THE INTERNAL/EXTERNAL QUESTION

Philip HUGLY & Charles SAYWARD
University of Nebraska/Lincoln

Introduction

Rudolph Carnap expresses his famous distinction between the internal question and external question thus:

> What is now the nature of the philosophical question concerning the existence or reality of numbers? To begin with, there is the internal question which, together with the affirmative answer can be formulated ... by "There are numbers" ... This statement follows from the analytic statement "five is a number" and is therefore itself analytic. Moreover it is rather trivial ... therefore nobody who meant the question "Are there numbers?" in the internal sense would either assert or seriously consider a negative answer. This makes it plausible to assume that those philosophers who treat the question of the existence of numbers as a serious philosophical problem and offer lengthy arguments on either side, do not have in mind the internal question ... Unfortunately, these philosophers have so far not given a formulation of their question in terms of the common scientific language. Therefore our judgement must be that they have not in giving the external question and to the possible answers any cognitive content.[1]

As a question internal to arithmetic, the question 'Are there numbers?' has a trivial affirmative answer. As a question external to arithmetic it has no cognitive content.

A fundamental question in the philosophy of mathematics is whether there are numbers. Realists produce 'There are numbers'; anti-realists produce 'There aren't any numbers'. These philoso-

1. Rudolph Carnap, "Empiricism, Semantics and Ontology". Reprinted from *Revue Internationale de Philosophie*, 4 (1950), in Irving M. Copi and James A. Gould (editors), *Contemporary Philosophical Logic* (New York, 1978), p. 153.

phers take themselves to be thereby making assertions on which they differ – one asserting what the other denies. For Carnap it is not the case that there is either assertion or denial in such exchanges, although, doubtlessly, it feels like assertion and denial.

This essay defends Carnap's view of the matter. We differ on details but agree on the bottom line that either 'There are numbers' has a content unsuitable for a philosophy of mathematics or it has no content at all.

2. *A realist objection*

A realist objection to Carnap goes like this: Carnap admits that 'There are numbers' is a true sentence of arithmetic. So it is true that there are numbers. This is the fundamental thesis of realism. So what Carnap says hardly is an objection to realism. Now it is true that Carnap also says that what 'There are numbers' says within arithmetic differs from what it says when asserted by a mathematical realist. But this is just obscure – too obscure to bother with. Carnap says that '5 is a number' is analytic, and, since 'There are numbers' follows from it, Carnap holds that 'There are numbers' is also analytic. So Carnap might have responded to part of this realist objection by saying that, since 'There are numbers' is analytic, it is not suitable for a metaphysical claim. This is not a persuasive reply to the objection. The thesis that the theorems of arithmetic are analytic is implausible, and it is not a point on which we would wish to defend Carnap.

3. *'Number' within mathematics*

Both Carnap and his realist opponent take it as obvious that 'There are numbers' is a sentence of arithmetic. But this is not obvious. After all 'There are numbers' contains no mathematical signs. In developing what we call natural number theory we use the signs for addition, multiplication, equality and the numerals, along with such letters as 'n' and 'm' together with signs for generality and sentential composition.

Among these signs is *not* to be found a sign corresponding to the word 'number' in the kind of grammatical application it is given in 'There are numbers'.

The *mathematical* sign 'number' is the word together with its use in numerical quantifiers (as in 'For every number n there is some number m such that m is greater than n'). *That* sign is absent from 'There are numbers'. And, apart from its use in numerical quantifiers, the word plays no role in arithmetic.

We need to distinguish between the primitive vocabulary of arithmetic and the defined vocabulary. The primitive vocabulary is very sparse, consisting of the signs for addition, multiplication, equality, the numerals, the signs for generality and sentential composition. (It can be rendered even more sparse with the numerals giving way to two signs: one for zero and one for the successor function.)

But, in addition, there are signs which are defined in terms of the primitive vocabulary. For example, to say n is even is to say that, for some m, $n = 2 \times m$.

So there is the option of introducing into arithmetic a predicative use of 'number'. All that is required is that we select some numerical formula Fn which is correct no matter what numeral we put for 'n'; for example,

for some m, $m = n + 1$

will serve as a definition of

n is a number

thus conferring a predicative use of 'number' within arithmetic.

But if 'n is a number' is to be taken as short for 'for some m, $m = n + 1$' then the philosopher's 'There are numbers' comes to nothing other than

For some n and m, $m = n + 1$

in *its* mathematical sense.

Now ask yourself whether the content of mathematical realism is entirely mathematical in nature so that the propositions of mathematical realism are nothing other than the theorems of arithmetic.

This picture of a person who from time to time – and perhaps in books and articles – writes down various well-known arithmetical quantifications and calls it a *philosophy* of mathematics fits nothing with which we are familiar.

Someone says 'There are no numbers'. The Zen master replies with a theorem of arithmetic, and perhaps lays out its proof. He meets every such challenge in just this way. He might also, from time to time, apply the conclusicns of certain proofs in some practical way. One might call him a philosopher, but one would not say that *what* he says constitutes a philosophy of mathematics.

In any case, the mathematical realist is no Zen master. The realist wants to *add* something – namely, that these theorems and proofs are not about *nothing* – that *there are* things (that there are *things*) which they are about – *numbers.*

But to add this is to utter a sentence *in addition* to those of mathematics, and so to utter a sentence *not* a part of mathematics, and thus *not* a sentence the sense of which is secured by its place within mathematics.

Realists may present various proofs within mathematics and make various assertions within mathematics. But then they want to add something, and try to do so with such words as 'There are numbers'. This is something they can't so much as *try* to do if they stick with the sentences of mathematics. So, we cannot show that the realist's sentence makes sense by noting that it is just another mathematical sentence for which it is unproblematic that it makes sense.

(And what holds for the realist holds as well for the antirealist who disputes realism by negation.)

If 'number' is short for some *mathematical* formula, then 'There are numbers' is just another quite ordinary mathematical sentence, one of a kind which goes virtually undisputed. And if 'number' is *not* short for some mathematical formula, then one certainly cannot show that 'There are rumbers' has a sense by showing that it is a sentence of mathematics!

4. *Model theory*

A realist objection to all of this goes as follows: Even if the realist assertion 'There are numbers' is not a sentence *of* mathematics, it is an assertion that is justified by thinking *about* mathematics. Model theory shows us that a simple arithmetic quantification such as

$$\exists n \quad n = 0$$

is true only if the class of numbers – the class over which the variable 'n' ranges – is non-empty, and that class is non-empty only if there are numbers.

Modal theory for arithmetic is done in different ways. The key idea is to define a function on the terms of the object language, and then to define truth relative to this function.

The function, call it 'f', satisfies these conditions: First, for any variable v of the object language, f(v) = n, for some number n; second, f('0') = 0; third, for any term t, f(⌜st⌝) = f(t) + 1; fourth, for any terms t and r, f(⌜t+r⌝) = f(t) + f(r), and f(⌜txr⌝) = f(t)xf(r)

Truth is then defined relative to f. First, for any terms t and r, (⌜t = r⌝) is true relative to f if and only if f(t) = f(r). Second, for any formulas A and B, and variable v, ⌜~A⌝ is true relative to f if and only if A is not true relative to f; ⌜AvB⌝ is true relative to f if and only if A is true relative to f or B is true relative to f; and, finally, ⌜∃vA⌝ is true relative to f if and only if, for some function g like f except at most that f(v) ≠ g(v),A is true relative to g.

Within this framework such familiar biconditionals as

'∃n n = 0' is true relative to f if, and only if, if for some number n, n = 0

are derivable. What has been set out is a model theory and in this theory it is nowhere said that numbers exist.

But, someone might object, we have managed to avoid the *word* 'exists', but not the *concept* of existence – for we used the word 'some'.

The word 'some' *is* used. But we could just as well have used 'there exists'. Nothing turns on using one phrase or another. No one thinks that arithmetic *changes* if we everywhere use 'some' instead of 'exists' or vice versa.

It makes no difference whether we use '∃ n', 'for some n', 'for some number n' or 'there exists a number n such that'. Unless there is a predicative use of 'number' in the language of the model theory – that is to say, ordinary mathematical English – we cannot infer 'There exists a number' from 'There exists a number n such that ... n ...'.

An examination of the model theory for arithmetic set out above reveals no sign corresponding to the word 'number' in the kind of grammatical application it is given in 'There are numbers' or in 'There exists a number'. The sign 'number' is used in numerical quantifiers. But apart from this use the sign plays no role in model theory.

5. *'Number' outside mathematics*

Perhaps it is of no significance that we can do mathematics without the predicate 'number'. Perhaps we actually use the word predictively outside of mathematics.

That appears to be how we use it when, for example, we distinguish between colors and numbers. We say that three is a number, but that red isn't.

Or just imagine the use of mathematics in a physical theory. We there can use 'number' in a predicative manner so as to distinguish, e.g., particles from numbers.

It is like the case with sets. *Pure* set theory does without any sign for sets – but only because in pure set theory our domain consists of sets alone. But the domains of those languages in which set theory finds an application are not thus limited-and within them a predicative sign for sets finds a use.

So we now need to consider such sentences as 'Colors are not numbers', 'Red is not a number', 'Particles aren't numbers', and 'Tables aren't sets':

Suppose that the following is a truth formulable in the language of some theory of color including at least elementary arithmetic:

For every x, if x is c

We will not want to inf

If red is odd, then re

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instead

For every x, if x is a

But if we do, we can th

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For every x, if x is odd, x is not even

We will not want to infer, e.g.,

If red is odd, then red is not even

and to bar this inference we could use the word 'number' and write instead

For every x, if x is a number then if x is odd, x is not even.

But if we do, we can then infer

If red is a number then if red is odd, then red is not even

which is equally unwanted. What we really need is

For every number x, if x is odd, then x is not even

which yields

If 3 is odd, then 3 is not even

but not

If red is odd, then red is not even.

So what is needed to guard against nonsense is not a *predicate* 'number', but the use of that word to delimit acceptable substitutes for letters used to express generalities. And what is essential here is the *practice* of replacing certain letters *only* by numerical terms. And so once again we do not need the word as a predicate – to draw a distinction among things.

The fundamental point would seem to be that the language games for color and number don't 'intersect'. A term with the use of a color word won't in general yield a sense when it replaces a term with the use of number word, and conversely. Neither 'Red is the product of 3 and 2' nor '3 is brighter than pink' make sense. We can of course count each not true, and in that sense false. But in that sense a can opener also is false.

Just as 'number' serves as an index to generalization, and thus is dispensable, so also for 'color'. Suppose we lacked this word. We yet might say that Joseph had a coat of many colors by using a sentence like

For many f, Joseph's coat was f

where it was our practice to recognize as instances of this generalization only such sentences as

Joseph's coat was red
Joseph's coat was blue

and the like – that is, as *we* would say, to recognize as instances of this generalization only sentences formed with words for colors or words with *that* kind of use.

6. *Further objections and replies*

Objection 1. This whole involuted enquiry is predicated on the supposition that there is a doubt whether 'There are numbers' makes sense. But there is no plausibility in this so-called doubt. The words at issue are plain English. You may *claim* not to understand the sentence 'There are numbers', but you understand it nonetheless. This doubt is just a pretense, and the enquiry to which it has led has no point.

Reply. The sentence 'There are numbers' has a normal grammar and its words are familiar ones. Does that show that it has a sense? If it did, it would show as well that the sentence 'Three is red' has a sense. And it isn't *obvious* that it has a sense.

We grant that it *may* be that everybody does perfectly well understand this sentence, that it does make sense and that everybody grasps the sense it makes. It *may* be that it is only a false philosophy which keeps us from seeing clearly that this sentence makes a sense we grasp. But – for whatever reason – it yet is not *clear* to us that

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it says anything. In *that* sense our doubt is a real one, even if it is founded on nothing but the obscurities in our own thought.

Objection 2. You need to relax and learn to accept the obvious. It may help if you look at the matter this way: We all grasp the concept of number, and so surely can conceive that something falls under it.

Reply. There is no doubt that we grasp the concept of number. But what kind of concept is it? Is it a concept of a kind such that there are concepts of that kind under which things fall?

Not every concept is a concept of that kind. The concept of existence (as expressed in a sentence like 'There are lions') is not a concept of that kind, as is shown by the fact that what expresses this concept is a *sign for generality* (e.g., a quantifier), not a general term (e.g., a noun).

It may well be that the concept of number is expressed by our arithmetical language. But *what* about that language expresses the concept of number? Might it not be the *letters* 'm', 'n' and the like? We construct formulas using these letters and replace these letters by numerals and carry out certain inferences with formulas resulting from these by replacing letters by numerals.

Won't it be (roughly) our grasp of all this which constitutes our grasp of the concept number – not our mastery of one or another mathematical *predicate*. (And what we would need for 'There are numbers' is the kind of concept expressed by a predicate.)

Remember: Even if we had 'number' as a predicate in the language of natural number theory, it would be entirely useless. Our *actual* use of 'number' is for the expression of generality. It's use is like that of a subscript which reminds us which expressions are to go in for the letters.

Objection 3. You asked for something which supports the claim that 'There are numbers' makes sense. But why? Could *you* produce something which supports the claim that 'There are beets' makes sense? And if you couldn't, would that at all sap your confidence that it does?

That we are not sure how to show that a sentence makes sense is *no reason whatsoever* for doubting that it *does* make sense.

Reply. Consider the case of beets. Someone might read a description using 'beet', and then ask to be shown that there are such roots.

We know how to respond to this request. We bring various roots an see whether any fits the description, and find one does. We the agree: Yes, there are beets.

There is something analogous for 'square of 27'. An easy calcu lation shows that 729 is a square of 27. Having carried it out, w will then agree that, yes, there are squares of 27.

But the case for *number* doesn't fit this familiar pattern. Wha description do we have for 'number', so that we can decide whethe 27 fits that description? Shall we say that a number is a timeles entity? By what method might we find out that 27 fits that descrip tion? Or, what calculation shows that 27 is a number? No *calculatio* shows any such thing.

A *calculation* shows e.g., that 27 is 13 + 4 + 10. That 27 is number is not something which can be brought out *within* mathe matics.

Someone unfamiliar with our notation might ask whether 27 i a number. We could then exhibit our use of that sign. They woul then be satisfied. '27 is a number' can be used to express a recogni tion about the use of a sign.

Objection 4. This is just so much palaver. The key point gets los in all this talk. It is *clear* that we *all* realize that three is a numbe and that red, for example, isn't. And this recognition can *easily* b expressed in words – as easily as it *has* just been expressed i words!

Reply. If there *is* a recognition here, then it might be lacking. S let us suppose that someone failed to recognize that three is a number. What would they have missed? And how might their failur be remedied?

We here imagine a person who counts, adds, multiplies, applie the results of adding and multiplying, etc. We imagine a perso reasonably competent in the empirical application of mathematical terms, and in the arithmetic of those terms. This is enough to hav him be one who grasps the idea of three. But he is supposed to *fail to recognize* that three is a number.

So now we tell him something about numbers. Our hope is that once he gets this information he'll recognize that three is a number.

What do we tell him? Shall we say that a number is a timeless, placeless entity?

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7. *Final remarks*

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The doubt whether anything is asserted by 'There are numbers' is matched by the doubt whether anything is asserted by 'Three is timeless' and the like.

7. *Final remarks*

Both Carnap and his (imaginary) realist critic had it wrong about 'number' within mathematics. They each thought that 'There are numbers' is a theorem of arithmetic. But an examination of how arithmetic is actually set out reveals no such thing. 'Number' has no predicative use within arithmetic. Distinctions such as those between odds and evens, for example, get made in familiar ways. And how these distinctions get made does not draw on any cooked up predicate like 'number'.

It is true that one *could* cook up such a predicate. There are infinitely many degree-one formulas

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which become true sentences no matter which numeral is put for 'n'. The formula

n = n

is an example. If 'number' is defined by some such formula, then we say, with Carnap, that it is implausible to suppose that the content of mathematical realism comes down to the assertion of such sentences as '$\exists$n n = n' or that the content of anti-realism comes down to the denial of such sentences.

Carnap was on target with his point that philosophers speak about numbers from a point of view external to mathematics. Claims about numbers such as that five is one of them, that they exist, or that they don't exist are unmathematical claims.

A final point Carnap made with which we agree is that these unmathematical claims lack content. As is the case inside mathematics, 'number' has no clear use as a predicate outside of mathematics; rather, it is a sign for generality.