

## The Theory of Computability Developed in Terms of Satisfaction

JAMES CAIN

**Abstract** The notion of computability is developed through the study of the behavior of a set of languages interpreted over the natural numbers which contain their own fully defined satisfaction predicate and whose only other vocabulary is limited to  $\mathbf{0}$ , individual variables, the successor function, the identity relation and operators for disjunction, conjunction, and existential quantification.

**1 Introduction** Techniques from recursion theory (the theory of computability) have proven to be invaluable in the study of the theory of truth. It is perhaps not as well known that recursion theory can in turn be studied in terms of the theory of truth (in particular the theory of satisfaction). I suspect that for one who has been introduced to the basic metatheorems of mathematical logic up to, say, Gödel's first incompleteness theorem, a nice introduction to the fundamental concepts and theorems of recursion theory is through the theory of satisfaction. Accordingly, this paper is directed to three audiences: (1) those familiar with recursion theory who want to see its connections to satisfaction, (2) those interested in theory of truth who want to see some of its application and methods, and (3) those, especially in philosophy, who do not have a background in recursion theory but are interested in seeing a development of some of its fundamental concepts and theorems. No familiarity with the theory of truth or satisfaction will be assumed and no knowledge of recursion theory will be assumed that goes beyond the techniques used in the proof of Gödel's first incompleteness theorem. In particular, we will assume that the reader knows that "Gödel numbers" can be used to code syntactically definable features of a language and that sets and relations of Gödel numbers defining some of these features are primitive recursive. Section 2 will introduce and briefly study some logical features of a very simple language which can be interpreted so that it contains its own satisfaction predicate. In terms of this language Section 3 will introduce some of the basic concepts of recursion theory and prove some of its fundamental theorems. Of course, my

*Received September 2, 1999; revised January 4, 2001*

treatment of recursion theory is not intended to be comprehensive; rather, the aim is to give the reader a feel for how recursion theory can be developed in terms of the theory of satisfaction. Since this paper will not assume that the reader is an expert in either of the areas of discussion, one who is will find that certain sections of the paper can simply be skimmed over.

**2 A simple language containing its own satisfaction predicate** Let  $L$  be an uninterpreted language whose vocabulary includes the following symbols:

$$, \quad ( ) \quad \mathbf{0} \quad \text{Sat} \quad s \quad \approx \quad \wedge \quad \vee \quad \exists$$

plus the infinite set of variables  $x_0, x_1, x_2, \dots$ . Note that  $L$  contains no symbol for negation or universal quantification. The *terms* of  $L$  include the individual constant  $\mathbf{0}$ , the variables, and, for any term  $t$ ,  $s(t)$  is a term. The *atomic formulas* of  $L$  include  $t_0 \approx t_1$  (for any terms  $t_0$  and  $t_1$ ) and  $\text{Sat}(t_0, \dots, t_n)$  (for any sequence of terms  $t_0, \dots, t_n$ , for  $n \geq 0$ ). The *formulas* (or *wffs*) of  $L$  include the atomic formulas, and whenever  $A$  and  $B$  are well-formed formulas so are  $(A \vee B)$ ,  $(A \wedge B)$ , and, for any  $i$ ,  $\exists x_i A$ .

An *interpretation*  $I$  of  $L$  provides a domain  $D_I$ , assigns to  $\mathbf{0}$  some element  $\mathbf{0}_I$  of  $D_I$ , assigns to  $s$  a 1-place function  $s_I$  mapping  $D_I$  into  $D_I$ , assigns identity to  $\approx$ , and assigns to  $\text{Sat}$ , a set  $\text{Sat}_I$  of finite sequences  $\langle d_0, \dots, d_n \rangle$  of elements of  $D_I$  (the sequences in  $\text{Sat}_I$  need not all be of the same length). We will restrict our attention to interpretations  $I$  of  $L$  such that  $D_I = \mathbb{N} = \{0, 1, 2, \dots\}$ ,  $\mathbf{0}_I = 0$ ; and  $s_I =$  the successor function. Thus to specify an interpretation it will suffice to specify  $\text{Sat}_I$ , the extension of  $\text{Sat}$ . We will indicate the interpreted language in which  $\text{Sat}_I = S$  by writing  $\mathcal{L}(S)$ .

Let  $r$  be any denumerable sequence,  $\langle r_0, r_1, r_2, \dots \rangle$ , of natural numbers. We let  $r(x_i)$  be a function mapping the variables into  $\mathbb{N}$  by setting  $r(x_i) = r_i$ . We may now define the notion of the *denotation of term  $t$  with respect to  $r$* ,  $\text{den}_r(t)$ . If  $t$  is a variable then  $\text{den}_r(t) = r(t)$ . If  $t = \mathbf{0}$ ,  $\text{den}_r(t) = 0$ . If  $t = s(t_0)$ , then  $\text{den}_r(t) =$  the successor of  $\text{den}_r(t_0)$ . Finally, we define *formula  $A$  is satisfied by sequence  $r$  in  $\mathcal{L}(S)$*  (written  $\models_S A[r]$ ).  $\models_S A[r]$  if and only if

- $A$  is  $t_0 \approx t_1$  and  $\text{den}_r(t_0) = \text{den}_r(t_1)$ , or
- $A$  is  $\text{Sat}(t_0, \dots, t_n)$  and  $\langle \text{den}_r(t_0), \dots, \text{den}_r(t_n) \rangle \in S$ , or
- $A$  is  $(B \vee C)$  and either  $\models_S B[r]$  or  $\models_S C[r]$ , or
- $A$  is  $(B \wedge C)$  and both  $\models_S B[r]$  and  $\models_S C[r]$ , or
- $A$  is  $\exists x_i B$  and  $\models_S B[r']$ , for some  $r'$  which is an  *$i$ -variant of  $r$*  (that is,  $r'$  differs from  $r$  at most in its assignment to  $x_i$ ).

Say that  $r$  and  $r'$  *agree on the free variables in a term  $t$ , or in a formula  $A$* , provided that whenever  $x_i$  is free in  $t$  or in  $A$ ,  $r(x_i) = r'(x_i)$ . The following useful lemma can be proven by induction on the complexity of term or a formula.

**Lemma 2.1** *If  $r$  and  $r'$  agree on the free variables in  $t$  then  $\text{den}_r(t) = \text{den}_{r'}(t)$ . And if  $r$  and  $r'$  agree on the free variables in  $A$ , then for any  $S$   $\models_S A[r]$  if and only if  $\models_S A[r']$ .*

It follows from the lemma that if  $A$  is a sentence then  $\models_S A[r]$  for some  $r$  if and only if  $\models_S A[r]$  for every  $r$ . We write  $\models_S A$  as short for  $\models_S A[r]$  for all  $r$ . We say that  $r$

extends  $\langle d_1, \dots, d_n \rangle$  provided that for  $0 \leq i < n$ ,  $r_i = d_{i+1}$ , that is,  $r(x_i) = d_{i+1}$ . We say that *formula A is satisfied by*  $\langle n_1, \dots, n_m \rangle$  in  $\mathcal{L}(S)$  ( $\models_S A[\langle n_1, \dots, n_m \rangle]$ ) if and only if  $\models_S A[r]$  for every  $r$  that extends  $\langle n_1, \dots, n_m \rangle$ .

Next we must consider the conditions under which Sat can be treated as a satisfaction predicate. Two approaches might be taken. The direct approach would require the domain of the interpretation to contain the formulas of the language. The more usual approach is in terms of Gödel numberings and this is the approach we will follow. We let a Gödel numbering be a one-to-one mapping from formulas of  $\mathcal{L}$  into  $\mathbb{N}$ . Generally when people consider Gödel numberings it is understood that the mapping will be 'effective', but, for the time being, we will not put any such restriction on the Gödel numbering. For a fixed Gödel numbering gn, we will define

$$\Phi(S) = \{ \langle d_0, \dots, d_n \rangle \mid \text{either (1) } n = 0 \text{ and } d_0 = \text{gn}(A) \text{ for some sentence } A \text{ of } \mathcal{L} \text{ such that } \models_S A, \text{ or (2) } n > 0 \text{ and } d_0 = \text{gn}(A) \text{ for some well-formed formula } A \text{ of } \mathcal{L} \text{ with at most } x_0, \dots, x_{n-1} \text{ free and } \models_S A[\langle d_1, \dots, d_n \rangle] \}.$$

With respect to a given Gödel numbering, we say that *Sat expresses a satisfaction predicate for*  $\mathcal{L}(S)$  if and only if  $\Phi(S) = S$ .

Thus, for example, if the Gödel number of the formula  $x_0 = x_1$  is 5, then if Sat is a satisfaction predicate,  $\text{Sat}(x_0, x_1, x_2)$  will be satisfied by a sequence  $\langle 5, m_1, m_2, m_3, m_4, \dots \rangle$  if and only if the formula  $x_0 = x_1$  is satisfied by the sequence  $\langle m_1, m_2, m_3, m_4, \dots \rangle$ , in other words, if and only if  $m_1 = m_2$ .

If  $\Phi(S) = S$ , we say that  $S$  is a *fixed point of*  $\Phi$ , and that  $\mathcal{L}(S)$  is a *fixed point language*. Let  $\Phi^0(S) = S$ ,  $\Phi^{\alpha+1}(S) = \Phi(\Phi^\alpha(S))$ , and, for limit ordinal  $\lambda$ ,  $\Phi^\lambda(S) = \bigcup_{\alpha < \lambda} \Phi^\alpha(S)$ . With respect to a given Gödel numbering, we call a fixed point a *least fixed point* provided it is a subset of every fixed point, and where  $S$  is a least fixed point, we speak of  $\mathcal{L}(S)$  as a *least fixed point language*. We let  $\Lambda$  be the empty set.

### Theorem 2.2

- (a) *With respect to any Gödel numbering, there exists a least fixed point language, namely  $\mathcal{L}(\Phi^\omega(\Lambda))$ .*
- (b) *In  $\mathcal{L}(\Phi^\omega(\Lambda))$ , Sat is a satisfaction predicate.*
- (c) *If  $S \subseteq \Phi(S)$ , then  $\Phi^\omega(S)$  is a fixed point; in fact,  $\Phi^\omega(S)$  is a subset of every fixed point containing  $S$ , and in  $\mathcal{L}(\Phi^\omega(S))$ , Sat is satisfaction predicate.*

Our proof of Theorem 2.2 will rely on the following lemma.

**Lemma 2.3** *With respect to any Gödel numbering the following hold.*

- (a) *Suppose  $S_1 \subseteq S_2$ ; then for any formula A and sequence r,  $\models_{S_1} A[r]$  entails  $\models_{S_2} A[r]$ .*
- (b)  *$\Phi$  is monotonic (that is, if  $S_1 \subseteq S_2$  then  $\Phi(S_1) \subseteq \Phi(S_2)$ ).*
- (c) *If  $S \subseteq \Phi(S)$ , then for any ordinal  $\alpha$ ,  $\Phi^\alpha(S) \subseteq \Phi^{\alpha+1}(S)$ .*
- (d) *If  $S \subseteq \Phi(S)$  and  $\alpha \leq \beta$  then  $\Phi^\alpha(S) \subseteq \Phi^\beta(S)$ .*
- (e) *If  $S \subseteq \Phi(S)$  then, for any formula A and sequence r,  $\models_{\Phi^\omega(S)} A[r] \implies \models_{\Phi^n(S)} A[r]$  for some finite n.*

*Proof of Lemma 2.3:* (a) can be shown by induction on the complexity of the formula A. (b) follows from (a) given the definition of  $\Phi$ . For (c), (d), and (e) we assume that  $S \subseteq \Phi(S)$ . To prove (c) we show by induction on  $\alpha$  that  $\Phi^\alpha(S) \subseteq \Phi^{\alpha+1}(S)$ . For

$\alpha = 0$ , note that  $\Phi^0(S) = S \subseteq \Phi(S) = \Phi^1(S)$ . Next we must show that if  $\Phi^\alpha(S) \subseteq \Phi^{\alpha+1}(S)$  then  $\Phi^{\alpha+1}(S) \subseteq \Phi^{(\alpha+1)+1}(S)$ . Suppose  $\Phi^\alpha(S) \subseteq \Phi^{\alpha+1}(S)$ . Then, by the monotonicity of  $\Phi$ ,  $\Phi(\Phi^\alpha(S)) \subseteq \Phi(\Phi^{\alpha+1}(S))$ , that is,  $\Phi^{\alpha+1}(S) \subseteq \Phi^{(\alpha+1)+1}(S)$ . Finally, suppose that  $\alpha$  is a limit ordinal and that  $d \in \Phi^\alpha(S)$ . We must show that  $d \in \Phi^{\alpha+1}(S)$ . For some  $\beta < \alpha$ ,  $d \in \Phi^\beta(S)$  since  $\Phi^\alpha(S) = \bigcup_{\gamma < \alpha} \Phi^\gamma(S)$ . By hypothesis  $\Phi^\beta(S) \subseteq \Phi^{\beta+1}(S)$ , so  $d \in \Phi^{\beta+1}(S)$ . By monotonicity, since  $\Phi^\beta(S) \subseteq \Phi^\alpha(S)$ ,  $\Phi^{\beta+1}(S) \subseteq \Phi^{\alpha+1}(S)$ . Thus  $d \in \Phi^{\alpha+1}(S)$ .

For (d) we show by induction on ordinal  $\delta$  that  $\Phi^\alpha(S) \subseteq \Phi^{\alpha+\delta}(S)$ . Clearly  $\Phi^\alpha(S) \subseteq \Phi^{\alpha+0}(S)$ . Suppose that  $\Phi^\alpha(S) \subseteq \Phi^{\alpha+\delta}(S)$ . By monotonicity  $\Phi^{\alpha+1}(S) \subseteq \Phi^{\alpha+\delta+1}(S)$ . By (c),  $\Phi^\alpha(S) \subseteq \Phi^{\alpha+1}(S)$ , and thus  $\Phi^\alpha(S) \subseteq \Phi^{\alpha+\delta+1}(S)$ . If  $\delta$  is a limit then so is  $\alpha + \delta$ , and  $\alpha < \alpha + \delta$ . Thus  $\Phi^{\alpha+\delta}(S) = \bigcup_{\gamma < \alpha+\delta} \Phi^\gamma(S)$ . Thus  $\Phi^\alpha(S) \subseteq \Phi^{\alpha+\delta}(S)$ .

(e) can be shown by induction on the complexity of formula  $A$ . We assume that  $\models_{\Phi^\omega(S)} A[r]$ . Suppose that  $A$  is atomic. The case is clear for  $A = t_1 \approx t_2$ , so suppose that  $A = \text{Sat}(t_0, \dots, t_n)$ . Then  $\langle \text{den}_r(t_0), \dots, \text{den}_r(t_n) \rangle \in \Phi^\omega(S)$ . But then, since  $\Phi^\omega(S) = \bigcup_{n < \omega} \Phi^n(S)$ , for some finite  $n$ ,  $\langle \text{den}_r(t_0), \dots, \text{den}_r(t_n) \rangle \in \Phi^n(S)$ , and thus  $\models_{\Phi^n(S)} A[r]$ . Suppose  $A = (B \vee C)$ . Then either  $\models_{\Phi^\omega(S)} B[r]$  or  $\models_{\Phi^\omega(S)} C[r]$ . By hypothesis, for some finite  $n$ , either  $\models_{\Phi^n(S)} B[r]$  or  $\models_{\Phi^n(S)} C[r]$  and thus  $\models_{\Phi^n(S)} (B \vee C)[r]$ , that is,  $\models_{\Phi^n(S)} A[r]$ . Suppose  $A = (B \wedge C)$ . Then  $\models_{\Phi^\omega(S)} B[r]$  and  $\models_{\Phi^\omega(S)} C[r]$ , and so for some finite  $j$  and  $k$ ,  $\models_{\Phi^j(S)} B[r]$  and  $\models_{\Phi^k(S)} C[r]$ , and thus  $\models_{\Phi^{j+k}(S)} (B \wedge C)[r]$  by (d) and (a) above. Suppose  $A = \exists x_i B$ .  $\models_{\Phi^\omega(S)} \exists x_i B[r] \implies \models_{\Phi^\omega(S)} B[r']$  for some  $r'$  which differs from  $r$  at most in its assignment to  $x_i \implies$  (by the induction hypothesis)  $\models_{\Phi^n(S)} B[r']$  for some finite  $n \implies \models_{\Phi^n(S)} \exists x_i B[r]$ .  $\square$

*Proof of Theorem 2.2:* We begin with (c). Suppose that  $S \subseteq \Phi(S)$ . We need to show (1)  $\Phi^\omega(S)$  is a fixed point, (2)  $\Phi^\omega(S)$  is contained in every fixed point that contains  $S$ , and (3) in  $\mathcal{L}(\Phi^\omega(S))$ ,  $\text{Sat}$  is a satisfaction predicate. Let's begin with (2). By induction we can see that, for all  $n \in \mathbb{N}$ ,  $\Phi^n(S)$  is contained in every fixed point containing  $S$  (if any exist): for  $\Phi^0(S) = S \subseteq$  every fixed point containing  $S$ , and whenever  $\Phi^n(S) \subseteq S^*$ , where  $S^*$  is a fixed point containing  $S$ , by monotonicity and the definition of a fixed point,  $\Phi^{n+1}(S) = \Phi(\Phi^n(S)) \subseteq \Phi(S^*) = S^*$ . Since, for each  $n$ ,  $\Phi^n(S)$  is contained in every fixed point that contains  $S$ ,  $\bigcup_{n \in \mathbb{N}} \Phi^n(S) = \Phi^\omega(S)$  must also be.

To show (1) we need to show that  $\Phi^\omega(S) = \Phi^{\omega+1}(S)$ . By (c) of Lemma 2.3,  $\Phi^\omega(S) \subseteq \Phi^{\omega+1}(S)$ , so we must show that  $\Phi^{\omega+1}(S) \subseteq \Phi^\omega(S)$ , that is,  $\langle m_0, \dots, m_n \rangle \in \Phi^{\omega+1}(S) \implies \langle m_0, \dots, m_n \rangle \in \Phi^\omega(S)$ . By the definition of  $\Phi$ , if  $\langle m_0, \dots, m_n \rangle \in \Phi^{\omega+1}(S)$  then  $m_0 = \text{gn}(A)$ , for some formula  $A$  containing at most  $x_0, \dots, x_{n-1}$  free, and  $\models_{\Phi^\omega(S)} A[r]$  for any  $r$  that extends  $\langle m_1, \dots, m_n \rangle$  (or, if  $n = 0$ , for any  $r$ ). Take any such  $r$ . By (e) of Lemma 2.3,  $\models_{\Phi^n(S)} A[r]$  for some finite  $n$ . But then by Lemma 2.1,  $\models_{\Phi^n(S)} A[r']$  for any  $r'$  that extends  $\langle m_1, \dots, m_n \rangle$  (or for any  $r'$  if  $n = 0$ ) since  $r'$  will agree with  $r$  on any free variables in  $A$ . Thus by the definition of  $\Phi$ ,  $\langle m_0, \dots, m_n \rangle \in \Phi^{n+1}(S)$ . Since  $\Phi^{n+1}(S) \subseteq \Phi^\omega(S)$ ,  $\langle m_0, \dots, m_n \rangle \in \Phi^\omega(S)$ . Thus (1) holds. (3) follows from (1) by the definition of satisfaction predicate.

To complete the proof of Theorem 2.2 it suffices to note that parts (a) and (b) follow from part (c) since  $\Lambda \subseteq \Phi(\Lambda)$  and  $\Lambda \subseteq$  any fixed point.  $\square$

It may be useful to pause and reflect on the construction of the least fixed point. Note that for no finite  $n$  is  $\mathcal{L}(\Phi^n(\Lambda))$  a fixed point. We can see this as follows. For any formula  $A$  with Gödel number  $n$ , let  $\ulcorner A \urcorner$  be the numeral for  $n$  (for example, if  $\text{gn}(A) = 2$ , then  $\ulcorner A \urcorner = s(s(\mathbf{0}))$ ). Let  $\text{Sat}^0(\ulcorner A \urcorner) = A$ , and let  $\text{Sat}^{n+1}(\ulcorner A \urcorner) = \text{Sat}(\ulcorner \text{Sat}^n(\ulcorner A \urcorner) \urcorner)$ . Then  $\text{Sat}^n(\ulcorner \mathbf{0} = \mathbf{0} \urcorner)$  first gets evaluated true in language  $\mathcal{L}(\Phi^n(\Lambda))$  (that is,  $\models_{\Phi^m(\Lambda)} \text{Sat}^n(\ulcorner \mathbf{0} = \mathbf{0} \urcorner)$  if and only if  $m \geq n$ ), and  $\langle \text{gn}(\text{Sat}^n(\ulcorner \mathbf{0} = \mathbf{0} \urcorner)) \rangle \in \Phi^{n+1}(\Lambda)$  but  $\notin \Phi^n(\Lambda)$ . Thus for no finite  $n$  is  $\mathcal{L}(\Phi^n(\Lambda))$  a fixed point language.

We will say that a formula  $A(x_0)$ , whose only free variable is  $x_0$ , is a *truth predicate for  $\mathcal{L}(S)$*  if and only if, for every sequence  $r$ ,

$$\models_S A(x_0)[r] \text{ if and only if } r(x_0) = \text{gn}(B) \text{ for some sentence } B \text{ and } \models_S B.$$

$\text{Sat}(x_0)$  will be a truth predicate in each fixed point language.

$\mathcal{L}(\Phi^\omega(\Lambda))$  will not be the only fixed point language. Consider the formula  $\text{Sat}(x_0, x_0)$ . Suppose that its Gödel number is  $n$ . Consider any interpretation  $S$  of  $\text{Sat}$ . Suppose that  $\langle n, n \rangle \in S$ . Then  $\models_S \text{Sat}(x_0, x_0)[r]$  for any  $r$  that extends  $\langle n \rangle$ . Thus  $\langle n, n \rangle \in \Phi(S)$ . In particular  $\{\langle n, n \rangle\} \subseteq \Phi(\{\langle n, n \rangle\})$ , and thus, by Theorem 2.2(c),  $\Phi^\omega(\{\langle n, n \rangle\})$  will be a fixed point containing  $\langle n, n \rangle$ . On the other hand, suppose  $\langle n, n \rangle \notin S$  then it is not the case that  $\models_S \text{Sat}(x_0, x_0)[r]$  for any  $r$  that extends  $\langle n \rangle$ , and thus  $\langle n, n \rangle \notin \Phi(S)$ . By induction it can be seen that, for all  $\alpha$ ,  $\langle n, n \rangle \notin \Phi^\alpha(S)$ . In particular, for all  $\alpha < \omega$ ,  $\langle n, n \rangle \notin \Phi^\alpha(\Lambda)$ , and thus  $\langle n, n \rangle \notin \Phi^\omega(\Lambda)$ . Thus  $\Phi^\omega(\{\langle n, n \rangle\})$  and  $\Phi^\omega(\Lambda)$  are distinct fixed points. As we shall see, for each Gödel numbering there are  $2^{\aleph_0}$  fixed points.

To make our terminology less cumbersome, we will abbreviate  $\mathcal{L}(\Phi^\alpha(S))$  as  $\mathcal{L}_{S,\alpha}$ . Our primary interest will be the least fixed point language  $\mathcal{L}_{\Lambda,\omega}$ . Rather than write  $\models_{\Phi^\omega(\Lambda)} A[r]$  we will write  $\models A[r]$ , and rather than write  $\models_{\Phi^\omega(\Lambda)} A$  we will write  $\models A$ .

Given a fixed Gödel numbering for the language, the theory of the least fixed point interpretation,  $\mathcal{L}_{\Lambda,\omega}$  admits of a rather straightforward axiomatization. We take as our sole axiom

$$\mathbf{0} \approx \mathbf{0}.$$

The set of *theorems* is the smallest set of *sentences* which includes the axiom and is closed under the following *rules of inference*. (Read  $\vdash A$  as ‘ $A$  is a theorem’.  $\vdash A \implies \vdash B$  means that if  $A$  is a theorem then so is  $B$ .)

- Rule 1  $\vdash t_0 \approx t_1 \implies \vdash s(t_0) \approx s(t_1)$
- Rule 2  $\vdash A \implies \vdash (A \vee B)$  (for any sentence  $B$ )
- Rule 3  $\vdash B \implies \vdash (A \vee B)$  (for any sentence  $B$ )
- Rule 4  $\vdash A$  and  $\vdash B \implies \vdash (A \wedge B)$
- Rule 5  $\vdash A(x_i/\mathbf{n}) \implies \vdash \exists x_i A$
- Rule 6  $\vdash A(x_0/\mathbf{n}_0, \dots, x_m/\mathbf{n}_m) \implies \vdash \text{Sat}(\ulcorner A \urcorner, \mathbf{n}_0, \dots, \mathbf{n}_m)$

Here we let  $\mathbf{n}$  stand for the numeral for  $n$  (for example, if  $n = 2$  then  $\mathbf{n} = s(s(\mathbf{0}))$ ), and  $A(x_0/t_0, \dots, x_m/t_m)$  is the formula that results from replacing all free occurrences of the variables  $x_i$  with  $t_i$  (for  $0 \leq i \leq m$ ) in formula  $A$ .

A *derivation* is a sequence of sentences of  $L$  in which each sentence is either  $\mathbf{0} \approx \mathbf{0}$  or else follows from a previous sentence (or sentences) in the sequence by one of the rules of inference. A sentence will be a theorem if and only if it occurs as the

last sentence of a derivation. The set of theorems will be relative to a Gödel numbering; that is, different Gödel numberings will generally result in a different set of theorems. For any fixed Gödel numbering, our axiomatization is a sound and complete axiomatization of the least fixed point language  $\mathcal{L}_{\Lambda, \omega}$ .

**Theorem 2.4** (Soundness Theorem) *With respect to any Gödel numbering, for any sentence  $A$ , if  $\vdash A$  then  $\models A$ ; in fact, if  $S$  is any fixed point of  $\Phi$  then if  $\vdash A$  then  $\models_S A$ .*

*Proof:* This holds since  $\models \mathbf{0} = \mathbf{0}$ ; Rules 1 through 5 preserve truth under any interpretation of  $L$ , and Rule 6 preserves truth in any fixed point interpretation,  $\mathcal{L}(S)$ , of the language.  $\square$

**Theorem 2.5** (Completeness Theorem) *With respect to any Gödel numbering, for any sentence  $A$ , if  $\models A$  then  $\vdash A$ .*

*Proof:* By Lemma 2.3(e), if  $\models A$  then, for some finite  $n$ ,  $\models_{\Phi^n(\Lambda)} A$ . Thus it will suffice to show by induction on  $n$  that if  $\models_{\Phi^n(\Lambda)} A$  then  $\vdash A$ . Suppose that  $n = 0$  and  $\models_{\Phi^0(\Lambda)} A$ , that is,  $\models_{\Lambda} A$ . We show by induction on the complexity of  $A$  that  $\vdash A$ . Suppose that  $A$  is atomic.  $A$  cannot be  $\text{Sat}(t_0, \dots, t_m)$  since the extension of  $\text{Sat}$  is empty in  $\mathcal{L}(\Lambda)$ . So  $A$  is  $t_0 \approx t_1$ . Since  $A$  is a true sentence,  $t_0$  and  $t_1$  will be the same numeral. Thus  $t_0 \approx t_1$  will either be the axiom  $\mathbf{0} \approx \mathbf{0}$  or be obtained from  $\mathbf{0} \approx \mathbf{0}$  by a finite number of applications of Rule 1. The induction step, for  $A$  nonatomic, is straightforward. So now we assume that, for any sentence  $A$ , if  $\models_{\Phi^n(\Lambda)} A$  then  $\vdash A$ , to show that, for any  $A$ , if  $\models_{\Phi^{n+1}(\Lambda)} A$  then  $\vdash A$ . We assume  $\models_{\Phi^{n+1}(\Lambda)} A$  and show by induction on the complexity of  $A$  that  $\vdash A$ . The argument is the same as for the case of  $n = 0$  except that now we must also consider the case where  $A$  is the sentence  $\text{Sat}(t_0, \dots, t_m)$ . Suppose  $\models_{\Phi^{n+1}(\Lambda)} \text{Sat}(t_0, \dots, t_m)$ .  $t_0, \dots, t_m$  will be numerals. We may assume that they denote respectively the numbers  $k_0, \dots, k_m$ .  $k_0$  is the Gödel number of a formula  $B$  with at most  $x_0, \dots, x_{m-1}$  free such that  $\models_{\Phi^n(\Lambda)} B[r]$  for every  $r$  that extends  $\langle k_1, \dots, k_m \rangle$ . Thus  $\models_{\Phi^n(\Lambda)} B(x_0/t_1, \dots, x_{m-1}/t_m)[r]$  for every  $r$ , that is,  $\models_{\Phi^n(\Lambda)} B(x_0/t_1, \dots, x_{m-1}/t_m)$ . Thus, by the induction hypothesis,  $\vdash B(x_0/t_1, \dots, x_{m-1}/t_m)$ . So, by Rule 6,  $\vdash \text{Sat}(\ulcorner B \urcorner, t_1, \dots, t_m)$ , that is,  $\vdash \text{Sat}(t_0, \dots, t_m)$ , which is what we wanted to show.  $\square$

Recall that so far we have put no restrictions on the way language  $L$  is to be Gödel numbered. If the Gödel numbering allows us either (1) to effectively decide for any formula  $A$  what its Gödel number is or (2) to effectively decide for any number (expressed as a numeral of  $L$ ) whether it is the Gödel number of a formula and, if so, of which formula, then we will be able to effectively decide whether a sentence in a sequence follows from previous sentences by one of the rules (and in particular we will be able to decide whether it follows by Rule 6), and thus it will be effectively decidable whether a given sequence is a derivation. If, however, neither (1) nor (2) is effectively decidable we will have no effective way to determine whether one sentence follows from another by Rule 6.

It will be useful to introduce the notion of the definability of a set or relation in  $\mathcal{L}(S)$ , for arbitrary  $S$ . We say that a formula  $F$  containing at most  $x_0, \dots, x_i$  as free variables *defines* the relation  $R$  in  $\mathcal{L}(S)$  provided that  $R = \{\langle n_0, \dots, n_i \rangle \mid \models_S F[\langle n_0, \dots, n_i \rangle]\}$ . For  $n \in \mathbb{N}$ , we identify  $\langle n \rangle$  with  $n$  itself and so treat a set of numbers as a one-place relation on the numbers. We identify an  $n$ -place function with

an  $n + 1$ -place relation. Thus our notion of definition applies to sets and functions. Say that a *condition*  $C$  determines an  $i$ -place relation  $R$  provided  $C$  holds of  $\langle n_1, \dots, n_i \rangle$  if and only if  $\langle n_1, \dots, n_i \rangle \in R$ . If condition  $C$  determines an  $i$ -place relation  $R$ , then we say that a given formula defines condition  $C$  if and only if it defines the relation determined by  $C$ . A condition or relation is *definable* in  $\mathcal{L}(S)$  provided that there is a formula which defines it in  $\mathcal{L}(S)$ .

Suppose that  $\{\langle n_1, \dots, n_i \rangle \mid \langle n_1, \dots, n_i \rangle \in R\}$  is any  $i$ -place relation on  $\mathbb{N}$  and suppose that  $1 \leq j \leq i$ ; then the relation  $\{\langle n_1, \dots, n_j, \dots, n_i \rangle \mid \forall m(m < n_j \implies \langle n_1, \dots, m, \dots, n_i \rangle \in R)\}$  is said to be obtained by *bounded universal quantification on the  $j^{\text{th}}$  term of  $R$*  (here we are using  $\langle n_1, \dots, m, \dots, n_i \rangle$  to stand for the result of replacing the  $j^{\text{th}}$  term in  $\langle n_1, \dots, n_j, \dots, n_i \rangle$  with  $m$ ). The following theorem will be useful.

**Theorem 2.6** *With respect to any Gödel numbering, for each  $i$  and for  $1 \leq j \leq i$ , there is a formula  $BU_j^i(x_0, \dots, x_i)$  of  $\mathcal{L}$  such that, for any fixed point language  $\mathcal{L}(S)$  and any formula  $A$  with at most  $x_0, \dots, x_{i-1}$  free which defines in  $\mathcal{L}(S)$  the  $i$ -place relation  $R$ ,  $BU_j^i(x_0, \dots, x_{i-1}, \ulcorner A \urcorner)$  defines the relation obtained by bounded universal quantification on the  $j^{\text{th}}$  term of  $R$ .*

*Proof:* Suppose that  $\mathcal{L}(S)$  is a fixed point language and formula  $A$ , with at most  $x_0, \dots, x_{i-1}$  free, defines the  $i$ -place relation  $\{\langle n_1, \dots, n_i \rangle \mid \langle n_1, \dots, n_i \rangle \in R\}$  in  $\mathcal{L}(S)$ , and let  $1 \leq j \leq i$ . Take the formula

$$(*) \quad x_{j-1} = \mathbf{0} \vee \exists x_{i+2}(s(x_{i+2}) = x_{j-1} \wedge \text{Sat}(x_i, x_0, \dots, x_{j-1}/x_{i+2}, \dots, x_{i-1}) \wedge \text{Sat}(x_{i+1}, x_0, \dots, x_{j-1}/x_{i+2}, \dots, x_i, x_{i+1})).$$

(Here I am using  $\text{Sat}(x_i, x_0, \dots, x_{j-1}/x_{i+2}, \dots, x_{i-1})$  to abbreviate the result of replacing  $x_{j-1}$  in  $\text{Sat}(x_i, x_0, \dots, x_{j-1}, \dots, x_{i-1})$  with  $x_{i+2}$ .) Suppose the Gödel number of  $(*)$  is  $k$ . Let  $BU_j^i(x_0, \dots, x_{i-1}, x_i)$  abbreviate the formula,

$$x_{j-1} = \mathbf{0} \vee \exists x_{i+2}(s(x_{i+2}) = x_{j-1} \wedge \text{Sat}(x_i, x_0, \dots, x_{j-1}/x_{i+2}, \dots, x_{i-1}) \wedge \text{Sat}(\mathbf{k}, x_0, \dots, x_{j-1}/x_{i+2}, \dots, x_i, \mathbf{k})),$$

that is,  $BU_j^i(x_0, \dots, x_i)$  results from  $(*)$  by replacing the variable  $x_{i+1}$  with the numeral for the Gödel number for  $(*)$ .  $BU_j^i(x_0, \dots, x_{i-1}, \ulcorner A \urcorner)$  defines the relation obtained by bounded universal quantification on the  $j^{\text{th}}$  term of  $R$ . To show this we need to prove that

$$\models_S BU_j^i(x_0, \dots, x_{j-1}, \dots, x_{i-1}, \ulcorner A \urcorner)[\langle n_1, \dots, n_j, \dots, n_i \rangle] \quad (1)$$

holds if and only if

$$\forall m(m < n_j \implies \models_S A[\langle n_1, \dots, m, \dots, n_i \rangle]) \quad (2)$$

(where, again,  $\langle n_1, \dots, m, \dots, n_i \rangle$  is used to stand for the result of replacing the  $j^{\text{th}}$  term in  $\langle n_1, \dots, n_i \rangle$  with  $m$ ). This will be proven by induction on  $n_j$ . Basis step: Suppose  $n_j = 0$ . Then (2) holds trivially, and (1) holds since the first disjunct of  $BU_j^i(x_0, \dots, x_{j-1}, \dots, x_{i-1}, \ulcorner A \urcorner)$  is  $x_{j-1} = \mathbf{0}$  and  $\models_S x_{j-1} = \mathbf{0}[\langle n_1, \dots, n_j, \dots, n_i \rangle]$  when  $n_j = 0$ . Induction step: Suppose that the induction hypothesis holds for  $n_j \leq p$ .

We need to show that for  $n_j = p + 1$ , (1) holds if and only if (2) holds. Suppose that  $n_j = p + 1$ . Then, since  $\langle n_1, \dots, n_j, \dots, n_i \rangle$  does not satisfy the first disjunct of  $BU_j^i(x_0, \dots, x_{j-1}, \dots, x_{i-1}, \ulcorner A \urcorner)$ , (1) holds if and only if  $\langle n_1, \dots, n_j, \dots, n_i \rangle$  satisfies

$$\begin{aligned} \exists x_{i+2}(s(x_{i+2}) = x_{j-1} \wedge \text{Sat}(\ulcorner A \urcorner, x_0, \dots, x_{j-1}/x_{i+2}, \dots, x_{i-1}) \wedge \\ \text{Sat}(\mathbf{k}, x_0, \dots, x_{j-1}/x_{i+2}, \dots, \ulcorner A \urcorner, \mathbf{k})) \end{aligned}$$

which holds, since  $n_j = p + 1$ , if and only if  $\langle n_1, \dots, p, \dots, n_i \rangle$  satisfies

$$\text{Sat}(\ulcorner A \urcorner, x_0, \dots, x_{j-1}, \dots, x_{i-1}) \wedge \text{Sat}(\mathbf{k}, x_0, \dots, x_{j-1}, \dots, \ulcorner A \urcorner, \mathbf{k})$$

which holds if and only if both (3) and (4) hold:

$$\models_S A[\langle n_1, \dots, p, \dots, n_i \rangle] \tag{3}$$

$$\forall m(m < p \implies \models_S A[\langle n_1, \dots, m, \dots, n_i \rangle]). \tag{4}$$

(We use the fact that  $\mathcal{L}(S)$  is a fixed point to get (3) and the induction hypothesis together with the fact that  $\mathcal{L}(S)$  is a fixed point to get (4).) (3) together with (4) in turn holds if and only if (2) holds since  $n_j = p + 1$ .  $\square$

**Corollary 2.7** *The conditions which determine definable  $i$ -place relations in any given fixed point are closed under bounded universal quantification.*

**3 Recursion theory** We now study recursion theory in terms of our fixed point languages, especially  $\mathcal{L}_{\Lambda, \omega}$ . We begin by looking at the partial recursive functions. Here we speak of a *partial function* if the domain and range are subsets of  $\mathbb{N}$ ; a partial function whose domain is the whole of  $\mathbb{N}$  will be called a *total function* or simply a *function*. The set of *partial recursive functions* includes the following basic functions defined on  $\mathbb{N}$ :

1. the *zero function*,  $z(n)$ : for all  $n$ ,  $z(n) = 0$ ;
2. the *successor function*,  $'$ : for all  $n$ ,  $n' = n + 1$ ;
3. the *identity functions*,  $id_j^i$  (for  $1 \leq j \leq i$ ): for all  $n_1, \dots, n_i$ ,  $id_j^i(n_1, \dots, n_i) = n_j$ .

Furthermore, the partial recursive functions are closed under the following operations:

4. *composition*: suppose that  $f^i$  is an  $i$ -place partial function and  $g_1^j, \dots, g_i^j$  are  $j$ -place partial functions; then if  $h(n_1, \dots, n_j) = f^i(g_1^j(n_1, \dots, n_j), \dots, g_i^j(n_1, \dots, n_j))$ , we say that  $h$  is the composition of  $f^i$  with  $g_1^j, \dots, g_i^j$ .
5. *primitive recursion*: suppose that  $f$  is an  $i$ -place partial function and  $g$  is an  $i + 2$ -place partial function; then we define the  $i + 1$ -place partial function  $h$  by primitive recursion on  $f$  and  $g$  as follows:

$$\begin{aligned} h(n_1, \dots, n_i, 0) &= f(n_1, \dots, n_i), \\ h(n_1, \dots, n_i, m') &= g(n_1, \dots, n_i, m, h(n_1, \dots, n_i, m)). \end{aligned}$$



6. *minimization*: suppose that  $f^{i+1}$  is an  $i + 1$ -place partial function; then we say that the  $i$ -place partial function  $g$  is the minimization of  $f$  provided that

$$g(n_1, \dots, n_i) = m \text{ if } f(n_1, \dots, n_i, m) = 0$$

and

$$\forall k (k \langle m \implies f(n_1, \dots, n_i, k) \rangle 0),$$

and  $g(n_1, \dots, n_i)$  is undefined if no such  $m$  exists. We will write  $g(n_1, \dots, n_i)$  as  $\mu m[f(n_1, \dots, n_i, m) = 0]$ .

A *recursive function* is a partial recursive function that is a total function. The partial functions that can be defined in terms of the three basic functions together with composition and primitive recursion but without minimization will be a subset of the recursive functions; such functions are called *primitive recursive functions*. We say that a *set or relation is (primitive) recursive* if and only if it has a (primitive) recursive characteristic function; that is, an  $i$ -place relation  $R$  is (primitive) recursive if and only if there is a (primitive) recursive function  $f$  such that  $f(n_1, \dots, n_i) = 0$  if  $\langle n_1, \dots, n_i \rangle \in R$  and  $f(n_1, \dots, n_i) = 1$  if  $\langle n_1, \dots, n_i \rangle \notin R$ . A *set (or relation) is recursively enumerable (abbreviated r.e.)* if and only if it is the domain of a partial recursive function. ( $\Lambda$  is r.e. since it is the domain of the totally undefined partial recursive function  $\mu m[(id_1^2(n_1, m))' = 0]$ .) Any recursive relation  $R$  will be r.e., for  $R = \{\langle n_1, \dots, n_i \rangle \mid f(n_1, \dots, n_i) = 0\}$ , for some recursive characteristic function  $f$ , and thus  $R$  is the domain of the partial recursive function  $\mu m[f(id_1^{i+1}(n_1, \dots, n_i, m), \dots, id_i^{i+1}(n_1, \dots, n_i, m)) = 0]$ .

The intuitive ideas these formal notions are intended to capture are roughly as follows. Say that an  $i$ -place relation  $R$  on  $\mathbb{N}$  has an *effective decision procedure* if and only if there is an algorithm such that, given any  $i$ -tuple  $\langle n_1, \dots, n_i \rangle$  of numbers (in some standard notation, for example, decimal notation), eventually the procedure will give an answer stating whether or not  $\langle n_1, \dots, n_i \rangle \in R$ . The notion of a recursive relation is widely thought to capture the idea of an effectively decidable relation on  $\mathbb{N}$ . The notion of an r.e. relation is supposed to capture the weaker notion of a relation for which there is a positive algorithmic test which will eventually produce an affirmative answer if and only if  $\langle n_1, \dots, n_i \rangle \in R$  (but may not produce any output if  $\langle n_1, \dots, n_i \rangle \notin R$ ). The notion of a partial recursive function is intended to capture the idea of a partial function,  $f^i$ , for which there exists an algorithmic procedure such that, given an input  $\langle n_1, \dots, n_i \rangle$ , the procedure will yield a resulting value,  $m$ , if and only if  $f^i(n_1, \dots, n_i)$  is defined (that is,  $\langle n_1, \dots, n_i \rangle \in \text{domain}(f^i)$ ) and  $f(n_1, \dots, n_i) = m$ . If  $f(n_1, \dots, n_i)$  is undefined then the algorithm yields no value (and the algorithmic procedure might not even terminate). Recursive functions cover the special case where for each input  $\langle n_1, \dots, n_i \rangle$  the procedure yields a value. We will not discuss the issue of whether recursion theory succeeds in providing an adequate formalization of these ideas.

**Theorem 3.1** *Given any fixed Gödel numbering,*

- (a) *for every partial recursive function,  $f$ , there is a formula,  $A$ , such that  $A$  defines  $f$  in every fixed point language;*
- (b) *there are formulas of  $L$ ,  $\mathbf{Z}(x_0, x_1)$ ,  $\mathbf{Suc}(x_0, x_1)$ , and for  $1 \leq j \leq i$ ,  $\mathbf{Id}_j^i(x_0, \dots, x_i)$ , such that, for every  $S$ , in  $\mathcal{L}(S)$ :*

$\mathbf{Z}(x_0, x_1)$  defines the zero function,

$\mathbf{Suc}(x_0, x_1)$  defines the successor function,

for  $1 \leq j \leq i$ ,  $\mathbf{Id}_j^i(x_0, \dots, x_i)$  defines the identity function  $id_j^i(n_0, \dots, n_{i-1}) = n_{j-1}$ ;

(c) for every  $i$  and  $j$ , there are formulas of  $L$ ,  $\text{Comp}_{i,j}(x_0, \dots, x_{j+i+1})$ ,  $\text{Pr}^{j+1}(x_0, \dots, x_{j+3})$ , and  $\text{Mn}^i(x_0, \dots, x_{i+1})$ , such that in every fixed point language:

- (i) if  $f^i$  is an  $i$ -place partial function defined by formula  $F$  and  $g_1^j, \dots, g_i^j$  are  $j$ -place partial functions defined by formulas  $G_1, \dots, G_i$  respectively, then  $\text{Comp}_{i,j}(x_0, \dots, x_j, \ulcorner G_1 \urcorner, \dots, \ulcorner G_i \urcorner, \ulcorner F \urcorner)$  defines the composition of  $f^i$  with  $g_1^j, \dots, g_i^j$ ;
- (ii) if  $f^j$  is a  $j$ -place partial function defined by formula  $F$  and  $g^{j+2}$  is a  $j+2$ -place partial function defined by formula  $G$ , then  $\text{Pr}^{j+1}(x_0, \dots, x_{j+1}, \ulcorner F \urcorner, \ulcorner G \urcorner)$  defines the  $j+1$ -place function defined by primitive recursion on  $f^j$  and  $g^{j+2}$ ;
- (iii) if the  $i+1$ -place partial function  $f^{i+1}$  is defined by the formula  $F(x_0, \dots, x_{i+1})$ , then the  $i$ -place partial function which is the minimization on  $f^{i+1}$  is defined by the formula  $\text{Mn}^i(x_0, \dots, x_i, \ulcorner F \urcorner)$ .

*Proof:* Part (a) follows from parts (b) and (c). Part (b): Take any interpretation  $\mathcal{L}(S)$ . The zero function is the relation  $\{\langle n_0, n_1 \rangle \mid n_1 = 0\}$ . Thus we may let  $\mathbf{Z}(x_0, x_1)$  be the formula  $x_1 = \mathbf{0}$  which defines this function in  $\mathcal{L}(S)$  since  $\models_S x_1 = \mathbf{0}[r]$  for every  $r$  that extends  $\langle n_0, 0 \rangle$ . We may take  $\mathbf{Suc}(x_0, x_1)$  to be the formula  $s(x_0) = x_1$ , and  $\mathbf{Id}_j^i(x_0, \dots, x_i)$  to be the formula  $x_{j-1} = x_i$ . Part (c): Let  $\mathcal{L}(S)$  to be any fixed point language. For composition, suppose that in  $\mathcal{L}(S)$  formula  $F$  defines  $f^i$  and  $G_1, \dots, G_i$  respectively define  $g_1^j, \dots, g_i^j$ . Let  $y_k$  abbreviate  $x_{j+i+1+k}$ . Then

$$\exists y_1 \dots \exists y_i (\text{Sat}(\ulcorner G_1 \urcorner, x_0, \dots, x_{j-1}, y_1) \wedge \dots \wedge \text{Sat}(\ulcorner G_i \urcorner, x_0, \dots, x_{j-1}, y_i) \wedge \text{Sat}(\ulcorner F \urcorner, y_1, \dots, y_i, x_j))$$

defines in  $\mathcal{L}(S)$  the function  $f(g_1^j(n_0, \dots, n_{j-1}), \dots, g_i^j(n_0, \dots, n_{j-1})) = n_j$ , the composition of  $f^i$  with  $g_1^j, \dots, g_i^j$ . Thus we may take  $\text{Comp}_{i,j}$  to be the formula,

$$\exists y_1 \dots \exists y_i (\text{Sat}(x_{j+1}, x_0, \dots, x_{j-1}, y_1) \wedge \dots \wedge \text{Sat}(x_{j+i}, x_0, \dots, x_{j-1}, y_i) \wedge \text{Sat}(x_{j+i+1}, y_1, \dots, y_i, x_j)).$$

Next consider partial functions defined by primitive recursion. For simplicity, suppose that the partial function  $h(n_0, n_1) = m$  is defined by primitive recursion from the partial functions  $f(n_0) = m$  and  $g(n_0, n_1, n_2) = m$ . (Our considerations can be easily generalized to cover cases where  $h$  is not a two-placed partial function.) Suppose that the formulas  $F(x_0, x_1)$  and  $G(x_0, x_1, x_2, x_3)$  define  $f$  and  $g$ , respectively. We need to show that there is a formula which defines  $h$ . Consider the following formula,

$$(x_1 = \mathbf{0} \wedge \text{Sat}(x_3, x_0, x_2) \vee \exists x_6 \exists x_7 (x_1 = s(x_6) \wedge \text{Sat}(x_5, x_0, x_6, x_7, x_3, x_4, x_5) \wedge \text{Sat}(x_4, x_0, x_6, x_7, x_2))).$$

Suppose that the Gödel number of this formula is  $k$ . The following formula,

$$\text{Sat}(\mathbf{k}, x_0, x_1, x_2, x_3, x_4, \mathbf{k}),$$

will be abbreviated as  $\text{Pr}^2(x_0, x_1, x_2, x_3, x_4)$ .  $\text{Pr}^2(x_0, x_1, x_2, \ulcorner F \urcorner, \ulcorner G \urcorner)$  defines the two-place partial function  $h$  that is obtained from  $f$  and  $g$  by primitive recursion. To show this we must show that  $h(n_0, n_1) = m$  if and only if  $\models_S \text{Pr}^2(x_0, x_1, x_2, \ulcorner F \urcorner, \ulcorner G \urcorner)[\langle n_0, n_1, m \rangle]$ . This we show by induction on  $n_1$ . First we note that, since  $\mathcal{L}(S)$  is a fixed point,  $\text{Pr}^2(x_0, x_1, x_2, \ulcorner F \urcorner, \ulcorner G \urcorner)$  is satisfied by the same sequences as

$$(x_1 = \mathbf{0} \wedge \text{Sat}(\ulcorner F \urcorner, x_0, x_2) \vee \exists x_6 \exists x_7 (x_1 = s(x_6) \wedge \text{Sat}(\mathbf{k}, x_0, x_6, x_7, \ulcorner F \urcorner, \ulcorner G \urcorner, \mathbf{k}) \wedge \text{Sat}(\ulcorner G \urcorner, x_0, x_6, x_7, x_2))).$$

Suppose  $n_1 = 0$ . Then this formula will be satisfied by  $\langle n_0, n_1, m \rangle$  if and only if the first disjunct is so satisfied, if and only if  $f(n_0) = m$ , if and only if  $h(n_0, n_1) = m$ . Suppose that  $n_1 = i'$ . Then the formula will be satisfied by  $\langle n_0, n_1, m \rangle$  if and only if the second disjunct is so satisfied. Note that the second disjunct is the formula,

$$\exists x_6 \exists x_7 (x_1 = s(x_6) \wedge \text{Pr}^2(x_0, x_6, x_7, \ulcorner F \urcorner, \ulcorner G \urcorner) \wedge \text{Sat}(\ulcorner G \urcorner, x_0, x_6, x_7, x_2))$$

and thus, by the induction hypothesis, it will be satisfied by  $\langle n_0, n_1, m \rangle$  (that is,  $\langle n_0, i', m \rangle$ ) if and only if  $m = g(n_0, i, h(n_0, i))$ , that is,  $m = h(n_0, i')$ , that is,  $m = h(n_0, n_1)$ , which is what we wanted to prove.

Finally we turn to minimization. Suppose that the partial function  $f^{i+1}(n_0, \dots, n_i) = n_{i+1}$  is defined by formula  $F(x_0, \dots, x_{i+1})$  and the partial function  $g$  is the minimization of  $f$ . We want to find a formula that defines  $g$ .  $g(n_0, \dots, n_{i-1}) = n_i$  if and only if the following condition is met:

$$(*) \quad \text{For any } j (j < n_i \implies f(n_0, \dots, n_{i-1}, j) > 0) \text{ and } f(n_0, \dots, n_{i-1}, n_i) = 0.$$

The condition  $f(n_0, \dots, n_{i-1}, n_i) > 0$  can be defined by  $\exists x_{i+2} \text{Sat}(\ulcorner F \urcorner, x_0, \dots, x_{i-1}, x_i, s(x_{i+2}))$ . Let  $k$  be the Gödel number of  $\exists x_{i+2} \text{Sat}(x_{i+1}, x_0, \dots, x_{i-1}, x_i, s(x_{i+2}))$ . Then (using formula  $BU_i^{i+1}$  from Theorem 2.6) we may let  $Mn^i(x_0, \dots, x_{i+1})$  be the formula

$$BU_{i+1}^{i+2}(x_0, \dots, x_{i+1}, \mathbf{k}) \wedge \text{Sat}(x_{i+1}, x_0, \dots, x_i, \mathbf{0}).$$

Then  $Mn^i(x_0, \dots, x_i, \ulcorner F \urcorner)$  will then be the formula

$$BU_{i+1}^{i+2}(x_0, \dots, x_i, \ulcorner F \urcorner, \mathbf{k}) \wedge \text{Sat}(\ulcorner F \urcorner, x_0, \dots, x_i, \mathbf{0}),$$

which defines condition  $(*)$ , completing the proof.  $\square$

**Corollary 3.2** *With respect to any Gödel numbering, in every fixed point, every set or relation that is r.e. (and thus any set or relation that is recursive) is definable.*

*Proof:* Take any  $i$ -place r.e. relation  $R$ . It is the domain of an  $i$ -place partial recursive function, which is definable by some formula  $A(x_0, \dots, x_i)$ .  $R$  is defined by the formula  $\exists x_i A(x_0, \dots, x_i)$ .  $\square$

Let's illustrate how we can construct formulas of L that define partial recursive functions in any fixed point. Take the primitive recursive function  $n_0 + n_1 = n_2$ . We may define this function by primitive recursion from appropriate functions  $f$  and  $g$  as follows:

$$\begin{aligned} n_0 + 0 &= f(n_0) = n_0; \\ n_0 + n'_1 &= g(n_0, n_1, n_0 + n_1) = (n_0 + n_1)'. \end{aligned}$$

$f$  is the function  $id_1^1$ .  $g(n_0, n_1, n_2) = n_3$  is the function formed by composition of the successor function with  $id_3^3$ . Let  $\text{Sum}(x_0, x_1, x_2)$  abbreviate

$$\text{Pr}^2(x_0, x_1, x_2, \ulcorner \text{Id}_1^1(x_0, x_1) \urcorner, \ulcorner \text{Comp}(x_0, x_1, x_2, x_3, \ulcorner \text{Suc}(x_0, x_1) \urcorner, \ulcorner \text{Id}_3^3(x_0, x_1, x_2, x_3) \urcorner) \urcorner).$$

$\text{Sum}(x_0, x_1, x_2)$  defines the relation  $\{(n_0, n_1, n_2) \mid n_0 + n_1 = n_2\}$  in every fixed point language. Readers not familiar with this technique might want to try as exercises constructing formulas  $\text{Prod}(x_0, x_1, x_2)$  and  $\text{exp}(x_0, x_1, x_2)$  to respectively define the functions  $n_0 \cdot n_1 = n_2$  and  $n_0^{n_1} = n_2$ . (Note that the product and exponentiation functions are primitive recursive:  $n_0 \cdot 0 = 0$ ,  $n_0 \cdot n'_1 = (n_0 \cdot n_1) + n_0$  and  $n_0^0 = 1$ ,  $n_0^{n'_1} = n_0 \cdot n_0^{n_1}$ .)

**Theorem 3.3** *There are enumerably many partial recursive functions, recursive functions, recursively enumerable relations, and recursive relations.*

*Proof:* Since all recursive functions are partial recursive functions and all recursive relations are r.e. relations it will suffice to show that there are at most enumerably many partial recursive functions and r.e. relations. That there are enumerably many partial recursive functions (and thus enumerably many r.e. relations) follows from the fact that (in any given fixed point) each partial recursive function can be defined by a formula and there are only enumerably many formulas of L.  $\square$

**Theorem 3.4** *There are functions that are not partial recursive functions and relations that are not r.e.*

*Proof:* This follows from the previous theorem since there are nondenumerably many functions and relations on  $\mathbb{N}$ .  $\square$

We have seen that in any fixed point language all r.e. relations are definable. One might wonder whether only r.e. relations are definable in the fixed point languages, or, at least, whether only r.e. relations are definable in the least fixed point languages for any given Gödel numbering. The following theorems will answer these questions.

**Theorem 3.5** *For any Gödel numbering:*

- (a) *there exists  $2^{\aleph_0}$  fixed points;*
- (b) *for any relation  $R$ , there is some fixed point in which  $R$  is definable; and*
- (c) *there are fixed points in which relations are definable that are not r.e.*

*Proof:*

- (a) Suppose that we have a fixed Gödel numbering. Consider the formula  $\text{Sat}(x_0, x_0, x_1, \dots, x_i)$ . Suppose its Gödel number is  $n_0$ . Then for any  $S$  and any  $\alpha$ ,  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in \Phi^\alpha(S)$  if and only if  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in S$ . This can be shown by induction on  $\alpha$ . For  $\alpha = 0$ , this holds since  $\Phi^0(S) = S$ . If it holds for  $\alpha$ , then  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in S$  if and only if  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in \Phi^\alpha(S)$  if and only if  $\models_{\Phi^\alpha(S)} \text{Sat}(x_0, x_0, x_1, \dots, x_i)[\langle n_0, n_1, \dots, n_i \rangle]$  if and only if  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in \Phi^{\alpha+1}(S)$ . Finally, let  $\alpha$  be a limit ordinal. If  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in S$  then  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in \bigcup_{\beta < \alpha} \Phi^\beta(S) = \Phi^\alpha(S)$ . If  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in \Phi^\alpha(S)$ , then for some  $\beta < \alpha$ ,  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in \Phi^\beta(S)$ , and thus by the induction hypothesis,  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in S$ .

Consider any  $i$ -place relation,  $R$ , on  $\mathbb{N}$ . Let  $R^* = \{\langle n_0, n_0, n_1, \dots, n_i \rangle \mid \langle n_1, \dots, n_i \rangle \in R\}$ . By the above considerations  $R^* \subseteq \Phi(R^*)$ . Thus, by Theorem 2.2(c),  $\Phi^\omega(R^*)$  is a fixed point. Furthermore,  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in \Phi^\omega(R^*)$  if and only if  $\langle n_0, n_0, n_1, \dots, n_i \rangle \in R^*$ . Thus to each  $i$ -place relation  $R$ , there exists a distinct fixed point  $\Phi^\omega(R^*)$ . There are thus at least  $2^{\aleph_0}$  fixed points since there are  $2^{\aleph_0}$   $i$ -place relations on  $\mathbb{N}$ . Furthermore, there are at most  $2^{\aleph_0}$  fixed points since there are at most  $2^{\aleph_0}$  ways of interpreting  $\text{Sat}$  by a set of finite sequences of natural numbers.

- (b) Any  $i$ -place relation  $R$  is defined by  $\text{Sat}(\mathbf{n}_0, \mathbf{n}_0, x_0, \dots, x_{i-1})$  in the fixed point language  $\mathcal{L}_{R^*, \omega}$ . (It is interesting to note that the formula  $\text{Sat}(\mathbf{n}_0, \mathbf{n}_0, x_1, \dots, x_i)$  works like a ‘truth teller’.)
- (c) Follows from part (b) and Theorem 3.4. □

**Theorem 3.6** *For any relation  $R$ , there exists a Gödel numbering for which  $R$  can be defined in the least fixed point language for that Gödel numbering.*

*Proof:* Let  $R$  be any  $i$ -place relation on  $\mathbb{N}$ .  $R$  is enumerable. Let  $\langle n_{0,1}, n_{0,2}, \dots, n_{0,i} \rangle, \langle n_{1,1}, n_{1,2}, \dots, n_{1,i} \rangle, \langle n_{2,1}, n_{2,2}, \dots, n_{2,i} \rangle, \dots$  be a nonredundant enumeration of the elements of  $R$ . (This enumeration need not be effective; recall that we are not requiring our Gödel numberings to be effective.)

*Case 1:* The enumeration is finite. In this case  $R$  can be defined under any Gödel numbering. If  $R$  is empty,  $\mathbf{0} = \mathbf{1}$  defines  $R$ . Suppose  $R$  is nonempty but finite. Let its enumeration be  $\langle n_{0,1}, n_{0,2}, \dots, n_{0,i} \rangle, \dots, \langle n_{k,1}, n_{k,2}, \dots, n_{k,i} \rangle$ . Then the formula

$$(x_0 = \mathbf{n}_{0,1} \wedge x_1 = \mathbf{n}_{0,2} \wedge \dots \wedge x_{i-1} = \mathbf{n}_{0,i}) \vee \dots \vee (x_0 = \mathbf{n}_{k,1} \wedge x_1 = \mathbf{n}_{k,2} \wedge \dots \wedge x_{i-1} = \mathbf{n}_{k,i})$$

defines  $R$ .

*Case 2:* The enumeration of  $R$  is infinite. For each  $m \in \mathbb{N}$ , we let  $2m$  be the Gödel number of the formula

$$(x_0 = \mathbf{n}_{m,1} \wedge x_1 = \mathbf{n}_{m,2} \wedge \dots \wedge x_{i-1} = \mathbf{n}_{m,i}) \vee \text{Sat}(2\mathbf{m} + \mathbf{2}, x_0, x_1, \dots, x_{i-1}).$$

(Here  $2\mathbf{m} + \mathbf{2}$  is the numeral for the number  $2m + 2$ .) To all the other formulas assign odd Gödel numbers.  $(x_0 = \mathbf{n}_{0,1} \wedge x_1 = \mathbf{n}_{0,2} \wedge \dots \wedge x_{i-1} = \mathbf{n}_{0,i}) \vee$

$\text{Sat}(\mathbf{2}, x_0, x_1, \dots, x_{i-1})$  defines  $\{\langle n_{0,1}, n_{0,2}, \dots, n_{0,i} \rangle\}$  in  $\mathcal{L}_{\Lambda,0}, \{\langle n_{0,1}, n_{0,2}, \dots, n_{0,i} \rangle, \langle n_{1,1}, n_{1,2}, \dots, n_{1,i} \rangle\}$  in  $\mathcal{L}_{\Lambda,1}, \dots, \{\langle n_{0,1}, n_{0,2}, \dots, n_{0,i} \rangle, \langle n_{1,1}, n_{1,2}, \dots, n_{1,i} \rangle, \dots, \langle n_{k,1}, n_{k,2}, \dots, n_{k,i} \rangle\}$  in  $\mathcal{L}_{\Lambda,k}$ , and so on. In the least fixed point language  $\mathcal{L}_{\Lambda,\omega}$ , the formula  $(x_0 = \mathbf{n}_{0,1} \wedge x_1 = \mathbf{n}_{0,2} \wedge \dots \wedge x_{i-1} = \mathbf{n}_{0,i}) \vee \text{Sat}(\mathbf{2}, x_0, x_1, \dots, x_{i-1})$  defines  $R$ .  $\square$

Taking for  $R$  a relation that is not r.e., Theorem 3.6 shows the following.

**Corollary 3.7** *There are Gödel numberings with respect to which the definable relations in  $\mathcal{L}_{\Lambda,\omega}$  exceed the r.e. relations.*

We see that part of the strength (measured in terms of power to define relations) of a least fixed point language may be determined by the Gödel numbering, but each fixed point has at least the strength to define all r.e. relations. We have put no restrictions on allowable Gödel numberings. Normally in setting out a particular Gödel numbering care is taken so that the numbering is effective: one can effectively go from an expression (say a sentence or formula) set out in appropriate notation to its Gödel number (expressed in appropriate notation), and one can effectively decide for a number (expressed in appropriate notation) whether it is the Gödel number for an expression and if so which expression (expressed in appropriate notation). I do not want the development of our theory to depend on the general notion of an effective Gödel numbering. On the other hand, I do not want to take the space to carefully set out a particular Gödel numbering. So I will merely make a few remarks about well-known features of the Gödel numbering of languages.

Using standard techniques, a Gödel numbering  $\text{gn}^*$  can be set out that has the following properties: (1) Not only are Gödel numbers assigned to formulas but they are also assigned to finite sequences of formulas, in particular a derivation will have a Gödel number; (2) The following functions,  $\text{Sub}_i(n_1, n_2)$  and  $\text{Proof}(n_1, n_2)$ , are primitive recursive. For each  $i$ ,  $\text{Sub}_i(n_1, n_2) = n_1$  if  $n_1$  is not a Gödel number for a well-formed formula; if  $n_1 = \text{gn}^*(A)$  for some well-formed formula  $A$  then  $\text{Sub}_i(n_1, n_2) = \text{gn}^*(A(x_i/\mathbf{n}_2))$ .  $\text{Proof}(n_1, n_2) = 0$  if  $n_2$  is the Gödel number for a derivation and  $n_1$  is the Gödel number for the last sentence of the derivation; otherwise  $\text{Proof}(n_1, n_2) = 1$ . Let

$$\text{Sub}_{0,\dots,k}(n_0, n_1, \dots, n_{k+1}) = \text{Sub}_0(\dots(\text{Sub}_{k-1}(\text{Sub}_k(n_0, n_{k+1}), n_k))\dots, n_1).$$

Thus, if

$$n_0 = \text{gn}^*(A), \text{Sub}_{0,\dots,k}(n_0, n_1, \dots, n_{k+1}) = \text{gn}^*(A(x_0/\mathbf{n}_1, \dots, x_k/\mathbf{n}_{k+1})).$$

$\text{Sub}_{0,\dots,k}(n_0, n_1, \dots, n_{k+1})$  is primitive recursive since it can be obtained from the  $\text{Sub}_i$  functions (along with the id functions) using composition.

Let  $\mathcal{L}^*$  be the least fixed point language under Gödel numbering  $\text{gn}^*$ .

**Theorem 3.8** *The relations and partial functions definable in  $\mathcal{L}^*$  are exactly the r.e. relations and the partial recursive functions.*

*Proof:* Theorem 3.1 and its corollary tell us that all r.e. relations and partial recursive functions are definable in  $\mathcal{L}^*$ . What we need to show then is that (a) each partial function definable in  $\mathcal{L}^*$  is a partial recursive function and (b) each relation that is definable in  $\mathcal{L}^*$  is r.e.

- (a) First we need to show that each partial function definable in  $\mathcal{L}^*$  is a partial recursive function. Suppose that, for  $i > 0$ ,  $R = \{\langle n_0, \dots, n_i \rangle \mid f(n_0, \dots, n_{i-1}) = n_i\}$  is the graph of an  $i$ -place partial function  $f$  defined in  $\mathcal{L}^*$  by the formula  $A(x_0, \dots, x_i)$ . We need to show that  $f$  is a partial recursive function. Define  $g$  by primitive recursion as follows:

$$\begin{aligned} g(r, n_0, \dots, n_{i-1}, 0, m) &= \text{Proof}(\text{Sub}_{0, \dots, i}(r, n_0, \dots, n_{i-1}, 0), m); \\ g(r, n_0, \dots, n_{i-1}, p', m) &= g(r, n_0, \dots, n_{i-1}, p, m) \cdot \\ &\quad \text{Proof}(\text{Sub}_{0, \dots, i}(r, n_0, \dots, n_{i-1}, p'), m). \end{aligned}$$

Note that  $g$  is primitive recursive (this relies on the fact we saw earlier that multiplication is primitive recursive).  $g(\text{gn}^*(A), n_0, \dots, n_{i-1}, n_i, m) = 0$  if and only if  $m$  is the Gödel number of a derivation of  $A(x_0/\mathbf{n}_0, \dots, x_{i-1}/\mathbf{n}_{i-1}, x_i/\mathbf{k})$  for some  $k \leq n_i$ ; otherwise the value of the function is 1. Note that, for any sentence  $B$ , if there is a derivation of  $B$  then there are an infinite number of derivations of  $B$  (one can take any derivation of length  $j$  of  $B$  and form a derivation of length  $j + 1$  of  $B$  by adding the sentence  $B$  to the end of the original derivation); thus there is no upper bound on the size of Gödel numbers of derivations of  $B$ . This tells us that if  $\vdash A(x_0/\mathbf{n}_0, \dots, x_i/\mathbf{n}_i)$  then for some  $m$ ,  $g(\text{gn}^*(A), n_0, \dots, n_{i-1}, m, m) = 0$ . For any such  $m$ ,  $m$  is the Gödel number of a derivation of  $A(x_0/\mathbf{n}_0, \dots, x_i/\mathbf{n}_i)$  since  $A$  defines a partial function. Thus the partial function  $f$  is the following partial function:

$$\begin{aligned} &\mu k[\text{Proof}(\text{Sub}_{0, \dots, i}(\text{gn}^*(A), n_0, \dots, n_{i-1}, k), \\ &\mu m[g(\text{gn}^*(A), n_0, \dots, n_{i-1}, m, m) = 0]) = 0]. \end{aligned} \tag{5}$$

Here we may take ‘ $\text{gn}^*(A)$ ’ to stand in for the  $i + 1$ -place constant function whose value is  $\text{gn}^*(A)$ ; this function is primitive recursive since it can be defined as  $(z(\text{id}_1^i(n_0, \dots, n_i)))' \dots'$  (where the number of applications of the successor function =  $\text{gn}^*(A)$ ). The function,  $\mu k[\text{Proof}(\text{Sub}_{0, \dots, i}(\text{gn}^*(A), n_0, \dots, n_{i-1}, k), \mu m[g(\text{gn}^*(A), n_0, \dots, n_{i-1}, m, m) = 0]) = 0]$  is thus a partial recursive function with arguments  $n_0, \dots, n_{i-1}$ , built up from the zero function, the identity functions, and the successor function using composition, primitive recursion, and minimization. Thus  $f$  is a partial recursive function.

- (b) Take any  $i$ -place relation,  $R$ , definable in  $\mathcal{L}^*$ . We must show that  $R$  is r.e.  $R$  is defined in  $\mathcal{L}^*$  by some formula  $A(x_0, \dots, x_{i-1})$ . The formula  $A(x_0, \dots, x_{i-1}) \wedge x_i = 0$  thus defines an  $i$ -place partial function. By part (a) this partial function is partial recursive.  $R$  is thus the domain of a partial recursive function, and so  $R$  is an r.e. relation, which completes the proof.  $\square$

Theorem 3.8, together with Theorem 3.1 and its corollary, gives us the following corollary.

**Corollary 3.9** *A relation is recursively enumerable, and a partial function is partial recursive, if and only if it is definable in every fixed point language with respect to every Gödel numbering.*

**Theorem 3.10**

- (a) If  $R_1$  and  $R_2$  are  $i$ -place r.e. relations, then so are  $R_1 \cup R_2$  and  $R_1 \cap R_2$ .
- (b) For any  $i$ -place relation  $R$ ,  $R$  is recursive  $\iff$  both  $R$  and  $R'$ , the complement of  $R$ , are r.e. (where the complement of  $R = \{\langle n_0, \dots, n_{i-1} \rangle \mid \langle n_0, \dots, n_{i-1} \rangle \notin R\}$ ).
- (c) If  $R_1$  and  $R_2$  are  $i$ -place recursive relations, then so are  $R'_1$ ,  $R_1 \cup R_2$ , and  $R_1 \cap R_2$ .

*Proof:*

- (a) Suppose  $R_1$  and  $R_2$  are  $i$ -place r.e. relations. Let  $A_1$  and  $A_2$  be formulas which respectively define  $R_1$  and  $R_2$  in  $\mathcal{L}^*$ . Then, in  $\mathcal{L}^*$ ,  $A_1 \vee A_2$  defines  $R_1 \cup R_2$ , and  $A_1 \wedge A_2$  defines  $R_1 \cap R_2$ . By Theorem 3.8,  $R_1 \cup R_2$  and  $R_1 \cap R_2$  are r.e.
- (b)  $\implies$  Suppose that  $R$  is a recursive relation. Then  $R$  has a recursive characteristic function  $c_R$  such that  $c_R(n_0, \dots, n_{i-1}) = 0$  if  $\langle n_0, \dots, n_{i-1} \rangle \in R$  and  $c_R(n_0, \dots, n_{i-1}) = 1$  otherwise. By Theorem 3.8,  $c_R$  is defined in  $\mathcal{L}^*$  by some well-formed formula  $C_R(x_0, \dots, x_i)$ . Then  $C_R(x_0, \dots, x_{i-1}, \mathbf{0})$  defines  $R$  and  $C_R(x_0, \dots, x_{i-1}, \mathbf{1})$  defines  $R'$ . Since  $R$  and  $R'$  are definable in  $\mathcal{L}^*$ ,  $R$  and  $R'$  are r.e.

$\Leftarrow$  Now suppose that  $R$  and  $R'$  are r.e. Then they are defined in  $\mathcal{L}^*$  by some well-formed formulas  $A_R(x_0, \dots, x_{i-1})$  and  $A_{R'}(x_0, \dots, x_{i-1})$ .  $(A_R(x_0, \dots, x_{i-1}) \wedge x_i = 0) \vee (A_{R'}(x_0, \dots, x_{i-1}) \wedge x_i = 1)$  defines in  $\mathcal{L}^*$  a total function which is the characteristic function,  $c_R$ , for  $R$ .  $c_R$  is thus a recursive function and so  $R$  is recursive.

- (c) By part (b),  $R_1$  is recursive  $\iff R_1$  and  $R'_1$  are r.e.  $\iff R'_1$  and  $R''_1$  are r.e.  $\iff R'_1$  is recursive. Suppose that  $R_1$  and  $R_2$  are  $i$ -place recursive relations. By part (b),  $R_1$ ,  $R'_1$ ,  $R_2$ , and  $R'_2$  are r.e. By part (a),  $R_1 \cup R_2$  and  $(R_1 \cup R_2)' (= R'_1 \cap R'_2)$  are r.e. Thus, by (b),  $R_1 \cup R_2$  is recursive. The proof for  $R_1 \cap R_2$  is similar.  $\square$

Let us return to some ideas developed in the proof of Theorem 3.8. Consider the recursive function obtained from (5) by replacing 'gn\*( $A$ )' with the variable ' $e$ ':

$$\mu k[\text{Proof}(\text{Sub}_{0, \dots, i}(e, n_0, \dots, n_{i-1}, k), \mu m[g(e, n_0, \dots, n_{i-1}, m, m) = 0]) = 0].$$

Abbreviate this as  $\text{enum}_i(e, n_0, \dots, n_{i-1})$ . We have seen that if  $e_0$  is the gn\* of a formula  $A$  which defines in  $\mathcal{L}^*$  an  $i$ -place partial function  $f$ , then  $\text{enum}_i(e_0, n_0, \dots, n_{i-1})$  just is the partial function  $f$ . If  $e_0$  is not the gn\* of a formula with at most  $x_0, \dots, x_i$  free, then  $\text{enum}_i(e_0, n_0, \dots, n_{i-1})$  is the  $i$ -place partial function which is totally undefined (that is, its range =  $\Lambda$ ). Suppose that  $e_0 = \text{gn}^*(A)$  for some formula  $A$  which defines in  $\mathcal{L}^*$  an  $i + 1$ -place relation  $R$  which is not a partial function (that is, for some  $n$  and some  $m \neq n$ , both  $\langle n_0, \dots, n_{i-1}, n \rangle$  and  $\langle n_0, \dots, n_{i-1}, m \rangle \in R$ ), then  $\text{enum}_i(e_0, n_0, \dots, n_{i-1})$  defines an  $i$ -place partial function whose graph is a subset of  $R$ . The value of  $\text{enum}_i(e_0, n_0, \dots, n_{i-1})$  may be described as follows: take the smallest number  $m$  for which  $m$  is the Gödel number of a derivation whose last sentence is of the form  $A(\mathbf{n}_0, \dots, \mathbf{n}_{i-1}, \mathbf{k})$  where  $k \leq m$ ; then  $\text{enum}_i(e_0, n_0, \dots, n_{i-1}) = k$



(if no such  $m$  exists, then  $\text{enum}_i(e_0, n_0, \dots, n_{i-1})$  is undefined for the arguments  $n_0, \dots, n_{i-1}$ ). Since  $\text{enum}_i(e, n_0, \dots, n_{i-1})$  is a partial recursive function,  $\text{enum}_i(e, n_0, \dots, n_{i-1})$  is an  $i + 1$ -place partial recursive function which enumerates all the  $i$ -place partial recursive functions. This gives us our next theorem.

**Theorem 3.11** *For each  $i$ , there is an  $i + 1$ -place partial recursive function,  $\text{enum}_i(e, n_0, \dots, n_{i-1})$  such that, for any  $i$ -place partial function  $f$ ,  $f$  is partial recursive if and only if for some  $e$ :*

$$f(n_0, \dots, n_{i-1}) = \text{enum}_i(e, n_0, \dots, n_{i-1}).$$

*Comment on Theorem 3.11:* A stronger version of Theorem 3.11 could have been derived if we had given a more fully developed treatment of the primitive recursive functions. It can be shown that there are primitive recursive functions  $P^2$ ,  $L^1$ , and  $R^1$  such that whenever  $P(n_0, n_1) = n_2$ , then the following hold:

$$\begin{aligned} L(n_2) &= n_0, \\ R(n_2) &= n_1. \end{aligned}$$

Rather than define  $\text{enum}_i(e, n_0, \dots, n_{i-1})$  as

$$\mu k[\text{Proof}(\text{Sub}_{0, \dots, i}(e, n_0, \dots, n_{i-1}, k), \mu m[g(e, n_0, \dots, n_{i-1}, m, m) = 0]) = 0],$$

we could instead have used

$$L(\mu k[\text{Proof}(\text{Sub}_{0, \dots, i}(e, n_0, \dots, n_{i-1}, L(k)), R(k)) = 0]).$$

This involves only one usage of minimization. If  $e = \text{gn}^*(A)$  for some formula which defines in  $\mathcal{L}^*$  a partial function  $f$ , then  $L(\mu k[\text{Proof}(\text{Sub}_{0, \dots, i}(e, n_0, \dots, n_{i-1}, L(k)), R(k)) = 0])$  is the partial function  $f$ ; if  $e \neq \text{gn}^*$  of a formula with at most  $x_0, \dots, x_i$  free then  $L(\mu k[\text{Proof}(\text{Sub}_{0, \dots, i}(e, n_0, \dots, n_{i-1}, L(k)), R(k)) = 0])$  is the  $i$ -place partial function with range  $= \Lambda$ , and if  $e = \text{gn}^*(A)$  for some formula  $A$  that defines in  $\mathcal{L}^*$  an  $i + 1$ -place relation  $R$  that is not the graph of an  $i$ -place partial function then the graph of  $L(\mu k[\text{Proof}(\text{Sub}_{0, \dots, i}(e, n_0, \dots, n_{i-1}, L(k)), R(k)) = 0])$  is a proper subset of  $R$ . Letting  $h(e, n_0, \dots, n_{i-1}, p)$  abbreviate  $\text{Proof}(\text{Sub}_{0, \dots, i}(e, n_0, \dots, n_{i-1}, L(p)), R(p))$ , we get a version of *Kleene's Normal Form Theorem*: there are primitive recursive functions  $L$  and  $h$  such that for every  $i$ -place partial function  $f$ ,  $f$  is partial recursive if and only if there is an  $e$  such that  $f(n_0, \dots, n_{i-1}) = L(\mu k[h(e, n_0, \dots, n_{i-1}, k)])$ .

Let  $W_i = \{n \mid \text{for some } m, \text{enum}_1(i, n) = m\}$ . From Theorem 3.11 we obtain the following.

**Theorem 3.12** *For any  $S \subseteq \mathbb{N}$ ,  $S$  r.e. if and only if there is an  $i$  such that  $S = W_i$ .*

**Theorem 3.13** *Under any given Gödel numbering, for any  $i$ ,*

- (a) *there is a formula  $\text{ENUM}_i(x_0, \dots, x_{i+1})$  which defines  $\text{enum}_i(e, n_0, \dots, n_{i-1})$  in every fixed point;*
- (b) *given any fixed point  $S$ , an  $i$ -place partial function  $f$  is a partial recursive function if and only if there is an index  $e$  such that*

$$f(n_0, \dots, n_{i-1}) = m \iff \models_S \text{ENUM}_i(\mathbf{e}, \mathbf{n}_0, \dots, \mathbf{n}_{i-1}, \mathbf{m});$$

(c) *there is a formula  $W(x_0, x_1)$  that enumerates, in every fixed point  $S$ , all and only the r.e. sets, that is, a set  $R \subseteq \mathbb{N}$  is r.e. if and only if there exists an index  $e$  such that*

$$n \in R \iff \models_S W(\mathbf{e}, \mathbf{n}).$$

*Proof:* (a) follows from Theorems 3.1(a) and 3.11. (b) follows from (a). For (c) we can take  $W(x_0, x_1)$  to be  $\exists x_2 \text{ENUM}_1(x_0, x_1, x_2)$ . Note that if we were only concerned with the least fixed point language  $\mathcal{L}^*$  we could take  $W(x_0, x_1)$  to be  $\text{Sat}(x_0, x_1)$ . Of course, the formulas  $\text{ENUM}_i$  and  $W(x_0, x_1)$  will vary depending on the Gödel numbering.  $\square$

*Department of Philosophy  
Oklahoma State University  
308 Hanner Hall  
Stillwater OK 74078-5064  
email: [jcain@okstate.edu](mailto:jcain@okstate.edu)*