Are Dynamic Shifts Dynamical Symmetries?

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Abstract

Shifts are a well-known feature of the literature on spacetime symmetries. Recently, discussions have focused on so-called dynamic shifts, which by analogy with static and kinematic shifts enact arbitrary linear accelerations of all matter (as well as a change in the gravitational potential). But in mathematical formulations of these shifts, the analogy breaks down: while static and kinematic shift act on the matter field, the dynamic shift acts on spacetime structure instead. I formulate a different, 'active' version of the dynamic shift which does act on matter.

1 Introduction

The literature on spacetime symmetries is replete with so-called 'shifts':[1]

> **Static shift**: a uniform translation of all material content of the universe.
>
> **Kinematic shift**: a uniform boost of the velocities of all material content of the universe.
>
> **Dynamic shift**: an arbitrary linear time-dependent acceleration of all material content of the universe, jointly with an appropriate transformation of the gravitational potential.

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[1]The static and kinematic shift were discussed by Leibniz and Clarke in their correspondence (Alexander, 1977). Saunders (2013) argues that Newton was well aware of the possibility of dynamic shifts. The terms 'static shift' and 'kinematic shift' were coined by Maudlin (1993), and Huggett (1999) first used 'dynamic shift'.

All three shifts are symmetries of NG: transformations that preserve the theory's dynamics. Moreover, since these transformations leave the theory's observable quantities—distances, relative velocities and relative accelerations—the same, the possibilities they relate are empirically equivalent.

As I have phrased them, all shifts involve a transformation of the universe's matter content, leaving the spacetime within which that matter is located the same. In this, the dynamic shift is analogous to the static and kinematic shifts. This correspondence is emphasised in Saunders' (2013, 37) paraphrase of Newton's Corollary VI:

> If bodies moved in any manner among themselves, are urged in the direction of parallel lines by equal gravitational forces due to outside bodies, they will all continue to move among themselves, after the same manner as if they had not all been urged by that force.

However, in contemporary mathematical treatments a discontinuity appears. In mathematical physics, shifts are expressed as transformations that act on certain geometric objects defined over a manifold M. For the first two shifts, these are pushforward maps of diffeomorphisms acting on objects that represent the universe's matter content: the matter field ρ, its associated gravitational potential φ, and the trajectories ξ^a of test particles. But the dynamic shift breaks with this trend. In standard expositions, dynamic shifts act not on the matter field ρ but on spacetime's standard of inertial motion: the covariant derivative operator ∇ (see, for instance, Friedman (1983) or Malament (2012)). The effect of such a transformation, literally understood, is *not* to accelerate the trajectories of all material bodies. Rather, it is to leave those trajectories the same while adopting a different convention for which trajectories to call 'accelerating'. In this sense, the contemporary version of the dynamic shift is a *passive* transformation: it only changes how we describe the motion of matter, which is itself left the same. (Of course, such transformations are not passive in the sense that they merely affect a change in coordinates: a substantivalist will understand a dynamic shift as changing the real structure of spacetime itself).

The lesson drawn from passive dynamic shifts is slightly different than that usually drawn from the static and kinematic shifts, too. The standard response to the static and kinematic shifts is that quantities that vary under them, namely absolute position and absolute velocity, are not physically real. But the lesson drawn from the dynamic shift is not that absolute acceleration is not real (indeed, the preferred 'successor theory', Newton-Cartan Theory, has its own absolute standard of acceleration).[2] Rather, it is that "the gravity/inertia split does not reflect physical structure," but is "a mere artefact of our representation" (Knox, 2014, 871).

[2] Saunders (2013) is an exception here; see also §6 below.

The aim of this paper is to formulate an 'active' version of the dynamic shift: one that acts on matter rather than spacetime. The rationale for this project is to strengthen the parallels between successive shifts and to arrive at a formulation that is more faithful to non-mathematical characterisations of the dynamic shift. In addition, the possibility of *passive* dynamic shifts has led to several reformulations of NG: on the one hand, Newton-Cartan Theory, and Maxwell Gravitation on the other. I will consider whether these theories fare well with respect to an active version of the dynamic shift. The answer is 'Yes', but in different ways: for the former, active dynamic shifts are neither dynamical nor spacetime symmetries, while for the latter they are both. I will use this difference to put pressure on recent claims that Newton-Cartan Theory and Maxwell Gravitation are in some sense equivalent. For instance, Weatherall (2016, 89) contends that "[i]n a sense, [Maxwell Gravitation] simply is Newton-Cartan theory", while Dewar (2018, 261) claims that "the two theories might be regarded as equivalent over the nonvanishing-mass sector".[3] In a similar vein, Wallace (2020) concludes that "there is essentially no difference between Newton-Cartan theory [...] and Saunders's relational version of Newtonian dynamics", the latter arguably being a version of Maxwell Graviation.

The plan is as follows. In §2, I briefly set out some background: the precise formulation of Newtonian Gravitation that I will employ, the mathematical form of shifts in this formalism, and an account of symmetries based on Earman (1989). In §3, I formulate the symmetry transformation that corresponds to an *active* dynamic shift. In §4 and §5, I consider the consequences of such shifts for Newton-Cartan Theory and Maxwell Gravitation respectively. In §6, I consider some important differences between these successor theories, in particular with respect to recent claims that Newton-Cartan Theory and Maxwell Gravitation are equivalent.

2 Gravitation, Shifts and Symmetries

I will assume the semantic view of theories, on which a theory is presented through a class of models. The kinematically possible models (KPMs) of Newtonian Gravitation (NG) set on Galilean spacetime are tuples of the form $\langle M, t_{ab}, h^{ab}, \nabla, \varphi, \rho, \{\xi^a\}\rangle$, where M is a four-dimensional smooth manifold diffeomorphic to $\mathbb{R}^4$; t_{ab} and h^{ab} are compatible temporal and spatial metrics ($t_{an}h^{nb} = 0$); ∇ is a derivative operator encoding a standard of uniform motion compatible with both metrics ($\nabla_a t_{bc} = \nabla_a h^{bc} = 0$); φ and ρ are scalar fields that represent the gravitational field and the matter distribution respectively; and $\{\xi^a\}$ is a set of timelike vector fields that represents the four-velocities

[3] Note that in this paper I only consider non-vanishing ρ.

of test particles.[4] The dynamically possible models (DPMs) or 'solutions' of NG in addition satisfy the following equations:

$$R^a{}_{bcd} = 0 \tag{1}$$

$$h^{ab}\nabla_a \nabla_b \varphi = 4\pi\rho \tag{2}$$

$$-\nabla^a \varphi = \xi^b \nabla_b \xi^a \tag{3}$$

where $R^a_{bcn}\xi^n = \nabla_{[b}\nabla_{c]}\xi^a$. Here, (1) imposes flatness on ∇; (2) is the Newton-Poisson equation; and (3) is the analogue of Newton's second law for test particles.

Following Earman (1989), we can distinguish between a theory's spacetime symmetries and dynamical symmetries. Suppose that we can partition the theory's geometric objects into *absolute* objects A and *dynamical* objects P. The former represent "fixed spacetime structure", while the latter represent "the physical contents of spacetime" (Earman, 1989, 45). Let $*$ denote the map such that for any diffeomorphism d and any geometric object O, $d_*O(d(p))$ has the 'same' value as $O(p)$.[5] Then a *spacetime symmetry* is a diffeomorphism d such that $d_*A = A$ for all absolute objects, while a *dynamical symmetry* is a diffeomorphism d such that $\langle M, A_1, ..., A_n, P_1, ..., P_n\rangle$ is a solution iff $\langle M, A_1, ..., A_n, d_*P_1, ..., d_*P_n\rangle$ is.

Earman also lays down a famous set of symmetry principles—SP1 and SP2—to the effect that every spacetime symmetry is a dynamical symmetry and vice versa. In particular, these principles entail that if a theory's dynamical symmetries outstrip its spacetime symmetries then that theory has 'too much' spacetime structure. In this way, one can use symmetries as a guide towards superfluous spacetime structure.

We can define static and kinematic shifts as follows. The relevant diffeomorphisms are defined indirectly in terms of a coordinate representation (cf. Earman (1989, Ch. 2)). Let ψ be an inertial coordinate system for M,[6] and let $\psi(p) = (t, \vec{x})$ for some arbitrary point p. Then the action of a translation-diffeomorphism is such that $\vec{x} \to \vec{x} + \vec{c}$ for some constant vector $\vec{c}$. Likewise, the action of a boost-diffeomorphism is such that $\vec{x} \to \vec{x} + \vec{v}t$ for some constant vector $\vec{v}$. (Of course, one can also combine boosts and translations. If one incorporates rotations and time-translations as well, then the result is the Galilei group of transformations. For simplicity, I will focus on 'pure' translations and boosts in what follows.)

We can use these diffeomorphisms to mathematically formulate the shifts:

[4] Here and in what follows, I use small-case $a, b, c, ...$ as 'abstract' indices, small-case $\mu, \nu, \sigma, ...$ as component indices ranging over both time and space, and small-case $i, j, k, ...$ as component indices ranging over space only.

[5] For scalar fields, this means that $d_*O(d(p)) = O(p)$; for tensor fields, $*$ is the pushforward map

[6] An inertial coordinate system is one in which the connection $\Gamma^a_{bc} = 0$, $t_{ab} = (1, 0, 0, 0)$ and $h^{ab} = (0, 1, 1, 1)$. Friedman (1983, Ch.2) shows that inertial coordinates always exist for flat spacetimes.

Static Shift: Where α is a diffeomorphism that enacts a uniform translation, a static shift is a transformation:

$$\langle M, A_1, ..., A_n, P_1, ..., P_n\rangle \rightarrow \langle M, A_1, ..., A_n, \alpha_* P_1, ..., \alpha_* P_n\rangle$$

Kinematic Shift: Where β is a diffeomorphism that enacts a uniform velocity boost, a kinematic shift is a transformation:

$$\langle M, A_1, ..., A_n, P_1, ..., P_n\rangle \rightarrow \langle M, A_1, ..., A_n, \beta_* P_1, ..., \beta_* P_n\rangle$$

In our formulation of NG, the absolute objects are t_{ab}, h^{ab} and ∇, while the dynamical objects are φ, ρ and $\{\xi^a\}$. The claim that static and kinematic shifts are dynamical symmetries of NG then means that, for these diffeomorphisms, $\langle M, t_{ab}, h^{ab}, \nabla, \varphi, \rho, \{\xi^a\}\rangle$ is a solution of the theory iff $\langle M, t_{ab}, h^{ab}, \nabla, \alpha_*\varphi, \alpha_*\rho, \{\alpha_*\xi^a\}\rangle$ respectively $\langle M, t_{ab}, h^{ab}, \nabla, \beta_*\varphi, \beta_*\rho, \{\beta_*\xi^a\}\rangle$ are solutions too. The static and kinematic shifts are also spacetime symmetries, since $\alpha_* t_{ab} = \beta_* t_{ab} = t_{ab}$ and same for h^{ab} and ∇. So, Galilean spacetime has the right amount of structure with respect to these symmetries.

Meanwhile, passive dynamic shifts are defined as follows (cf. Malament (2012, Prop. 4.2.5):[7]

Passive Dynamic Shift: For $\nabla' = (\nabla, t_b t_c \nabla^a(\varphi' - \varphi))$ and φ' such that $\nabla^a\nabla^b(\varphi' - \varphi) = 0$, a dynamic shift is a transformation:

$$\langle M, t_{ab}, h^{ab}, \nabla, \varphi, \rho, \{\xi^a\}\rangle \rightarrow \langle M, t_{ab}, h^{ab}, \nabla', \varphi', \rho, \{\xi^a\}\rangle$$

The contrast with static and kinematic shifts is stark: a passive dynamic shifts leaves ρ and $\{\xi^a\}$ alone, and instead acts on the covariant derivative ∇ in addition to φ. It is this disanalogy that motivates the search for an alternative formulation of the dynamic shift in the next section.

In particular, it is unclear where passive dynamic shifts fall with respect to Earman's symmetry principles. Firstly, such shifts are not generated by diffeomorphisms, and secondly, they act neither solely on absolute nor solely on dynamical objects. The latter

[7] The notation here derives from Malament (2012, Prop 1.7.3): $\tilde{\nabla} = (\nabla, C^a_{bc})$ iff for all smooth tensor fields $\alpha^{a_1...a_r}_{b_1...b_s}$,

$$(\tilde{\nabla}_m - \nabla_m)\alpha^{a_1...a_r}_{b_1...b_s} = \alpha^{a_1...a_r}_{nb_2...b_s} C^n_{mb_1} + ... + \alpha^{a_1...a_r}_{b_1...b_{s-1}n} C^n_{mb_1} - \alpha^{na_2...a_r}_{b_1...b_s} C^{a_1}_{mn} - ... - \alpha^{a_1...a_{r-1}n}_{b_1...b_s} C^{a^r}{}_{mn}$$

point in particular means that it is difficult to apply Earman's principles. Does the fact that passive dynamic shifts are symmetries of NG entail that the theory has surplus spacetime structure? The answer is not necessarily, since such transformations also affect spacetime structure itself. We cannot easily separate out the effect on spacetime from the effect on matter. Of course, the possibility of dynamic shifts has still led philosophers to conclude that Galilean spacetime has too much structure with respect to the dynamics of NG all the same—and rightly so! But I contend that an active version of the dynamic shift, which acts solely on the universe's matter content, will prove the same point more explicitly. Therefore, the applicability of Earman's symmetry principles provides a further motivation for the formulation of an active version dynamic shift.

3 Active Dynamical Shifts

In this section, I will define active dynamic shifts and show that they, too, are symmetries of NG. First, we define an *acceleration-diffeomorphism*: in an inertial coordinate system ψ, such a diffeomorphism enacts the map $\vec{x} \to \vec{x} + \vec{a}(t)$ where $\vec{a}(t)$ is an arbitrary twice-differentiable vector-valued function of t. (Note that this class of transformations subsumes both boosts and translations. As before, we can expand the class of transformations to include rotations and translations in time: the result is what Earman (1989, §2.3) calls the Maxwell group. Here we will only consider 'pure' accelerations for simplicity.)

We can then formulate an active dynamic shift as follows:

> **Active Dynamic Shift**: Let δ be a diffeomorphism that enacts an arbitrary linear acceleration. If $P_i = \varphi$ for some i, a dynamic shift is a transformation:
>
> $$\langle M, A_1, ..., A_n, P_1, ..., \varphi, ..., P_n\rangle \to \langle M, A_1, ..., A_n, \delta_* P_1, ..., \delta_* \varphi', ..., \delta_* P_n\rangle$$
>
> where φ' is such that $\nabla^a \nabla^b (\varphi' - \varphi) = 0$. Otherwise, a dynamic shift is a transformation:
>
> $$\langle M, A_1, ..., A_n, P_1, ..., P_n\rangle \to \langle M, A_1, ..., A_n, \delta_* P_1, ..., \delta_* P_n\rangle$$

This definition is perhaps a little clumsy, but the benefit is that it applies to theories which posit a gravitational potential, such as NG, as well as to theories from which φ has disappeared, such as Newton-Cartan Theory and Maxwell Gravitation.

The static, kinematic and passive dynamic shift are symmetries of NG. So is the active dynamic shift:[8]

[8] Both Pooley (2013, 551) and Read and Møller-Nielsen (2018, 271) assert versions of this proposition, but without proof.

Proposition 1. *If δ is a diffeomorphism that enacts an arbitrary linear acceleration $\vec{x} \to \vec{x} + \vec{a}$, then $\langle M, t_{ab}, h^{ab}, \nabla, \varphi, \rho, \{\xi^a\}\rangle$ is a solution of NG iff $\langle M, t_{ab}, h^{ab}, \nabla, \delta_*\varphi', \delta_*\rho, \{\delta_*\xi^a\}\rangle$ is a solution for some φ'.*

Proof. We use two facts. Firstly, from the possibility of passive dynamics shifts it follows that (Malament, 2012, Prop. 4.2.5):

$$\begin{gathered}\langle M, t_{ab}, h^{ab}, \nabla, \varphi, \rho, \{\xi^a\}\rangle \text{ is a solution} \\ \Updownarrow \\ \langle M, t_{ab}, h^{ab}, \nabla', \varphi', \rho, \{\xi^a\}\rangle \text{ is a solution}\end{gathered} \tag{4}$$

whenever (i) $\nabla^a\nabla^b(\varphi' - \varphi) = 0$ and (ii) $\nabla' = (\nabla, t_{bc}h^{an}\nabla_n(\varphi' - \varphi))$.

Secondly, from the diffeomorphism invariance of the above formulation of NG it follows that:

$$\begin{gathered}\langle M, t_{ab}, h^{ab}, \nabla', \varphi', \rho, \{\xi^a\}\rangle \text{ is a solution} \\ \Updownarrow \\ \langle M, t_{ab}, h^{ab}, \delta_*\nabla', \delta_*\varphi', \delta_*\rho, \{\delta_*\xi^a\}\rangle \text{ is a solution}\end{gathered} \tag{5}$$

where we have used the fact that $\delta_* t_{ab} = t_{ab}$ and $\delta_* h^{ab} = h^{ab}$.

From (4) and (5), it then follows that whenever (i) and (ii) are satisfied:

$$\begin{gathered}\langle M, t_{ab}, h^{ab}, \nabla, \varphi, \rho, \{\xi^a\}\rangle \text{ is a solution} \\ \Updownarrow \\ \langle M, t_{ab}, h^{ab}, \delta_*\nabla', \delta_*\varphi', \delta_*\rho, \{\delta_*\xi^a\}\rangle \text{ is a solution}\end{gathered} \tag{6}$$

This is almost identical to the proposition, except that the bottom model in (6) contains $\delta_*\nabla'$ instead of ∇'.

But note that ∇' is the pushforward $d'_*\nabla$ of ∇ for some acceleration-diffeomorphism δ', since from (ii): $\nabla'_b\xi^a = \nabla_b\xi^a + t_{bc}h^{an}\nabla_n(\varphi' - \varphi)\xi^c$.[9] When we let $\nabla'_b\xi^a = 0$, this is just the geodesic equation for ∇'. In a coordinate representation the connection is $\Gamma^i_{00} = \partial_i(\varphi' - \varphi)$ and zero otherwise, so:

$$\frac{d^2x^i}{dt^2} + \partial_i(\varphi' - \varphi) = 0 \tag{7}$$

It is easy to see that when we set $\varphi' = \varphi - \ddot{b}^i x^i$, the diffeomorphism δ' enacts the coordinate transformation $\vec{x} \to \vec{x} + \vec{b}$. If we let $\vec{b} = -\vec{a}$, then $\delta' = \delta^{-1}$ and hence $\delta_*\nabla' = \delta_*\delta_*^{-1}\nabla = \nabla$.

We then prove the proposition by substituting ∇ for $\delta_*\nabla'$ in (6). ☐

[9]Cf. Malament (2012, Problem 1.7.3).

Unlike the passive dynamic shift, but like the static and kinematic shifts, an active dynamic shift acts solely on the matter content of the universe. I claim that this is an improvement. Firstly, the active definition is more faithful to intuitive characterisations of the dynamic shift (cf. Saunders' paraphrase of Corolllary VI). Secondly, it is now straightforward to apply Earman's symmetry principles: since δ is a dynamical symmetry but not a spacetime symmetry, Galilean spacetime has too much structure with respect to the dynamics of NG. (The fit is not quite perfect, since φ is transformed to φ' in addition to accelerated. I will skip over this fact in what follows.) In particular, ∇ provides a standard of absolute acceleration, even though the dynamics are invariant under arbitrary linear accelerations.

The response to similar arguments in the case of kinematic shifts has led to the move from Newtonian to Galilean spacetime. And the possibility of passive dynamic shifts has likewise led to novel spacetimes with less structure: Newton-Cartan spacetime and Maxwell spacetime. In the following sections, I will consider how these reformulations of NG fare with respect to the active dynamic shift. Both Newton-Cartan Theory and Maxwell Gravitation provide a satisfactory home for the dynamics of NG, but the approaches differ over their treatment of Earman's symmetry principles.

4 Newton-Cartan Theory

In response to the gauge freedom of NG, one common move is to 'geometrise' the gravitational interaction. This leads to *Newton-Cartan Theory* (NCT). Define a new, invariant quantity, $\tilde{\nabla} = (\nabla, t_{bc}\nabla^a\varphi)$. The KPMs of NCT are then of the form $\langle M, t_{ab}, h^{ab}, \tilde{\nabla}, \rho, \{\xi^a\}\rangle$. The DPMs in addition satisfy the following set of equations:

$$R^a_{bcd} = 0 \tag{8}$$

$$R_{ab} = 4\pi\rho t_{ab} \tag{9}$$

$$\xi^b\tilde{\nabla}_b\xi^a = 0 \tag{10}$$

$$R^{a\ c}_{\ b\ d} = R^{c\ a}_{\ d\ b} \tag{11}$$

In particular, note that (9) is the 'geometrised' version of the Newton-Poisson equation (2).

The main motivation behind NCT is that it removes the redundancy exemplified in passive dynamic shifts. In particular, one can prove that for any pair of models of NG related by a dynamic shift, there exists a unique 'geometrisation' in NCT (Malament, 2012, Prop. 4.2.1). Vice versa, one can prove that for every model of NCT, there exists an equivalence class of 'recovered' models in NG closed under passive dynamic shifts (Malament, 2012, Prop. 4.2.5). In more detail, if $\langle M, t_{ab}, h^{ab}, \nabla, \varphi, \rho, \{\xi^a\}\rangle$ is a model of

NG, then its geometrisation $\langle M, t_{ab}, h^{ab}, \tilde{\nabla}, \rho, \{\xi^a\}\rangle$ is a model of NCT, where $\tilde{\nabla} = (\nabla, -t_{bc}\nabla^a\varphi)$. Conversely, if $\langle M, t_{ab}, h^{ab}, \tilde{\nabla}, \rho, \{\xi^a\}\rangle$ is a model of NCT, then $\langle M, t_{ab}, h^{ab}, \nabla, \varphi, \rho, \{\xi^a\}\rangle$ is a model of NG. Finally, any other model $\langle M, t_{ab}, h^{ab}, \nabla', \varphi', \rho, \{\xi^a\}\rangle$ of NG for which (i) and (ii) are satisfied has the same geometrisation as $\langle M, t_{ab}, h^{ab}, \nabla, \varphi, \rho, \{\xi^a\}\rangle$.

The aim of this section is to show that these results do not hold in quite the same form for dynamic shifts conceived of actively. It is not the case that there exists a unique geometrisation for a pair of active dynamic shift-related models of NG. Nevertheless, I will argue that this does not stymie NCT's capacity to satisfy Earman's principles. Indeed, active dynamic shifts clarify in what way NCT achieves this.

I will comment on the significance of these result in §6. First, we prove a pair of theorems:

Proposition 2. *Where $\mathcal{M}$ and $\mathcal{M}_{dyn}$ are pairs of models related by an active dynamic shift, their geometrizations $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_{dyn}$ are distinct yet isomorphic.*

Proof. We have $\tilde{\mathcal{M}} = \langle M, t_{ab}, h^{ab}, \tilde{\nabla}, \rho, \{\xi^a\}\rangle$ and $\tilde{\mathcal{M}}_{dyn} = \langle M, t_{ab}, h^{ab}, \tilde{\nabla}', \delta_*\rho, \{\delta_*\xi^a\}\rangle$. Distinctness follows from the fact that generally $\delta_*\rho \neq \rho$. But more importantly, the Newton-Cartan spacetimes underlying these models are also distinct: $\tilde{\nabla}' \neq \tilde{\nabla}$, because $\tilde{\nabla}' = (\nabla, t_{bc}\nabla^a(\delta_*\varphi')) \neq (\nabla, t_{bc}\nabla^a\varphi)$.

For isomorphism, note that $\tilde{\nabla} = (\nabla, -t_{bc}\nabla^a\varphi) = (\nabla', -t_{bc}\nabla'^a\varphi')$, so

$$\delta_*\tilde{\nabla} = (\delta_*\nabla', -t_{bc}\delta_*(\nabla'^a\varphi')) = (\nabla, -t_{bc}\nabla^a(\delta_*\varphi') = \tilde{\nabla}' \tag{12}$$

We also know that $\delta_* t_{ab} = t_{ab}$ and $\delta_* h^{ab} = h^{ab}$. Therefore, $\delta_*\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_{dyn}$ and so these models are isomorphic. □

Proposition 3. *Active dynamic shifts are neither dynamical symmetries nor spacetime symmetries of NCT.*

Proof. To show that δ is not a dynamical symmetry, note that (9) is not invariant under active dynamic shifts since generally $\delta_*\rho \neq \rho$. Therefore, δ does not map solutions to solutions, so it is not a dynamical symmetry.

To show that d is not a spacetime symmetry, recall from Proposition 2 that $\delta_*\tilde{\nabla} = \tilde{\nabla}' \neq \tilde{\nabla}$. Therefore, δ does not preserve the theory's absolute objects, so it is not a spacetime symmetry. □

Proposition 2 shows that NCT eliminates the surplus structure of NG. While dynamic shift-related models of the latter theory are non-isomorphic, their geometrisations are. Although isomorphic models *prima facie* seem to represent distinct yet observably equivalent states of affairs, it is widely-agreed that one can interpret isomorphic models as physically equivalent. For Weatherall (2018), this follows directly from the fact that isomorphism is our standard of 'sameness' for models. For a sophisticated substantivalist such as **?**, this follows from the rejection of primitive identities for spacetime points. Since isomorphic models differ *only* as to which point occupies which structural role within the model, the rejection of primitive identities entails that such models must represent the same state of affairs. Either way, isomorphism is a satisfactory 'stopping point' for the removal of redundant structure. NCT offers a satisfactory response to active dynamic shifts.

Proposition 3 confirms this: it shows that NCT satisfies Earman's principles, and hence has neither too much nor too little spacetime structure. But the way in which NCT achieves this is unusual. Since dynamic shifts are symmetries of NG but not of NCT, the latter has in effect 'removed' these symmetries from the theory. This stands in contrast to the move from Newtonian to Galilean spacetime in response to the kinematic shift. In Galilean spacetime, kinematic shifts remain symmetries of the theory; the difference is that they are now *also* spacetime symmetries, unlike for Newtonian spacetime. As we will see below, the move to Galilean spacetime is more similar to the move to Maxwell spacetime in this respect.

5 Maxwell Gravitation

Recently, Maxwell Gravitation (MG) has been put forward as an alternative to NCT.[10] Unlike NCT, MG does not geometrise spacetime. Instead, MG weakens the structure of spacetime to include a standard of rotation without a standard of linear acceleration. Instead of a single covariant derivative ∇, Maxwell spacetime has an equivalence class of 'rotationally equivalent' derivative operators $[\nabla]$: for any $\nabla, \nabla' \in [\nabla]$ and any unit timelike field θ^a, $\nabla^{[a}\theta^{b]} = 0$ iff $\nabla'^{[a}\theta^{b]} = 0$.[11] Following Weatherall (2016), I will assume that the KPMs of MG are of the form $\langle M, t_{ab}, h^{ab}, [\nabla], \rho, \{\xi^a\}\rangle$. Furthermore, I assume that the dynamics of MG are as follows: $\langle M, t_{ab}, h^{ab}, [\nabla], \rho, \{\xi^a\}\rangle$ is a DPM of MG iff for all $\nabla \in [\nabla]$, there exist some φ_∇ such that (1)–(3) are satisfied for ρ, ξ, ∇ and φ_∇.[12] In particular, note that φ is not itself included in the theory's models. This is possible because one can show that for any $\nabla \in [\nabla]$, an appropriate φ_∇ always exists

[10] For discussions of Maxwell spacetime, see Saunders (2013); Knox (2014); Weatherall (2016); Dewar (2018); Wallace (2020).

[11] Weatherall (2017) offers a more 'intrinsic' definition of such a standard of rotation.

[12] Dewar (2018) offers a different, more comprehensive account of MG, in which the mass-momentum tensor T^{ab} represents both source and test matter. For ease of comparison, I will not consider Dewar's account here. However, I suspect that analogues of Propositions 4 and 5 will also hold for it.

(Weatherall, 2016, 88). In this sense, the traditional lesson of (passive) dynamic shifts—that the gravity/inertia split is arbitrary—is carried over to MG.

We can compare the behaviour of our version of MG to that of NCT under active dynamic shifts. Just as models of NG have a geometrisation, so they have a 'de-acceleration'. If $\langle M, t_{ab}, h^{ab}, \nabla, \varphi, \rho, \{\xi^a\}\rangle$ is a model of NG, then its de-acceleration is $\langle M, t_{ab}, h^{ab}, [\nabla], \rho, \{\xi^a\}\rangle$, where $[\nabla]$ is the closure of ∇ under rotational equivalence. On the one hand, it is obvious from the dynamics specified above that the de-acceleration of a solution of NG is a solution of MG. Conversely, one can move from models of MG to models of NG: if $\langle M, t_{ab}, h^{ab}, [\nabla], \rho, \{\xi^a\}\rangle$ is a solution of MG, then its recoveries form an equivalence class of models $\langle M, t_{ab}, h^{ab}, \nabla, \varphi_\nabla, \rho, \{\xi^a\}\rangle$ such that (1)–(3) are satisfied. In particular, just as in NCT, any pair of models $\langle M, t_{ab}, h^{ab}, \nabla', \varphi', \rho, \{\xi^a\}\rangle$ and $\langle M, t_{ab}, h^{ab}, \nabla, \varphi, \rho, \{\xi^a\}\rangle$ for which (i) and (ii) are satisfied have the same de-acceleration.

We can now prove propositions analogous to Proposition 2 and 3:

Proposition 4. *Where $\mathcal{M}$ and $\mathcal{M}_{dyn}$ are related by an active dynamic shift, their de-accelerations $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}}_{dyn}$ are distinct but isomorphic.*

Proof. We have $\bar{\mathcal{M}} = \langle M, t_{ab}, h^{ab}, [\nabla], \rho, \{\xi^a\}\rangle$ and $\bar{\mathcal{M}}_{dyn} = \langle M, t_{ab}, h^{ab}, [\nabla], \delta_*\rho, \{\delta_*\xi^a\}\rangle$. For distinctness, it once again suffices to note that generally $\delta_*\rho \neq \rho$. But unlike in Proposition 2, the underlying Maxwell spacetimes of $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}}_{dyn}$ are clearly identical.

For isomorphism, we only need to show that $[\delta_*\nabla] = [\nabla]$. This follows directly from the claim that d is a spacetime symmetry (since $[\nabla]$ is an absolute object), which is proven in Proposition 5. □

Proposition 5. *Active dynamic shifts are both dynamical symmetries and spacetime symmetries of MG.*

Proof. To show that δ is a dynamical symmetry, we need to show that $\langle M, t_{ab}, h^{ab}, [\nabla], \rho, \{\xi^a\}\rangle$ is a solution iff $\langle M, t_{ab}, h^{ab}, [\nabla], \delta_*\rho, \{\delta_*\xi^a\}\rangle$ is. $\langle M, t_{ab}, h^{ab}, [\nabla], \rho, \{\xi^a\}\rangle$ is a solution iff for all $\nabla \in [\nabla]$, there exists a φ such that $\langle M, t_{ab}, h^{ab}, \nabla, \varphi_\nabla, \rho, \{\xi^a\}\rangle$ satisfies (1)–(3). From Proposition 1, the latter is the case iff $\langle M, t_{ab}, h^{ab}, \nabla, \delta_*\varphi'_\nabla, \delta_*\rho, \{\delta_*\xi^a\}\rangle$ satisfies (1)–(3). But since this holds for all ∇, it is true iff $\langle M, t_{ab}, h^{ab}, [\nabla], \delta_*\rho, \{\delta_*\xi^a\}\rangle$ is a solution. Therefore, δ is a dynamical symmetry.

To show that δ is a spacetime symmetry, we need to show that $[\delta_*\nabla] = [\nabla]$. We will use the fact that $\nabla, \nabla' \in [\nabla]$ iff $\nabla' = (\nabla, t_{bc}\nabla^a\varphi)$ for some φ such that $\nabla^a\nabla^b\varphi = 0$ (Dewar, 2018, Prop. 1). Recall that $\nabla' = (\nabla, t_{bc}\nabla^a\varphi)$ iff $\nabla' = \delta_*\nabla$ for some δ; it follows that $[\delta_*\nabla] = [\nabla]$. Therefore, δ is a spacetime symmetry. □

Like Proposition 2 showed for NCT, Proposition 4 shows that MG effectively removes the surplus spacetime structure of NG. Like Proposition 3, Proposition 5 confirms that the theory satisfies Earman's symmetry principles. But whereas passive dynamic shifts are neither spacetime nor dynamical symmetries of NCT, they are both for MG. In this sense, the move from Galilean to Maxwellian spacetime is akin to the move from Newtonian to Galilean spacetime. In both cases, spacetime structure is weakened, and as a result the class of spacetime symmetries broadened. NCT, on the other hand, does not remove surplus *spacetime* structure: the group of spacetime symmetries of NCT is not a subset of the group of spacetime symmetries of NG. In the next section, I will argue that this puts pressure on the claim that these theories are, in some sense, equivalent to each other.

6 Close

The propositions concerning active dynamic shifts reveal some interesting similarities and dissimilarities between NCT and MG.

Consider the similarities first. For both theories, there is a one-to-one correspondence between models of NG related by an active dynamic shift on the one hand, and geometrised or de-accelerated models of NCT and MG on the other. Moreover, while such models are not isomorphic in NG, they are in both NCT and MG. As I mentioned above, it is widely agreed that one can—or even ought to—interpret isomorphic models as representing the same state of affairs, If we follow this route, both NCT and MG successfully avoid the underdetermination of NG revealed by such shifts.

However, the *way* they do so is different. In MG, we have removed spacetime structure by replacing ∇ with $[\nabla]$. As a result, Maxwell spacetime has more symmetries than Galilean spacetime. This broader class of spacetime symmetries now matches the theory's dynamical symmetries. Compare this with the move from Newtonian to Galilean spacetime: by removing the standard of rest one removes spacetime structure in order to match the theory's dynamical structure. NCT, on the other hand, has removed the dynamical symmetries themselves from the theory. In NCT, active dynamic shifts are neither dynamical nor spacetime symmetries. In particular, this means that the latter theory has a different class of spacetime symmetries.

This puts some pressure on recent claims that NCT and MG are theoretically equivalent (Weatherall, 2016; Dewar, 2018; Wallace, 2020). In particular, the considerations above suggest that these theories differ in both their dynamical *and* their spacetime structure. On the dynamical side, consider what happens when we apply an active dynamic shift to models of MG and NCT respectively. In the former case, we obtain another dynamically possible model: one that differently describes the very same state of affairs. But in the latter case, we obtain a model that is *not* a solution. In physical terms, this means that MG and NCT differ over their *modal profiles*. For the former theory, it is true that if we were to accelerate all matter, the laws of nature would remain satisfied. But for the latter theory, this is false. Arguably, this is a real physical difference between these theories.

With respect to spacetime symmetries, the fact is that Newton-Cartan spacetime and Maxwellian spacetime have different symmetries, and hence different structures.[13] In particular, Newton-Cartan spacetime comes attached with an absolute standard of acceleration in the form of $\tilde{\nabla}$, while Maxwellian spacetime only has an equivalence class $[\nabla]$ that determines a unique standard of rotation. However, there is a sense in which MG *also* has an absolute standard of acceleration: from any solution of MG in which ρ does not vanish, one can uniquely define a covariant derivative $\tilde{\nabla}$ such that all particle trajectories ξ^a are geodesics of $\tilde{\nabla}$ (Weatherall (2016, Prop. 4); Dewar (2018, Prop. 6)). This claim seems to stand in tension with the fact that active dynamic shifts are symmetries of Maxwell spacetime. But note (as Dewar also points out) that one can only recover $\tilde{\nabla}$ from a model of MG *when that model satisfies the equations of motion*. In other words, one cannot recover $\tilde{\nabla}$ from the structure of Maxwell spacetime alone, but one also requires dynamical structure in the form of the set of dynamically allowed particle trajectories. In that sense, Maxwell spacetime *by itself* has no standard of acceleration. Again, this seems a real physical difference between NCT and MG.

What, then, has led some to suggest that these are equivalent theories after all? This claim is motivated by the fact that there exists a one-to-one correspondence between solutions of Newton-Cartan Theory and Maxwell Gravitation. But a theory is *more* than its solutions, or DPMs: one also has the kinematically possible models to reckon with.[14] The latter reveal the difference between Newton-Cartan Theory and Maxwell Gravitation. In the case of dynamical symmetries, Newton-Cartan Theory distinguishes itself from Maxwell Gravitation in that active dynamic shifts can move us outside the space of DPMs in the former but not in the latter theory. In the case of spacetime symmetries, the fact that one cannot recover a unique standard of acceleration from any KPM of Maxwell Gravitation means that that theory has less spacetime structure than Newton-Cartan Theory. These differences are only apparent when we consider the full space of KPMs, whether solutions or not. The claim that Newton-Cartan Theory and Maxwell Gravitation are theoretically equivalent results from paying too much attention to a theory's space of solutions, and too little to the entire space of models.

[13] For the connection between symmetries and (amount of) structure, see Barrett (2015).
[14] For a convincing case for the importance of kinematical structure, see Curiel (2016).

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