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Alessandro Torza *Editor*

Quantifiers, Quantifiers, and Quantifiers: Themes in Logic, Metaphysics, and Language

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Chapter 1
Quantification and Logical Form

(postprint version)

Andrea Iacona

This paper deals with the logical form of quantified sentences. Its purpose is to elucidate one plausible sense in which a considerably wide class of quantified sentences can be expressed in a classical first order language. Sections 1.1 and 1.2 provide some preliminary clarifications. Section 1.3 illustrates by means of familiar examples how the truth conditions of quantified sentences can formally be represented. Sections 1.4 and 1.5 show that the method of formalization suggested is consistent with some established undefinability results, and that it can easily be extended to a broad variety of cases. Section 1.6 draws a distinction between logical and non-logical quantifier expressions. Finally, section 1.7 adds some concluding remarks.

1.1 Two questions instead of one

The line of thought that underlies this paper stems from the idea that there is a crucial ambiguity in the question of what is the logical form of quantified sentences. This question can be construed in at least two ways:

(Q1) How are quantified sentences to be formally represented in order to account for the logical relations involving them?
(Q2) How are quantified sentences to be formally represented in order to provide a compositional account of their meaning?

At least *prima facie*, (Q1) and (Q2) are independent questions: one thing is to provide a formal explanation of the logical relations involving certain sentences, quite another thing is to provide a compositional account of the meaning of those sentences. However, the most common attitude towards (Q1) and (Q2) is to think that they are closely related, in that one and the same notion of logical form can provide an answer to both. As it will be explained in this section, the line of thought advanced here differs from two major views characterized by that attitude: one is old, the other is new.

According to the old view, which goes back to Frege, (Q1) is prior to (Q2), in that the notion of logical form that proves adequate to address (Q1) also provides an answer to (Q2). Consider the following sentences:

(1) All philosophers are rich
(2) Aristotle is rich

Frege suggested that there is a substantial difference between (1) and (2): although (1) is superficially similar to (2), its logical form substantially differs from that of (2). The difference that Frege had in mind turns out clear if (1) and (2) are formalized in a classical first order language. Let L be a first order language whose vocabulary includes a set of predicate letters $P, Q, R...$, a set of individual constants $a, b, c...$, a set of variables $x, y, z...$ and the connectives $\sim, \supset, \vee, \wedge, \forall, \exists$. (1) and (2) can be represented in L as follows:

(3) $\forall x(Px \supset Qx)$
(4) Pa

Here P stands for 'philosopher' and Q stands for 'rich'. If one regards this formalization as a guide to a compositional account of the meaning of (1), one will be inclined to think that, once we have an answer to (Q1), we also get an answer to (Q2)[1].

However, some doubts might be raised in connection with this view. First of all, it is not clear how (3) can figure as part of a compositional account of the meaning of (1), given that it does not explain the apparent semantic analogy between (1) and (2). (1) contains a noun phrase, 'all philosophers', which in many respect resembles 'Aristotle', while it does not contain the expression 'if...then...'. Secondly, even if (3) were regarded as the real semantic structure of (1), in spite of such disanalogies, it would still be an open question how a compositional account of (3) could be given. As it is well known, a definition of truth for the sentences of L can be provided in the way suggested by Tarski, assuming that the truth value of any formula $\forall x\alpha$ depends on the satisfaction conditions of α. However, Tarski's method does not guarantees compositionality. Since $\forall x\alpha$ is formed by adding $\forall x$ to α in accordance with the usual syntactic rule, in order for compositionality to hold, the truth value of $\forall x\alpha$ should result from the combination of the meaning of $\forall x$ with the meaning of α. But if truth in L is defined in the way outlined by Tarski, it is quite natural to read $\forall x$ as an expression that does not have meaning in isolation[2].

[1] This line of thought originates from Frege [**?**].

[2] Note that no clear alternative to this reading is provided by Frege's notion of "second-level function". One might be tempted to say that $\forall x$ denotes a second-level function F, so that the truth value of $\forall x\alpha$ is obtained by combining F with the meaning of α. But this is not a viable route. Let $\forall x\alpha$ be $\forall xPx$ and consider a variable y distinct from x. Do $\forall x$ and

According to the new view, which is currently adopted within formal approaches to natural language, (Q2) is prior to (Q1), in that the notion of logical form that proves adequate to address (Q2) also provides an answer to (Q1). In this case it is assumed that logical form is determined by syntactic structure, where syntactic structure is understood as LF, that is, as a formal representation that is distinct from surface structure and is the input of semantic interpretation. The LF of (1) and (2) may be represented as follows in order to provide a compositional account of their meaning:

(5) [Every philosopher$_1$[t_1 is rich]]
(6) [Aristotle[is rich]]

If one assumes that the logical form of a sentence is determined by its LF, one will be inclined to think that that the inferences involving (1) must be explainable in terms of (5). This is why now it is quite common to claim, against Frege, that the logical form of (1) does not substantially differ from that of (2)[3].

However, it might be argued that this view is not immune to troubles. If one assumes that the logical form of a sentence is determined by its LF, one will be unable to provide a formal explanation of all the logical relations in which the sentence may be involved. For some of those relations hold in virtue of the content expressed by the sentence. This turns out clear if we consider context sensitive sentences, which express different contents in different contexts even though their LF remains the same. To illustrate, consider (1) and the following sentence:

(7) Not all philosophers are rich

Imagine that you utter (1) with the intention to assert that all philosophers in your university are rich, while I utter (7) with the intention to assert that some philosophers in my university are not rich. There is an obvious sense in which we are not contradicting each other. But if the formal representation of (1) and (7) does not take into account the content they express, the apparent absence of contradiction is not formally explained. For the formula assigned to (7) must be the negation of the formula assigned to (1). More generally, let Γ be a set of sentences such that some of its members contain context sensitive expressions. In order to provide a formal explanation of the logical relations in Γ, the formal representation of Γ must display the semantic relations

$\forall y$ denote the same function? On the one hand, it seems that they should. If two functions assign the same values to the same arguments, as it is presumable in this case, then they are the same function. On the other, however, it seems that they should not. If $\forall x$ and $\forall y$ have the same meaning, then their meaning must be combinable in the same way with the meanings of other expressions. But $\forall yPx$ does not have the same meaning as $\forall xPx$. As a matter of fact $\forall yPx$ is not even a sentence, so it cannot be evaluated as true or false.

[3] The formal approaches to natural language derive to a good extent from Montague [**?**]. The view that a unique syntactic notion of logical form is able to provide both a compositional account of meaning and a formal explanation of logical properties emerges in several recent works such as Neale [**?**], Stanley [**?**] and Borg [**?**].

between the contents expressed by the sentences in Γ. However, this is not possible if logical form is individuated in terms of LF. For according to such a criterion of individuation, the logical form of each of the sentences in Γ does not depend on the content it expresses. Arguably, this is a serious limitation, which prevents any syntactic notion of logical form from being ideal for the purpose of formal explanation[4].

The misgivings considered suggest that neither of the two views is entirely satisfactory: on the one hand, it is not obvious how a compositional account of the meaning of quantified sentences can be provided by their representation in a classical first order language; on the other, it is not obvious how an adequate formal explanation of the logical relations involving quantified sentences can be provided in terms of their syntactic structure. Therefore, unless such misgivings are dispelled, it is reasonable to presume that there is something wrong with the uniqueness assumption that underlies both views, namely, the assumption that one and the same notion of logical form can provide answers to both (Q1) and (Q2).

In what follows it will be taken for granted that different notions of logical form can be employed to address (Q1) and (Q2). More specifically, the hypothesis that will be held about (Q1), which is the focus of this paper, is that the notion of logical form that suits the purpose of formal explanation is *truth conditional*, that is, it is a notion according to which logical form is determined by truth conditions. Since no uniqueness assumption about (Q1) and (Q2) will be adopted, this is compatible with there being a different notion of logical form that is suitable for (Q2). So it is compatible with the hypothesis that a syntactic notion of logical form is to be adopted to answer (Q2).

The truth conditional notion of logical form stems from the idea that an adequate formalization of a sentence s must provide a representation of what is said by uttering s. For what is said by uttering s cannot be represented unless the truth conditions of s are exhibited. Obviously, this does not mean that what is said by uttering s is reducible to the truth conditions of s, at least if truth conditions are understood as sets of possible worlds, and sameness of truth conditions is rendered as sameness of truth value in every structure. It is reasonable to presume that only some of the formulas that preserve the truth conditions of s in that sense adequately formalize s. For example, it is usually taken for granted that Fa is better than $\sim\sim Fa$ or $Fa \wedge (Gb \vee \sim Gb)$ as a representation of 'Fido is a dog': even though $\sim\sim Fa$ and $Fa \wedge (Gb \vee \sim Gb)$ have the same truth value as Fa in every structure, they do not capture what is said by using 'Fido is a dog' in the relevant sense of 'what is said'. Nonetheless, preservation of truth conditions may plausibly be regarded as a necessary condition of adequate formalization[5].

[4] Iacona [**?**] provides a more articulated defence of this claim.

[5] Sainsbury [**?**] suggests a criterion of adequate formalization that rests on the idea that formalization must preserve what is said, pp. 161-162.

1.2 Some terminology

In order to provide a formal account of quantified sentences based on a truth conditional notion of logical form, a principled distinction must be drawn between the meaning of quantified sentences and their truth conditions. This section introduces some terminology that will be employed to phrase the distinction.

In the first place, the term 'quantifier expression' will be used to refer to expressions such as 'all' or 'some', which occur in noun phrases as determiners of nominal expressions. In accordance with this use, we will restrict attention to simple quantified sentences that contain expression of this category, such as (1) or the following:

(8) Some philosophers are rich

In the second place, the term 'domain' will be used to refer to the totality of things over which a quantifier expression is taken to range. In ordinary talk, quantifier expressions often carry a tacit restriction to a set of contextually relevant objects. For example, on one occasion (1) may be used to assert that all philosophers in a university U are rich, while on another occasion it may be used to assert that all philosophers in another university U' are rich. So it is presumable that in the first case 'all' ranges over a set of people working or studying in U, while in the second it ranges over a set of people working or studying in U'. In order to take into account contextual restrictions of this kind it will be assumed that, whenever a quantifier expression is used, some domain is associated with its use, that is, the domain over which the quantifier expression is taken to range[6].

In the third place, the term 'quantifier' will be used to refer to functions from domains to binary relations. In accordance with this use, the meaning of 'all' may be defined as a quantifer *all*, that is, as a function which, for any domain D, denotes a binary relation that satisfies the following condition:

Definition 1. $all_D(A, B)$ if and only if $A \subseteq B$.

Here A and B are sets whose members belong to D, and the left-hand side is read as 'the relation denoted by 'all' relative to D obtains between A and B'.

The meaning of 'some' may be defined in similar way as a quantifier *some*, that is, as a function which, for any D, denotes a binary relation that satisfies the following condition:

[6] This assumption leaves unsettled the question of how the restriction is determined in the context. More specifically, it is neutral with respect to the divide between semantic and pragmatic accounts of domain restriction. The accounts of the first kind represent domains by some sort of parameters in the noun phrase, either in the determiner or in the noun. Those of the second kind, instead, leave the determination of domains to pragmatic factors which determine the communicated content as distinct from what is literally said.

Definition 2. $some_D(A, B)$ if and only if $A \cap B \neq \emptyset$[7].

The relativization to domains involved in definitions 1 and 2 accounts for the fact that the extension of a quantifier expression may vary from occasion to occasion, even though its meaning does not change. If e is a quantifier expression that means Q, then Q_D is the extension of e relative to D. Thus if D is a set of people working or studying in U and D' is a set of people working or studying in U', 'all' denotes different relations relative to D and D'. So there is a sense in which 'all' means the same thing on both occasions, yet the relations denoted differ. The same goes for 'some'. More generally, a distinction may be drawn between *global* quantifiers and *local* quantifiers, that is, between quantifiers as functions from domains to binary relations and quantifiers as values of such functions. If Q is a global quantifier and D is a domain, then Q_D is the local quantifier assigned by Q to D[8].

If the meaning of quantifier expressions is defined in the way outlined, and it is assumed that nominal expressions denote sets, the meaning of quantified sentences is easily obtained by composition. Let A and B be sets denoted by 'philosophers' and 'rich' relative to D. For example, if D is a set of people working or studying in U, A and B are subsets of that set. Given definition 1, all_D fixes truth conditions for (1) relative to D, that is, (1) is true if and only if $A \subseteq B$. So the meaning of (1) may be described as a function from domains to truth conditions, which results from the combination of *all* with the meanings of 'philosophers' and 'rich'. The case of (8) is similar. Assuming that A and B are sets denoted by 'philosophers' and 'rich' relative to D, the meaning of (8) may be described as a function from domains to truth conditions which results from the combination of *some* with the meanings of 'philosophers' and 'rich'. More generally, the meaning of a quantified sentence s that contains a quantifier expression e that means Q is a function from domains to truth conditions that is obtained by combining Q with the meaning of the nominal expressions in s. The value of the function for each D is determined by Q_D, that is, by the local quantifier assigned by Q to D.

1.3 Formalization and interpretation

Section 1.2 shows how a principled distinction can be drawn between the meaning of quantified sentences and their truth conditions. The meaning of a quantified sentence s results from the composition of the meanings of its constituent expressions, so it belongs to s independently of how s is understood on this or that occasion. The truth conditions of s, instead, are fixed

[7] Definitions 1 and 2 are as in Peters and Westerståhl [**?**], pp. 62-64.

[8] The distinction between global quantifiers and local quantifiers is drawn in Peters and Westerståhl [**?**], p. 48.

by the domain associated with the quantifier expression that occurs in s, so they depend just on how s is understood on this or that occasion.

Let an *interpretation* of a sentence be an assignment of semantic properties that determines definite truth conditions for the sentence in accordance with the meaning of its constituent expressions. On the formal account of quantified sentences that will be suggested, quantified sentences have logical form relative to interpretations. For interpretations fix domains for the quantifier expressions occurring in them.

The hypothesis that will be adopted is that quantified sentences can be formalized in L by means of formulas that represent their truth conditions relative to interpretations. To illustrate, consider (1). The simplest way to represent (1) in L is by means of (3). The representation provided by (3) includes no restriction on the domain. Note that the assumption that quantifier expressions are used in association with domains does not entail that, whenever one uses a quantifier expression, one has in mind a set of contextually relevant objects. It is consistent with that assumption to say that there are contexts in which nothing is excluded as irrelevant. So (3) represents (1) as used in such a context. In other words, (3) represents the truth conditions of (1) relative to an interpretation whose domain is the totality of everything.

In order to deal with a context in which some things are excluded as irrelevant, the intended restriction may be stated as part of the formula. Suppose that (1) is used to assert that all philosophers in U are rich. In this case, (1) may be represented as follows:

(9) $\forall x(Rx \supset (Px \supset Qx))$

Here R stands for a condition that applies to a set of people working or studying in U. So if two utterances of (1) differ in the intended restriction on the domain, they may be represented by means of different predicate letters. Suppose that (1) is used in one context to assert that all philosophers in U are rich and in another context to assert that all philosophers in U' are rich. This difference may be represented in terms of the difference between (9) and the following formula:

(10) $\forall x(Sx \supset (Px \supset Qx))$

Here S stands for a condition that applies to a set of people working or studying in U'. From (9) and (10) it turns out clear that (1) has different truth conditions relative to different interpretations. Note that if (1) and (7) are formalized in this way, the example considered section 1.1 can easily be handled as a case where no contradictory pair of formulas is involved.

The case of (8) is similar. The simplest way to represent (8) in L is the following:

(11) $\exists x(Px \wedge Qx)$

Again, this representation includes no restriction on the domain. In order to deal with a context in which some things are excluded as irrelevant, the intended restriction may be stated as part of the formula. From now on, however, considerations about restricting conditions will be avoided for the sake of simplicity.

1.4 The issue of first order definability

A major implication of the thesis that quantified sentences can be formalized in L in virtue of their truth conditions concerns a fact that is usually regarded as decisive for the issue of the expressive power of classical first order logic. The fact is that some quantifier expressions are not *first order definable*, in the sense that they do not denote quantifiers that satisfy the following condition:

Definition 3. A quantifier Q is *first order definable* if and only if there is a formula α of L containing two unary predicate letters such that, for every set D and $A, B \subseteq D$, $Q_D(A, B)$ if and only if α is true in a structure with domain D where the predicate letters in α denote A and B.

As it is easy to verify, 'all' is first order definable, because (3) is a formula of L containing two unary predicate letters such that, for every set D and $A, B \subseteq D$, $all_D(A, B)$ if and only if (3) is true in a structure with domain D where its predicate letters denote A and B. The same goes for 'some', given that (8) can be represented as (11).

However, not all quantifier expressions are like 'all' and 'some'. Consider the following sentence, which contains the quantifier expression 'more than half of':

(12) More than half of philosophers are rich

The quantifier *more than half of* may be defined as a function which, for any D, denotes a binary relation that satisfies the following condition:

Definition 4. $more\ than\ half\ of_D(A, B)$ if and only if $\mid A \cap B \mid > 1/2 \mid A \mid$

Although this definition differs from definitions 1 and 2 in that it involves a proportional relation that applies to the cardinality of A and B, *more than half of* is a function from domains to binary relations exactly like *all* and *some*. So (12) is semantically similar to (1) and (8), in that it is formed by expressions of the same semantic categories combined in the same way. However, there is no formula of L that translates (12) in the same sense in which (3) and (11) translate (1) and (8). This is to say that 'more than half of' is not first order definable[9].

Many are inclined to think that this fact constitutes a serious limitation of the expressive power of first order logic. If it is assumed that formalization

[9] Barwise and Cooper [?], pp. 213-214, provides a proof of the first order undefinability of 'more than half of'.

is a matter of translation, understood as meaning preservation, then it is natural to think that there is no way to formalize (12) in L. More generally, one may be tempted to think that a quantified sentence can be formalized in L only if the quantifier expressions it contains are first order definable[10].

Without that assumption, however, there is no reason to think that the first order undefinability of more than half of' rules out the possibility that (12) is formalized in L. Certainly, it undermines the claim that there are sentences of L that have the same meaning as (12). But if logical form is determined by truth conditions, such a claim makes little sense anyway, even in the case of (1) and (8). For formalization is not a matter of translation, but a matter of representation of truth conditions.

Instead of asking whether a quantifier expression is first order definable, one may ask whether it is *first order expressible*, that is, whether it denotes a quantifier that satisfies the following condition:

Definition 5. A quantifier Q is *first order expressible* if and only if, for every set D and $A, B \subseteq D$, there is an adequate formula α of L containing two unary predicate letters such that $Q_D(A, B)$ if and only if α is true in a structure with domain D where the predicate letters denote A and B.

The sense in which α is required to be adequate is the same sense in which a formalization is expected to be adequate, as explained in section 1.1: α must represent what is said, relative to D, by a sentence which contains a quantifier expression that denotes Q and two predicates for A and B. Clearly, adequacy so understood cannot be phrased in formal terms, as the notion of what is said is irreducibly vague. However, the condition that α is adequate is clear enough for the purposes at hand, or so it will be assumed.

To see how adequacy matters, it suffices to think that a trivial proof of the existence of α can easily be provided if no such condition is imposed on α. For it easy to find some α that has the required truth value in the structure for independent reasons. For example, if $Q_D(A, B)$ and α is a logical truth, then $Q_D(A, B)$ if and only if α is true in the structure. However, it is clear that in this case α is not adequate. The same goes for similar trivial proofs of the existence of α. What is not trivial, instead, is to prove the existence of an adequate α. As it will be shown, 'more than half of' is first order expressible, in that for every D and $A, B \subseteq D$, there is an adequate sentence α of L containing two predicate letters such that *more than half of*$_D(A, B)$ if and only if α is true in a structure with domain D where the predicate letters denote A and B.

The proof that will be provided rests on two assumptions. The first is that A and B are finite. This is an assumption that one can plausibly make when one restricts attention to natural language, for 'more than half of' is normally used to state relations between finite quantities, as indicated by the proportion 1/2 that occurs in definition 4. This is not to deny that 'more than

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[10] As in Barwise and Cooper [?], p. 159.

half of' can be used in some intelligible way for infinite domains. Presumably, some technical or semi-technical meaning can be specified for that purpose. However, infinitary uses of 'more than half of' will not be considered in what follows. Independently of how such uses relate to the ordinary understanding of the expression, the reasoning simply will not apply to them[11].

The second assumption is that, if what is said by s relative to D is that at least n As are Bs, then a formula of L that contains n occurrences of $\exists$ and two unary predicates P and Q can provide an adequate representation of s. More precisely, let the symbol $\exists_{\geq n}$ be used to abbreviate formulas of L in the following way: $\exists_{\geq n}\bar{x}\alpha(\bar{x})$ means $\exists x_1 ... \exists x_n(\alpha(x_1) \wedge ... \wedge \alpha(x_n) \bigwedge_{1\leq i<j\leq n} x_i \neq x_j)$, where $\alpha(x_i)$ is a formula in which x_i occurs free, and in the second part of the conjunction every x_i is said to differ from every other. Then, if what is said by s relative to D is that at least n As are Bs, then the following formulas adequately represents s:

(13) $\exists_{\geq n}\bar{x}P(\bar{x}) \wedge Q(\bar{x})$

For example, suppose that D includes some persons, and that three of them are philosophers. Then what is said by (12) relative to D is that at least two philosophers are rich, which is adequately represented by the formula $\exists x\exists y(Px \wedge Qx \wedge x \neq y)$.

Given these two assumptions, the first order expressibility of 'more than half of' can be proved in two steps.

Theorem 1. *If $A, B \subseteq D$, there is an n such that $\mid B \mid > 1/2 \mid A \mid$ if and only if $\mid B \mid \geq n$.*

Proof. Let F be a function defined as follows. If $m = 0$, then $F(m) = 1$. If $m > 0$ and m is even, then

$$F(m) = \frac{m+2}{2}$$

If $m > 0$ and m is odd, then

$$F(m) = \frac{m+1}{2}$$

Let $\mid A \mid = m$ and $n = F(m)$. n is such that $\mid B \mid > 1/2 \mid A \mid$ if and only if $\mid B \mid \geq n$. Suppose that $m = 0$. Then $1/2 \mid A \mid = 0$ and $F(m) = 1$, so $\mid B \mid > 0$ if and only if $\mid B \mid \geq 1$. Suppose that $m > 0$ and m is even. Then there is a k such that $m = 2k$, hence $\mid B \mid > 1/2 \mid A \mid$ if and only if $\mid B \mid > k$. Moreover,

$$F(m) = \frac{m+2}{2} = \frac{2k+2}{2} = \frac{2(k+1)}{2} = k+1$$

Therefore, $\mid B \mid > k$ if and only if $\mid B \mid \geq k+1$. Finally, suppose that $m > 0$ and m is odd. Then there is a k such that $m = 2k+1$, hence $\mid B \mid > 1/2 \mid A \mid$

[11] Barwise and Cooper [?], p. 163, consider infinitary uses of 'more than half of'.

if and only if $\mid B \mid > k + 1/2$. By hypothesis, $\mid B \mid$ is a natural number, so $\mid B \mid > k + 1/2$ if and only if $\mid B \mid > k$. Moreover,

$$F(m) = \frac{m+1}{2} = \frac{2k+1+1}{2} = \frac{2(k+1)}{2} = k+1$$

Therefore, $\mid B \mid > k$ if and only if $\mid B \mid \geq k + 1$.

Theorem 2. *For every D and $A, B \subseteq D$, there is an adequate sentence α of L that contains two unary predicate letters such that more than half of$_D(A, B)$ if and only if α is true in a structure with domain D where the predicate letters denote A and B.*

Proof. Let $A, B \subseteq D$. From theorem 1, replacing B with $A \cap B$, it turns out that there is an n such that $\mid A \cap B \mid > 1/2 \mid A \mid$ if and only if $\mid A \cap B \mid \geq n$. By definition 4, there is an n such that *more than half of*$_D(A, B)$ if and only if $\mid A \cap B \mid \geq n$. The condition that $\mid A \cap B \mid \geq n$ is adequately expressed in L by (13). Moreover, (13) is true in a structure with domain D where P and Q denote A and B, and more than half the As are Bs[12].

Theorem 1 expresses the obvious truth that, for every finite set, there is an n such that saying 'more than half of' amounts to saying 'at least n'. This guarantees that, although the global quantifier *more than half of* is characterized by a proportional relation, each local quantifier *more than half of*$_D$ fixes a non-proportional relation expressible in L. Theorem 2, accordingly, "squeezes" a proportional relation on a set of non-proportional relations. So we get that, for any domain, (12) has a logical form representable in L relative to that domain. This means that, for any interpretation, (12) has a logical form representable in L relative to that interpretation.

1.5 Generalization

The account of 'more than half of' suggested in section 1.4 may easily be extended to other quantifier expressions whose meaning is definable in terms of proportional relations, such as 'most', 'few' and 'many'. Even though 'most', 'few' and 'many' exhibit a kind of indeterminacy that does not affect 'more than half of', in that they admit multiple admissible readings, this difference does not prevent them from being amenable to the same kind of treatment that applies to 'more than half of'.

To illustrate, let us focus on 'most'. A basic fact about its meaning seems to be that the condition stated in definition 4 must be satisfied for the intended relation to obtain. Consider the following sentence:

[12] The number triangle method outlined by Peters and Westerståhl in [**?**], pp. 160-161, provides a clear visual representation of the fact that *more than half* determines an n on every finite domain.

(14) Most philosophers are rich

If one utters (14), one says at least that more than half of philosophers are rich. However, this is a necessary but not a sufficient condition. Although 'most' may be used as synonymous of 'more than half of', its meaning seems to allow for variation in the proportion between the size of $A \cap B$ and the size of A. In order to account for this variation, a definition of *most* may be given along the following lines:

Definition 6. $most_D(A, B)$ if and only if $| A \cap B | > n/m | A |$

Here $0 < n < m$ and $n/m \geq 1/2$. For example, 1/2 and 2/3 are equally admissible values for n/m. In other words, *most* is defined as a class of quantifiers rather than as a single quantifier. Consequently, the meaning of (14) may be described as a class of functions from domains to truth conditions that is obtained by combining *most* with the meanings of 'philosophers' and 'rich'. This means that (14) differs from (12), in that the determination of its truth conditions involves a parameter other than the domain. Let A and B be the sets denoted by 'philosophers' and 'rich' relative to D. Whether $most_D$ obtains between A and B depends on the values assigned to n and m. For example, if $n = 2$ and $m = 3$, then it obtains just in case $| A \cap B | > 2/3 | A |$. In order to determine definite truth conditions for (14), we need both a domain and a value of the additional parameter[13].

If *most* is defined in the way suggested, the distinction between first order definability and first order expressibility drawn in section 1.4 can be applied to (14). Although it is a fact that 'most' is not first order definable, on the assumption that logical form is determined by truth conditions (15) can be formalized in L independently of this fact. For what matters is that 'most' is first order expressible[14].

To show that (14) can be formalized in L, it suffices to prove a squeezing result similar to theorem 2. This can be done by means of a generalization of theorem 1: if $A, B \subseteq D$ and $0 < n < m$, there is a k such that $| B | > n/m | A |$ if and only if $| B | \geq k$. From such generalization it follows that, for every D and $A, B \subseteq D$, there is an adequate sentence α of L that contains two unary predicate letters such that $most_D(A, B)$ if and only if α is true in a structure with domain D where the predicate letters denote A and B.

As in the case of 'most', the meanings of 'few' and 'many' may be defined as classes of quantifiers *few* and *many*. So it may be assumed that the meaning of the following sentences is obtained by combining *few* and *many* with the meanings of 'philosophers' and 'rich':

(15) Few philosophers are rich

[13] Definition 6 is in line with the suggestion in Barwise and Cooper [**?**], p. 163, and the account in Westerståhl [**?**], pp. 405-406. In the latter work, two readings of 'most' are considered. But if definition 6 is adopted there seems to be no reason to do that.

[14] Peters and Westerståhl, in [**?**], pp. 466-468, outline a proof method that can be employed to show that 'most' and other proportional quantifiers are not first order definable.

(16) Many philosophers are rich

The meaning of (15) and (16) may thus be described as a class of functions from domains to truth conditions. This suggests that, as in the case of *most*, a squeezing argument can be provided to the effect that *few* and *many* are first order expressible[15].

In substance, (14)-(16) can be treated in the same way as (12), with the only difference that in the case of (14)-(16) some parameter other than the domain must be taken into account as relevant to the determination of truth conditions. Therefore, on the assumption that an interpretation of (14)-(16) includes both a domain and a value for such a parameter, it turns out that, for every interpretation of (14)-(16), there is a formula of L that represents the truth conditions of (14)-(16) relative to that interpretation.

1.6 Logicality

The point that emerges from sections 1.4 and 1.5 is that it must not be assumed that first order definability is the property to be considered in order to settle the question whether quantified sentences can adequately be formalized in a classical first order language. On the formal account of quantified sentences suggested here, the property to be considered is first order expressibility. This does not mean, however, that first order definability is not a significant property. As it will be suggested, there is a straightforward relation between first order definability and *logicality*.

The quantifier expressions traditionally studied by logicians, such as 'all' or 'some', have always been regarded as paradigmatic examples of logicality. However, there are many more quantifier expressions than those traditionally studied by logicians. So it is natural to ask whether all quantifier expressions must be classified as logical. According to Barwise and Cooper they must not, in that there is no reason to think that the meaning of every quantifier expression is to be "built into the logic". A distinction must be drawn between logical and non-logical quantifier expressions: 'all' and 'some' belong to the first category, while 'more than half', 'most', 'many' and 'few' belong to the second. The method of formalization adopted here provides one way to substantiate this distinction[16].

We saw that, for every interpretation of a quantified sentence s, there is a formula of L that represents the truth conditions of s on that interpretation. Therefore, different formulas of L may represent s on different interpretations. But there are basically two ways in which the formal representation of s can vary as a function of its interpretation. Consider (1) and (12). In the case

[15] The case of 'few' and 'many' is definitely more controversial. For example, Keenan and Stavi [**?**] excludes that 'few' and 'many' can be treated in this way.

[16] Barwise and Cooper [**?**], p. 162.

of (1), the variation concerns at most the non-logical vocabulary of L, as in (9) and (10). In the case of (12), instead, it may also concern the logical vocabulary of L. For example, the following formulas of L represent the logical form of (12) on different interpretations:

(17) $\exists_{\geq 3}\bar{x}P(\bar{x}) \wedge Q(\bar{x})$
(18) $\exists_{\geq 4}\bar{x}P(\bar{x}) \wedge Q(\bar{x})$

One thing is to say that more than half of five things have a certain property, quite another thing is to say that more than half of six things have that property.

The contrast between the two cases considered may be described in terms of two kinds of variation in the formal representation of a sentence *s*. A *weak variation* in the formal representation of *s* depends on some difference in the non-logical vocabulary of the formulas assigned to *s*. Instead, a *strong variation* in the formal representation of *s* depends on some difference in the logical vocabulary of the formulas assigned to *s*. So, the first case may be described as one in which a difference between two interpretations entails weak variation in the formal representation of (1), as in (10) and (11), while the second may be described as one in which a difference between two interpretations entails strong variation in the formal representation of (13), as in (18) and (19).

There is a plausible sense in which weak variation, unlike strong variation, does not entail difference in logical form. This is to say that *sameness of logical form* may be understood in terms of weak variation: *s* has the same logical form on two interpretations if and only if the difference between them entails at most weak variation in the formal representation of *s*. Logicality may be defined in terms of sameness of logical form so understood:

Definition 7. A quantifier expression is *logical* if and only if every quantified sentence in which it occurs has the same logical form on all interpretations[17].

From definition 7 it turns out that ‘all’ is logical. For (1) has the same logical form on all interpretations, whether or not its formalization includes a restricting condition. The same goes for (8). By contrast, ‘more than half of’, ‘most’, ‘many’ and ‘few’ are non-logical, for (12) and (14)-(16) have different logical forms on different interpretations.

Note that the sense of ‘logical’ provided by definition 7 is essentially relative, in that it depends on the choice of logical constants that underlies the language in which logical forms are expressed. On the assumption that logical forms are expressed in L, ‘logical’ is to be read as relative to L. This, however, should not be regarded as a flaw. Definition 7 is neutral with respect to the notoriously controversial question of whether an absolute criterion of logical constancy can be specified in non-circular way. If the answer to that question

[17] Note that, given the restriction mentioned in section 1.2, ‘quantified sentence’ refers to simple quantified sentences such as (1) or (12). This rules out obvious counterexamples such as ‘Most but not all philosophers are rich’.

is affirmative, then it is presumable that some independent justification of the choice of logical constants that underlies L can be provided. If it is negative, instead, then the choice of logical constants that underlies L is itself in need of justification, so an account of logicality based on L is definitely circular. Even though it is arguable that only in the first case we can get an interesting distinction between logical and non-logical quantifier expressions, in any case the relativity involved in definition 7 causes no trouble by itself.

There is a straightforward relation between logicality so defined and first order definability:

Theorem 3. *Every logical quantifier expression is first order definable.*

Proof. Let us assume that e is a logical quantifier expression that denotes a quantifier Q, and that s is a quantified sentence in which e occurs. Let α be a formula of L which contains two predicate letters and represents the truth conditions of s on some interpretation with domain D. Then it must be the case that, for $A, B \subseteq D$, $Q_D(A, B)$ if and only if α is true in a structure with domain D where the predicate letters in α denote A and B. Now take any domain D'. For some interpretation with domain D', there is a formula α' of L such that α' represents the truth conditions of s, so that, for $A', B' \subseteq D'$, $Q_{D'}(A', B')$ if and only if α' is true in a structure with domain D' where the predicate letters in α' denote A' and B'. But since e is logical, s has the same logical form on all interpretations. This means that α and α' differ at most in the predicate letters. Therefore, α' is true in a structure with domain D' where the predicate letters in α' denote A' and B' if and only if α is true in a structure with D' where the predicate letters in α denote A' and B'. This is to say that α satisfies the condition required by definition 3, so that e is first order definable.

Theorem 3 characterizes logical quantifier expressions as first order definable quantifier expressions. This characterization entails that every quantifier expression that is not first order definable is not logical. So, the point that has been made in sections 1.4 and 1.5 may be refined as follows. Quantifier expressions such as 'more than half of', 'most', 'few' and 'many' are not first order definable. But this does not entail that the quantified sentences in which they occur cannot be formalized in a classical first order language. What it entails is at most that they are not logical[18].

[18] There is an interesting convergence between the account of logical quantifier expressions suggested here and the independently motivated account outlined in Feferman [**?**], see p. 140. As it is noticed in that work, pp. 144-145, it is not as obvious as it might seem that the converse of theorem 3 is guaranteed to hold.

1.7 Conclusion

From the analysis of quantified sentences suggested in the previous sections it turns out that there is something right and something wrong in each of the two views considered in section 1.1. On the one hand, there is a sense in which it is right to say that (1) and (2) are structurally different, namely, that in which (1) and (2) are adequately represented as (3) and (4) in order to formally explain the inferences involving them. On the other, there is a sense in which it is right to say that (1) and (2) are structurally similar, namely, that in which (1) and (2) are adequately represented as (5) and (6) in order to provide a compositional account of their meaning. What is wrong is to think that there must be a unique sense in which either (1) and (2) are structurally different or they are structurally similar. On the understanding of logical form that is suitable to address (Q1) they are structurally different, while on the understanding of logical form that is suitable to address (Q2) they are structurally similar. This is just another way of saying that there is no unique answer to the question of what is the logical form of quantified sentences.