}^n, v) \not\models \psi[\alpha]$. By the corollary to Lemma 5, $\mathfrak{A}^n \models \Phi(\mathscr{L})$. So by Lemma 4, $(\mathfrak{A}^n, u) \not\models \phi[\alpha]$. This proves the last claim. Q.E.D.

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