Kant vs. Legendre on *Symmetry*: Mirror Images in Philosophy and Mathematics

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Abstract. In 1768, Kant published a short essay in which he inquired into the possibility of determining the directionality of space. Kant's central argument invokes the strategy that if one were to demonstrate directionality, then the relational view of space that Leibniz propounded would be refuted. This paper has been considered a major turning point in Kant's philosophical development towards his critical philosophy of transcendental idealism. I demonstrate that in this study, Kant came very close to the modern concept of *symmetry*. His novel construction of incongruent counterpart (*inkongruentes Gegenstück*) contains elements essential to the modern notion of *symmetry*. However, Kant does not consider the incongruent counterparts, which he designates as 'Right' and 'Left', symmetric; rather, he holds the French encyclopaedist view that symmetry is a kind of balance. This study convinced Kant that the solution to the problem of the nature of space lies not in mathematics but in metaphysics. He was wrong in this conclusion, at least with respect to symmetry. The solution was found within the framework of mathematics, that is, solid geometry. In 1794, Legendre recast the traditional encyclopaedist concept of *symmetry* by calling a certain property of polyhedra symmetrical. The view of Kant is contrasted with that of Legendre by comparing their usages of mirror image as an aid for understanding. While in both cases mirror images are not considered illusions—perhaps for the first time in the history of mirror reflections—the differences are substantial, highlighting the limitation of Kant's position and the great potential of Legendre's new concept of symmetry.

1. Introduction: Kant's Incongruent Counterparts and His Use of Symmetry

In 1768, Immanuel Kant (1724–1804) published a short essay in which he inquired into the possibility of determining the directionality of space. In the essay, 'Concerning the ultimate ground of the differentiation of directions in space', Kant sought to undermine Leibniz's proposal for a new mathematical discipline, namely, *analysis situs*, and to demonstrate the validity of Newton's assertion concerning the nature of space as an absolute entity—an entity that can act but cannot be acted upon (Kant [1768]/1912; Walford and Meerbote 1992, pp. lxviii–lxx). Kant's central argument invokes the strategy that if one were to demonstrate the real existence of a fundamental, essential, and unanalysable spatial quality, such as directionality, in the absence of which certain phenomena would be either unintelligible or impossible, then the relational view of space that Leibniz propounded would be refuted

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(Walford and Meerbote 1992, p. lxix). This paper has been considered a major turning point in Kant's philosophical development towards his critical philosophy of transcendental idealism (Buroker 1981; Walford and Meerbote 1992, p. lxx; Rusnock and George 1995, p. 274).

In a crucial stage of the argument, Kant provides a constructive account of how a mirror image of a hand is formed:

> From all the points on its surface let perpendicular lines be extended to a plane surface set up opposite to it; and let these lines be extended the same distance behind the plane surface, as the points on the surface of the hand are in front of it; the ends of the lines, thus extended, constitute, when connected together, the surface of a corporeal form (Kant [1768]/1912, p. 398; Walford and Meerbote 1992, p. 370).

In the same passage, Kant names this constructed form the 'incongruent counterpart' of the original form: 'I shall call a body which is exactly equal and similar [*völlig gleich und ähnlich*] to another, but which cannot be enclosed in the same limits as that other, its *incongruent counterpart* [*inkongruentes Gegenstück*]' (Kant [1768]/1912, p. 398 italics in the original; Walford and Meerbote 1992, p. 370).[1] Thus, if the hand in question is a right hand, then its incongruent counterpart is a left hand. Kant comments that, 'the reflection of an object in a mirror rests upon exactly the same principle' (Kant [1768]/1912, p. 398; Walford and Meerbote 1992, p. 370).

For a modern reader, it appears somewhat surprising that Kant does not refer to this mirror image as a symmetry phenomenon—a bilateral, mirror-image symmetry to use the technical expression (e.g. Weyl 1952, pp. 4–5). The temptation is strong to cast Kant's analysis into modern terminology and to proceed by rendering the incongruent counterparts as elements that could be superposed through some symmetry transformation. One may feel justified in doing so by the fact that it is quite clear that Kant is groping for a new concept that would facilitate the analysis of directionality of space. Indeed, Kant refers twice to *symmetry* in this context:

> . . . the most common and clearest example is furnished by the limbs of the human body, which are symmetrically [*symmetrisch*] arranged relative to the vertical plane of the body. The right hand is similar and equal to the left hand . . . (Kant [1768]/1912, p. 398; Walford and Meerbote 1992, p. 370).

But proceeding in this way would have us using an analyst's category, while Kant is an actor in this story. Note that a few lines later Kant writes that

> in order to demonstrate the possibility of such a thing [that is, an incongruent counterpart], let a body be taken consisting, not of two halves which are symmetrically [*symmetrisch*] arranged relatively to a single intersecting plane but rather, say, a *human hand* (Kant [1768]/1912, p. 398 italics in the original; Walford and Meerbote 1992, p. 370).

Thus, it becomes clear that Kant holds a categorically different conception of *symmetry* from the conception which the historically insensitive modern reader may wish to impose on Kant's position. In terms of the actor's category, Kant does not conceive of his newly invented concept of incongruent counterpart as related to symmetry. If one were to adhere systematically and faithfully to Kant's own usage of the term *symmetry*, and furthermore keep strictly to the context and within the bounds of the discussion which Kant pursues, then one would have to admit that Kant's concept of *symmetry* conveys some sense of balance, of equality, but certainly not the modern conception that implies superposition via transformation. To be sure, incongruent counterparts would be symmetrical under certain transformations, but Kant severs this concept from symmetry. Notice further that Kant has no concern for any aesthetic aspect of symmetry—the concept he uses conveys only a spatial quality.

Consider now the fact that these two occurrences of the term *symmetry* are, to the best of my knowledge, the only occurrences of *symmetry* or *symmetrical* in the entire Kantian corpus in reference to spatial features. This fact lends credence to the claim that Kant was not developing an argument concerning the nature of space based on some symmetry principle. *Symmetry*, it should be underlined, is not mentioned, let alone discussed, in the famous sections, §§12 and 13, of the *Prolegomena* which Kant published in 1783, 15 years after the essay on directions in space. I therefore consider Carus's 1902 rendering of the celebrated phrase, *widersinnig gewundener Schnecken* (Kant [1783]/1920, p. 41), as 'two symmetric helices' (Beck 1950, p. 34) completely misleading. A faithful translation would be 'oppositely spiralled snails' (Hatfield 1997, p. 38). It is noteworthy that in his essay of 1768, Kant remarks that 'almost all snails, with the exception of perhaps, only three species, have shells which, when viewed from above, that is to say when their curvature is traced from the apex to the embouchure, coil from left to right' (Kant [1768]/1912, p. 396; Walford and Meerbote 1992, p. 368).[2] Hence in 1783, Kant was surely talking literally about 'snails' and not about the abstract notion of 'spirals.'[3] Carus is therefore no exception in imposing the modern conception of symmetry on texts, ancient and early modern alike, that were written before the introduction of the modern concept of *symmetry* into the scientific domain (Hon and Goldstein 2005a).

2. *Kant and* Symmetry *in the French Architectural Tradition*

There is strong evidence that Kant was using the concept of *symmetry* in the French architectural tradition proclaimed by Claude Perrault (1613–1688) in the latter half of the 17th century. In his translation of Vitruvius's *De architectura*, Perrault drew the following definition of symmetry: 'Symmetry [*Symmetrie*], in French, signifies only, a Relation of Parity and Equality [*parité & d'égalité*]' (Perrault 1674, pp. 39–41; Anon. 1692, pp. 29–31). Together with Bernard R. Goldstein, I have argued that this is the first explicit definition of symmetry which breaks away from the long-standing tradition of the Vitruvian conception which considers symmetry a specific kind of proportion, namely, the relation between the module and whole edifice (Hon and Goldstein 2005b). Kant is alluding to the French usage, and his term *gleich und ähnlich* corresponds to that in the *Encyclopédie*.

Kant holds the French encyclopaedist view that symmetry is a kind of a balance which expresses parity and equality. 'In architecture', so a standard French architectural definition at the turn of the 18th century goes, '*uniform symmetry* is that in which order rules the entirety (*pourtour*) in a single way, and *respective symmetry* is that in which opposite sides are like (*pareils*) each other' (Daviler 1691, p. 821 italics in the original). The former definition, *uniform symmetry*, relates to the old Vitruvian tradition, while the latter, *respective symmetry*, depicts the new conception that is concerned with similar elements placed equally across a discernable axis. In his *Essay on Taste*, which was written at the request of D'Alembert for the *Encyclopédie* and published in 1757, Charles-Louis de Secondat, Baron de Montesquieu (1689–1755), reflected on this new conception and introduced the productive metaphor of the balance.

> There is yet another consideration that pleads in favour of *symmetry*, and that is the desire, so natural to the mind, of seeing everything finished and brought to perfection. In all complex objects there must be a sort of counterballance, or equilibrium [*une espece de pondération ou de balancement*] between the various parts that terminate in one whole; and an edifice with one wing, or with one wing shorter than the other, would be as unfinished and imperfect a production as a body with only one arm, or with two of unequal length (Montesquieu 1757; Diderot *et al.* 1751–1776, Vol. 7 [1757], p. 764; Gerard 1759, p. 280).

Whereas previously, *symmetry* was presented as relation of *correspondence*, Montesquieu invokes the physical analogy of *balance*. The conception is that the two halves of a structure, set to the right and to the left sides of a discernible axis, not only correspond but also figuratively weigh the same, taking the axis as an abstract fulcrum of a formal balance. This image was very powerful and immediately gained currency so that by the latter half of the 18th century, symmetry was clearly considered balancing, that is, an equilibrium of equal weights, as it were, on the two sides of an axis.

Thus, no mirror image is involved in Kant's usage of *symmetry* in these crucial passages where he explores the directionality of space. In spite of his construction of the incongruent counterpart following the principles of mirror reflection, he does not link it to the idea of *symmetry* in the sense of its contemporary meaning, i.e., balance. The word *symmetry* is simply not put to use by Kant in this context, and it is therefore worth drawing attention to the possible source of the term he invents: 'incongruent counterpart'.

Although Kant does not explain the reasons why he chose the expression in German, *das inkongruente Gegenstück*, I surmise that he had in mind *correspondence* (or, indeed, counterpart) that was already in the tradition for understanding symmetry, and combined it with *incongruent*, where *congruent* in the plane had long been understood as allowing superposition.[4] Hence, *incongruent* would mean 'not superposable'. Thus, other than his terminology that, as far as I can tell, was not accepted by any of his successors, one may say that Kant had a version of the modern notion of *symmetry*. Since his successors did not notice Kant's insight, it is fair to say that he did not contribute to the introduction of *symmetry* into scientific discourse and, in fact, Kant had other issues in mind.

3. Kant's Move From Mathematics to Metaphysics

As indicated above, Kant gropes for a new concept, and a scientific concept at that—with no concern for any aesthetic aspect. But he has not connected the French architectural idea of *symmetry* (which was aesthetic) with his geometrical conception of an incongruent counterpart to form the modern idea of *symmetry*, that is, the identity relation via transformation.

Kant's study of 1768 on the directionality of space convinced him that the solution to the problem of the nature of space lies not in mathematics but in metaphysics. As Walford explains the argument

> Incongruent counterparts (such as left and right hands) show the real existence of the quality of directionality because, although equal in magnitude and similar in form, they cannot be contained within each other's spatial limits (except, of course, by being rotated through an extra dimension). Their congruency is prevented by their differing directionality. Directionality must, therefore, be a real quality of space. The Leibnizian account of congruency, which underlies the *analysis situs*, wholly fails to take account of this essential spatial quality (Walford and Meerbote 1992, pp. lxix–lxx).[5]

Focusing on the essential quality of space led Kant to abandon mathematics as a tool for probing conceptually the quantitative nature of space, and to concentrate on the metaphysical analysis of space as pure intuition. At stake were not the equalities of magnitude but rather the similarities of form. The *Prolegomena* of 1783 attests convincingly to this transition from mathematics to metaphysics.

In my view, advances in mathematics were required for obtaining the necessary linkage of quantity with form. Indeed, the emergence of the modern concept of *symmetry* dependent on mathematical insights, both in geometry and algebra. The modern concept of *symmetry* allows, with the appropriate transformation, precisely for this linkage of quantity and quality, that is, of magnitude and form. Kant's incongruent counterparts could then be brought together via a mathematically defined transformation. Indeed, as Weyl observed

> ... in [Kant's] opinion only transcendental idealism offers a solution for this riddle. No doubt the meaning of congruence is based on spatial intuition, but so is similarity. Kant seems to aim at some subtler point, but just this point is one which can be completely clarified by an analysis in terms of a group Γ and its invariant subgroups Δ.... The phenomenon about which Kant wonders can thus be most satisfactorily subsumed under general and abstract "concepts" (Weyl and Helmer [1927]/1949, p. 80).

It turns out that this transformation—in modern parlance, the symmetry groups—reflects an essential quality of the space in which the elements in question are embedded.

As a rule, one has to proceed cautiously when recasting the writings of historical figures in terms of modern concepts. I submit that in 1783 the modern concept of *symmetry* had not yet been formulated. At that time, the concept only conveyed the idea of balance but not the additional crucial element of transformation. Thus, analyses—historical and philosophical—which impute the modern concept of *symmetry* to historical actors before 1783 are anachronistic.

4. The Modern Concept of Symmetry

This result suggests that we need to determine the time, and the context, when the modern concept of *symmetry* emerged. Together with Bernard R. Goldstein, I claim that this juncture occurred in 1794 when Adrien-Marie Legendre (1752–1833), the celebrated French mathematician, put forward a new definition of symmetry in a textbook on the elements of geometry (Hon and Goldstein 2005a). Discussing the features of solid angles, Legendre proposed the following definition of *symmetric solid angles*:

> ... two equal solid angles which are formed (by the same plane angles) but in the inverse order will be called *angles equal by symmetry* [*angles égaux par symmétrie*], or simply *symmetric angles* (Legendre [1794]/1813, p. 155).

Having defined symmetric solid angles (solid angles being the essential elements of polyhedra), Legendre was in a position to define *symmetric polyhedra*. In Book VI, *Les polyèdres*, he writes

> Def. XVI. I will call two polyhedra *symmetric polyhedra* which, having a common base, are constructed similarly such that one is above the plane of this base and the other is below it, with the condition that the summits of the corresponding solid angles are located at equal distances from the plane of the base, on the same line perpendicular to this plane (Legendre [1794]/1817, p. 163, Book VI, Definition XVI).

And adds

> For example, if the line ST is perpendicular to the plane ABC, and at point O, where it meets this plane, it is divided into two equal parts, then the two pyramids SABC and TABC, which have base ACB in common, will be two symmetric polyhedra [*polyèdres symétriques*] (Legendre [1794]/1817, p. 163) (see Figure 1).

The figure that Legendre draws illustrates two symmetric polyhedra which share the same base. The faces that form the solid angles at S and T can be identified by the sides of triangle ABC: in the pyramid with apex T, the order of these sides is ABC, whereas in the pyramid with apex S, the order is ACB, or vice versa. But it is not evident in this passage how we can determine these orders.

This is the first instance of the modern concept of symmetry applied in a scientific domain. With this definition in hand, Legendre made a discovery: a convex polyhedron has, in this newly minted term, a *symmetric* polyhedron with congruent faces that is equal to it in volume and yet they cannot be superposed.

The notion that elements organized in a certain order that repeats itself in reverse comes to fruition in Legendre's conception of mutually symmetric polyhedra. Legendre thus recast the meaning of a word that previously had an entirely different connotation. In recognising the importance of the inverse order of the faces of a polyhedron when they are equal and similar to those of a given polyhedron, he searched for a new and distinctive term for it. In a certain sense, Legendre's choice of the term *symmetry* was arbitrary: if he had decided to use a different word, we—moderns—might have used it; for example,

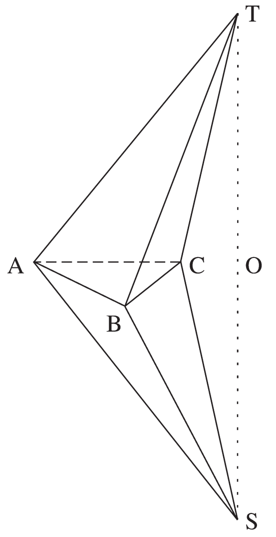

Fig. 1. Legendre's figure illustrating symmetric polyhedra (redrawn after Legendre [1794]/1813, Plate 9, fig. 202).

he could have chosen the Kantian *incongruent counterparts*. As it happened, Legendre, in a moment of inspiration, chose *symmetry* for something that had real possibilities. Since he did not discuss the contents of Book X of Euclid's *Elements*, where *symmetry* means *commensurability*, this term was free for a new usage.

5. *Mirror Image: an Illusion or an Aid for Understanding*

Not surprisingly, Legendre's innovative use of the term *symmetry* in the analysis of the geometrical relations of polyhedra led him to add several explanatory notes in the appendix to his *Éléments*. A decisive note for clarifying his choice of the term is the following:

> Note VII. *On symmetric polyhedra* .... One can... get a very correct idea [*une idée très-juste*] of the set-up for these two solids, by considering one of the two as the image of the other formed in a plane mirror, which takes the place of the plane of which we were just speaking (Legendre [1794]/1817, p. 305: Note VII).

A plane mirror is used here as an analogy, like Montesquieu's appeal to the *balance*. In the case of the balance, neither of the two sides with respect to the vertical axis is privileged; similarly, with a mirror, a reflective plane plays a role corresponding to that of the vertical axis in the balance, i.e., in both cases the relationship between the two sides is mutual. Hence, the two, three-dimensional mathematical objects, are mirror images of each other such that one object does not have any ontological priority over the other—the two objects have in fact the same status, and what is more, they are similar and equal. As we shall see, in Legendre's conception of symmetry, the 'reflected' object stands on the same geometrical footing as the original object. Referring to the image as an illusion is no longer appropriate, for each of the two solids is the mirror image of the other—to repeat, neither one is privileged and the relationship is mutual. This understanding was new in the latter half of the 18th century; it had never been invoked in the case of mirror images.

Consider the view of Ptolemy that the image in the mirror is an illusion. Ptolemy refers in his *Optics* to the 'illusion' of right and left being interchanged in an image as seen in a plane mirror. He mentions this phenomenon without, however, introducing a technical term for it (indeed, there is no occurrence of the term *symmetria* in the *Optics*):

> This is what happens in [the perception] of position when we look into plane mirrors and the visible object [i.e., the viewer himself] faces the mirror directly. In that case our sight shows us our [right-hand and left-hand] sides in the way that is natural for it to show objects viewed directly: i.e., what is seen by right-hand rays appears to the right, while what is seen by left-hand rays appears to the left. Our mind, however, shows us right as left and left as right, because objects that actually face us are so disposed that their right is opposite to our left, while their left is opposite to our right. And this is why, when we move one of our hands [in front of a mirror] our sight tells us that the hand that moves [in the mirror] is the one facing it [i.e., right to right or left to left], while our mind tells us the opposite (Smith 1996, p. 126; Ptolemy, *Optics*, Bk. II §138).

Observe that for Ptolemy, mirror image is an optical illusion: he does not discuss this phenomenon in nature, and he certainly does not turn it into a methodological principle.

This view is still found in Newton's *Opticks*, a millennium and a half later. Newton (1643–1727) clearly considers the image in the mirror as a kind of illusion, in the same way that Ptolemy does: one imagines that the object is in a certain place, but it is really elsewhere.

> Ax. VIII. *An Object seen by Reflexion or Refraction, appears in that place from whence the Rays after their last Reflexion or Refraction diverge in falling on the Spectator's Eye*. If the Object A ... be seen by Reflexion of the Looking-glass *mn*, it shall appear, not in its proper place A, but behind the Glass at *a*,.... For these Rays do make the same Picture in the bottom of the Eyes as if they had come from the Object really placed at *a* without the Interposition of the Looking-glass; ... (Newton [1730]/1952, p. 18 italics in the original).

Note the key expressions in Newton's analysis: *as if* the rays had come from the object which is *really* placed in front of the mirror. Clearly, Newton distinguishes sharply between the real object and the reflected image, assigning different ontological status to the two visualized elements.

Denis Diderot (1713–1784) maintains the same view. What does a blind person understand by a mirror? Diderot engages this question in his 'Letter on the Blind'. He reports that a blind person, an acquaintance of his, responded with the following words:

> A device that puts things into relief at a distance, provided they are in the right relative position. It is like a hand that can feel an object without touching it.... Not comprehending why he was unable to feel the "relief copy" which according to him was made by a mirror, he exclaimed, "Here is a device that brings two senses into conflict; a more perfect device would reconcile the two—except that, even so, the objects would be no more real. Perhaps a still more perfect and less-deceiving

> device would make them disappear and would advise us of the error" (Diderot [1749]/1963, pp. 4–5; Morgan 1977, pp. 33–34 slightly modified).

For Diderot, an image in a mirror is not 'real', for it cannot be touched. Thus, what the blind man considers a deception, that the mirror brings the senses of sight and touch into conflict, is for Diderot a straightforward, explicable illusion. Unlike the blind man, Diderot can 'see' that the mirror does not throw into relief the images it 'creates'.

Kant's construction of the incongruent counterpart on the principle of mirror reflection stands in contrast to the age-old understanding of the mirror image as an illusion. Recall that Kant demonstrates how to construct the incongruent counterpart of a given body: a body which is exactly equal and similar to another, but which cannot be enclosed in the same limits as that of the other, is the *incongruent counterpart* of the original body. Kant explicitly says that a mirror image rests exactly upon this principle. Thus, if the hand in question is a right hand, then its incongruent counterpart is a left hand, and both are real objects. *Prima facie* Kant's usage resembles that of Legendre, and especially so since the incongruent counterparts are similar and equal, the very feature which Legendre associates with symmetric polyhedra. Nevertheless, there appears to be no connection between these two innovative usages of plane mirrors in the second half of the 18th century.

6. Kant vs. Legendre

In the first place, Kant's problem was different from that of Legendre. For Kant, the issue was the possible determination of the directionality of space, thereby vindicating Newton's absolutist position. Kant, in other words, was interested in the geometrical character of physical space. In contrast, Legendre plays the role of a pure geometer who has no interest in the physics of the problem, and so his treatment is the first instance in the history of pure geometry in which a mirror is used as an aid to clarify geometrical relations—in this case, the relation between similar and equal polyhedra. Thus, Legendre integrates *symmetry* into a discussion of geometry, whereas Kant only alludes to geometrical arguments without proof. Kant, for example, claims that two spherical triangles can be exactly equal and similar, and yet cannot be made to coincide, but does not attempt a formal proof of the claim. By comparison, Legendre investigates the properties of spherical triangles. In a scholium, where he demonstrates how two spherical triangles may be equal, Legendre writes

> The equality of these two [spherical] triangles is not, however, absolute or of superposition, for it would be impossible to place one on the other exactly unless they were isosceles. The equality in question is the equality that we have already called equality by symmetry, and for this reason we call these triangles … *symmetric triangles* (Legendre [1794]/1817, pp. 214–215 italics in the original).

Here Legendre finds a good opportunity to extend the newly minted term *symmetry* to triangles on the same sphere or on equal spheres. He then explores the properties of symmetric spherical triangles. Triangles drawn on a surface of a sphere may have properties different from those of plane triangles, and Legendre demonstrates here that his new notion of *symmetry* is applicable and, indeed, helpful in elucidating some basic properties of spherical triangles. The equality that these triangles exhibit is precisely the *equality by symmetry* which he defined in his study of solid angles earlier in Book V and which he applied in Book VI. It is noteworthy that Legendre seems to be unaware that Kant had already drawn attention to this case of spherical triangles in 1768, almost 30 years earlier: 'a spherical triangle can be exactly equal and similar [*völlig gleich und ähnlich*] to another such triangle, and yet still not coincide with it' (Kant [1768]/1912, p. 398; Walford and Meerbote 1992, p. 370). Kant's claim is puzzling, for he seems to be aware of an important geometrical discovery, and yet he neither demonstrates it formally nor provides the source for his knowledge of it. Kant, however, does not see a relationship between spherical polygons and convex polyhedra which is central to Legendre's deep insight into solid geometry. Moreover, Kant does not connect this property of spherical triangles to *symmetry*.

Legendre's symmetric solids are really where they appear to be. His purpose is to characterize symmetric solids for which there is a mutual relationship. No mirror is referred to in the text of Book V or VI of his *Elements*; the mirror is only introduced in Note VII for explanatory purposes. Furthermore, there is no mirror in any of his figures that illustrate symmetric solids. The mirror is mentioned, therefore, simply as an aid to understanding. It should be stressed that Legendre invokes the case of images in a mirror as if one of the objects were seen in a mirror, but either object works for symmetric solids; this is not the case for an image in a mirror which does not have the same status as the object. It is important to reiterate that the so-called reflected solid is not an image but a real polyhedron which cannot be superposed onto the original solid body.

A second important difference between Kant and Legendre is the fact that Legendre does not distinguish between a 'Right' polyhedron and a 'Left' polyhedron while maintaining that the two polyhedra are symmetric. Once again, we see that Legendre, unlike Kant, is not interested in the directionality of physical space. Indeed, in his treatise, Legendre does not address the physics of space. Kant, by contrast, does not render the incongruent counterparts, which he designates 'Right' and 'Left', as symmetric. We have already seen that he holds the French encyclopaedist view that *symmetry* is a kind of a *balance*.

Moreover, Kant appeals to a 'feeling' that the right and left sides of the human body are different, an argument that is completely alien to Legendre's approach. Indeed, Kant seems to say that the distinction between right and left is intrinsic to nature where *right* has the advantage. In his essay of 1768 on the ultimate ground for the differentiation of directions in space, Kant writes

> Since the distinct feeling [*Gefühl*] of the right and the left side is of such great necessity for judging directions, nature has established an immediate connection between this feeling and the mechanical organisation [*mechanische Einrichtung*] of the human body.... The right side ... enjoys an indisputable advantage over the other in respect of skill.... Hence, all the peoples of the world are right-handed (apart from a few exceptions which ... do not upset the universality of the regular nat-

> ural order [*natürlichen Ordnung*]. . . . And thus it is that the two sides of the body are, in spite of their great external similarity, sufficiently distinguished from each other by a clear feeling . . . . We are trying to demonstrate . . . [that the] ground of the complete determination of a corporeal form does not depend simply on the relation and positions of its parts to each other; it also depends on the reference of that physical form to universal absolute space [*allgemeinen absoluten Raum*]. . . . The thread of a screw which winds round its pin from left to right will never fit a nut of which the thread runs from right to left (Kant [1768]/1912, pp. 396–398; Walford and Meerbote 1992, p. 369).

In his later work, when Kant alludes to the directionality of helices, he again appeals to the example in nature that he had already invoked in his essay of 1768, 'oppositely spiralled snails' (Kant [1783]/1920, p. 41; see n. 2). In other words, he is aware of the difference in directionality in both natural and man-made objects. Given also the fact that Kant is aware of the different directionality of spherical triangles, he appears then to have associated phenomena in three distinct domains, namely, geometry, natural objects, and man-made devices, that display a pattern, what he calls *incongruent counterparts*.[6] Kant's notion of *incongruent counterparts* may therefore be considered a precursor of the modern use of the term *symmetry*.

However, it seems that Kant's preoccupation with his search for *inner* characteristics to demonstrate absolute space interfered with his recognition of the significance of what he had discovered about *outer* characteristics. According to Kant, directionality should be perceived as an *inner* characteristic of the object. The *inner* feature is part of the argument to prove the existence of absolute space because such a feature is not dependent on any external relation. Thus, for Kant the difference between similar and equal objects which cannot be superposed is of an *inner* nature:

> . . . the shape of the one body may be perfectly similar to the shape of the other, and the magnitudes of their extensions may be exactly equal, and yet there may remain an *inner* difference [*ein* innerer *Unterschied*] between the two, this difference consisting in the fact, namely, that the surface which encloses the one cannot possibly enclose the other (Kant [1768]/1912, pp. 398–399; Walford and Meerbote 1992, pp. 370–371 italics added).[7]

In Kant's argument, it is the absolute space that provides in the first place this *inner ground* [*innere Grund*] that makes comparing the objects possible at all:

> it is only in virtue of absolute and original space [*absoluten und ursprünglichen Raum*] that the relation of physical [*körperlicher*] things to each other is possible. . . . our considerations make the following point clear: absolute space is not an object of *outer* sensations [äußeren *Empfindung*]; it is rather a fundamental concept which first of all makes possible all such outer sensation (Kant [1768]/1912, p. 399; Walford and Meerbote 1992, p. 371 italics added).

Nevertheless, Kant's construction of incongruent counterparts has an *outer* characteristic, the construction being dependent on a continuous external comparison of the mutual relations of the respective elements that comprise the two objects. It seems then that Kant has found an *outer* characteristic and then tried to use it as if it were an *inner* one. Kant's appeal to a 'feeling', namely, that the right and left sides of the human body are different, is part of his attempt to demonstrate the existence of *inner* characteristics of the directionality of absolute space. To be sure, 'feeling' is *inner*—a characteristic of a single body—and

Kant seems to suggest that this is the case with spherical triangles, screws, snails, and right and left hands. These cases, however, are all exemplars of *outer* relations, for they depend on the mutual relations of two bodies.[8]

Having considered this distinction between *inner* and *outer* characteristic of spatial directionality, I note that Legendre's notion of *symmetry* is properly an *outer* characteristic which, for that reason, would not have appealed to Kant even if he had been aware of this concept. Legendre, however, was oblivious to the issue of the directionality of space and thus did not characterize symmetry as an *outer* feature. All in all, it seems that these two cases are distinct, even though Legendre and Kant both proposed the analogical use of mirrors in a scientific context, for they had different objectives in mind.

7. *Conclusion*

Legendre says that if we considered the common plane of two symmetric solids a mirror, the result would be that one shape is the image of the other. But in the case of symmetric solids, neither one is an 'image', since both are real. We speak elliptically when we say that one solid is a mirror image of the other, for there is no mirror and no image in a mirror. We mean that it is *as if* there were a mirror that produced an image, although we are talking about real objects. Symmetric objects retain a special relation. Such objects can be transformed from one into the other by a certain operation. In the case of symmetric polyhedra, one polyhedron can be transformed into the other by a mirror reflection. The reason for this lies, of course, in the fact that the two symmetric polyhedra have faces that are similar and equal but, as we have seen, they cannot be superposed by the very definition which Legendre puts in place; in fact, it is due to the inverse order of the plane angles that form the solid angles.

I have emphasized that Kant did not consider the incongruent counterparts to be symmetric, even though they are produced by a mirror reflection. I thus claim that with Legendre for the first time a mirror image becomes 'real', as the mirror 'links' the two symmetric polyhedra. There appears to be no precedent for this claim. Admittedly, Legendre is only concerned with mathematical-geometrical objects and not with material-physical objects. But the idea is novel, ready for applications in mathematics and physics.

With his concern for the possible directionality of space, Kant suggests making this distinction, hinting at the principle of ordering without presenting it in mathematical terms: 'The distinctive characteristic in question consists in the particular direction in which the order of the parts is turned' (Kant [1768]/1912, p. 396; Walford and Meerbote 1992, p. 368). Kant, however, does not apply his rule to polyhedral angles. He appears to have had many useful insights but, on occasion, he is truly 'undisciplined' when it comes to discussions that involve matters other than philosophy. He moves quickly from mathematics to physics to psychology to invertebrate zoology, etc. This is probably one reason for his minor impact (if any) on discussions of *symmetry*. For example, in the passage on ordering, he does not work in a mathematical tradition, and it is likely that for this reason his analysis was not cited by mathematicians (such as Legendre). And he makes no attempt to describe the difference between a right- and a left-turning screw mathematically. Kant's goal of proving absolute space was quite different from the specific geometrical issue addressed by Legendre and may have led him to base his argument on examples drawn from a variety of disciplines.

This may be an historical irony. In 1768, Kant expressed the idea of the principle of ordering, i.e., the order and its inverse, but did not apply it in a formal, mathematical way. In 1794, a quarter of a century later, Legendre put to use the idea of the inverse order in a geometrical context and indeed gave it a name: *symmetry*. But he did not ground it formally, that is, Legendre did not offer a formal technique for designating the order and its inverse. It remains a puzzle how Kant came to know of the directionality in the ordering of the sides of spherical triangles, and it is equally unclear why Legendre did not address in detail the principle of ordering which lies at the core of his revolutionary use of the term *symmetry*.

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NOTES

1. *Similarity* is a qualitative notion signifying likeness in form, shape, and structure. *Equality* is a quantitative notion signifying equivalence in magnitude where magnitude may relate to the length of a line, the size of an angle, and the area of a figure. See, e.g. Kant ([1768]/1912, p. 398); Walford and Meerbote 1992, p. 370: 'It is apparent from the ordinary example of the two hands that the figure [*Figur*: 'form', 'shape' or 'structure'] of one body may be perfectly similar to the figure of the other, and the magnitudes of their extensions may be exactly equal....' Hence, Kant distinguished between form and magnitude.
2. The specific information about snails in Kant and in the *Encyclopédie* (see n. 3) may need to be interpreted by experts in descriptions of such animals but here the point of interest is that both sources state that some snails coil to the right and some to the left.
3. Cf. Diderot *et al.* 1751–1776, Vol. 4 [1754], p. 185, *Coquilles de terre* (*Terrestrial shellfish* [land snails]): 'By holding snails in such a way that the apex is up, the mouth down, and the opening in front, one sees that in most [instances] the cavity turns about the core from right to left, but in some from left to right'. In other words, this property of snails was well known at the time when Kant wrote his essay.
4. See, e.g. D'Alembert 1767, vol. 4, pp. 165–166: 'Superposition ... consists in imagining one figure transported onto another and in concluding, from the assumed equality of certain parts of the two figures, the coincidence of these parts, respectively, and [in concluding] from their coincidence the coincidence of the rest [of the two figures]: from which perfect equality and similarity of the entire figures result. This way of demonstration has then the advantage not only of rendering the truths evident, but of being the most rigorous and simplest possible; in a word, of satisfying the mind by speaking to the eyes'. For *congruence* in French as a term in geometry, see Diderot *et al.* 1751–1776, Vol. 3 [1753], p. 869: 'Congruence. Equality and similarity of two things. For example, two triangles that are similar and equal are congruent'. Cf. the entry *Géométrie* in Diderot *et al*. 1751–1776, Vol. 7 [1757], p. 634: 'By superposition I understand here not only the application of one figure on another, but that of a part of a figure on another part of the same figure, in order to compare them with each other; and this aforementioned way of applying the principle of superposition is an extremely simple usage in the elements of geometry. See *Congruence*'.
5. The distinction between magnitude and form lies at the root of the problem of superposition in three dimensions and, as we have seen, Kant was aware of these two distinct conceptions. Cf. Falkenburg (2001).
6. This discovery is not a minor achievement. D'Alembert's entries in the *Encyclopédie* for 'screw' (*vis*) and for 'spiral' (*spirale*) do not mention the right/left distinction that Kant describes. See Diderot *et al.* 1751–1776, Vol. 15 [1765], p. 474: *Spirale*; and Vol. 17 [1765], p. 331: *Vis*. This suggests that the significance of the association of phenomena which exhibit a pattern of directionality was not widely recognized, if it was recognized at all.
7. On the origin of the distinction, inner/outer, in Leibniz's *analysis situs*, see Rusnock and George (1995, pp. 261–262).
8. Rusnock and George (1995, p. 265) suggest that Kant 'seems to have wanted it both ways'.