Visual Geometry
Author(s): James Hopkins
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VISUAL GEOMETRY

I

KANT thought Euclid's geometry true of everything spatially intuitable. This implied that only Euclid's geometry—Euclidean figures or a Euclidean space—could be seen, imagined, or visualized. To many, including modern philosophers, this has seemed true. Thus Frege:

Empirical propositions hold good of what is physically or psychologically actual, the truths of [Euclidean] geometry govern all that is spatially intuitable, whether actual or product of our fancy. The wildest visions of delirium, the boldest inventions of legend or poetry ... all these remain, so long as they remain intuitable, still subject to the axioms of geometry. Conceptual thought can after a fashion shake off that yoke, when it assumes, say, a space . . . of positive curvature. To study such conceptions is not useless by any means; but it is to leave the ground of intuition entirely behind. If we do make use of intuition even here as an aid, it is still the same old intuition of Euclidean space, the only space of which we can have any picture [*Foundations*, p. 20].[1]

Bennett:

If we restricted ourselves to what could be "imagined" or seen at a glance, then perhaps we should be bound to regard space as Euclidean . . . it is not clear how we could see at a glance that two straight lines intersect twice: it seems that if both intersections are seen at once, then at least one of the lines must look curved [*Kant's Analytic*, p. 31].[2]

And Strawson:

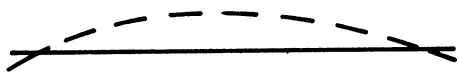
Consider the proposition that not more than one straight line can be drawn between two points. The natural way to satisfy ourselves of the

[1] Frege, *The Foundations of Arithmetic*, trans. by Austin (Oxford, 1950).

[2] Jonathan Bennett, *Kant's Analytic* (Cambridge, 1967). Bennett's position is more complex than might be inferred from the passage quoted. He is opposing "that preoccupation with the visual which has weakened and narrowed epistemology for centuries," and he says of the passage, "I am not sure this is right, perhaps because I am not sure what I mean by 'must look curved.' "

truth of this axiom of phenomenal geometry is to consider an actual or imagined figure. When we do this it becomes evident that we cannot, either in the imagination or on paper, give ourselves a picture such that we are prepared to say of it both that it shows two distinct straight lines, and that it shows these lines as drawn between the same two points [*The Bounds of Sense*, p. 283].[3]

Surely what Bennett and Strawson say here is true. We cannot see or picture two definitely straight lines between two points. Given two points we can picture one definitely straight line between them; but any other we picture will be curved. For example:



So it seems we can form the Euclidean but not the non-Euclidean picture. Similarly in other cases: we picture triangles of the same shape but different sizes, whose angles equal two right angles. Such are Euclid's, and we can imagine no others. So our pictures do suggest, as Frege believed, that we see, imagine, or picture anything whatever as Euclidean—as spatially disposed, and hence geometrically describable, in no other than Euclid's terms.

This belief is part of the content of the Kantian theory that the form of outer sense is Euclidean. Also it was a source of Kant's conviction that Euclid's propositions were known true a priori. Kant assumed that geometric proof required construction on a figure. He thought the proved propositions synthetic because this construction was a synthesis to be contrasted with the analysis of concepts. He thought them known a priori because the construction was not taken from experience. And the construction, the picture, inevitably yielded Euclidean results.

[3] P. F. Strawson, *The Bounds of Sense* (London, 1966). All quotations from Strawson are from this book, and only their page number is cited in the text. This view illustrated by Strawson and Bennett has been surprisingly common. See Ewing's *A Short Commentary to Kant's Critique of Pure Reason* (London, 1938), p. 45, for his statement and that from Johnson's *Logic*; also the text and notes to "Empiricism and the Geometry of Visual Space" in Grunbaum, *Philosophical Problems of Space and Time* (New York, 1963), for references to Carnap and others. W. & M. Kneale seem to discuss it in *The Development of Logic* (Oxford, 1962), p. 385.

[M]athematical knowledge . . . is knowledge gained by reason from the construction of concepts . . . I construct a triangle by representing the object which corresponds to the concept either in the imagination alone, in pure intuition, or in accordance therewith also on paper, in empirical intuition, in both cases completely *a priori*, without having borrowed the pattern from any experience [*Critique*, Kemp-Smith translation, p. 577].[4]

Frege knew of modern developments in geometry and had more sophisticated reasons for regarding Euclidean propositions as synthetic. Yet he seems to have thought them known a priori, solely because they alone could be intuited. "In calling the truths of geometry synthetic and a priori [Kant] revealed their true nature."[5]

Now it is commonplace that Kant's beliefs about geometry have been superseded. The refinement of geometry as an abstract science has made clear that construction on a figure has no such role in proof as Kant supposed. The discovery of non-Euclidean geometries has been taken to show that the truth or falsity of Euclid's description of space is an empirical matter. And it is widely accepted that the successful use of a non-Euclidean geometry in Einstein's theory of relativity—in which, for example, there may be two straight lines (two paths as short as, or shorter than, any other) between two points—has established that Euclidean geometry is as a matter of fact false of physical space.

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[4] Kant, *Critique of Pure Reason*, trans. by Kemp-Smith (London, 1963). Compare Mill, *System of Logic* (London, 1843), II, v, 5: "The foundations of geometry would therefore be laid in direct experience, even if the experiments (which in this case consist merely in attentive contemplation) were practiced solely upon what we call our ideas, that is, upon the diagrams in our minds."

[5] Frege, *Foundations*, pp. 101 f. According to Reichenbach the shadow of this view remains.

"The relativity of geometry has been used by neo-Kantians as a back door through which the *a priorism* of Euclidean geometry was introduced into Einstein's theory: if it is always possible to select a Euclidean geometry for the description of the universe, then the Kantian insists that it is this description which should be used, because Euclidean geometry, for a Kantian, is the only one that can be visualized" (*A. Einstein, Philosopher-Scientist*, ed. by Schilpp [New York, 1959]).

On the relativity of geometry see below, pp. 27-28; the argument of the paper shows the irrelevance of this Kantian insistence.

Euclidean and non-Euclidean geometries contain inconsistent statements about straight lines. So on a consistent interpretation of "straight line" only one geometry can be true. On interpretations which are common, plausible, and scientifically useful, the geometry of space according to Einstein's theory is not in general Euclidean. Einstein writes:

Euclidean geometry does not hold even to a first approximation in the gravitational field, if we wish to take one and the same rod, independently of its place and orientation, as a realization of the same interval.[6]

And Barker describes the situation as follows:

Suppose a closed three-sided figure is laid out, its sides being determined by light rays, or by paths along which measuring rods need be laid down the fewest times, or by paths along which stretched strings lie. According to the theory of relativity we must predict that in the presence of a gravitational field the sum of the angles of this figure will be greater than two right angles. We must also predict that between any two separate points there will, in the presence of gravitational fields, be more than one path divisible into overlapping sub-segments along which a measuring rod need be laid down a minimum number of times to get from end point to end point, and so forth.

Many . . . would say that the theory of relativity proves space to be Riemannian rather than Euclidean in its general form. Einstein himself is the outstanding representative of this viewpoint, which he expressed in a more general form in his often-quoted dictum "As far as the laws of mathematics refer to reality they are not certain; as far as they are certain they do not refer to reality."[7]

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[6] Einstein, *The Principle of Relativity* (London, 1923), p. 161.

[7] S. F. Barker, *Encyclopedia of Philosophy* (New York, 1968), III, 288. I *think* there are not multiple paths between every pair of points, but only certain pairs. Also, Barker is describing only one kind of field. In others the metrical situation is neither so definite nor so direct an extension of everyday technique. According to Reichenbach there are gravitational fields in which the geometry given by light rays differs from that given by rods; and in some fields in which there is not a unique light path between points, the metrical situation is so indeterminate it hardly seems useful to speak of lines in an ordinary sense at all. (See *The Philosophy of Space and Time* [New York, 1958], ch. 27; all textual references to Reichenbach are to this book, and only their page number is given.)

But here, surely, is a problem. It seems that science has given reason for believing that Euclidean geometry is false, that physical space may most accurately be described by a non-Euclidean geometry. Yet examples lead us to suppose that the only space we can imagine, picture, or visualize, is one described by Euclidean geometry. But the space it seems we must picture as Euclidean is the same space as that which, on scientific grounds, is judged non-Euclidean. And why, one might ask, can we not picture our space as science gives reason to believe it is? How are we constrained to see, imagine, or visualize it in terms of a theory inconsistent with what we might believe true of it on scientific grounds?

II

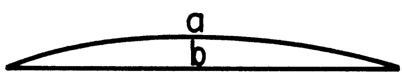
It may be thought relevant that there are familiar ways of representing a non-Euclidean space. Popular scientists and mathematicians sometimes draw gently curving arcs, which may intersect twice, to represent non-Euclidean straight lines. Again, the surface of a sphere provides a model for a Riemannian (non-Euclidean) space. An arc of a great circle is the shortest path between two points on the surface of a sphere; and many propositions about lines and figures in a Riemannian space hold for great circles and spherical figures. Thus there may be two great circle paths between (antipodal) points, the sum of the angles of a closed figure bounded by three great circles is more than two right angles, and so forth.

Diagrams and models of this kind are often invoked in connection with the problems of picturing non-Euclidean space. But clearly they cannot help us picture space as non-Euclidean. For arcs on paper and great circles on a sphere both are, and are seen or pictured as, curved lines. So in using such a diagram or model, we picture curved lines, but not straight lines, intersecting twice. And it gets us no further to try to picture space as *in accord with* the diagram, or *on the model* of the sphere. In failing to picture distinct straight lines intersecting twice, we fail to imagine straight lines with the relevant characteristics of the diagram or model. So we thereby fail to picture in accord with the diagram or on the model of a sphere.

There is a deeper and more sophisticated approach to visualizing the non-Euclidean. Philosophers and mathematicians, among them Reichenbach,[8] describe visibly non-Euclidean worlds. Their descriptions are meant to enable us to form imaginative pictures of the worlds.

Since in either Euclidean or non-Euclidean geometry a straight line is the shortest path between two points, it is clear that which lines in a given manifold are straight will be determined by measurement. So it is possible, for example, to describe a world in which we see two paths between two points and find both to be straight, say by measuring with rods rigid by every test to show both equal and any alternative longer.

With descriptions of this kind in mind, persons often claim to form non-Euclidean pictures. But on investigation it seems we cannot really do so. Consider, for example, lines such as *a* and *b*.



We have an interpretation, consistent with Euclid, of how *a* and *b* look. On the face of it, and without any special story in mind, we see or picture *a* as curved, *b* as (approximately) straight. Now suppose we try to imagine a world in which both are straight—that is, equal and shorter than any alternative between their intersections.

We can, for example, picture *a* and *b* as being measured to give this result. But this is not yet imagining them straight. For it is no different from imagining rods to grow in measuring *a* or shrink in measuring *b* to give the result. So imagining *a* equal to *b* is not yet distinguished from imagining *a* curved but measured equal to *b* by unstable instruments. As regards the length or straightness of *a* and *b* we have imagined nothing new. Our picture, our way of seeing their length or straightness, has not changed. So no different description is justified.

[8] See Reichenbach, esp. chs. 9, 10, 11, 13. He does not discuss the problem mentioned above, although sometimes he seems close (pp. 47, 91) and provides material for its solution.

It adds nothing to *speak* of the rods, or the unit of measure, as staying the same in measuring *a* and *b* to show them equal. What is in question is whether we actually picture this. Even if we add other pictures, and imagine rods rigid by other tests in other contexts, we are still required to picture them rigid in measuring *a* and *b* equal. As long as our picture of *a* and *b* does not change, we cannot do so.

So it seems no matter what story we tell, or what additional pictures we form, we do not picture anything different from that with which we began—namely, *a* curved and *b* straight. So we do not picture a situation different from the Euclidean. Thus even by means of this more sophisticated approach, we do not succeed in picturing a non-Euclidean situation.

A comparison may make this clearer. There are cases in which two ways of seeing, between which we can change, are available. We can picture other lines on paper—a duck/rabbit, say—first one way (as a duck), then another (as a rabbit). Here the way of seeing does change, and there is an experience of change of sight. The change can be induced by giving descriptions or by interpolating other pictures. With our way of seeing geometrically it is not like this. We cannot first see *a* as longer than *b*, then see them as equal. Here no change occurs and no pictures or descriptions induce one.

Reichenbach calls such a change in the way of picturing lengths as would constitute non-Euclidean picturing an adjustment in congruence or an emancipation from Euclidean congruence. Thus he says in a parallel case that during the adjustment "one can forget that from the viewpoint of Euclidean geometry these distances are different in length" (p. 56). But we cannot forget that *a* and *b* are different in length and see them as equal. This shows, in Reichenbach's terms, that no adjustment or emancipation from Euclidean congruence takes place. And Reichenbach says "so long as we cannot emancipate ourselves from Euclidean congruence . . . non-Euclidean relations can only be mapped on the visualized Euclidean space" (p. 57). It follows that our visualized space remains Euclidean.

So we are left with the difficulty of picturing space as other than Euclidean. This may explain what Frege meant, when he said of the study of non-Euclidean geometry:

If we do make use of intuition even here, it is still the same old intuition of Euclidean space, the only space of which we can have any picture

for, as he continues,

only here the intuition is not taken at its face value, but as symbolic of something else; for example we call something straight or plane, which we actually intuit as curved.

III

This difficulty, among others, is sharply presented by the final section of *The Bounds of Sense.* Strawson there espouses a carefully qualified analogue of Kant's theory of geometry. He introduces the notion of a phenomenal figure as a correlative of Kant's object in pure intuition.

Kant said it did not matter whether "construction of a (spatial) concept in pure intuition" took place with the aid of a figure drawn on paper or *simply in the imagination.* Now the visual imagination cannot supply us with physical figures. But it can supply us with what, for want of a better word, I will call phenomenal figures. The straight lines which are the objects of pure intuition are not physical straight lines. They are, perhaps, phenomenal straight lines. They are not physical objects or physical edges which, when we see them, look straight. They are rather the looks physical things have when, and in so far as, they look straight. An arrangement of physical lines or edges may look triangular. But it is not the physical lines, so arranged, which constitute the triangle which is the object of pure intuition. It is rather the triangular look they have, the phenomenal triangle which they present [p. 282].

This suggests that "*X* is a phenomenal straight line (triangle, etc.)" is to be read as "*X* is the look a thing has if it looks a straight line." A similar account can be given for Strawson's other phenomenal items, such as visual images (pp. 282, 287) and objects of sight, described as seen (the picture we give ourselves when we draw a geometric diagram).

These are distinct items—visual images, for example, are not the appearances of things—and their assimilation has caused confusion in the philosophy of perception. Here the assimilation serves a likeness. In a large range of cases we should apply or withhold many of the same predicates, among them descriptions of shape and color, in describing the appearance of *X*; *X*, as it appears, is seen, imagined, or visualized; the visual image had in seeing or the image had in remembering, imagining, or visualizing *X*; and so forth. This links the disparate items brought under the concept of a phenomenal object, and gives the examples under discussion their suggestion of scope.

Strawson wishes to use phenomenal figures, so far as possible, to provide an interpretation for geometry. This essentially requires that for "straight line" in a geometry we read "phenomenal straight line," and so forth, so far as possible. On this interpretaton a geometry is true so far as statements in it concerning straight lines are true of phenomenal straight lines and so forth; and this will be determinable when the corresponding facts about the way things appear, and so forth, are.

Strawson shows Euclidean statements true of phenomenal figures by the exercise in visualization similar to Bennett's already quoted. He concludes:

It seems that Euclidean geometry may also be interpreted as a body of unfalsifiable propositions about phenomenal straight lines, triangles etc.; as a body of *a priori* propositions about appearances of these kind, and hence, of course, as a theory whose application is restricted to such appearances [p. 286].

Only phenomenal geometry, he stresses, is necessarily Euclidean. Physical geometry is not:

according to modern physics, the possibility that the structure of space is non-Euclidean is something more than a bare possibility . . . it appears that the findings of astro-physics are more easily accommodated by a theory of space inconsistent with the Euclidean [p. 280].

Thus according to Strawson one geometry is true of physical space while another, inconsistent with it, is true of phenomenal space, of the way things look or are seen spatially. Indeed he criticizes Kant for not providing for such a possibility.

Kant's fundamental error lay in not distinguishing between Euclidean geometry in its phenomenal interpretation and Euclidean geometry in its physical interpretation. Because he did not do this, he supposed that the necessity which truly belongs to Euclidean geometry in its phenomenal interpretation also belongs to it in its physical interpretation. He thought the geometry of physical space *had* to be identical with the geometry of phenomenal space [p. 285].

In contrast, on Strawson's account, physical and phenomenal geometry need not be the same. One is the apparent geometry of spatial things or the geometry of their appearances; the other is the geometry of spatial things themselves. For the two to differ, then, is for things to appear systematically to have one geometry while in fact having another. This is the possibility it was Kant's fundamental error to overlook.

But again, such a possibility is surely a strange one. Surely if it were so, if there were such a systematic contradiction between spatial appearance and spatial reality, it would require explanation. How could the form of outer sense differ from the form of outer things?

IV

The source of difficulty is the assertion that we should apply Euclidean predicates and withhold non-Euclidean ones in describing phenomenal figures. Philosophical writers have suggested, alternatively, that this assertion is false, or that it is true, but so explicable in familiar and un-Kantian terms as to remove the difficulty.

Lucas writes:

Ewing (1938) and Strawson (1966) have attempted to save Kant's account of geometry by maintaining that it is *a priori* true at least of phenomenal geometry—the geometry of our visual experience—that is Euclidean. But this is just what the geometry of appearances is not. Let the reader look up at the four corners of the ceiling of his room, and judge what the apparent angle at each corner is; that is, at what angle the two lines where the walls meet the ceiling appear to him to intersect each other. If the reader imagines himself sketching each corner in turn, he will soon convince himself that all the angles are more than right angles, some considerably so. And yet the ceiling appears to be a quadrilateral. From which it would seem that the geometry of appearance is non-Euclidean, with the angles of a quadrilateral adding up to more than 360°. And so it is.[9]

The impression that phenomenal geometry is Euclidean should not be yielded so easily. Really this case is more naturally described in Strawson's (Euclidean) than Lucas' (non-Euclidean) terms. We can, as Lucas implies, describe an image or appearance by means of a sketch or two-dimensional projection.[10] In sketching (the appearance of) the corners of his ceiling, the reader may draw, produce on paper, angles visibly greater than right angles; and in this sense he can say the corners appear obtuse angles. In sketching (the appearance of) his ceiling, the reader may draw a quadrilateral; and in this sense he can say his ceiling appears quadrilateral. But in no sketch will the reader draw a quadrilateral with four visibly obtuse angles. None can be drawn. So the reader will not, in this sense, describe his ceiling, or anything else, as having the appearance of a quadrilateral whose angles add up to more than 360 degrees. This description might be got only by mixing different descriptions of different projections. Rather it

[9] J. R. Lucas, *British Journal for the Philosophy of Science*, 20 (1969), 6. On one interpretation of what Lucas says, the case is comparable to that treated by Strawson at pp. 290-291. Lucas also distinguishes the space of our ordinary experience, which presumably we inhabit, from that which our physical theories are about. But he gives no reason for this distinction, and on the face of it, it is implausible.

[10] Craig, *British Journal for the Philosophy of Science*, 20 (1969), 121-134, explains the use of projection in this context. "Suppose we are looking at some figure. Whatever it may be, and irrespective of whether it 'looks Euclidean' or not, there will be a projection of it, as we see it, on to a Euclidean plane. . . . So whatever the nature of the figure, our sense impressions of it could, by this criterion, be said to be Euclidean, or at any rate, not to be non-Euclidean. This is a necessary proposition; but a very weak one." I do not wish to claim that there could be no use for "looks non-Euclidean" or that no picture whatever might some way be describable as showing two straight lines intersecting twice. Escher's drawings, as Craig points out, might be said to look geometrically or logically impossible, or to show impossible situations. I don't know what pictures like this a clever artist might produce, or what arguments a philosopher might give to support that description. But I think such a case will be distinguishable from those I discuss.

seems that the reader who goes by projections will say his ceiling appears a quadrilateral of 360 degrees. The most natural description of any diagram or projection will be Euclidean; the lines will not be found to deviate from Euclid's description.

The natural account for descriptions given by means of drawings or projections holds more generally. The notion of looking *X*, for example, is fundamentally connected with that of looking *to be X*, so that how a thing looks is connected with how it might be judged to be. Where whether something is *X* can be told by looking, "looks *X*" often simply means either "looks to be *X*" or "looks like a thing which looks to be *X*, in ways relevant to judging whether it is *X*." The latter holds trivially with the former; and often it can be expanded to "looks as things which are *X* typically do, in ways relevant to judging whether they are *X*." Such appearance-world relating uses of "looks," "appears," and so forth, are complex and varied. But it seems that in none of them should we say that something looked non-Euclidean. We do not know what it would be for something to look to be non-Euclidean, or to look so as to lead us to judge it non-Euclidean, or to look as non-Euclidean things typically do, in ways relevant to judging them non-Euclidean.

Nagel, by contrast, does allow that "we find we cannot form our images except in conformity with the Euclidean axioms." He thinks this explicable in familiar terms:

When we perform experiments in imagination upon straight lines, in what manner are these lines envisaged? We cannot employ any arbitrary images of lines in the experiment. We must construct our images in a certain manner. However, if we examine the mode of construction in those cases in which we allegedly intuit the imagined figures as Euclidean, we soon notice that the Euclidean assumptions are tacitly being used as *the rules of construction*. For example, we can certainly imagine two distinct lines with two points in common. But such lines do not count as straight lines, simply because they do not satisfy the Euclidean requirement of straightness, so that we seek to form our images so as to satisfy those requirements . . . accordingly, if the Euclidean postulates serve as rules for constructing our mental experiments, it is not at all surprising that the experiments inevitably conform to the rules. In short, if Euclidean axioms are used as implicit definitions, they are indeed *a priori* and necessary because they then specify what sorts of things are to be counted as their own instances.[11]

The suggestion is that our images satisfy Euclidean axioms because we use the axioms as implicit definitions of the characteristics of the items we construct, and hence as rules to which their construction conforms. As Reichenbach says, "the images by which we visualize geometry are always so adjusted as to correspond to the laws we read from them" (p. 44). This must mean, for example, that we refuse to count imagined lines as straight, solely because they violate Euclid's axioms of straightness. This suggestion is unfounded. Consider Nagel's example:

> we can certainly imagine two distinct lines between two points. But such lines do not count as straight lines, simply because they do not satisfy the Euclidean requirement for straightness.

If we do draw or imagine such lines, we can judge both curved, or one straight and the other curved. These judgments do accord with the Euclidean axiom that straight lines do not intersect twice. They cannot, however, be based solely on the axiom. The axiom alone entails only that not both lines are straight; it could not yield the judgment that both were curved, or that one was straight or another curved. Clearly we can make such judgments. We therefore judge in accord with some criterion other than the Euclidean axiom. The criterion seems obvious: length. When we judge that both lines are curved, say, we judge that both are longer than some other which might be produced; in other cases we judge that only one is. In general, so far as visual judgments (including those we have discussed) are concerned, one criterion of straightness suffices. We can say a line between two points is straight if it is a line than which no other is shorter. This criterion of straightness is common to both Euclidean and non-Euclidean geometries. So its use does not constitute the use of Euclid's axioms as implicit definitions. The fact that our images satisfy

[11] Nagel, *The Structure of Science* (London, 1961), pp. 224-225. Nagel gives a lucid account of what Strawson calls the "positivist view" of geometry. He and Reichenbach may have felt that only in this way could the Euclideanness of phenomenal geometry fit the positivist view.

Euclid's axioms cannot, therefore, be explained by saying that in forming images we use the axioms as implicit definitions.[12] Some other explanation is required.

V

Strawson also attempts to explicate the Euclideanness of phenomenal geometry as a priori; and in this he is involved in an unconventional account of some geometric propositions.

He outlines what he calls "the positivist account" of geometry, developed to clarify the role of analytic and empirical propositions in geometry to oppose the Kantian suggestion that geometric propositions are both synthetic and a priori. The formalization of a geometry shows the theorems deducible from the axioms on the strength of their logical expressions alone; so the conditionals connecting axioms and theorems can be regarded as logical, necessary, or a priori propositions, but not as empirical or synthetic. The axioms and theorems may be taken as uninterpreted, and hence not determinate propositions; or they may be regarded as, for example, physically interpreted, and hence as empirical or synthetic propositions, but not necessary or a priori. (Strawson mentions in passing "a variant of the positivist view," in which "we do indeed secure the necessity of our axioms and theorems; but only by qualifying our announced physical interpretation of the non-logical expressions of the theory by the rule that nothing whatever is to count as a falsification of the axioms or theorems." This is similar to the approach Nagel describes as using the axioms as implicit definitions.) Thus, according to the positivists, in geometry there are necessary (logical, a priori) propositions and there

[12] Someone might wish to urge that the proper description of some seen or imagined lines was that the two looked (were imagined) equally short and shorter than any alternative. Here one could hardly use the Euclidean axiom to force the judgment that one or both must look curved, or say that the image had been constructed in accord with the axiom; for both are conceded to look or be straight (short) in Euclidean terms, and equal. Such a description fits with the indeterminacy thesis argued below, p. 22 ff. It might be claimed that someone who refused to give this non-Euclidean description in an appropriate case was using the Euclidean axioms implicitly; but this would be difficult to establish.

are empirical (synthetic) ones, but none which can be regarded as both.

Now Strawson says that the positivist account is "in a sense true." But he believes also that a consideration of phenomenal geometry shows it to be in a sense inadequate. He summarizes his outline:

The positivist view offers us two ways of looking at the propositions of Euclidean geometry: as formulae in an uninterpreted calculus; or as the body of logically connected empirical propositions which result from a physical interpretation of the fundamental expressions of the formulae [p. 285].

and to it opposes his description of phenomenal geometry:

what we have had to notice is that there is a third way, different from either of these, which is also possible and which the positivist view neglects . . . Euclidean geometry may also be interpreted as a body of unfalsifiable propositions about phenomenal straight lines, circles, etc. As a body of *a priori* propositions about spatial appearances of these kinds and hence as a theory whose application is restricted to such appearances [p. 286].

Two connected features here distinguish Strawson's third way of regarding geometry from the positivist view. First, the geometrical terms in the axioms are given a phenomenal, as opposed to a physical, interpretation. Second, the phenomenally interpreted axioms are unfalsifiable, a priori propositions, rather than falsifiable, empirical ones. Strawson emphasizes this latter feature. Elsewhere he speaks of the "phenomenally analytic," and of Kant's proper recognition of "the necessity which truly belongs to Euclidean geometry in its phenomenal interpretation."

It is unclear precisely how, or how far, Strawson takes these features as marking off a genuinely distinct and neglected way of regarding geometry. On the surface they do not. The positivists neglected neither phenomenal geometry nor the possibility that its propositions were a priori. Reichenbach wrote at length about geometric visualization; and he and Nagel, as we saw, describe a phenomenal interpretation of Euclidean geometry as true a priori within the framework of what Strawson calls the positivist view.

Strawson's description of phenomenal analyticity does distinguish his from the positivist view. He says, in ostensibly familiar terms, that phenomenal propositions are true in virtue of meanings. But these, unfamiliarly, are "essentially phenomenal, visual meanings . . . essentially pictureable meanings." Proof in phenomenal geometry involves "a phenomenal exhibition of meanings," in which "phenomenal figure-patterns can be elaborated to exhibit an extensive system of relations between phenomenal spatial concepts" (p. 286).

We cannot, either in the imagination or on paper, give ourselves a picture. . . . Such an impossibility used to be expressed by saying that such axioms are necessarily true because self-evident. This left the character of the necessity, of the impossibility, insufficiently explained. We can explain it by saying that the axioms are true solely in virtue of the meanings attached to the expressions they contain, but these meanings are essentially phenomenal, visual meanings, essentially pictureable meanings. Any picture we are prepared to give ourselves of the meaning of "two straight lines" is different from any picture we are prepared to give ourselves of the meaning of "two distinct lines between two points" [p. 283].

This is not Nagel's positivistic unfalsifiability, and a positivistic account of truth in virtue of meaning (rules of use, and so forth) would render irrelevant the visual images Strawson emphasizes.

But where it is distinct, Strawson's account is elusive. The expressions "phenomenal exhibition of meanings," and so forth, by themselves convey little beside echoes of the notion that an object such as a mental image could be the meaning of a term. Strawson's use takes us no further. He passes from, for example, "picture showing two straight lines" to "picture of the meaning of 'two straight lines,' " without separating the two. This leaves us unable to distinguish a phenomenal exhibition of meanings from a plain exhibition of phenomenal figures.

This in turn leaves obscure the purported grounding of phenomenally analytic truths. For an exhibition of phenomenal figures, like one of physical objects, could naturally be taken to support no more than the claim that a certain geometry was contingently true of the exhibited objects. Again, the visualizing which Strawson calls a phenomenal exhibition of meaning Nagel calls an experiment in the imagination. It is hard to see why Strawson's description should be preferred.[13]

The difficulty with Strawson's account at this point closely resembles that of Kant's account of the intuitional foundation of a synthetic a priori proposition. This is an exegetic felicity. Strawson intends his account to mirror Kant's. He hopes partly to vindicate Kant's belief that "The construction of concepts in pure (i.e. non-Empirical) intuition" is a source of geometric knowledge, by showing how "Kant's theory of empirical intuition can be construed as a reasonable account of the nature of geometry in its phenomenal interpretation" (pp. 277; 283-284). It is to this end that he compares proof by consideration of a phenomenal figure to Kant's construction in intuition. Now, the synthesis or empirical element in construction is the source of Kant's mathematical *synthetic*; so no wonder we feel it to conflict with Strawson's geometric a priori.

A belief that geometric proof can in part essentially be accomplished by construction or exhibition of a figure may explain some features of Strawson's and Kant's descriptions. If construction were part of demonstration it would perhaps be appropriate to speak of the construction or exhibition of concepts, meanings, or conceptual connections. If construction were essential to demonstration, if it could not be eliminated or replaced by statement, it would be difficult to distinguish proof from experiment, or exhibition of meanings from exhibition of objects.

Of course it seems that construction or exhibition can have no role at all in proof. By a proof we understand a set of statements, premises and conclusion. If the premises do not entail it, the conclusion is not yet proved, construction or no; if they do, nothing further is needed. So a construction or exhibition is either impotent or otiose.

[13] Another move to meaning in this case is quoted by Mill from Bain (*System*, II, v, 5): "We cannot have the full meaning of straightness, without going through a comparison of straight things among themselves, and with their opposites, bent or crooked objects. The result of this comparison is, *inter alia*, that straightness in two lines is seen to be incompatible with enclosing a space; the enclosure of a space involves a crookedness in one of the lines." The idea that understanding meaning involves being able to treat cases is a good one; but the feature here attributed to meaning was surely induced from the cases.

More generally, it seems, as on the positivist view, that there is no reason to differentiate kinds of necessity among a priori propositions; and that we could find no foundation outside language for the necessity there is in proof. Consequently it seems there is no specific phenomenal necessity, nor could there be any such explanation of it as Strawson attempts to provide.

This means we find no explanation of phenomenal necessity either within or without the positivist view. There is no problem here. For there is no reason to accept the assumption, underlying both Strawson's account and that of Nagel and Reichenbach, that phenomenal propositions are necessarily true.

Rather it seems that we should take phenomenal geometry simply on a par with physical geometry, and hold that its propositions, if true, are contingently true. For, schematically, if putting "physical straight line" for "straight line" in a geometry produces a contingent theory about physical straight lines, then putting "phenomenal straight line" should produce a contingent theory about phenomenal straight lines. Nothing in the nature of the case forestalls this.

We cannot picture two straight lines between two points. A color-blind person, or one lacking certain experiences, may be unable to picture anything red, and no one can picture other than certain colors. We should presumably say these latter were empirical propositions, contingent on persons' experience or powers of discrimination. Why not the former?

Statements about the imagination may suggest the a priori. They are verifiable by introspection, not examination of the world (compare Strawson, p. 282). No alternative to their truth can be imagined (in the sense that we cannot, even by trying very hard, imagine what in fact we cannot imagine; nor what another whose power exceeds ours imagines; and so forth). These resemble Kant's grounds for calling propositions a priori and give the designation a certain fitness. But as features of contingent propositions about imagination they cannot make a proposition a priori in any sense contrasting with the contingent or empirical.

Thus we can regard phenomenal propositions as contingent, and hence as fitting in the positivist framework. A further description of their contingency follows upon the resolution of the paradox with which we began.

VI

This paradox was that it seems we must picture space as Euclidean, whereas on scientific grounds we may judge it non-Euclidean. It seems odd that we cannot picture things as they are, that rather we are constrained to picture the contradictory of what we might have scientific reason to think true. This was exemplified by Strawson's acceptance of a necessarily Euclidean geometry of the visual, together with a non-Euclidean geometry of space. So it will be appropriate to begin by noting what he says about the latter.

> The testing of Euclidean geometry by observation and measurement shows its theorems to be verified with an acceptable degree of accuracy for extents of space less than those with which astro-physics is concerned; but for astro-physics itself, a different physical geometry, inconsistent with the Euclidean, is found to accommodate observation and measurement more easily [p. 286].

The situation alluded to here is not that one geometry is true of small regions while another, inconsistent with it, is true of large regions. This could not be the case. Large regions are composed of small regions; and if one spatial region is Euclidean, and another adjoining region is Euclidean, then the larger region composed of the combined adjoining regions must also be Euclidean. So if we regard large regions as non-Euclidean, we cannot regard the small regions composing them as Euclidean. We must regard them as strictly, if undetectably, non-Euclidean.[14]

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[14] Lucas argues that "it is a necessary condition of our being able to apply the concept 'same shape though different size' that our geometry should be Euclidean," and that hence "the price of abandoning Euclidean geometry would be the loss of an important respect in which things can be similar to or different from one another . . . we should no longer be able to classify by shape." But clearly we can regard Euclidean geometry as false and still take things as comparable in respect of shape and indeed, for all practical purposes, as having the same shape but different sizes. It suffices to regard things as (non-Euclidean

The fact is that the inconsistencies between the two geometries typically yield empirically detectable differences only in application to very large regions. In a small region considered in isolation, observation and measurement may fit equally well with either geometry. Here one can loosely say of either, as Strawson says of the Euclidean, that it is "verified with an acceptable degree of accuracy." At this degree of accuracy, however, one geometry is not verifiable in opposition to another which contradicts it. The purported verification is equally the verification of contradictory theories. Nevertheless, there may be good reason for regarding such a region as genuinely, if (locally) undetectably, non-Euclidean, as opposed to Euclidean. For it may be part of a large region which is detectably non-Euclidean. And this, as Strawson says, appears to be the case.

With this in mind, let us reconsider the assertion that phenomenal geometry is Euclidean. The relevant phenomenal figures are visual and mental images and the looks of things. The latter are Strawson's paradigms; their geometry is easy to determine. Thus phenomenal straight lines are "the looks physical things have when, and insofar as, they look straight." By "physical things" here are meant "physical lines or edges" examples of which are taut strings, light paths, and lines on paper. Three intersecting physical lines form a physical triangle, the look of which is a phenomenal triangle. The phenomenal triangle is Euclidean if the look of the physical one is; and this, presumably, is true if the physical triangle looks Euclidean.

It follows at once that phenomenal geometry is *not* Euclidean. For such a phenomenal figure as the look of a physical triangle is not. No physical triangle looks Euclidean as opposed to non-Euclidean. The difference made by the assumption that a physical triangle is Euclidean as opposed to non-Euclidean is visually undetectable. It therefore looks just as much non-Euclidean as

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and) approximately Euclidean. The strength of the approximation, in fact, makes the Euclidean concepts as usable as any.

The facts and connections Lucas cites do not prove his contention that we must regard things as Euclidean. Together with the approximate truth of Euclidean geometry, however, they partly explain its outstanding naturalness and historical pre-eminence. Some such explanation is surely better than Strawson's in terms of phenomenal necessity.

Euclidean. Local observation and measurement fit equally with Euclidean and non-Euclidean assumptions; so it is not surprising that the looks of things fit equally with both assumptions. To say this is to say that these phenomenal figures fit both equally. Similarly for images: just as visually indiscriminable items have the same look, so the same image represents them indifferently. Phenomenal figures are therefore no more Euclidean than non-Euclidean. So phenomenal geometry is not Euclidean. Rather it is neutral or indeterminate.

(This line of thought requires the geometry of phenomenal items to be tied, as in Strawson's account, to the geometry that things are seen or imagined to have. Otherwise it is quite opaque what geometrical ascriptions to phenomenal items would mean, or how they could non-arbitrarily be made—let alone made with precision sufficient to differentiate geometries visibly indistinguishable in application. So it could hardly be argued that if the geometrical properties of phenomenal items were made independent, phenomenal geometry might still prove Euclidean.)

How, then, can we account for the plausibility of the suggestion that phenomenal geometry is Euclidean, and for the thought-experiments which seemed to establish phenomenal axioms?

Partly the explanation is simple. Euclid's geometry is familiar and approximately true. We naturally describe in familiar terms, and where measurement is concerned we correctly speak more or less imprecisely. We therefore naturally and correctly describe figures in Euclidean terms—just as, say, we call a line segment an inch long, despite the fact that its length may not be very precisely determinable, and not excluding the possiblity that N such segments, where N is large, should produce a line greater than N inches long. "Euclidean" here really means no more than "approximately Euclidean"; and although "Euclidean" and "non-Euclidean" are contradictories, "approximately Euclidean" and "non-Euclidean" are here true together. It is easy to forget that approximation is involved and so to suppose, erroneously, that in this use "Euclidean" contradicts "non-Euclidean." Hence a belief that things as they are look Euclidean, and that to look non-Euclidean they would have to be or look different, so as to fit contradictory descriptions. Or that our images represent

Euclidean figures only, so that a different geometry would require different images.

Thus, for example, someone might try to picture a non-Euclidean triangle by starting with an image of a triangle assumed to be of 180 degrees, and trying to increase an angle without bending a side. This would prove somewhat frustrating; and it would be to overlook the fact that an image determines no exact angular sum for an imagined triangle. Since points and lines can be pictured only in terms of areas or their (imperfectly determinable) boundaries, any geometric image will be ambiguous. Here the same image can equally represent invisibly dissimilar alternatives, Euclidean and non-Euclidean. So no further picturing is required.

The indeterminacy may be overlooked also because of unreflecting exaggeration of the ability to picture. We can picture what we cannot see, either the very small or the very large, far or near. The capacity may seem unrestricted by size or distance; we can choose whatever scale for our pictures we please. One may feel, for example, that it should be possible to draw or imagine any simple figure, of any size, in space; and hence to picture figures on the astral scale, or with regard to the minute differences, relevant to the verification of non-Euclidean geometry.

One may also feel that we ought somehow be able to form pictures, different from any we have, of the non-Euclidean; or, failing this, that our imagery is unambiguously Euclidean. If it is possible to picture with a precision or on a scale relevant to detecting non-Euclidean phenomena, it should be possible to picture detectably non-Euclidean phenomena; and our imagery remains as Euclidean when constructed with reference to an astral scale as when referred to the middle-sized or the very small.

In fact we simply do not picture relevantly here. It is true we can imagine or draw what we cannot see; but what we can imagine accurately, or picture accurately in general, has limitations connected with sight.

Suppose, for example, we wish to represent two stars and the distance between them by dots and a blank space on a sheet of paper.[15] The dots can be related in circumference as the stars. But

[15] I explicitly treat only two-dimensional pictures seen from straight on. This seems adequate to account for visualization of geometric figures, Mill's

if the stars are sufficiently far apart in relation to their size, we will be able to form no picture in which their size is shown accurately in relation to their separation—in which, that is, the size of the dots is to the distance between them as the size of the stars is to the distance between them. For the dots may be so related that from any given point, if they are large enough to be seen then they will be so far apart that both cannot be seen at once. So if dots and distances are in scale, the picture cannot be taken in. If we want a picture which like a mental or visual image can be taken in at once, we can make it only by enlarging the dots in relation to their separation. The picture will then show stars larger in relation to their separation than those we set out to represent. Here, owing to the imperfection of sight, the only picture we can have is out of scale.

Or consider a very long pair of straight railroad tracks, to be pictured, as from above, by parallel lines. The rails will be a few inches wide and a few feet apart, but thousands of miles long. No picture will show us their width and separation in relation to their length. In no picture, that is, will the relations of length, thickness, and separation of the lines be the same as those of the rails. We are not capable of seeing lines related in length and thickness as such rails; we can see only relatively thicker lines. Consequently, any picture we can take in will show lines thicker in relation to their length than the rails we set out to picture; and similarly the separation of the rails will be shown out of proportion to their width or length.

A simple principle is involved. If a picture is to be taken in, the elements (for example, dots, lines) which compose it must be simultaneously visible. They will therefore have certain spatial properties and relations. Scale pictures like geometric diagrams show spatial situations by the spatial characteristics of their elements. Those characteristics required by considerations of scale may conflict with those needed for visibility. A distortion results from the sacrifice of scale to visibility. Similarly for images. Just as, say, there will be a maximum ratio of length to thickness consistent with the visibility of (the representation of) a line, there will be

"diagrams in the mind." I think analogous considerations would apply to other kinds of pictures and models.

such a relation for any visualized line.[16] And as the maintenance of this ratio for visible pictures means that certain spatial relations cannot be pictured accurately, for images it means that they cannot be imagined accurately. This systematic possibility of distortion entails, among other things, that the ambiguity of images between Euclidean and non-Euclidean cannot be resolved by change of scale.

Someone may, for example, think he can picture Euclidean parallel straight lines. For simplicity, and to fix what is meant by a line, suppose he pictures such a pair of lines as could be drawn on a blackboard, a few feet apart and a few yards long, at the maximum ratio of length to thickness. Now it can be pointed out that his picture of these lines does not differ from one of lines which would meet if extended, say for a few miles. The picture does not exclude this possibility, so it does not show the lines as parallel. He may reply that he can regard the lines as extended; he can exclude the possibility that the lines he pictures would meet if extended, by picturing them as long as he likes. This is really the assertion that he can change the scale of his image to represent longer lines. But as the scale is changed, the picture ceases to show the disposition of *lines*. As the length represented increases so does the width and hence the area shown covered by what was to be a line; and nothing in the changed picture will be capable of showing how lines such as could be drawn on a blackboard are disposed. If a picture is to show lines of a certain kind its scale must be limited; if its scale is limited it cannot show the lines as parallels, or in general as Euclidean. So pictures, like sight, remain geometrically indeterminate, whatever our intentions as to their precision and scale.

In fact the limits of accuracy in imagination seem directly tied to those of picturing by visible pictures, and so indirectly to sight. Roughly, we should not expect to find a person capable of imagining a spatial situation accurately unless he was, or had been, capa-

[16] Lines of this kind are components of the most familiar geometrical diagrams. You might have another kind of visual geometry, say using areas shown by color patches, and represent lines by color edges. Similar considerations regarding accuracy still apply. There will be limits on the kind of color areas visualizable and the indeterminacy of the visible location of a color edge will mean that it can be treated as I treat (areas representing) lines here.

ble of seeing an accurate visible representation of it. If a person were unable to see any pair of dots related in size and separation as a pair of stars or bits of dust he was attempting to visualize, we should expect to find that he could not visualize their size and separation accurately either. If he said he could not, the matter would presumably be settled. It would puzzle us if, knowing what was involved, he said he could; and in default of very special testimony, we should have no reason to accept his claim—we should reject it, or not know what to do with it. From this it seems we are justified in assuming that what persons can imagine accurately is limited to what they can see accurate pictures of, which in turn is determined by their powers of sight.

VII

The limits of geometric imaginability and their connection with contingency and necessity can now be more fully set out. The main points are perhaps as follows.

It seems (again roughly) that if a person can see items as of a certain kind, there is reason to accept his claim so to picture them. So if things are seen as they are, phenomenal geometry will depend upon how things are and how precisely they can be seen. The phenomenal geometry of someone able simply to see the non-Euclidean character of our space would be accordingly non-Euclidean. That of someone with perfect sight in a Euclidean world would be Euclidean; and of someone even with imperfect sight but in a visibly non-Euclidean world, non-Euclidean. As stressed, the geometry of imperfect sight in an unobvious world will be indeterminate.

These other phenomenal geometrics are, explicably, not ours. As has been shown we cannot form the non-Euclidean pictures of our space we might have with more powerful sight. Nor can we simply alter our way of seeing distances and shapes to what it might be in a different world. But now this latter inability can be described a bit further by reference to some abstract features of geometry.

Determining the geometry of space requires comparing distinct spatial intervals. This is typically thought of as accomplished by the use of standards of length, such as a portable rod which realizes a certain interval and whose coincidence relations with other objects and intervals provides their measure. Measurement then consists in the establishing of these coincidence relations.

A given set of coincidence relations among rods, objects, and intervals generally can be interpreted in terms of measure and geometry in various ways. In particular, the relations can yield one set of measurements and one geometry if the interval realized by a standard rod is taken as everywhere the same, other measurements and another geometry if the interval is taken to vary with the position and orientation of the rod. The differences in measurements will result in the relations' determining different sets of intervals congruent or equal. And with one set of intervals congruent the geometry will be Euclidean, with another, non-Euclidean.[17]

Since this indeterminacy arises in interpreting the facts of coincidence on the basis of which measurements are assigned and geometry assessed, it cannot be resolved by any further recourse to measurement or geometry. Still, the coincidence relations themselves may fix the geometry, by practically ruling out the assumption that the length of the standard varies. For to retain the same size can be little more than to retain the same size in relation to things in general. So if coincidence relations among the standard, bodies, and items with size in general are unvarying, the standard is (to be regarded as) rigid. Given such rigidity, congruence becomes simply coincidence with a standard.

(Here, as one might say, the harmony of things with size can fix geometry despite indeterminacy. Einstein was inclined to assimilate the non-Euclidean geometry of a gravitational field to such a case,[18] and philosophers of science, among others, have followed him. But the cases are not entirely comparable. For in Einstein's theory a gravitational field changes the coincidence relations—the relative shapes and sizes—of bodies of different shape and size.[19] In the field things are non-Euclidean measured

[17] See Grunbaum for exegesis of these matters.

[18] *The Theory of Relativity* (London, 1920), pp. 85-86. Einstein's simplified analogue treats only of the coincidences of rods, and so ignores other bodies.

[19] See Swinburne, *Space and Time* (London, 1968), pp. 92-93. I do not think his description of "the original interpretation" of the general theory applies to the paper of Einstein's to which he refers.

by small rods where these are taken as rigid. But no harmony of coincidence relations forces us to take small rods as rigid. Consequently the choice between geometries must be made on less obvious grounds.

Perhaps Einstein can be seen partly as giving the simplest account of local measurement and its most direct embodiment in geometry. Given the fundamental role of local measurement in Einstein's physics and in the verification of physical theory generally, this procedure seems appealing; its justification would be complex.)

Now as an expression of the indeterminacy of geometry, we have

> (*a*) It is possible to describe a world as Euclidean or non-Euclidean, depending upon which of its intervals are taken as congruent or equal

while also a world will be fixed as Euclidean or as non-Euclidean if found so by measurement using standards whose relative size, like the relative size of things in general, stays constant. And this is not arbitrary: the bodies, and so forth, of such a world are to be regarded as rigid on conceptual grounds.

Still, in consequence of (*a*),

> (*b*) It is possible to describe a non-Euclidean (Euclidean) world of rigid bodies as a Euclidean (non-Euclidean) world of bodies changing dimensions with position and orientation, but in such a way that their coincidence relations stay constant.

Since they stress only the relations of geometric descriptions, these principles might loosely be called logical. Now suppose we apply them to the descriptions under which things are seen, and so to visual geometry, by putting "see" and "seen" for "describe" and "taken" in (*a*) and "see" for "describe" in (*b*). We then have modified, visual principles, to the effect (*a*) that it is possible to see things as Euclidean or non-Euclidean depending upon which intervals are seen as congruent and (*b*) that it is possible to see a non-Euclidean rigid world as Euclidean changing and vice versa.

These visual principles could be variously interpreted. The possibility of seeing something under a description might be taken simply as given by its being so describable. The visual principles would then be versions or rephrasings of (*a*) and (*b*), and hence (loosely) logical statements. Or they might be taken as substantive claims—for example, about persons' abilities to see things as falling under alternative geometric descriptions. Here the principles would be contingent statements, and possibly false.

Now Reichenbach's discussion of geometric visualization pivots on such principles. He says, in accord with (*a*):

> Space as such is neither Euclidean nor non-Euclidean . . . it becomes Euclidean if a certain definition of congruence is assumed for it . . . if a different definition is introduced. . . . Space becomes non-Euclidean.

He holds that we visualize with a Euclidean definition of congruence. This is only because our world is so nearly Euclidean: if things were different our way of seeing would change in accord with the possibilities of (*b*), interpreted visually:

> if in daily life we dealt occasionally with rigid bodies that adjusted themselves to a non-Euclidean geometry. . . . At first we would have the feeling that objects changed when transported . . . After some time we would lose this feeling and no longer perceive any change . . . we would have adjusted our visualization [to a non-Euclidean geometry; pp. 54-55].

This is an example of the essential change. Since we are adjusted to Euclidean congruence, and since non-Euclidean visualization means visualization adjusted to a non-Euclidean congruence, we need only undergo such a change of sight or visualization, such an adjustment of congruence, to accomplish non-Euclidean visualization. Thus Reichenbach gives a principle of visual adjustment of congruence corresponding to (*a*):

> Whoever has successfully adjusted himself to a different congruence is able to visualize non-Euclidean structures as easily as Euclidean [p. 55].

He takes this as supporting a substantive claim:

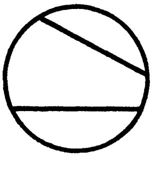
The mathematician is thus correct in saying that he has become accustomed to visualize non-Euclidean geometry [p. 53].

This he applies, as we saw earlier, to the interpretation of geometric drawings.

Although this account of what is involved in non-Euclidean visualization requires a number of assumptions (for example, about the identification of images, the propagation of light, and so forth) it is appealing and seems informative. But it does not support Reichenbach's belief that persons can actually visualize non-Euclidean geometry. Neither the fact that our way of seeing depends upon how things are, nor reasonable speculation about how we should adapt if things were different, shows that *as things are* we can see or visualize in any other than the familiar approximately Euclidean (or weakly non-Euclidean) mode. We cannot.

For, as I have argued, the change in visual congruence on which this account of non-Euclidean visualization pivots does not occur. No one in fact experiences a change of sight relevant to seeing or visualizing in non-Euclidean terms. It seems in consequence that those who claim non-Euclidean visualization do not actually accomplish it. Rather they visualize in familiar terms while describing their images non-Euclideanly.

(This is easily recognized in one of Reichenbach's own examples. He says the small drawing known as Klein's model of a non-



Euclidean space can be "truly a visualization of Lobatchewsky's space" since "it is possible to adjust to the other congruence." But

in order to accomplish the visualization we must forget everything outside the circle . . . we must imagine ourselves in the circle and remember that the periphery cannot be reached in a finite number of steps [p. 58].

There is no telling in what such a visualization might consist—to what visual image could such descriptions [involving infinity, being inside, and so forth] meaningfully be applied? and how might it relate to the drawing?—and we simply have no idea of a change of sight which might accomplish it. Really, Reichenbach must picture the circle just as we do. Hence the notion of change to an alternative visualization, and with it the notion of non-Euclidean visualization, has been given no content. Its emptiness is perhaps hidden by the ornamentation of Reichenbach's analysis.)

This conclusion might of course be refuted by the testimony of visualizers; but so far as I know, no testimony of any weight has been given.

VIII

There remains the fact which caught the attention of Bennett and Strawson—that we cannot picture two straight lines between two points.

It may seem obvious that since the visibly determinable spatial features of objects fit both geometries, appearance must also; and it follows naturally that picturing will be consistent with both. Some images are clearly neutral in this way. It is easy to regard an image or picture of a triangle, for example, as consistent with both geometries and hence as showing either equally. But no image shows equally the Euclidean situation of one straight line and the non-Euclidean situation of two straight lines, between two points. The pictures we have on the face of it show only the Euclidean phenomenon.

This perhaps explains the peculiar impression of intuitive self-evidence associated with the axiom; and its consequent central role in producing the conviction that phenomenal geometry is Euclidean.[20] But it is itself explicable in terms of the kind of distortion in picturing encountered already. In consequence it can

[20] This axiom is usually given as a likely candidate for intuitive self-evidence. See, e.g., Einstein, in Feigl and Sellars, *Readings in the Philosophy of Science* (New York, 1953), p. 196.

be seen to have no bearing, either on the axiom or on the claim that phenomenal geometry is Euclidean.

Consider the situation Barker describes: between two very distant points are two paths measuring equally short and shorter than any other, which light rays follow and along which taut cords lie.[21]

Suppose we wished to picture it. One way, illustrating the principle involved, would be to stretch a suitably large sheet of paper between the points and make a picture by drawing along the lines. Now clearly this picture could not be seen. Someone far enough away to see both points would be too far off to see the lines, which would be minute in comparison to the sheet. The only way to make anything visible here would be to thicken the lines. But then they would overlap before becoming large enough to be seen. So *two* lines could not be seen. Owing to the distortion required to make the lines visible, the only way to make two lines visible would be to bend one away from the other. Then one line would be and appear curved. Hence the only usable (visible) pictures fail to show two lines, or show one curved. The same is true, for like reasons, of our images and other pictures.

So really there is no accurate picture of the situation described. Paths of the required ratio cannot be pictured. Because of their relative thickness, the areas which can be pictured cannot mirror the disposition of lines; and in this case the particular form of distortion leaves no alternative but pictures easily interpreted as showing Euclidean lines. It is like the transformation of a delicate design painted over with a thick brush.

Since no picture here is capable of showing the disposition of lines in space, none shows these lines as Euclidean. Just as no picture could show two straight lines between two points if they were there, so no picture shows the one and only straight line there is. At this scale and in this case we can only disregard our images; we cannot take them as showing how things are. So despite the impression, our images are not really Euclidean; rather they are too crude to serve.

[21] Only very long lines would be empirically distinguishable. But presumably the argument would apply locally as well.

IX

So, finally, nothing constrains us to picture space in terms of a superseded theory. The impression is only the result of misleading pictures. We can neither picture every spatial situation nor change our way of picturing at will; but still we see and picture consistently with Euclidean and non-Euclidean theories. Possibly this is not obvious, but it ought not be surprising. It has always been clear that the observations required to tell between physical geometries could not be made by unaided sight.

James Hopkins

King's College, Cambridge