DOCUMENT ID: **678259**

LENDER: **COF :: Colorado State University**

BORROWER: **COF**

BORROWER ADDRESS: 129.82.31.248

CROSS REF ID: 130758

Processed by RapidX: 10/10/2014 2:54:57 PM

ARISTOTLE'S MODAL SYLLOGISMS

Fred Johnson

> Considering Aristotle's discussion of syllogisms as a whole, the most striking point is that its focus is the modal syllogisms – This is the point on which the logical tradition has diverged most completely from Aristotle, as a rule giving no attention to modal syllogisms . . ..   Paul Henle

> Aristotle's system of modal syllogisms, to be found in chapters 3 and 8-22 of the first book of the *Prior Analytics,* has been open to public inspection for over 2300 years. And yet perhaps no other piece of philosophical writing has had such consistently bad reviews.   Storrs McCall

> . . . by raising the [completeness] problem, Aristotle earns the right to be considered not only the father of logic, but also the (grand)father of meta- logic.   Jonathan Lear

Storrs McCall [1963] developed the first formal system, the L-X-M calculus, for which a decision procedure for assertion or rejection of formal sentences is given that has any chance of matching Aristotle's judgments about which of the n-premised (for $n \geq 2$) "apodeictic syllogisms" are valid or invalid. McCall's remarkable results were achieved by extending Jan Łukasiewicz's [1957] decision procedure for assertion or rejection of expressions in his formal system, ŁA, that is designed to capture Aristotle's judgments about which of the "assertoric (or plain[1]) syllogisms" are valid or invalid.

Łukasiewicz also considers using his four-valued modal system, the $\mathrm{Ł}M$ system, to present Aristotle's syllogistic but finds that the match is not very good. Peter Geach also proposes a system for dealing with the apodeictics. But, again, the match is not very good. After examining McCall's L-X-M system and work related to it we shall turn to his work on the "contingent syllogisms". His purely syntactic system, Q-L-X-M, has some unAristotelian features that lead us to develop a modified system, QLXM'. A semantics for QLXM' is developed that enables us to provide formal countermodels for a large percentage of the assertoric, apodeictic or contingent syllogisms that Aristotle explicitly considered to be invalid.

1   ŁUKASIEWICZ'S ASSERTORIC SYSTEM, ŁA

For Łukasiewicz, Aristotle's syllogisms are "implicational" rather than "inferential". He says in [1957, p. 21]:

---

[1] In [1964] P. T. Geach prefers 'plain' over 'assertoric'.

Syllogisms of the form:

> All B is A;
> all C is B;
>   therefore
> all C is A

are not Aristotelian. We do not meet them until Alexander. This transference of the Aristotelian syllogisms from the implicational form into the inferential is probably due to the Stoics.

So, Łukasiewicz claims Aristotle construed the above syllogism, with traditional name 'Barbara', as a conditional claim:

> If all B are A then if all C are B then all C are A.

Robin Smith's [1989, p.4] translation of Barbara at *Prior Analytics* 25b37-40 seems to conform with Łukasiewicz's view:

> ... if A is predicated of every B and B of every C, it is necessary for A to be predicated of every C ....

But see [Corcoran, 1972] and [Smiley, 1973] for the view that Aristotle developed natural deduction systems rather than the axiomatic systems of the sort Łukasiewicz envisages.

Łukasiewicz uses Polish notation, a parenthesis-free notation, to express the well-formed formulas (wffs) in his formal system, which we refer to as ŁA. We replace his notation with current "standard" notation when giving the basis for it.[2] So, for example, his Cpq ('If p then q') is our $(p \rightarrow q)$. His Np ('not p') and Kpq ('p and q') are our $\neg p$ and $(p \wedge q)$, respectively.

Łukasiewicz's assertions and rejections are marked by $\vdash$ and $\dashv$, respectively. The system that is essentially Łukasiewicz's will be called ŁA.

So, for example, $\vdash (Aba \rightarrow (Acb \rightarrow Aca))$ says that Barbara is asserted in ŁA, which is true. $\dashv Aba$ says that Aba is rejected in ŁA, which is true. Assertions and rejections are relative to systems. We shall avoid using $\vdash$ŁA , say, and rely on the context to indicate that the assertion is relative to system ŁA.

Primitive symbols

term variables	$a, b, c, \ldots$ (with or without subscripts)
monadic operator	$\neg$
dyadic operator	$\rightarrow$
quantifiers	A, I
parentheses	$(,)$

---

[2]The manner of presentation of this system is heavily influenced by Hughes and Cresswell's presentations of various systems in [1996].

Formation rules

FR1 If Q_u is a quantifier and x and y are term variables then $Q_u xy$ is a *wff*.

FR2 If p and q are wffs then $\neg p$ and $(p \rightarrow q)$ are *wffs* .

FR3 The only wffs are those in virtue of FR1 and FR2.

So, for example, Aab, Iab and $(Abc \rightarrow \neg Ibc)$ are wffs. Read them as 'All a are b', 'Some a are b' and 'If all b are c then it is not true that some b are c', respectively.

Definitions

Def $\wedge$ $(p \wedge q) =_{df} \neg(p \rightarrow \neg q)$

Def $\leftrightarrow$ $(p \leftrightarrow q) =_{df} ((p \rightarrow q) \wedge (q \leftarrow p))$

Def E $Exy =_{df} \neg Ixy$

Def O $Oxy =_{df} \neg Axy$

Eab and Oab may be read as 'No a are b' and 'Some a are not b', respectively.

Łukasiewicz's ŁA contains theses that are "assertions" (indicated by $\vdash$) as well as theses that are "rejections"(indicated by $\dashv$). We begin with the former, which are generated by assertion axioms and assertion rules.

Assertion axioms

A0 (PC).   If p is a wff that is valid in virtue of the propositional calculus (PC) then $\vdash p$ (that is, p is asserted). (So, for example, $\vdash (Aab \rightarrow Aab)$ since it is not possible that the antecedent Aab is true and the consequent Aab is false. And $\vdash ((Aab \rightarrow Iab) \rightarrow (\neg Iab \rightarrow \neg Aab))$ since it is not possible that all of these conditions are met: $(Aab \rightarrow Iab)$ is true, $\neg Iab$ is true and $\neg Aab$ is false.)

A1 $\vdash Aaa$

A2 $\vdash Iaa$

A3 (Barbara) $\vdash (Abc \rightarrow (Aab \rightarrow Aac))$

A4 (Datisi) $\vdash (Abc \rightarrow (Iba \rightarrow Iac))$

Transformation rules for assertions

AR1 (Uniform substitution for assertions, US) From $\vdash p$ infer $\vdash q$ (that is, from the assertion of p infer the assertion of q) provided q is obtained from p by uniformly substituting variables for variables. (So, for example, from $\vdash (Aab \rightarrow Iba)$ we may infer $\vdash (Acb \rightarrow Ibc)$ and $\vdash (Abb \rightarrow Ibb)$, by rule US. But rule US does not permit us to infer that $\vdash (Aab \rightarrow Iba)$ given that $\vdash (Abb \rightarrow Ibb)$.

AR2 (Modus Ponens, MP) From $\vdash (p \rightarrow q)$ and $\vdash p$ infer $\vdash q$.

AR3 (*Definiens* and *definiendum* interchange for assertions, DDI) From $\vdash (\ldots \alpha \ldots)$ and $\alpha =_{df} \beta$ infer $\vdash (\ldots \beta \ldots)$, and vice versa. (So, for example, from $\vdash (\neg Iab \rightarrow \neg Aab)$ infer $\vdash (Eab \rightarrow Oab)$ by two uses of DDI, given definitions Def E and Def O. Typically a use of DDI will be indicated by simply referring to a definition that is used. So, from $(\neg Iab \rightarrow \neg Aab)$ infer $\vdash (Eab \rightarrow \neg Aab)$ by Def E. It is to be understood that DDI is also used.)

Given the assertion portion of the basis for ŁA, we shall give some "assertion deductions" — sequences of wffs such that each member of the sequence is either an assertion axiom or is entered from a prior member of the sequence by using a transformation rule for assertions — that capture some Aristotelian principles involving conversions, subordinations, and oppositions.

Theorem 1.1. (Assertoric conversions, Con) $\vdash (Iab \to Iba)$ and $\vdash (Eab \to Eba)$.

Proof.

1. $\vdash (Abc \to (Iba \to Iac))$ (by A4)
2. $\vdash (Abb \to (Iba \to Iab))$ (from 1 by US)
3. $\vdash Abb$ (by A1 and US)
4. $\vdash (Iba \to Iab)$ (from 2 and 3 by MP)
5. $\vdash (Iab \to Iba)$ (from 4 by US)
6. $\vdash ((Iba \to Iab) \to (\neg Iab \to \neg Iba))$ (by A0)
7. $\vdash (\neg Iab \to \neg Iba)$ (from 6 and 4 by MP)
8. $\vdash (Eab \to Eba)$ (from 7 by Def E, using DDI) ∎

The above reasoning may be presented more succinctly by using the following derived rule for assertions.

DR1   (Reversal, RV) i) From $\vdash (p \to q)$ infer $\vdash (\neg q \to \neg p)$; ii) from $\vdash (p \to (q \to r))$ infer $\vdash (p \to (\neg r \to \neg q))$; and iii) from $\vdash (p \to (q \to r))$ infer $\vdash (\neg r \to (p \to \neg q))$.

Proof. i) Suppose $\vdash (p \to q)$. By A0 $\vdash ((p \to q) \to (\neg q \to \neg p))$. By MP $\vdash (\neg q \to \neg p)$.
ii) Suppose $\vdash (p \to (q \to r))$. By A0 $\vdash ((p \to (q \to r)) \to (p \to (\neg r \to \neg q)))$. By MP $\vdash (p \to (\neg r \to \neg q)))$. Use similar reasoning for iii). ∎

So, the annotation for line 7 in the above deduction may read: '(from 4 by RV)'. Line 6 may be deleted.

The following derived rules are useful in generating other principles.

DR2   (Assertion by antecedent interchange, AI) From $\vdash (p \to (q \to r))$ infer $\vdash (q \to (p \to r))$.

Proof. Assume $\vdash (p \to (q \to r))$. By A0 $\vdash ((p \to (q \to r)) \to (q \to (p \to r)))$. By MP $\vdash (q \to (p \to r))$. ∎

DR3   (Assertion by antecedent strengthening (or equivalence), AS) From $\vdash (p \to (q \to r))$ and $\vdash (s \to q)$ infer $\vdash (p \to (s \to r))$; and from $\vdash (p \to (q \to r))$ and $\vdash (s \to p)$ infer $\vdash (s \to (q \to r))$.[3]

---

[3] 'Cut' is also used to refer to these rules.

DR4 (Assertion by consequent weakening (or equivalence), CW) From $\vdash$(p →q) and
$\vdash$(q →r) infer $\vdash$(p →r); and from $\vdash$(p →(q →r)) and $\vdash$(r →s) infer $\vdash$(p →(q →s)).[4]

To prove **DR3** and **DR4** use A0 and MP.

Theorem 1.2. (Assertoric subalternations, Sub-a) i) $\vdash(Aab \to Iab)$; and ii) $\vdash(Eab \to Oab)$.

Proof.

1. $\vdash(Abc \to (Iba \to Iac))$ (by A4)
2. $\vdash(Iba \to (Abc \to Iac))$ (from 1 by AI)
3. $\vdash(Iaa \to (Aac \to Iac))$ (from 2 by US)
4. $\vdash Iaa$ (by A2)
5. $\vdash(Aac \to Iac)$ (from 3 and 4 by MP)
6. $\vdash(Aab \to Iab)$ (i, from 5 by US)
7. $\vdash(\neg Iab \to \neg Aab)$ (from 6 by RV)
8. $\vdash(Eab \to Oab)$ (ii) from 7 by DDI, using Def E and Def O) ∎

Theorem 1.3. (Assertoric conversion *per accidens* , Con(pa)) i) $\vdash(Aab \to Iba)$; and ii) $\vdash(Eab \to Oba)$.[5]

Proof.

1. $(Aab \to Iab)$ (by Sub-a)
2. $(Iab \to Iba)$ (by Con)
3. $(Aab \to Iba)$ (i, from 1 and 2 by CW)
4. $(Eab \to Eba)$ (by Con)
5. $(Eba \to Oba)$ (from 4 by Sub-a and US)
6. $(Eab \to Oba)$ (ii, from 4 and 5 by CW) ∎

The following derived rule, proven by using AO and MP, is useful in proving the next theorem.

DR5 (Biconditional rule, BIC) From $\vdash(p \to q)$ and $\vdash(q \to p)$ infer $\vdash(p \leftrightarrow q)$.[6]

Proof. Suppose $\vdash(p \to q)$ and $\vdash(q \to p)$. By A0, $\vdash((p \to q) \to ((q \to p) \to (p \leftrightarrow q)))$.
By two uses of MP, $\vdash(p \leftrightarrow q)$. ∎

Theorem 1.4. (Assertoric oppositions, Opp) i) $\vdash(\neg Aab \leftrightarrow Oab)$; ii) $\vdash(\neg Eab \leftrightarrow Iab)$;
iii) $(\neg Iab \leftrightarrow Eab)$; and iv) $(\neg Oab \leftrightarrow Aab)$.

---

[4]'Transitivity' and 'Hypothetical syllogism' are also used to refer to the first of these two rules.
[5]I. M. Bocheński, on p. 212 of [1963], states that the "law of accidental conversion of the universal negative is not in Aristotle". He is not saying that Aristotle considered inference ii) to be invalid.
[6]This rule is discussed, but not named, on p. 29 of [Hughes and Cresswell, 1996].

Proof

1. $\vdash(\neg Aab \to \neg Aab)$ (by A0)
2. $\vdash(\neg Aab \to Oab)$ (from 1 by Def O)
3. $\vdash(Oab \to \neg Aab)$ (from 1 by Def O)
4. $\vdash(\neg Aab \leftrightarrow Oab)$ (i, from 2 and 3 by BIC)
5. $\vdash(\neg Eab \to \neg Eab)$ (by A0)
6. $\vdash(\neg Eab \to \neg\neg Iab)$ (by Def E)
7. $\vdash(\neg\neg Iab \to Iab)$ (by A0)
8. $\vdash(\neg Eab \to Iab)$ (from 6 and 7 by CW)
9. $\vdash(\neg\neg Iab \to \neg Eab)$ (from 5 by Def E)
10. $\vdash(Iab \to \neg\neg Iab)$ (by A0)
11. $\vdash(Iab \to \neg Eab)$ (from 10 and 9 by CW)
12. $\vdash(\neg Eab \leftrightarrow Iab)$ (ii, from 11 by BIC)
13. $\vdash((\neg Aab \leftrightarrow Oab) \to (\neg Oab \leftrightarrow Aab))$ (by A0)
14. $\vdash(\neg Oab \leftrightarrow Aab)$ (iv, from 4 and 13 by MP)
15. $\vdash((\neg Eab \leftrightarrow Iab) \to (\neg Iab \leftrightarrow Eab))$ (by A0)
16. $\vdash(\neg Iab \leftrightarrow Eab)$ (iii, from 12 and 15 by MP)   ■

The following derived rule is useful in conjunction with the assertoric oppositions.

DR6 (Substitution of equivalents, SE) From $\vdash(p \leftrightarrow q)$ and $\vdash(\dots p \dots)$ infer $\vdash(\dots q \dots)$.

Proof. Use mathematical induction.   ■

So, for example, from $\vdash(Aab \to (Abc \to (\neg Aad \to \neg Acd)))$ infer $\vdash(Aab \to (Abc \to (Oad \to Ocd)))$ by SE, given the oppositions Opp.

On table 1 assertions corresponding to the familiar two-premised syllogisms are listed. In the right column a method of deducing the assertion is given. So, for example, Barbara is trivially asserted by using axiom A3. Celarent is asserted since the assertion of 11 (Disamis) may be transformed into $\vdash(\neg Iac \to (Aba \to \neg Ibc))$ (by RV), which may be transformed into $\vdash(Eac \to (Aba \to Ebc))$ (by SE, since $\vdash(Eac \leftrightarrow \neg Iac)$ and $\vdash(Ebc \leftrightarrow \neg Ibc)$), which may be transformed into 2 (by US, putting 'b' in place of 'a' and 'a' in place of 'b'). Darii is asserted since the assertion of 12 may be transformed into $\vdash(Abc \to (Iab \to Iac))$ (by AS, since $(Iab \to Iba)$).

1.1 Rejection in ŁA

Łukasiewicz uses the notion of "rejection" to develop his formal system.[7] He shows that the invalid syllogistic forms expressed by "elementary wffs" may be rejected by augmenting his formal system for assertions by adding one rejection axiom and four transformation rules that generate rejections. We shall illustrate this claim but not give a full account

---

[7]Smiley, in his influential article [1996], points out that Carnap and Łukasiewicz were the first logicians to formalize the notion of rejection. Smiley attributes the shunning of rejection by most logicians to Frege's [1960]. Smiley effectively argues that Frege's rejection of rejection, using Occam's razor, was unfortunate, and Smiley shows how rejection may be put to good use in ways other than those envisioned by Carnap or Łukasiewicz. For recent work on rejection that is stimulated by Smiley's article see [Rumfitt, 1997] and [Johnson, 1999b].

Table 1. Deductions in system ŁA

Figure 1	Barbara (1)	$\vdash(Abc \to (Aab \to Aac))$	A3
	Celarent (2)	$\vdash(Ebc \to (Aab \to Eac))$	11,RV,SE,US
	Darii (3)	$\vdash(Abc \to (Iab \to Iac))$	12,AS
	Ferio (4)	$\vdash(Ebc \to (Iab \to Oac))$	12,RV,SE,US
Figure 2	Cesare (5)	$\vdash(Ecb \to (Aab \to Eac))$	12,RV,SE,AI,US
	Camestres (6)	$\vdash(Acb \to (Eab \to Eac))$	3,RV,SE,US
	Festino (7)	$\vdash(Ecb \to (Iab \to Oac))$	11,RV,SE,AI,US
	Baroco (8)	$\vdash(Acb \to (Oab \to Oac))$	1,RV,SE,US
Figure 3	Darapti (9)	$\vdash(Abc \to (Aba \to Iac))$	12,AS
	Felapton (10)	$\vdash(Ebc \to (Aba \to Oac))$	20,RV,SE,US
	Disamis (11)	$\vdash(Ibc \to (Aba \to Iac))$	12,AI,US,CW
	Datisi (12)	$\vdash(Abc \to (Iba \to Iac))$	A4
	Bocardo (13)	$\vdash(Obc \to (Aba \to Oac))$	1,RV,SE,US
	Ferison (14)	$\vdash(Ebc \to (Iba \to Oac))$	3,RV,SE,US
Figure 4	Bramantip (15)	$\vdash(Acb \to (Aba \to Iac))$	20,AI,US,CW
	Camenes (16)	$\vdash(Acb \to (Eba \to Eac))$	17,RV,SE,AI,US
	Dimaris (17)	$\vdash(Icb \to (Aba \to Iac))$	3,AI,US,CW
	Fresison (18)	$\vdash(Ecb \to (Iba \to Oac))$	17,RV,SE,AI,US
	Fesapo (19)	$\vdash(Ecb \to (Aba \to Oac))$	15,RV,SE,AI,US
Subalterns	Barbari (20)	$\vdash(Abc \to (Aab \to Iac))$	1,CW
	Celaront (21)	$\vdash(Ebc \to (Aab \to Oac))$	9,RV,SE,US
	Cesaro (22)	$\vdash(Ecb \to (Aab \to Oac))$	9,RV,SE,AI,US
	Camestrop (23)	$\vdash(Acb \to (Eab \to Oac))$	20,RV,SE,US
	Camenop (24)	$\vdash(Acb \to (Eba \to Oac))$	15,RV,SE,AI,US

of Łukasiewicz's work on rejections, which would require showing that all wffs may be "reduced" to sets of elementary wffs.

Definition 1.5. (elementary wff and simple wff) x is an *elementary wff* iff x has form $(x_1 \to (x_2 \to (x_3 \to \ldots x_n) \ldots))$, where each x_i is a *simple wff*, a wff of form Apq, Ipq, Opq or Epq.

Rejection axioms for ŁA

R1 $\dashv(Acb \to (Aab \to Iac))$

Rejection transformation rules for ŁA

RR1 (Rejection by uniform substitution, R-US) If $\dashv x$ and x is obtained from y by uniform substitution of terms for terms, then $\dashv y$.

RR2 (Rejection by detachment (or Modus Tollens), R-D) From $\vdash(x \to y)$ and $\dashv y$ infer $\dashv x$.

RR3 (Słupecki's rejection rule, R-S) From $\dashv(x \to z)$ and $\dashv(y \to z)$ infer $\dashv(x \to (y \to z))$ provided: i) x and y have form $\neg Apq$ or $\neg Ipq$; and ii) z has form $(x_1 \to (x_2 \to (x_3 \to \ldots x_n)\ldots))$ where each x_i is a simple sentence.

RR4 (*Definiens* and *definiendum* interchange for rejections, R-DDI) From $\dashv(\ldots \alpha \ldots)$ and $\alpha =_{df} \beta$ infer $\dashv(\ldots \beta \ldots)$, and vice versa. (So, for example, from $\dashv(\neg Aab \to \neg Iab)$ infer $\dashv(Eab \to Oab)$ by two uses of R-DDI, given definitions Def O and Def E.)

The following derived rules for rejections, which are counterparts of derived rules for assertions, are useful in simplifying presentations of rejection deductions — sequences of wffs in which each member of the sequence is either an (assertion or rejection) axiom or is entered by an (assertion or rejection) transformation rule, where the last member of the sequence is a rejection.

R-DR1 (Rejection by reversal, R-RV) i) From $\dashv(p \to q)$ infer $\dashv(\neg q \to \neg p)$; ii) from $\dashv(p \to (q \to r))$ infer $\dashv(p \to (\neg r \to \neg q))$; and iii) from $\dashv(p \to (q \to r))$ infer $\dashv(\neg r \to (p \to \neg q))$.

Proof. i) Suppose $\dashv(p \to q)$. By A0 (or PC) $\vdash((\neg q \to \neg p) \to (p \to q))$. By R-D $\dashv(\neg q \to \neg p)$. ii) Suppose $\dashv(p \to (q \to r))$. By A0 $\vdash((p \to (\neg r \to \neg q)) \to (p \to (q \to r)))$. By R-D $\dashv(p \to (\neg r \to \neg q))$. Use similar reasoning for iii). ∎

R-DR2 (Rejection by antecedent interchange, R-AI) From $\dashv(p \to (q \to r))$ infer $\dashv(q \to (p \to r))$.

Proof. Assume $\dashv(p \to (q \to r))$. By A0 $\vdash((q \to (p \to r)) \to (p \to (q \to r)))$. By R-D $\dashv(q \to (p \to r))$. ∎

R-DR3 (Rejection by antecedent weakening (or equivalence), R-AW) i) From $\dashv(p \to (q \to r))$ and $\vdash(q \to s)$ infer $\dashv(p \to (s \to r))$; and ii) from $\dashv(p \to (q \to r))$ and $\vdash(p \to s)$ infer $\dashv(s \to (q \to r))$.

Proof. Suppose $\dashv(p \to (q \to r))$ and $\vdash(q \to s)$. By AO $\vdash((q \to s) \to ((p \to (s \to r)) \to (p \to (q \to r))))$. By MP $\vdash((p \to (s \to r)) \to (p \to (q \to r)))$. By R-D $\dashv(p \to (s \to r))$. Use similar reasoning for ii). ∎

Proofs for the following two derived rules are easily constructed and will be omitted.

R-DR4 (Rejection by consequent strengthening (or equivalence), R-CS) From $\dashv(p \to q)$ and $\vdash(r \to q)$ infer $\dashv(p \to r)$; and from $\dashv(p \to (q \to r))$ and $\vdash(s \to r)$ infer $\dashv(p \to (q \to s))$.

R-DR5 (Rejection by substitution of equivalents, R-SE) From $\vdash(p \leftrightarrow q)$ and $\dashv(\ldots p \ldots)$ infer $\dashv(\ldots q \ldots)$.

R-DR6   (Rejection by implication introduction, R-II) From $\vdash p$ and $\dashv q$ infer $\dashv (p \to q)$.

Proof. Suppose $\vdash p$ and $\dashv q$. By A0 $\vdash (p \to ((p \to q) \to q))$. By MP $\vdash ((p \to q) \to q)$. By R-D $\dashv (p \to q)$. ■

Given the above apparatus we are able to show how the four syllogisms referred to at *Prior Analytics* 26a2-9 are rejected in ŁA. This is Łukasiewicz's translation from [1957, p. 67].

> If the first term belongs to all the middle [Aba], but the middle to none of the last [Ecb], there will be no syllogism of the extremes; for nothing necessary follows from the terms being so related; for it is possible that the first should belong to all as well as to none of the last, so that neither a particular nor a universal conclusion is necessary. But if there is no necessary consequence by means of these premises, there cannot be a syllogism. Terms of belong to all: animal, man, horse; to none: animal, man, stone.

The four syllogisms are $(Aba \to (Ecb \to x))$, where x is Ica, Oca, Aca or Eca. We shall give rejection deductions to establish the rejection of the first two (AEI-1 and AEO-1) and then use derived rule R-CS to show the last two (AEA-1 and AEE-1) are rejected.[8]

Theorem 1.6.   (Rejection of AEI-1) $\dashv (Aba \to (Ecb \to Ica))$.

Proof.

1. $\dashv (Acb \to (Aab \to Iac))$ (by R1)
2. $\vdash (Iac \to (Acb \to (Aab \to Iac)))$ (by A0)
3. $\dashv Iac$ (from 1 and 2 by R-D)
4. $\vdash Acc$ (by A1 and US)
5. $\dashv (Acc \to Iac)$ (from 3 and 4 by R-II)
6. $\dashv (Acb \to Iab)$ (from 5 by R-US)
7. $\dashv (Eab \to Ocb)$ (from 6 by R-RV and R-SE)
8. $\dashv (Acb \to Iac)$ (from 5 by R-US)
9. $\dashv (Eac \to Ocb)$ (from 8 by R-RV and R-SE)
10. $\dashv (Eab \to (Eac \to Ocb))$ (from 7 and 9 by R-S)
11. $\dashv (Acb \to (Eac \to Iab))$ (from 10 by R-RV)
12. $\dashv (Aba \to (Ecb \to Ica))$ (from 11 by R-US)   ■

---

[8]In the above passage Aristotle uses the semantic counterpart of this two-stage syntactic process. First, he shows by his counterexample that $\{Aba, Ecb, Aca\}$ and $\{Aba, Ecb, Eca\}$ are semantically consistent, from which it follows that neither of the particulars Oca and Ica is a semantic consequence of $\{Aba, Ecb\}$. Secondly, since the universal claims Eca and Aca are stronger than Oca and Ica, respectively, they cannot be a semantic consequence of $\{Aba, Ecb\}$. Aristotle is using what W. D. Ross [1949, p. 302] calls a "proof by contrasted instances," to show a pair of premises is, in Jonathan Lear's [1980, p. 54] terms, "semantically sterile".

Theorem 1.7. (Rejection of AEO-1) $\dashv$(Aba $\to$(Ecb $\to$Oca)).

Proof.

1. $\dashv(Acb \to (Aab \to Iac))$ (By R1)
2. $\dashv((Acb \to (Eac \to Oab))$ (from 1 by R-RV and SE)
3. $\dashv(Aba \to (Ecb \to Oca))$ (from 2 by R-US)   ∎

Theorem 1.8. (Rejection of AEA-1 and AEE-1) i) $\dashv(Aba \to (Ecb \to Aca))$; and ii) $\dashv(Aba \to (Ecb \to Eca))$.

Proof.

1. $\dashv(Aba \to (Ecb \to Ica))$ (by theorem 1.6)
2. $\vdash(Aca \to Ica)$ (by Sub-a, US)
3. $\dashv(Aba \to (Ecb \to Aca))$.   (i, from 1 and 2 by R-CS)
4. $\dashv(Aba \to (Ecb \to Oca))$ (by theorem 1.7)
5. $\vdash(Eca \to Oca)$ (by Sub-a, US)
6. $\dashv(Aba \to (Ecb \to Eca))$ (ii, from 4 and 5 by R-CS)   ∎

The following passage clearly shows that Ross favors Łukasiewicz's method of rejecting the AEx-1s over Aristotle's. On p. 302 of [1949] Ross says:

> ... [Aristotle] gives *no reason* (my italics) for this [claim that no conclusion is yielded by the premises of AEx-1], e.g. by pointing out that an undistributed middle or an illicit process is involved; but he often points to an empirical fact. ...instead of giving the reason why All B are A, No C is B yields no conclusion, he simply points to one set of values for A, B, C (animal, man, horse) for which, all B being A and no C being B, all C is in fact A, and to another set of values (animal, man, stone) for which, all B being A and no C being B, no C is in fact A. Since in the one case all C is A, a negative conclusion cannot be valid; and since in the other case no C is A, an affirmative conclusion cannot be valid. Therefore there is no valid conclusion (with C as subject and A as predicate).

Aristotle is reasoning as follows. It is true that all men are animals, it is true that no horses are men, and it is true that all horses are animals (and thus false that no horses are animals and false that some horses are not animals). So neither Eca nor Oca is a logical consequence of Aba together with Ecb. Since it is true that all men are animals, it is true that no stones are men, and it is true that no stones are animals (and thus false that all stones are animals and false that some stones are not animals), it follows that neither Aca nor Ica is a logical consequence of Aba together with Ecb.

Łukasiewicz also objects to Aristotle's reasoning, claiming in [1957, p. 72] that it:

> introduces into logic terms and propositions not germane to it. 'Man' and 'animal' are not logical terms, and the proposition 'All men are animals' is not a logical thesis. Logic cannot depend on concrete terms and statements. If we want to avoid this difficulty, we must reject some forms axiomatically.

But Aristotle's procedures have support among modern logicians. Robin Smith [1989, p. 114] regards Aristotle's reference to animals, men and horses as a reference to a "countermodel" and says "countermodels are the paradigmatic means of proving invalidity for modern logicians." In the surrounding text Smith refers to Jonathan Lear [1980, pp. 54-61 and pp. 70-75] who defends Aristotle's techniques against criticisms by Łukasiewicz and Geach [1972]. In the following sections we shall make extensive use of formal countermodels to show the invalidity of apodeictic and contingent syllogisms. Such models may also be used to show the invalidity of assertoric syllogisms.

The following passage from the *Prior Analytics 27b12-23*, quoted and discussed by Łukasiewicz on p. 70 of [1957], illustrates another method Aristotle uses to reject inferences. Ross [1949, p. 304] calls it an argument "from the ambiguity of a particular proposition." A better name for the reasoning is "rejection by premise weakening". Ross points out that this method of rejection is also used by Aristotle at 26b14-20, 27b27-28, 28b28-31, 29a6 and 35b11.   ← ꞁꞁ

> Let M belong to no N, and not to some X. It is possible then for N to belong either to all X or to no X. Terms of belonging to none: black, snow, animal. Terms of belonging to all cannot be found, if M belongs to some X, and does not belong to some X. For if N belonged to all X, and M to no N, then M would belong to no X; but it is assumed that it belongs to some X. In this way, then, it is not possible to take terms, and the proof must start from the indefinite nature of the particular premise. For since it is true that M does not belong to some X, even if it belongs to no X, and since if it belongs to no X a syllogism is not possible, clearly it will not be possible either.

Given the semantic consistency of {No snow is black, Some animals are not black, No animal is snow } we know by half of the "contrasted instances" argument that neither 'Some animal is snow' nor 'All animals are snow' is a logical consequence of 'No snow is black' together with 'Some animals are not black.' So, a "countermodel" is given for the inferences from Enm and Oxm to Ixn or Axn. To show that neither Oxn nor Exn is a semantic consequence of Enm and Oxm, Aristotle relies on two facts: i) neither Oxn nor Exn is a semantic consequence of Enm and Exm; and ii) Oxm is a semantic consequence of Exm.

In ŁA a purely syntactic rejection of the "implicational syllogisms" ($Enm \rightarrow (Oxm \rightarrow Oxn)$) and ($Enm \rightarrow (Oxm \rightarrow Exn)$) is given by using R-AW.

Theorem 1.9. (Rejection of EOO-2 and EOE-2) i) $^\dashv(Enm \rightarrow (Exm \rightarrow Oxn))$; and ii) $^\dashv(Enm \rightarrow (Oxm \rightarrow Exn))$.

Proof.

1. $^\dashv(Aba \rightarrow (Ecb \rightarrow Ica))$ (by theorem 1.6)
2. $^\dashv(Eca \rightarrow (Ecb \rightarrow Oba))$ (from 1 by R-RV and R-SE)
3. $^\vdash(Ecb \rightarrow Obc)$ (by Con(pa) and US)
4. $^\dashv(Eca \rightarrow (Obc \rightarrow Oba))$ (from 2 and 3 by R-AW)
5. $^\vdash(Eca \rightarrow Eac)$ (by Con and US)

6. $\neg(Eac \rightarrow (Obc \rightarrow Oba))$ (from 4 and 5 by R-AW)
7. $\neg(Enm \rightarrow (Oxm \rightarrow Oxn))$ (i, from 4 by R-US)
8. $\vdash(Exn \rightarrow Oxn)$ (by Sub-a and US)
9. $\neg(Enm \rightarrow (Oxm \rightarrow Exn))$ (ii, from 7 and 8 by R-CS)   ∎

Up to this point we have rejected elementary wffs of form $(x_1 \rightarrow (x_2 \rightarrow \ldots (x_n \rightarrow y) \ldots)$ where $n \le 2$. For Łukasiewicz's system to be fully Aristotelian he must show how elementary sentences, where $n > 2$, are rejected. We illustrate such a rejection.

Theorem 1.10. (Rejection of an AAAA mood) $\neg$(Aab →(Abc →(Adc →Aad))).

Proof.

1. $\neg(Acb \rightarrow (Aab \rightarrow Iac))$ (by R1)
2. $\vdash(Acb \rightarrow (Aba \rightarrow Iac))$ (by Bramantip)
3. $\neg((Acb \rightarrow (Aba \rightarrow Iac))mc(Acb \rightarrow (Aab \rightarrow Iac))$ (from 2 and 1 by R-II)
4. $\vdash((Aba \rightarrow Aab) \rightarrow ((Acb \rightarrow (Aba \rightarrow Iac))mc(Acb \rightarrow (Aab \rightarrow Iac)))$ (by A0)
5. $\neg(Aba \rightarrow Aab)$ (from 3 and 4 by R-D)
6. $\vdash Aaa$ (by A1)
7. $\neg(Aaa \rightarrow (Aba \rightarrow Aab))$ (from 6 and 5 by R-II)
8. $\neg(Aaa \rightarrow (Aaa \rightarrow (Aba \rightarrow Aab)))$ (from 7 and 5 by R-II)
9. $\neg(Aaa \rightarrow (Aaa \rightarrow (Ada \rightarrow Aad)))$ (from 8 by R-US)
10. $\neg(Aaa \rightarrow (Aac \rightarrow (Adc \rightarrow Aad)))$ (from 9 by R-US)
11. $\neg(Aab \rightarrow (Abc \rightarrow (Adc \rightarrow Aad)))$ (from 10 by R-US)   ∎

Łukasiewicz's system for the assertoric syllogistic has "100% Aristotelicity", to use McCall's expression. This means that every 2-premised syllogism deemed valid by Aristotle is asserted in Łukasiewicz's system, and every 2-premised syllogism deemed invalid by Aristotle is rejected in Łukasiewicz's system. We shall see below that McCall's L-X-M calculus also has 100% Aristotelicity though his Q-L-X-M calculus does not.

2 ŁUKASIEWICZ'S MODAL SYSTEM, ŁM

Łukasiewicz developed his system for the assertoric syllogistic by using the non-modal propositional calculus, what he calls the "theory of deduction," as a "base logic". Following the procedure used in Hughes and Cresswell's [1968] and [1996], we simplified Łukasiewicz's presentation of his system by simply using axiom A0 to provide his "basis". Łukasiewicz's approach to Aristotle's modal logic is to develop a modal propositional logic (with quantifiers), which we refer to as the "ŁM system", that will enable him to present Aristotle's work on the modal syllogisms.[9]

The following sentences are tautologies in ŁM, modifying Łukasiewicz's notation in a natural way: 1) $((p \rightarrow q) \rightarrow (Mp \rightarrow Mq))$ and 2) $((p \rightarrow q) \rightarrow (Lp \rightarrow Lq))$, reading M and L as 'it is possible that' and 'it is necessary that', respectively. The following passages on p. 138 of [Łukasiewicz, 1957] attempt to show that the "M-law of extensionality" (1) and the "L-law of extensionality" (2) are endorsed by Aristotle.

---

[9] See [1961] for Smiley's extensions of Łukasiewicz's work on ŁM.

ꞇ)   First it has to be said that if (if α is, β must be), then (if α is possible, β must be possible too).   [34a5-7]

ꞇꞇ   If one should denote the premises by α, and the conclusion by β, it would not only result that if α is necessary, then β is necessary, but also that if α is possible, then β is possible.   [34a22-24]

It has been proved that if (if α is, β is), then (if α is possible, then β is possible).   [34a29-31]

A more natural reading of these passages is that they show that Aristotle endorsed both 3) $(L(p \to q) \to (Mp \to Mq))$ and 4) $(L(p \to q) \to (Lp \to Lq))$.[10]
That 1) - 4) are tautologies in ŁM is seen by considering the following four-valued truth tables.

Table 2. Four-valued truth tables for $\to$, $\neg$, L and M

$\to$	1	2	3	4	$\neg$	M	L
*1	1	2	3	4	4	1	2
2	1	1	3	3	3	1	2
3	1	2	1	2	2	3	4
4	1	1	1	1	1	3	4

Among the four truth values 1 to 4, 1 is the only designated value, marked with an asterisk in its entry in the first column on the table. A sentence x in the ŁM-system is a *tautology* iff for every input of values the output value is always the designated value 1.

Theorem 2.1. (L-law of extensionality) $((p \to q) \to (Lp \to Lq))$ is a tautology.

Proof. Suppose $((p \to q) \to (Lp \to Lq))$ is assigned a value other than 1. Then i) $(p \to q)$ is not assigned 4 and ii) $(Lp \to Lq)$ is not assigned 1, and iii) the value assigned to $(p \to q)$ is not the value assigned to $(Lp \to Lq)$. By i) p is not assigned 1 and q is not assigned 4. By ii) Lp is not assigned 4 and thus p is assigned neither 3 nor 4. And by ii) Lp is not assigned the same value as Lq. So p is assigned the value 2 and q is assigned the value 3. Then $(p \to q)$ and $(Lp \to Lq)$ are assigned the same value, which conflicts with iii).   ∎

Proofs that 1), 3) and 4) are tautologies are not required for our purposes, and we omit the straightforward proofs.
McCall [1963, pp. 31-32] points out that Łukasiewicz's use of the L-law of extensionality yields highly unAristotelian results. For example, using McCall's notation, Camestres LXL ('Necessarily all c are b; no a are b; so (necessarily) necessarily no a

---

[10]See [Hughes and Cresswell, 1968, pp. 29-30] for a discussion of this sentence, an axiom in Robert Feys's *System T*.

are c'), Baroco LXL ('Necessarily all c are b; some a are not b; so necessarily some a are not c'), Barbara XLL ('All b are c; necessarily a are b; so necessarily all a are c') and Ferio XLL ('No b are c; necessarily some a are b; so necessarily some b are c'), when construed as "implicational syllogisms", are asserted in Łukasiewicz's Ł-system even though Aristotle rejects all of them.

Following McCall we use 'XXX' after the name of a syllogism to indicate that the syllogism is a plain, assertoric syllogism. So, for example, Camestres XXX has form 'All c are b; no a are b; so no a are c'. Camestres XXX, Baroco XXX, Barbara XXX and Ferio XXX are asserted in Łukasiewicz's assertoric system. So, given the following theorem, Camestres LXL, Baroco LXL, Barbara XLL and Ferio XLL are asserted in Łukasiewicz's Ł-system.

Theorem 2.2. i) $\vdash ((p \to (q \to r)) \to (p \to (Lq \to Lr)))$; and ii) $\vdash ((p \to (q \to r)) \to (Lp \to (q \to Lr)))$.

Proof.

1. $\vdash ((q \to r) \to (Lq \to Lr))$ (by theorem 2.1)
2. $\vdash (((q \to r) \to (Lq \to Lr)) \to ((p \to (q \to r)) \to (p \to (Lq \to Lr))))$ (by A0)
3. $\vdash ((p \to (q \to r)) \to (p \to (Lq \to Lr)))$ (i, from 1 and 2 by MP)
4. $\vdash (((p \to (q \to r)) \to (p \to (Lq \to Lr))) \to ((q \to (p \to r)) \to (Lq \to (p \to Lr))))$ (by A0)
5. $\vdash ((q \to (p \to r)) \to (Lq \to (p \to Lr)))$ (from 3 and 4 by MP)
6. $\vdash ((p \to (q \to r)) \to (Lp \to (q \to Lr)))$ (ii, from 5 by US)   ∎

One of the virtues of McCall's L-X-M calculus, discussed below, is that Camestres LXL, Baroco LXL, Barbara XLL and Ferio XLL are rejected in it. But before we examine McCall's system we look briefly at some recent systems of modal predicate logic that have been used to attempt to understand Aristotle's work on the modal syllogisms.

3   MODERN MODAL PREDICATE LOGIC

It is natural to try to view Aristotle's modal logic through the eyes of modern modal monadic first order predicate logic.[11] On pp. 18-22 McCall refers to Albrecht Becker's [1933][12] and works by others who have tried to do this. On pp. 176-181 Patterson discusses Ulrich Nortmann's [1990] attempt to do this. Patterson points out that the Kripkean "possible worlds semantics" used by Nortmann does not conform with Aristotle's ontological principles. I agree. McCall argues that all uniform readings of Aristotle's modal propositions as sentences in a modal first order predicate logic will make some valid Aristotelian syllogisms invalid or will make some invalid Aristotelian syllogisms valid. I also agree with McCall and will give some examples that support his position.

---

[11]For recent books that contain sections on modal predicate logic see [Hughes and Cresswell, 1996], [Fitting and Mendelsohn, 1998], [Girle, 2000] and [Bell *et al.*, 2001].

[12]See [Bocheński, 1963, pp. 57-62] for a useful discussion of Becker's work.

To illustrate how invalid Aristotelian inferences may be made valid consider Bocardo LXL, (that is, '$LObc, Aba$; so $LOac$', using McCall's notation). Suppose we translate it into modal predicate logic as: '$\exists x(Bx \land \Box\neg Cx); \forall x(Bx \rightarrow Ax)$; so $\exists x(Ax \land \Box\neg Cx)$' (that is, 'There is an x such that x is a B and x is necessarily not a C; for all x if x is a B then x is an A; so there is an x such that x is an A and x is necessarily not a C'). We are using one of Becker's two methods for translating LO sentences. Using "singular sentences" such as Bm (read as m is a B, for 'Max is a bear', for example) and familiar rules such as Existential Instantiation (EI)[13], Universal Instantiation (UI) and Existential Generalization (EG) together with propositional calculus (PC) inferences we may construct a deduction for Bocardo LXL, which Aristotle considered to be invalid.[14]

Proof.

1. $\exists x(Bx \land \Box\neg Cx)$ (premise)
2. $\forall x(Bx \rightarrow Ax)$ (premise)
3. $(Bm \land \Box\neg Cx)$ (from 1 by EI)
4. $(Bm \rightarrow Am)$ (from 2 by UI)
5. $(Am \land \Box\neg Cx)$ (from 3 and 4 by PC)
6. $\exists x(Ax \land \Box\neg Cx)$ (from 5 by EG) ■

To illustrate how valid Aristotelian inferences may be made invalid, consider Bocardo LLL, (that is, '$LObc$; $LAba$; so $LOac$', using McCall's notation). Using another Becker translation of LO sentences and a Becker translation of LA sentences the argument amounts to this: '$\forall x(Cx \rightarrow \Box Bx); \exists x(\Box Ax \land \Box\neg Bx)$; so $\exists x(\Box Ax \land \Box\neg Cx)$, call it the "the MPredC argument". Aristotle at [30a6-14] gives a proof by *ecthesis* to show that Bocardo LLL is valid. But using the semantics for the modal system, S5, the translated argument is S5-invalid. For suppose there are only two possible worlds w_1 and w_2, where each world "sees" each world (including itself). If "the MPredC argument" is S5-valid then the following modal propositional calculus argument is S5-valid, call it the "the MPropC argument": '$((Cm \rightarrow \Box Bm) \land (Cn \rightarrow \Box Bn)); ((\Box Am \land \Box\neg Bm) \lor (\Box An \land \Box\neg Bn))$; so $((\Box Am \land \Box\neg Cm) \lor (\Box An \land \Box\neg Cn))$'. But then a countermodel is constructed by: i) letting Am, Bn and Cn be true in world w_1; ii) letting Bm, Cm and An be false in w_1; iii) letting Am, Cm and Bn be true in world w_2; and iv) letting Bm, An and Cn be false in world w_2. Then in w_1 $(Cm \rightarrow \Box Bm)$ is true, $(Cn \rightarrow \Box Bn)$ is true, $(\Box Am \land \Box\neg Bm)$ is true, $\Box Am \land \Box\neg Cm)$ is false, and $(\Box An \land \Box\neg Cn)$ is false. So "the MPropC argument" is S5-invalid. So "the MPredC argument" is invalid.

The same countermodel may be used to invalidate the argument that results by replacing the premise $\forall x(Cx \rightarrow \Box Bx)$ in "the MPredC" argument with $\forall x(Cx \rightarrow \Box Bx)$.

Geach [1964, p. 202] makes the following remarks about McCall's comments list of seven "Becker-type interpretations":

---

[13] In [Johnson, 1993] Aristotle's proofs by *ecthesis* are treated as essentially proofs by Existential Instantiation. For alternative accounts of proofs by *ecthesis* see [Thom, 1993] and [Smith, 1982].

[14] Paul Thom in [1991] argues that Aristotle made a mistake in regarding Bocardo LXL as valid. Thom contrasts his views with those in [Johnson, 1989], [Patterson, 1989], [Patterson, 1990] and [van Rijen, 1989].

Here McCall has not proved what he claims: namely that no Becker-type in-
terpretation will secure simultaneously the validity of Barbara LLL and LXL,
the invalidity of Barbara XLL, and the simple conversion of LI propositions
(C LIab LIba). For all of these results are obtained if we combine reading (i)
of LA from McCall's list with reading (iii) or equivalently (iv) of LI.

McCall's list on p. 21 of Becker type interpretations is given on table 3.

Table 3. Seven Becker-type interpretations

	Universal	Particular
(i)	$\forall x(Ax \to \Box Bx)$	$\exists x(Ax \wedge \Box Bx)$
(ii)	$\Box \forall x(Ax \to Bx)$	$\Box \exists x(Ax \wedge Bx)$
(iii)	$\forall x \Box(Ax \to Bx)$	$\exists x \Box(Ax \wedge Bx)$
(iv)	$\forall x(\Box Ax \to \Box Bx)$	$\exists x(\Box Ax \wedge \Box Bx)$
(v)	$\forall x(\Diamond Ax \to \Box Bx)$	$\exists x(\Box Ax \wedge \Box Bx)$
(vi)	$\forall x(\Diamond Ax \to Bx)$	$\exists x(\Box Ax \wedge \Box Bx)$
(vii)	$\forall x(\Box Ax \to Bx)$	$\exists x(\Box Ax \wedge Bx)$

McCall finds interpretations (i) and (ii) in [Becker-Freyseng, 1933], (ii) in [von Wright,
1951], (i) to (v) in [Sugihara, 1957a] and [Sugihara, 1957b], and all but (v) in [Rescher,
1963].
This is what McCall says about these seven interpretations:

> None of these interpretations does justice to Aristotle's system. Not one of
> them even simultaneously provides for the validity of Barbaras LLL, the
> invalidity of Barbara XLL, and the convertibility of the particular premise
> 'Some A is necessarily B' into 'Some B is necessarily A'.

And McCall is correct. Geach is in effect proposing two more interpretations in ad-
dition to the seven on the list. Let us call one of them (viii), where $LAab$ is translated
as $\forall x(Ax \to \Box Bx)$ and $LIab$ is translated as $\exists x \Box(Ax \wedge Bx)$. As Geach says, the
other one is essentially the same as it. But interpretation (viii) produces results that
are not Aristotelian. For example, if Darii-LXL, valid for Aristotle, is translated using
interpretation (viii) the resulting argument is S5-invalid. McCall is looking for an in-
terpretation that provides "100% Aristotelicity". Geach (p. 202) invites the reader to
consider an interpretation of McCall's LAab and LOab as sentences of an extended as-
sertoric syllogistic, call it the "G-system", that allows sentences to be formed by using
complex terms, terms of form λp (necessarily p) and μp (possibly p), where p is a simple
term. McCall's $LAab, LEab, LIab$ and $LOab$ are translated into the G-system as $Aa\lambda b$,
$Ea\mu b, I\lambda a\lambda b$ and $Oa\mu b$ respectively. Geach (p. 202) says:

> A decision procedure for this calculus can easily be devised: write every
> formula so that λ-terms and μ-terms appear instead of categoricals prefaced

Figure 1. The invalidity of Darii LXL in the G-system

λc a

b λa
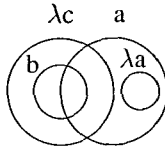
with L, add an antecedent of the form $CA\lambda aa$ [that is, $(A\lambda aa \to]$ for each λ-term and one of the form $CAa\mu a$ [that is, $(Aa\mu a \to]$ for each μ-term, and apply Łukasiewicz's decision procedure for the plain syllogistic to the resulting formula.

So, for example, to determine whether Bocardo LXL (that is, 'LObc; Aba; so LOac') is syntactically accepted or syntactically rejected we form the following sentence in the G-system: $(Ac\mu c \to (Ob\mu c \to (Aba \to Oa\mu c)))$. Following Łukasiewicz's decision procedure on pp. 121-126 of [1957], we form an elementary sentence consisting of affirmative simple sentences that is deductively equivalent to it: $(Ac\mu c \to (Aa\mu c \to (Aba \to Ab\mu c)))$ or (by interchanging terms) $(Ab\mu b \to (Ac\mu b \to (Aac \to Aa\mu b)))$. The latter sentence fits subcase (d) of the fifth case (p. 124):

The consequent is Aab, and there are antecedents of the type Aaf with f different from a. If there is a chain leading from a to b the expression is asserted on the ground of axiom 3 [our A3, above], the mood Barbara; if there is no such chain, the expression is rejected.

Since a is linked to b by the chain $\{Aac, Ac\mu b\}$, $(Ab\mu b \to (Ac\mu b \to (Aac \to Aa\mu b)))$ is accepted. So Bocardo LXL is accepted in the G-system. But for Aristotle Bocardo LXL is valid.

Since questions of validity in the G-system are reduced to questions of validity in the assertoric syllogistic, the familiar Euler diagrams provide a technique for determining whether or not arguments are valid. So, for example, the diagram in figure 1 displays the invalidity of Darii LXL, $(LAbc \to (Iab \to LIac))$. Since circle b is included in circle λc, $LAbc$ is true. Since circle a overlaps circle b, Iab is true. Since circle λa does not overlap λc, $LIac$ is false. When constructing such diagrams these conditions must be met: for every term x, the λx circle is included in or equal to the x circle, which is included in or equal to the μx circle. These conditions are natural since whatever is necessarily x is x, and whatever is x is possibly x.

The diagram in figure 2 displays the invalidity of Cesare LLL, $(LEcb \to (LAab \to LEac))$. $LEcb$ is true since circle c does not overlap circle μb; $LAab$ is true since circle a is included in circle λb, which is identical to circle μb; and $LEac$ is false since circle a overlaps circle μc.

Figure 2. The invalidity of Cesare LLL in the G-system

μc $\mu b, \lambda b$
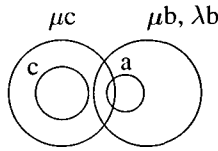
Geach does not claim that his G-system has "100 percent Aristotelicity". He says on p. 203 of [1964] that it "can fit in most of Aristotle's results about syllogisms *de necessario* ". But table 4 shows that the G-system does not get high marks. "V" occurs in a cell if and only if the relevant syllogism is valid for Aristotle, and "Gc" occurs in a cell if and only if the G-system's judgment about the acceptance or rejection of the relevant syllogism is in conflict with Aristotle's. So, for example, the "Gc" in the Darii/LXL cell means that Darii LXL is rejected in the G-system though Aristotle accepts it. The "Gc" in the Bocardo/LXL cell means that Bocardo LXL is accepted in the G-system though Aristotle rejects it. The G-system's Aristotelicity is $((3 \times 14) - 13) \div (3 \times 14)$ or about 69%.

Table 4. Aristotle's system vs. the G-system

		LLL	LXL	XLL
Figure 1	Barbara	V	V	
	Celarent	V	V	
	Darii	V	V,Gc	
	Ferio	V	V	
Figure 2	Cesare	V,Gc	V,Gc	
	Camestres	V,Gc		V,Gc
	Festino	V,Gc	V,Gc	
	Baroco	V,Gc		
Figure 3	Darapti	V	V,Gc	V,Gc
	Felapton	V	V	
	Disamis	V		V,Gc
	Datisi	V	V,Gc	
	Bocardo	V	Gc	
	Ferison	V	V	

Geach's G-system and Łukasiewicz's ŁM illustrate two approaches to understanding Aristotle's work on modal logic. Martha Kneale on p. 91 of [Kneale and Kneale, 1962] poses a dilemma for students of Aristotle given her belief that there are only two approaches to Aristotle's work.

If modal words modify predicates [Geach's *de re* approach is taken], there is no need for a special theory of *modal* syllogisms. For these are only ordinary assertoric syllogisms of which the premises have peculiar predicates. On the other hand, if modal words modify the whole statements to which they are attached [Łukasiewicz's *de dicto* approach is taken], there is no need for a special modal *syllogistic* since the rules determining the logical relations between modal statements are independent of the character of the propositions governed by the modal words.

McCall agrees with Kneale that the two approaches described above are inadequate. And he devises a third approach that is designed to "catch the fine distinctions Aristotle makes between valid and invalid syllogisms (p. 96)".

4   Mc CALL'S L-X-M SYSTEM

The basis for L-X-M includes that of ŁA together with the following primitive symbols, formation rules, definitions, axioms and transformation rules. Only some of the rejection axioms are given here. The partial list is big enough to illustrate how rejection deductions are constructed in L-X-M. For the full list of rejection axioms see [McCall, 1963] or [Johnson, 1989].

Primitive symbols

monadic operator L

Formation rules

FR1′ If Q_u is a quantifier and x and y are term variables then $Q_u xy$ is a *categorical expression* .

FR2′ If p is a categorical expression then $\neg p$ is a *categorical expression* and Lp is a *wff* .

FR3′ Categorical expressions are *wffs* .

FR4′ If p and q are wffs then $\neg p$ and $(p \to q)$ are *wffs* .

FR5′ The only *wffs* are those in virtue of FR1′ to FR4′.

So, for example, Aab is a categorical expression by FR1′, so $\neg Aab$ is a categorical expression by FR2′, so $\neg\neg Aab$ is a categorical expression by FR2′, so $L\neg\neg Aab$ is a wff by FR2′, so $\neg L\neg\neg Aab$ is a wff by FR4′. Note that $LLAab$ is not a wff.

Definitions

Def M $Mp =_{df} \neg L\neg p$

Assertion axioms

Use A0, A1, A3 and A4 from system ŁA. Change A2 for ŁA from $\vdash Iaa$ to $\vdash LIaa$. Then add the following axioms.

A5 (Barbara LXL) $\vdash (LAbc \rightarrow (Aab \rightarrow LAac))$
A6 (Cesare LXL) $\vdash (LEcb \rightarrow (Aab \rightarrow LEac))$
A7 (Darii LXL) $\vdash (LAbc \rightarrow (Iab \rightarrow LIac))$
A8 (Ferio LXL) $\vdash (LEbc \rightarrow (Iab \rightarrow LOac))$
A9 (Baroco LLL) $\vdash (LAcb \rightarrow (LOab \rightarrow LOac))$
A10 (Bocardo LLL) $\vdash (LObc \rightarrow (LAba \rightarrow LOac))$
A11 (LI conversion) $\vdash (LIab \rightarrow LIba)$
A12 (LA subordination) $\vdash (LAab \rightarrow Aab)$
A13 (LI subordination) $\vdash (LIab \rightarrow Iab)$
A14 (LO subordination) $\vdash (LOab \rightarrow Oab)$

Assertion transformation rules

Use the assertion transformation rules AR1 to AR3 from ŁA and add the following rule.

AR4 (Assertions involving doubly negated categorical expressions, DN) From $\vdash (\ldots p \ldots)$ infer $\vdash (\ldots \neg\neg p \ldots)$ and vice versa, if p is a categorical expression. (So, for example, from $\vdash (LAab \rightarrow LAab)$ infer $\vdash (LAab \rightarrow L\neg\neg Aab)$ by DN. By using SE we may infer that $\vdash (LAab \rightarrow \neg\neg LAab)$ given $\vdash (LAab \rightarrow LAab)$.)

Rejection axioms

Use R1 from system ŁA and add the following rejection axioms.

R2 (*5.21, p. 58) $\dashv$(LAbb →(MAab →(Aac →(LAca →(LAbc →LAac)))))
R3 (*5.3, p. 58) $\dashv$(LAaa →(LAcc →(MAac →(LAca →Aac))))[15]
R4 (*5.6, p. 64) $\dashv$(LAaa →(LAbb →(LAcc →(LAab →(MAba →
                    (MAbc →(LAcb →Iac)))))))

Page references are to McCall's [1963]. McCall uses asterisks to refer to rejections.

Rejection transformation rules

Use rejection transformation rules RR1-RR4 as well as the following rule.

RR5 (Rejections involving doubly negated categorical expressions, R-DN) From $\dashv (\ldots p \ldots)$ infer $\dashv (\ldots \neg\neg p \ldots)$ and vice versa, if p is a categorical expression. (So, for example, from $\dashv L\neg\neg Iab$ infer $\dashv LIab$.)

We imitate the discussion of Łukasiewicz's ŁA system by proving various "immediate inferences". Oppositions, conversions, subalternations and subordinations are listed.

Theorem 4.1. (Apodeictic oppositions, Ap-opp) i) $\vdash (\neg LAab \leftrightarrow MOab)$; ii) $\vdash (\neg MOab \leftrightarrow LAab)$; iii) $\vdash (\neg LEab \leftrightarrow MIab)$; iv) $\vdash (\neg MIab \leftrightarrow LEab)$; v)

---
[15]Correction: on p. 273 of [1989] change → $LAac$ in *5.3 to → Aac.

$^\vdash(\neg LIab \leftrightarrow MEab)$; vi) $^\vdash(\neg MEab \leftrightarrow LIab)$; vii) $^\vdash(\neg LOab \leftrightarrow MAab)$; and viii) $^\vdash(\neg MAab \leftrightarrow LOab)$.

Proof.

1. $^\vdash(\neg LAab \leftrightarrow \neg LAab)$ (by A0)
2. $^\vdash(\neg LAab \leftrightarrow \neg L\neg\neg Aab)$ (from 1 by DN)
3. $^\vdash(\neg LAab \leftrightarrow MOab)$ (i, from 2 by DDI, given Def M and Def O)
4. $^\vdash(\neg MOab \leftrightarrow \neg\neg LAab)$ (from 3 by RV)
5. $^\vdash(\neg\neg LAab \leftrightarrow LAab)$ (by A0)
6. $^\vdash(\neg MOab \leftrightarrow LAab)$ (ii, from 4 and 5 by SE)
7. $^\vdash(\neg LEab \leftrightarrow \neg LEab)$ (by A0)
8. $^\vdash(\neg LEab \leftrightarrow \neg L\neg\neg Eab)$ (from 7 by DN)
9. $^\vdash(\neg LEab \leftrightarrow \neg L\neg\neg\neg Iab)$ (from 8 by DDI, given Def E)
10. $^\vdash(\neg LEab \leftrightarrow \neg L\neg Iab)$ (from 9 by DN)
11. $^\vdash(\neg LEab \leftrightarrow MIab)$ (iii, from 10 by DDI, given Def M)
12. $^\vdash(\neg MIab \leftrightarrow \neg\neg LEab)$ (from 11 by RV)
13. $^\vdash(\neg\neg LEab \leftrightarrow LEab)$ (by A0)
14. $^\vdash(\neg MIab \leftrightarrow LEab)$ (iv, from 12 and 13 by SE) ∎

Use similar reasoning for the other four asserted biconditionals.

Theorem 4.2. (Apodeictic conversions, Ap-con) i) $^\vdash(LEab \rightarrow LEba)$; ii) $^\vdash(LIab \rightarrow LIba)$; iii) $^\vdash(MEab \rightarrow MEba)$; and iv) $^\vdash(MIab \rightarrow MIba)$.

Proof.

1. $^\vdash(LIab \rightarrow LIba)$ (ii, by A11)
2. $^\vdash(\neg LIba \rightarrow \neg LIab)$ (from 1 by RV)
3. $^\vdash(MEba \rightarrow MEab)$ (from 2 by SE, Ap-opp)
4. $^\vdash(MEab \rightarrow MEba)$ (iii, from 3 by US)
5. $^\vdash(LEcb \rightarrow (Aab \rightarrow LEac))$ (A6)
6. $^\vdash Aaa$ (A1)
7. $^\vdash(LEca \rightarrow LEac)$ (from 5 and 6 by AI, US, MP)
8. $^\vdash(LEab \rightarrow LEba)$ (i, from 7 by US)
9. $^\vdash(MIab \rightarrow MIba)$ (iv, from 8 by RV, SE, US) ∎

Theorem 4.3. (Apodeictic subalternations, Ap-sub-a) i) $^\vdash(LAab \rightarrow LIab)$; ii) $^\vdash(LEab \rightarrow LOab)$; iii) $^\vdash(MAab \rightarrow MIab)$; and iv) $^\vdash(MEab \rightarrow MOab)$.

Proof.

1. $^\vdash(LAab \rightarrow LIba)$ (by A11)
2. $^\vdash(LIba \rightarrow LIab)$ (by Ap-con, US)
3. $^\vdash(LAab \rightarrow LIab)$ (i, from 1 and 2 by CW)

4. $\vdash (MEab \rightarrow MOab)$ (iv, from 3 by RV, SE)

5. $\vdash (LEbc \rightarrow (Iab \rightarrow LOac))$ (by A8)

6. $\vdash (LEac \rightarrow (Iaa \rightarrow LOac))$ (from 5 by US)

7. $\vdash (LEab \rightarrow LOab)$ (ii, from 6 by AI, MP, US)

8. $\vdash (MAab \rightarrow MIab)$ (iii, from 7 by RV, SE) ∎

Theorem 4.4. (Apodeictic conversions *per accidens*, Ap-con(pa)) i) $\vdash (LAab \rightarrow LIba)$; ii) $\vdash (MAab \rightarrow MIba)$; iii) $\vdash (LEab \rightarrow LOba)$; and iv) $\vdash (MEab \rightarrow MOba)$.

Proof.

1. $\vdash (LAab \rightarrow LIab)$ (by Ap-sub-a)

2. $\vdash (LIab \rightarrow LIba)$ (by Ap-con)

3. $\vdash (LAab \rightarrow LIba)$ (i, from 1 and 2 by CW)

4. $\vdash (LEab \rightarrow LEba)$ (by Ap-con)

5. $\vdash (LEba \rightarrow LOba)$ (by Ap-sub-a, US)

6. $\vdash (LEab \rightarrow LOba)$ (iii, from 4 and 5 by CW)

7. $\vdash (MAab \rightarrow MIba)$ (ii, from 6 by RV, SE, US)

8. $\vdash (MEab \rightarrow MOba)$ (iv, from 3 by RV, SE, US) ∎

Theorem 4.5. (Subordinations, Sub-o) i) $\vdash (LAab \rightarrow Aab)$; ii) $\vdash (Aab \rightarrow MAab)$; iii) $\vdash (LEab \rightarrow Eab)$; iv) $\vdash (Eab \rightarrow MEab)$; v) $\vdash (LIab \rightarrow Iab)$; vi) $\vdash (Iab \rightarrow MIab)$; vii) $\vdash (LOab \rightarrow Oab)$; and viii) $\vdash (Oab \rightarrow MOab)$.

Proof.

1. $\vdash (LOab \rightarrow Oab)$ (vii, by A14)

2. $\vdash (Aab \rightarrow MAab)$ (ii, from 1 by RV, SE)

3. $\vdash (Aaa \rightarrow MAaa)$ (from 2 by US)

4. $\vdash Aaa$ (by A1)

5. $\vdash MAaa$ (from 3 and 4 by MP)

6. $\vdash (LEbc \rightarrow (Iab \rightarrow LOac))$ (by A8)

7. $\vdash (MAac \rightarrow (Iab \rightarrow MIbc))$ (from 6 by RV, SE)

8. $\vdash (MAaa \rightarrow (Iab \rightarrow MIba))$ (from 7 by US)

9. $\vdash (Iab \rightarrow MIba)$ (from 5 and 8 by MP)

10. $\vdash (MIba \rightarrow MIab)$ (by Ap-con, US)

11. $\vdash (Iab \rightarrow MIab)$ (vi, from 9 and 10 by CW)

12. $\vdash (LEab \rightarrow Eab)$ (iii, from 11 by RV, SE, US)

Proofs of the other four subordinations are straightforward and are omitted. ∎

We show that all of the entries marked with 'V' on table 4 and all of the entries marked with a blank on table 5 correspond to asserted wffs in L-X-M. Proofs are streamlined by assuming immediate inferences established above and any immediate inferences obtainable from them by US. So, for example, in the proof of Barbari LXL from Barbara LXL by CW in theorem LXL the subalternation $\vdash (LAac \to LIac)$ is assumed.

Theorem 4.6. All unmarked LXL and XLL cells on table 5 represent asserted wffs.

Proof.

1. $\vdash (LAbc \to (Aab \to LAac))$ (Barbara LXL, by A5)
2. $\vdash (LAbc \to (Aab \to LIac))$ (Barbari LXL, from 1 by CW)
3. $\vdash (Acb \to (LAba \to LIac))$ (Bramantip XLL, from 2 by AI, CW, US)
4. $\vdash (LAbc \to (Iab \to LIac))$ (Darii LXL, by A7)
5. $\vdash (LAbc \to (Iba \to LIac))$ (Datisi LXL, from 4 by AS)
6. $\vdash (LAbc \to (Aba \to LIac))$ (Darapti LXL, from 5 by AS)
7. $\vdash (Ibc \to (LAab \to LIac))$ (Disamis XLL, from 5 by AI, CW, US)
8. $\vdash (Icb \to (LAba \to LIac))$ (Dimaris XLL, from 4 by AI, CW, US)
9. $\vdash (Acb \to (LAba \to LIac))$ (Darapti XLL, from 7 by AS)
10. $\vdash (LEbc \to (Iab \to LOac))$ (Ferio LXL, by A8)
11. $\vdash (LEcb \to (Iab \to LOac))$ (Festino LXL, from 9 by AS)
12. $\vdash (LEbc \to (Iba \to LOac))$ (Ferison LXL, from 9 by AS)
13. $\vdash (LEbc \to (Aba \to LOac))$ (Felapton LXL, from 11 by AS)
14. $\vdash (LEcb \to (Iba \to LOac))$ (Fresison LXL, from 9 by AS)
15. $\vdash (LEcb \to (Aba \to LOac))$ (Fesapo LXL, from 13 by AS)
16. $\vdash (LEbc \to (Aab \to LEac))$ (Cesare LXL, by A6)
17. $\vdash (LEbc \to (Aab \to LEac))$ (Celarent LXL, from 15 by AS)
18. $\vdash (Acb \to (LEab \to LEac))$ (Camestres XLL, from 16 by AI, CW, US)
19. $\vdash (Acb \to (LEba \to LEac))$ (Camenes XLL, from 17 by AS)
20. $\vdash (LEbc \to (Aab \to LEac))$ (Celaront LXL, from 16 by CW)
21. $\vdash (LEcb \to (Aab \to LOac))$ (Cesaro LXL, from 15 by CW)
22. $\vdash (Acb \to (LEab \to LEac))$ (Camestrop XLL, from 18 by CW)
23. $\vdash (Acb \to (LEba \to LEac))$ (Camenop XLL, from 19 by CW)   ∎

Theorem 4.7. All unmarked LLL cells on table 5 represent asserted wffs.

Proof. Use A9, A10 and AS with theorem 4.6. So, for example, Barbara LLL is asserted, since Barbara LXL is asserted and $\vdash (LAab \to Aab)$. Disamis LLL is assserted, since Disamis XLL is asserted and $\vdash (LIab \to Iab)$.   ∎

Theorem 4.8. All unmarked MXM, XMM, LMX and MLX cells on table 5 represent asserted wffs.

Proof. Use theorem 4.6 and RV. So, for example, the assertion of Darii MXM is generated from the assertion of Ferison LXL as follows. $\vdash(MAbc \rightarrow (Iab \rightarrow MIac))$ since $\vdash(LEac \rightarrow (Iab \rightarrow LObc))$ (by RV and SE), since $\vdash(LEbc \rightarrow (Iba \rightarrow LOac))$ (by US). The assertion of Festino LMX is generated from the assertion of Celarent LXL as follows. $\vdash(LEcb \rightarrow (MIab \rightarrow Oac))$ since $\vdash(LEcb \rightarrow (Aac \rightarrow LEab))$ (by RV and SE), since $\vdash(LEbc \rightarrow (Aab \rightarrow LEac))$ (by US). The assertion of Camenes MLX is generated from the assertion of Fresison LXL as follows. $\vdash(MAcb \rightarrow (LEba \rightarrow Eac))$, since $\vdash(Iac \rightarrow (LEba \rightarrow LOcb))$ (by RV and SE), since $\vdash(LEba \rightarrow (Iac \rightarrow LOcb))$ (by AI), since $(LEcb \rightarrow (Iba \rightarrow LOac))$ (by US). ∎

4.1 Rejections in L-X-M

To reject the syllogisms not marked with a "V" on table 4, as well as other invalid inferences, McCall adds twelve rejection axioms to the list of rejection axioms for the ŁA-system. We shall illustrate how some of these rejection axioms are used to reject some wffs.

Theorem 4.9. (Rejection of Barbara XLL) $\dashv(Abc \rightarrow(LAab \rightarrow LAac))$.

Proof. Recall that R2 = $\dashv\sigma$, where $\sigma = (LAbb \rightarrow (MAab \rightarrow (Aac \rightarrow (LAca \rightarrow (LAbc \rightarrow LAac)))))$.

1. $\dashv\sigma$ (by R2)
2. $\vdash((Aac \rightarrow (LAca \rightarrow LAac)) \rightarrow \sigma)$ (by A0)
3. $\dashv(Aac \rightarrow (LAca \rightarrow LAac))$ (from 1 and 2 by R-D)
4. $\vdash(Aac \rightarrow (LAca \rightarrow LAaa))$ (by A5 and US)
5. $\vdash((Aac \rightarrow (LAca \rightarrow LAaa)) \rightarrow ((Aac \rightarrow (LAaa \rightarrow LAac)) \rightarrow (Aac \rightarrow (LAca \rightarrow LAac))))$ (by A0)
6. $\vdash((Aac \rightarrow (LAaa \rightarrow LAac)) \rightarrow (Aac \rightarrow (LAca \rightarrow LAac)))$ (from 4 and 5 by MP)
7. $\dashv(Aac \rightarrow (LAaa \rightarrow LAac))$ (from 3 and 6 by R-D)
8. $\dashv(Abc \rightarrow (LAab \rightarrow LAac))$ (from 7 by R-US) ∎

Theorem 4.10. Baroco XMM and Bocardo MLX are rejected.

Proof.

1. $\dashv(Abc \rightarrow (LAab \rightarrow LAac))$ (by theorem 4.9)
2. $\dashv(Abc \rightarrow (MOac \rightarrow MOab))$ (from 1 by theorem R-RV and R-SE)
3. $\dashv(Acb \rightarrow (MOab \rightarrow MOac))$ (Baroco XMM, from 2 by R-US)
4. $\dashv(MOac \rightarrow (LAab \rightarrow Obc))$ (from 1 by R-RV and R-SE)
5. $\dashv(MObc \rightarrow (LAba \rightarrow Oac))$ (Bocardo MLX, from 4 by R-US) ∎

Theorem 4.11. Barbara LMX, Baroco LXL and Bocardo XMM are rejected.

Proof. Recall that R3 = $\dashv\sigma$, where σ is $(LAaa \rightarrow (LAcc \rightarrow (MAac \rightarrow (LAca \rightarrow Aac))))$.

1. $\dashv\sigma$ (by R3)
2. $\vdash ((LAcc \to (MAac \to Aac)) \to \sigma)$ (by A5)
3. $\dashv (LAcc \to (MAac \to Aac))$ (from 1 and 2 by R-D)
4. $\dashv (LAbc \to (MAab \to Aac))$ (Barbara LMX, from 3 by R-US)
5. $\dashv (LAbc \to (Oac \to LOab))$ (from 4 by R-RV and R-SE)
6. $\dashv (LAcb \to (Oab \to LOac))$ (Baroco LXL, from 5 by R-US)
7. $\dashv (Oac \to (MAab \to MObc))$ (from 4 by R-RV)
8. $\dashv (Obc \to (MAba \to MOac))$ (Bocardo XMM, from 7 by R-US) ∎

Theorem 4.12. Barbari MLX, Bramantip LMX, Felapton XLL and Baroco XLL are rejected.

Proof. Recall that R4 = $\dashv\sigma$, where σ = (LAaa $\to$(LAbb $\to$(LAcc $\to$LAab $\to$(MAba $\to$(MAbc $\to$(LAcb $\to$Iac))))))) .

1. $\dashv\sigma$ (by R4)
2. $\vdash ((MAbc \to (LAab \to Iac)) \to \sigma)$ (by A0)
3. $\dashv (MAbc \to (LAab \to Iac))$ (Barbari MLX, from 1 and 2 by R-US)
4. $\dashv (LAab \to (MAbc \to Iac))$ (from 3 by R-AI)
5. $\vdash (Ica \to Iac)$ (by Con)
6. $\dashv (LAab \to (MAbc \to Ica))$ (from 4 and 5 by R-CS)
7. $\dashv (LAcb \to (MAba \to Iac))$ (Bramantip LMX, from 6 by R-US)
8. $\dashv (Eac \to (LAab \to LObc))$ (from 3 by R-RV and SE)
9. $\dashv (Ebc \to (LAba \to LOac))$ (Felapton XLL, from 8 by R-US)
10. $\dashv (Ebc \to Obc)$ (by Sub-a and US)
11. $\dashv (Obc \to (LAba \to LOac))$ (Baroco XLL, 9 and 10 by R-AW) ∎

Our purpose in this section has been to illustrate how McCall's rejection apparatus works. In the next section we discuss this result: whatever is rejected by using McCall's rejection apparatus may be shown invalid by using countermodels. McCall's [1963] contains no discussion of models.

5   SEMANTICS FOR L-X-M

In [Johnson, 1989] a semantics for McCall's L-X-M is given. Validity is defined by using models, asserted wffs in L-X-M are shown to be valid (that is, system L-X-M is *sound*), and rejected sentences are shown to be invalid. So valid wffs in X-L-M are shown to be accepted (that is, system L-X-M is *complete*) since, as McCall shows, every wff in L-X-M is either accepted or rejected. The presentation of the semantics here will benefit from comments about it in Thom's [1996] and Thomason's [1993] and [1997].[16]

---

[16]For example, I borrow Thom's use of "base conditions" and "superstructural conditions" to present what he calls a "two-layered semantics". And I borrow Thomason's use of "V_M" to refer to a valuation relative to a model.

The semantics for L-X-M extends the familiar semantics for the assertoric syllogistic that assigns non-empty sets of objects to terms. To define the semantic notion of validity we refer to models and valuations relative to models.

Definition 5.1. (model) $\mathcal{M}$ is a *model* iff $\mathcal{M} = \langle W, n^+, q^+, n^-, q^- \rangle$, where W is a non-empty set and n^+, q^+, n^-, and q^- are functions that map terms into subsets of W and satisfy the following "base conditions", where $^+(x)$ is short for $n^+(x) \cup q^+(x)$, and $x \circ y$ (x overlaps y) is short for $x \cap y \neq \emptyset$:

B1 If f and g are any of the functions n^+, q^+, q^- or n^- and $f \neq g$, then, for every term x, $f(x) \cap g(x) = \emptyset$; and for every x, $n^+(x) \cup q^+(x) \cup q^-(x) \cup n^-(x) = W$
B2 For every x, $n^+(x) \neq \emptyset$
B3 (For every x, y and z) if $^+(z) \subseteq n^-(y)$ and $^+(x) \subseteq {}^+(y)$ then $^+(x) \subseteq n^-(z)$
B4 If $^+(y) \subseteq n^+(z)$ and $^+(x) \circ {}^+(y)$ then $n^+(x) \circ n^+(z)$
B5 If $^+(y) \subseteq n^-(z)$ and $^+(x) \circ {}^+(y)$ then $n^+(x) \circ n^-(z)$
B6 If $^+(z) \subseteq n^+(y)$ and $n^+(x) \circ n^-(y)$ then $n^+(x) \circ n^-(z)$

For an intuitive grasp of the notion of a model think of W as the world, $n^+(a)$ as the set of things in W that are essentially a, $q^+(a)$ as the set of things in W that are contingently a and are a, $n^-(x)$ as the set of things in W that are essentially non-a, and $q^-(a)$ as the set of things in W that are contingently not a and are not a.

Definition 5.2. (valuation) A *valuation* V is a function that assigns t or f, but not both, to sentences, where: i) $V(\neg p) = t$ iff $V(p) = f$; and ii) $V(p \to q) = t$ iff $V(\neg p) = t$ or $V(q) = f$; and iii) $V(L\neg\neg p) = t$ iff $V(Lp) = t$.

Definition 5.3. (valuation relative to model M) Let V_M, a *valuation relative to a model M*, be a valuation that satisfies the following "superstructural conditions":

S1 (For every x and y) $V_M(Axy) = t$ iff $^+(x) \subseteq {}^+(y)$
S2 $V_M(Ixy) = t$ iff $^+(x) \circ {}^+(y)$
S3 $V_M(LAxy) = t$ iff $^+(x) \subseteq n^+(y)$
S4 $V_M(LIxy) = t$ iff $n^+(x) \circ n^+(y)$
S5 $V_M(L\neg Axy) = t$ iff $n^+(x) \circ n^-(y)$
S6 $V_M(L\neg Ixy) = t$ iff $^+(x) \subseteq n^-(y)$

Definition 5.4. (valid) Let σ be an L-X-M sentence. σ is *valid* ($\models \sigma$) iff, for every model M, every valuation relative to M assigns t to σ. σ is *invalid* iff σ is not valid.

In this section we shall construct models that show the invalidity of all of the syllogisms that correspond to marked cells on table 5. Exactly four models suffice to show the invalidity of the invalid LXL and XLL models marked on these tables. Models constructed by interchanging rows in these four models suffice to invalidate the remaining invalid syllogisms mentioned on the table.

Table 5 agrees with table 7 on p. 43 of [McCall, 1963]. A cell on the former is marked if and only if it is unmarked on the latter. The marks on McCall's table indicate the relevant syllogism is syntactically asserted in L-X-M. McCall's discussion of L-X-M is

totally syntactic. He gives no formal semantics and thus no formal definition of validity. But as shown in [1989], the syllogisms that are syntactically asserted in L-X-M are the syllogisms that are valid in L-X-M and vice versa. The above theorems 4.6 and 4.7 pertain to the unmarked cells on table 5.

Table 5. Countermodels for L-X-M syllogisms

		LLL	LXL	XLL	MXM	XMM	LMX	MLX
Figure 1	Barbara			1ac	4bc	4ab	3ab	2bc
	Celarent			2ac	1ba	2ab		
	Darii			1ac			3ac	2bc
	Ferio			2ac		2ab		1bc
Figure 2	Cesare			2ac	1ba	2ab		
	Camestres		3ac		2ba	1ab		
	Festino			2ac		2ab		1bc
	Baroco		3ac	4ac	2ba	1ab		4ba
Figure 3	Darapti						2cb	2bc
	Felapton			2ac		3bc		1bc
	Disamis		1ca				2cb	2bc
	Datisi			1ac			2cb	2bc
	Bocardo		4ac	2ac		3cb	4cb	1bc
	Ferison			2ac		3cb		1bc
Figure 4	Bramantip		1ca				2cb	3ba
	Camenes		3ac		2ab	1ab		
	Dimaris		1ca				2cb	3ba
	Fresison			2ac		3cb		1bc
	Fesapo			2ac		3cb		1bc
Subalterns	Barbari			1ac			3ab	2bc
	Celaront			2ac		2ab		
	Cesaro			2ac		2ab		
	Camestrop		3ac		2ba	1ab		
	Camenop		3ac		2ba	1ab		

We begin by constructing a model $\mathcal{M}_1$, presented by table 6, that shows the invalidity of Barbara LXL. When giving such tables we use the following conventions: set brackets are omitted when giving the range of a function, a blank cell indicates the range of the relevant function is the empty set, for terms x other than those explicitly mentioned on the table, $n^+(x) = n^+(a)$, $q^+(x) = q^+(a)$, $n^-(x) = n^-(a)$ and $q^-(x) = q^-(a)$, and $W = n^+(a) \cup q^+(a) \cup n^-(a) \cup q^-(a)$.

So, for example, given table 6 the set of things that are essentially a has only one member, namely 1. The set of things that are c and are contingently c has two members: 1 and 2. The set of things that are essentially not b has no members. And the set of things that are not d and are contingently not d has 3 as its only member. $W = \{1, 2, 3\}$.

Table 6 expresses a model. Base conditions **B1** and **B2**, here and below, do not require a comment. **B3**, **B5** and **B6** are trivially satisfied since, for every x and y, $^+(x) \cap n^-(y) = \emptyset$.

Table 6. Model $\mathcal{M}_1$

	n^+	q^+	n^-	q^-
a	1	2		3
b	1,2,3			
c	3	1,2		

Suppose $^{(}y) \subseteq n^+(z)$. Then $z = b$. For all x, $n^+(x) \circ n^+(b)$. So **B4** is satisfied.

Given model $\mathcal{M}_1$: i) $V_{\mathcal{M}_1}(Abc) = t$ since $^+(b) \subseteq {}^+(c)$; ii) $V_{\mathcal{M}_1}(LAab) = t$ since $^+(a) \subseteq n^+(b)$; and iii) $V_{\mathcal{M}_1}(LAac) = f$ since $^+(a) \not\subseteq n^+(c)$. So $V_{\mathcal{M}_1}(Abc \to (LAaB \to LAac)) = f$. So $\not\models (Abc \to (LAab \to LAac))$. So Barbara XLL is invalid. The invalidity of Barbara XLL is marked on table 5 by putting '1ac' in the Barabara/XLL cell.

Aristotle's informal counterexample for Barbara XLL at 30a28-30 uses terms 'motion', 'animal' and 'man'. For Aristotle, Barbara XLL, construed as an inferential syllogism, is invalid given the inference 'All animals are (accidentally) in motion; all men are necessarily animal; so all men are necessarily in motion'. Aristotle takes the premises to be true and the conclusion false, making Barbara XLL invalid.

By interchanging rows a and b in table 6 we may construct a model $\mathcal{M}_{1bc}$ expressed by table 7 that shows that Ferio MLX invalid.

Table 7. Model $\mathcal{M}_{1bc}$

	n^+	q^+	n^-	q^-
a	1,2,3			
b	1	2		3
c	3	1,2		

In general, if a table satisfies conditions **B1** to **B6** so will a table that results from the interchanging of its rows. For none of these conditions requires a particular ordering of rows. Note that $V_{\mathcal{M}_{1bc}}(MEac) = t$, $V_{\mathcal{M}_{1bc}}(LIab) = t$ and $V_{\mathcal{M}_{1bc}}(Obc) = f$. So $\not\models (MEac \to (LIab \to Oac))$. That is, Ferio MLX is invalid.

This is the recipe for constructing a table T_2 for model $\mathcal{M}_{Nxy}$ (where x and y are a, b or c) from a table T_1 for model $\mathcal{M}_N$ (where T_1 has rows a, b and c): make row a in T_1 be row x in T_2, make row c in T_1 be row y in T_2, and make row b in T_1 be the third row in T_2. Every row in T_2 must be an a-row, a b-row or a c-row. So, for example, consider the Baroco/XMM cell on table 5, which is marked with '1ab'. Use the recipe to construct table 8 for model $\mathcal{M}_{1ab}$, which invalidates Baroco XMM. (The a-row of table 6 becomes the a-row of table 8; the c-row of 6 becomes the b-row of table 8; and the b-row of 6 becomes the c-row of table 8.) Since $\mathcal{M}_{1ab}(Acb) = t$, $\mathcal{M}_{1ab}(MOab) = t$ and $\mathcal{M}_{1ab}(MOac) = f$, $\not\models (Acb \to (MOab \to MOac))$.

Table 8. Model $\mathcal{M}_{1ab}$

	n^+	q^+	n^-	q^-
a	1	2		3
b	3	1,2		
c	1,2,3			

Model $\mathcal{M}_2$ expressed by table 9 may be used to show that Celarent XLL is invalid.

Table 9. Model $\mathcal{M}_2$

	n^+	q^+	n^-	q^-
a	1		2	3
b	1		2	3
c	2	3		1

Table 9 expresses a model. For all x and y, $^+x \not\subseteq n^-(y)$. So conditions **B3** and **B5** are trivially satisfied. For all x and y if $^+(x) \circ {}^+(y)$ then $n^+(x) \circ {}^+(y)$. So **B4** is satisfied. For all x and y, if $^+(x) \subseteq n^+(y)$ then $n^-(y) \subseteq n^-(x)$. So **B6** is satisfied.

Celarent XLL is invalid since: i) $V_{\mathcal{M}_2}(Ebc) = t$ since $^+(b)$ does not overlap $^+(c)$; ii) $V_{\mathcal{M}_2}(LAab) = t$ since $^+(a) \subseteq n^+(b)$; and iii) $V_{\mathcal{M}_2}(LEac) = f$ since $^+(a) \not\subseteq n^-(c)$. So $V_{\mathcal{M}_2}(Ebc \to (LAab \to LEac)) = f$. So $\not\models (Ebc \to (LAab \to LEac))$.

Model $\mathcal{M}_3$ expressed by table 10 may be used to show that Camestres LXL is invalid.

Table 10. Model $\mathcal{M}_3$

	n^+	q^+	n^-	q^-
a	1	2		3
b	3		2	1
c	3		2	1

Table 10 expresses a model. **B3** and **B5** are trivially satisfied since, for every x and y, $^+(x) \not\subseteq n^-(y)$. For **B4** note that if $^+(x) \circ {}^+(y)$ then $n^+(x) \circ {}^+(y)$. For **B6** note that if $^+(z) \subseteq n^+(y)$ then $n^-(y) \subseteq n^-(z)$.

Camestres LXL is invalid since: i) $V_{\mathcal{M}_3}(LAcb) = t$ since $^+(c) \subseteq n^+(b)$; ii) $V_{\mathcal{M}_3}(Eab) = t$ since $^+(a) \cap {}^+(b) = \emptyset$; and iii) $V_{\mathcal{M}_3}(LEac) = f$ since $^+(a) \not\subseteq n^-(c)$. So $V_{\mathcal{M}_3}(LAcb \to (Eab \to LEac)) = f$. So $\not\models (LAcb \to (Eab \to LEac))$.

For Aristotle, Camestres LXL is invalid since Celarent XLL is invalid. A "semantic rule" that underwrites this reduction of an invalidity to an invalidity may be stated as follows.

R$^{\nvDash}$-DR3 i) From $\nvDash (p \to (q \to r))$ and $\models (p \to s)$ infer $\nvDash (s \to (q \to r))$; and ii) from $\nvDash (p \to (q \to r))$ and $\models (q \to s)$ infer $\nvDash (p \to (s \to r))$.

Proof. For i) Suppose a) $\nvDash (p \to (q \to r))$ and b) $\models (p \to s)$. By a) there is a model M such that $V_M(p) = t$, $V_M(q) = t$ and $V_M(r) = f$. By b) $V_M(s) = t$. So $\nvDash (s \to (q \to r))$. Use similar reasoning for ii). ∎

R$^{\nvDash}$-AW is the semantic counterpart of the syntactic rule **R-DR3**, which is called rejection by antecedent weakening (R-AW). Given the rejection of Celarent XLL ($Ebc \to (LAab \to LIac)$) and the conversion principle $^{\vdash}(Ebc \to Ecb)$, Camestres LXL is rejected by R-AW. Likewise, given the invalidity of Celarent XLL and the semantic conversion principle $\models (Ebc \to Ecb)$, Camestres LXL is invalid by **R$^{\nvDash}$-AW**.

Semantic counterparts of other syntactic rejection rules may be put to use to establish invalidities. We illustrate this point by considering the semantic counterpart of R-RV.

R$^{\nvDash}$-RV i) From $\nvDash (p \to q)$ infer $\nvDash (\neg q \to \neg p)$; ii) from $\nvDash (p \to (q \to r))$ infer $\nvDash (p \to (\neg r \to \neg q))$; and iii) from $\nvDash (p \to (q \to r))$ infer $\nvDash (\neg r \to (p \to \neg q))$.

Proof. For i) suppose $\nvDash (p \to q)$. So there is a model M such that $V_M(p) = t$ and $V_M(q) = f$. So $V_M(\neg q) = t$ and $V_M(\neg p) = f$. So $\nvDash (\neg q \to \neg p)$. Use similar reasoning for ii) and iii). ∎

By **R$^{\nvDash}$-RV**, since $\nvDash (Ebc \to (LAab \to LEac))$ (Celarent XLL is invalid), $\nvDash (Ebc \to (\neg LEac \to \neg LAab))$. By using semantic counterparts of other syntactic principles stated above it is easy to conclude that Festino XMM is invalid.

To show that Baroco XLL is invalid we use model $\mathcal{M}_4$, presented on table 11.[17]

Table 11. Model $\mathcal{M}_4$

	n^+	q^+	n^-	q^-
a	1,2	3,4		
b	1,3	4	2	
c	1	4	3	2

Table 11 expresses a model. Base conditions **B3** and **B5** are trivially satisfied since, for every x and y, $^+(x) \not\subseteq n^-(y)$. **B4** and **B6** are trivially satisfied since, for every x and y, $n^+(x) \not\subseteq n^+(y)$.

Given model $\mathcal{M}_4$: i) $V_{\mathcal{M}_4}(Acb) = t$ since $^+(c) \subseteq {}^+(b)$; ii) $V_{\mathcal{M}_4}(LOab) = t$ since $n^+ a \circ n^- b$; and iii) $V_{\mathcal{M}_4}(LOac) = f$ since $n^+(a)$ does not overlap $n^-(c)$. So $V_{\mathcal{M}_4}((Acb \to (LOab \to LOac))) = f$. So $\nvDash (Acb \to (LOab \to LOac))$. Following the pattern indicated above we record on table 5 the invalidity of Baroco XLL by

---

[17]Thomason [1993, p. 127] uses this table to invalidate Baroco XLL and Bocardo LXL though his definition of "validity" is not identical to that which we are currently discussing. Thomason models are discussed below.

putting '4ac' in the Baroco/XLL cell, where '4' refers to model $\mathcal{M}_4$ and 'ac' indicates that 'a' and 'c' are taken as minor and major terms, respectively.

Aristotle's counterexample for Baroco XLL is controversial. According to Thom on p. 148 of [1991] Aristotle used terms 'animal', 'man' and 'white', generating the purported counterexample: 'All men are animals; some whites are necessarily not animals; so some whites are necessarily not men.' Thom says:

> The problem with this counter-example is not (as van Rijen supposes [1989]) that the major premise is necessarily true. It is that, if the minor is taken to be true then the conclusion will be true also.

In agreement with Thom, Aristotle did not provide a good counterexample for Baroco XLL. A better informal counterexample is found in [Johnson, 1993, p. 179]: 'All things that are chewing are bears (Acb); some animals (dogs, say) are necessarily not bears $(LOab)$; so some animals are necessarily not chewing $(LOac)$'. We do not follow Thom in developing formal systems that take Baroco XLL to be invalid.

Though models $\mathcal{M}_1$ to $\mathcal{M}_4$ and variants of them constructed by interchanging rows in them suffice to give countermodels for the invalid syllogisms marked on table 5, other models are needed to invalidate all of McCall's rejection axioms and thus all of the invalid wffs. The model used in [1989] to invalidate McCall's (LAbb →(LAff →(Aad →(LAda →(MAae →(LAcb →(LAbd →(LAce →(Aec →(LAfc →(MAdf →MAac)))))))))))) (*5.41 on p. 59) has four members. It is presented on table 12.

Table 12. Model for *5.41

	n^+	q^+	n^-	q^-
a	1,2,3,4			
b	3,4			1,2
c	4	3	1	2
d	3,4	1,2		
e	3,4			1,2
f	4		2	1,3

An implication of [Johnson, 1989] is that every invalid L-X-M wff of form $(p_1 \rightarrow (p_2 \rightarrow \ldots \rightarrow p_n)\ldots)$, where each p_i (for $1 \leq i \leq n$) is a simple wff or the negation of a simple wff, may be shown invalid by using a model $\langle W, \ldots \rangle$ in which W has no more than 6 members. [18]

In the next section we shall examine valuable attempts by Thomason to improve on the semantics discussed in this section.

---

[18][Johnson, 1991] shows that W does not require more than 3 members if all simple sentences are assertoric and all terms are "chained".

5.1   Thomason models

In [Thomason, 1993] three notions of models are defined that enable Thomason to obtain soundness and completeness results for McCall's L-X-M calculus. In contrast to the soundness and completeness proofs given in [Johnson, 1989] no use is made of rejection axioms and rejection rules. One of these models comes close to the notion of a model defined above. We call it a "T3-model" (his "model$_3$") and define it as follows.

Definition 5.5. (T3-model) $\mathcal{M}$ is a *T3-model* iff $\mathcal{M} = \langle W, n^+, q^+, n^-, q^- \rangle$, where W is a non-empty set and n^+, q^+, n^-, and q^- are functions that map terms into subsets of W and satisfy the following "base conditions", where $^+(x)$ is short for $n^+(x) \cup q^+(x)$:

B1   If f and g are any of the functions n^+, q^+, q^- or n^- and $f \neq g$, then, for every term
         x, $f(x) \cap g(x) = \emptyset$; and for every x, $n^+(x) \cup q^+(x) \cup q^-(x) \cup n^-(x) = W$
B2   For every x, $n^+(x) \neq \emptyset$
BT3 (For every x and y) if $^+(x) \circ {}^+(y)$ then $^+(x) \circ n^+(y)$
BT4 If $^+(x) \subseteq n^-(y)$ then $^+(y) \subseteq n^-(x)$
BT5 If $^+(x) \subseteq n^+(y)$ then $n^-(y) \subseteq n^-(x)$

To define "valuation relative to a model" and "validity" Thomason uses the same superstructural conditions as used above.

Thomason, on p. 133 of [1997], says that in his [1993] he "tried to find simpler, and apparently weaker, requirements for models" than those given in [Johnson, 1989]. In the motivating section of [Thomason, 1993] he says "Johnson ... provided a semantics that has the right validities, but the latter is in some sense contrived." No doubt conditions **BT3, BT4** and **BT5** are more easily understood than **B3, B4, B5** and **B6** but Thomason is not correct in saying that the former, taken collectively, are weaker than the latter, taken collectively. We use the following theorem to show the relationship between T3-models and "J-models", the models defined above that satisfy base conditions **B1** to **B6**.

Theorem 5.6. i) Every T3-model is a J-model, but ii) there are J-models that are not T3-models.

Proof. For i) suppose $\mathcal{M}$ is a T3-model. First, suppose $^+(z) \subseteq n^-(y)$ and $^+(x) \subseteq {}^+(y)$. Then, by **BT4**, $^+(y) \subseteq n^-(z)$. Then $^+(x) \subseteq n^-(z)$. Then $\mathcal{M}$ satisfies **B3**. Next, suppose $^+(y) \subseteq n^+(z)$ and $^+(x) \circ {}^+(y)$. Then $^+(y) \circ {}^+(x)$ and, by **BT3**, $^+(y) \circ n^+(x)$. Then $n^+(x) \circ n^+(z)$. Then $\mathcal{M}$ satisfies **B4**. Next, suppose $^+(y) \subseteq n^-(z)$ and $^+(x) \circ {}^+(y)$. Then, by **BT3**, $n^+(x) \circ n^+(z)$. Then $\mathcal{M}$ satisfies B5. Next, suppose $^+(z) \subseteq n^+(y)$ and $n^+(x) \circ n^-(y)$. Then, by **BT5**, $n^-(y) \subseteq n^-(z)$. Then $n^+(x) \circ n^-(z)$. Then $\mathcal{M}$ satisfies **B6**.

For ii) note that $\mathcal{M}_4$, specified in table 11, is a J-model but not a T3-model since condition **BT3** is not satisfied. Note that $^+(a) \circ +(b)$ but $^+(a)$ does not overlap $n^+(b)$.   ∎

Though both T3-models and J-models, with the superstructural conditions defined above, will reveal the invalidity of any invalid syllogism with any finite number of antecedents (or premises), it is not clear that **BT3** and **BT5** are Aristotelian principles. Certainly **BT4** is Aristotelian, given 25a27-28. And J-models may be simplified by replacing **B3** with **BT4**, given the following theorem.

Theorem 5.7. i) **B3** is derivable from **BT4**; and ii) **BT4** is derivable from **B3**.

Proof. For i) suppose that a) if $^+(z) \subseteq n^-(y)$ then $^+(y) \subseteq n^-(z)$ and that b) $^+(z) \subseteq n^-(y)$ and $^+(x) \subseteq {}^+(y)$. Then $n^+(y) \subseteq n^-(z)$. Then $^+(x) \subseteq n^-(z)$.
For ii) suppose that c) if $^+(z) \subseteq n^-(y)$ and $^+(x) \subseteq {}^+(y)$ then $^+(x) \subseteq n^-(z)$ and d) $^+(z) \subseteq n^-(y)$. Since $+(y) \subseteq^+ (y)$, $^+(y) \subseteq n^-(z)$.   ∎

B4 (Darii LXL), **B5** (Ferio LXL) and **B6** (Baroco LLL) are Aristotelian given 30a37-b2 and 30a6-14 of the *Prior Analytics* .

5.2   Variants of the L-X-M system

Paul Thom in [1991, p. 137] points out that condition B2, used in the definitions of J-models and T3-models to guarantee that McCall's axiom $LIaa$ is valid, is unAristotelian. He says that it is unAristotelian to think that there are walkers that are essentially walkers and whites that are essentially white. Johnson's [1993] and [1995] provide variants of McCall's L-X-M that are sound and complete systems, where condition **B2** is omitted. Both systems have 100% Aristotelicity. The systems deviate from McCall's in that lines in deductions need not be axioms or lines that are ultimately derived from axioms by rules of inference. The systems are "natural deduction systems" rather than "axiomatic systems". Proofs of completeness assume that the inferences under discussion satisfy what Smiley calls the "chain condition" in [Smiley, 1994, p. 27]. And the systems attempt to accommodate Aristotle's proofs by *ecthesis* .[19] In the remainder of this section we illustrate proofs by ecthesis and then discuss the chain condition in the next section.

In addition to sentences such as Abc and $Labc$ discussed above we count $m \in a$ (m is an a), $m \in_n a$ (m is necessarily an a), $m \notin_n a$ (m is necessarily not an a), etc. The latter are singular sentences. In contrast to Thom's view, to present proofs by *ecthesis* singular sentences are required.[20] Consider this proof of Darapti XXX taken from Smith's [1989, p. 9] with my additions in square brackets:

> When they [terms] are universal, then when both P [that is, c] and R [that is, a] belong to every S [that is, b], it results of necessity that P will belong to some R. . . . It is . . . possible to carry out the demonstration through . . . the setting out [that is, by *ecthesis*]. For if both terms belong to every S, then if some one of the S's is chosen (for instance N [that is, m], then both P and R will belong to this; consequently, P will belong to some R. (28a18-26)

Aristotle's proof by *ecthesis* may be formalized as follows:

1. Abc (Premise)
2. Aba (Premise)

---

[19]The systems proposed by [Łukasiewicz, 1957], [Corcoran, 1972] and [Smiley, 1973] do not attempt to accommodate Aristotle's proofs by *ecthesis* . According to Thom's [Thom, 1991] account of *ecthesis* both Baroco XLL and Bocardo LXL are valid, though Aristotle regarded them as invalid.

[20]For an alternative method of working with singular sentences in the context of syllogistic reasoning see [Johnson, 1999a].

3. $m \in b$ (By *ecthesis* from 1. Since all b are c there must be a b that may be referred to as m.)
4. $m \in c$ (From 1 and 3. Since all b are c and m is a b it follows that m is a c.)
5. $m \in a$ (From 2 and 3 by the reasoning for line 4.)
6. Iac (From 4 and 5 by "Existential Generalization" – if a particular object m is both an a and a c then something is both an a and a c.)

Aristotle proves that Baroco LLL is valid in the following passage, taken from Smith's [1989, p. 13]:

> ... it is necessary for us to set out that part [m] to which each term [b and c] does not belong and produce the deduction about this [m]. For it will be necessary in application to each of these; and if it is necessary of what is set out, then it will be necessary of some part [a] of the former term (for what is set out is just a certain "that". (30a9-15)

His proof by *ecthesis* may be formalized as follows:

1. LAcb (Premise. Whatever is c is necessarily b.)
2. LOab (Premise. There is something that is necessarily a but necessarily not b.)
3. $m \in_n a$
4. $m \notin_n b$ (Lines 3 and 4 come from line 2 by *ecthesis* . This is a use of "Existential Instantiation".)
5. $m \notin_n c$ (From 1 and 4. If whatever is c is necessarily b and m is necessarily not in c then m is necessarily not in c.)
6. LOac (From 3 and 5 by Existential Generalization.)

6   THE CHAIN CONDITION, RELEVANCE LOGIC AND THE AP SYSTEM

The following remarks by Smiley from two of his papers show that Aristotle held views endorsed by contemporary "relevance logicians".[21]

> By building onto the propositional calculus Łukasiewicz in effect equates syllogistic implication with strict implication and thereby commits himself to embracing the novel moods corresponding to such theorems as

---

[21] It is very surprising that Aristotle is scarcely mentioned in [Anderson and Belnap, 1975 and 1992], which provides authoritative discussions of relevance logic. See McCall's discussion of "connexive implication" [1975 and 1992, pp. 434-452] for the one reference to Aristotle. In [Johnson, 1994] a syllogistic logic is developed that is a "connexive logic". Pleasing relevance logicians, the logic satisfies both *Aristotle's thesis* (If y is the logical consequence of a non-empty set of premises, X, then X is semantically consistent) and *Boethius's Thesis* (If z is the logical consequence of a set of premises, $X \cup y$, then z is not the logical consequence of a set of premises $X \cup y'$, where y' contradicts y). Ironically, neither Aristotle's nor Boethius's thesis holds for what is now known as the "classical propositional calculus". In [Johnson, 1994] a theorem is proven that has as a corollary this interesting result due to C. A. Meredith in [1953]: The number of valid n-premised assertoric syllogisms (for $n \geq 2$) is $3n^2 + 5n + 2$. There is no question that in Chapter 25 of Book I of the *Prior Analytics* Aristotle was looking for such a general result. Given the chain condition such counts are possible.

$((Aab \wedge Oab) \rightarrow Icd)$ or $((Aab \wedge Acd) \rightarrow Aee)$. On the other hand Aristotle's own omission of these syllogisms of strict implication, as they may be called can hardly be written off as an oversight. For they violate his dictum that a syllogism relating this to that proceeds from premises which relate this to that' (41a6). This dictum is part of a principle which is absolutely fundamental to his syllogistic, namely the principle that the premises of a syllogism must form a chain of predications linking the terms of the conclusion. Thus his doctrine of the figures, which provides the framework for his detailed investigation of syllogistic, is founded on this principle (40b30 ff.) Not less important is that the chain principle is essential to the success of his attempt at a completeness proof for the syllogistic. By this I mean his attempt to show that every valid syllogistic inference, regardless of the number of premises, can be carried out by means of a succession of two-premise syllogisms. [Smiley, 1973, pp. 139-140]

Probably the easiest way to formulate this 'chain condition' is to use the notation AB to denote any of the forms a, e, i, o regardless whether the subject is A or B. Then the condition is that a valid argument must be of the form '$AC, CD, DE, EF, \ldots GH, HB$; therefore AB'. The chain condition dramatically alters the character of the completeness problem (for a start, it excludes the possibility of anything following from an infinite number of premises) and it permits simple strategies for the proof that would otherwise be inconceivable. It is therefore not surprising that Aristotle's proof should fail to fit the same picture as, for example, Corcoran's own completeness proof for syllogistic logic without the chain condition [Corcoran, 1972].[22] [Smiley, 1994, p. 27]

Aristotle's case for the chain condition is redolent of relevance — the need for some overt connection of meaning between premises and conclusion as a prerequisite for deduction. [Smiley, 1994, p. 30]

Since McCall's presentation of the L-X-M calculus imitates Łukasiewicz's, it also embraces "novel moods" of the sort mentioned by Smiley. $(LAab \rightarrow (\neg LAab \rightarrow Icd))$ is asserted in L-X-M even though neither c nor d occurs in the antecedent (and thus the consequent is irrelevant to the antecedents). This follows from the completeness result, mentioned above, for L-X-M. Note that for every model $\mathcal{M}$ either $V_{\mathcal{M}}(LAab) = f$ or $V_{\mathcal{M}}(\neg LAab) = f$. So $\models (LAab \rightarrow (\neg LAab \rightarrow Ide))$. And $(LEab \rightarrow (LEcd \rightarrow (LEef \rightarrow Igg)))$ is asserted in L-X-M. For in every model $\mathcal{M}$, $V_{\mathcal{M}}(Igg) = t$. So $\models (LEab \rightarrow (LEcd \rightarrow (LEef \rightarrow Igg)))$. So, by completeness, $\vdash (LEab \rightarrow (LEcd \rightarrow (LEef \rightarrow Igg)))$ even though g does not occur in any of the antecedents.

---

[22]Corcoran gives a Henkin-style completeness proof for the assertoric syllogistic. His system validates inferences such as 'Eab; so Acc', inferences eschewed by relevance logicians. This inference is valid for Corcoran since the conclusion is logically true, even though the premise is irrelevant to the conclusion.

By using the chain condition in [1973], Smiley formulates an elegant decision procedure for the assertoric syllogistic. In [Johnson, 1994] a system is developed for Aristotle's apodeictic syllogisms, call it the "AP system", that uses the chain condition. A decision procedure is given for it that yields Smiley's decision procedure as a corollary.[23] Both decision procedures are given below.

Definition 6.1. (chain condition) Let Pr_i refer to "prefixes" of assertoric or apodeictic sentences: $A, E, I, O, LA, LE, LI, LO, MA, ME, MI$ and MO. A *chain* is a set of sentences whose members can be arranged as a sequence $\langle Pr_1[x_1x_2], Pr_2[x_2x_3], \ldots, Pr_n[x_nx_1] \rangle$, where $Pr_i[x_ix_j]$ is either $Pr_ix_ix_j$ or $Pr_ix_jx_i$ and $x_i \neq x_j$ if $i \neq j$.

So, for example, $\{LAab, MAcb, LIcd, Ead\}$ and $\{Oba, LEbc, LEdc, LAda\}$ are chains. But neither $\{LAaa\}$ nor $\{LAab, Aba, MAac, Aca\}$ is a chain.

Definition 6.2. (abbreviations for subsets of chains) $X/LAxy$ refers to Axy or $LAxy$. $X/LAx - y$ refers to $\emptyset$ if $x = y$; otherwise, it refers to $\{X/LAz_1z_2, X/LAz_2z_3, \ldots, X/LAz_{n-1}z_n\}$, a subset of a chain, where $x = z_1, y = z_n$ and $n > 1$. $LAx - y$ refers to $\emptyset$ if $x = y$; otherwise it refers to $X/LAx - z, LAzy$, a subset of a chain. $X/LExy$ refers to Exy or $LExy$. $X/LIxy$ refers to Ixy or $LIxy$. And $X/LOxy$ refers to Oxy or $LOxy$.

So, for example, $LAab, LAbc, Acd$ has form $X/LAa - d$, but does not have form $LAa - d$. $LAab, LAbc, Acd, LAd - e$ has form $X/LAa - e$ and form $LAa - e$.

Definition 6.3. (contradictory of, cd) Let $cd(Axy) = Oxy$ where 'cd' may be read as 'the contradictory of'. Let $cd(Ixy) = Exy$, $cd(LAxy) = MOxy$, $cd(LExy) = MIxy$, $cd(LIxy) = MExy$, and $cd(LOxy) = MAxy$. And let $cd(cd(x)) = x$. So, for example, $cd(Exy) = Ixy$.

Theorem 6.4. (Johnson [1994], decision procedure for "AP-validity") Suppose "valid$_{AP}$" (apodeictic syllogistic validity) is defined as in [1994]. Consider an inference in the "AP system" from premises $P_1, P_2, \ldots, P_n$ to conclusion C. This inference is valid$_{AP}$ if and only if $\{P_1, P_2, \ldots, P_n, cd(C)\}$ is a chain that has one of the following eleven forms:

1. X/LAx-y, X/LOxy
2. LAx-z, MAzu, LAu-y, LOxy
3. X/LAx-z, LAzy, MOxy
4. X/LAz-x, X/LAz-y, X/LExy
5. X/LAz-x, X/LAz-u, MAuv, X/LAv-y, LExy (or LEyx)
6. X/LAz-x, X/LAz-u, LAuy, MExy (or MEyx)
7. X/LAz-x, X/LAu-y, X/LIzu, X/LExy (or X/LEyx)
8. X/LAz-x, X/LAu-v, MAvw, X/LAw-y, X/LIzu (or X/LIuz), LExy (or LEyx)
9. X/LAz-x, X/LAu-y, MIzu, LExy (or LEyx)
10. LIxy, MExy (or MEyx)

---

11. X/LAz-x, X/LAu-v, LAvy, X/LIzu (or X/LIuz), MExy (or MEyx)

So, for example, $\{LAab, LAbc, Acd, cd(Aad)\}$ has form 1. So '$LAab, LAbc, Acd$; so Aad' is valid. $\{LAab, LAbc, Acd, cd(MAad)\}$ has form 1. So '$LAab, LAbc, Acd$; so $MAad$' is valid. $\{Aab, cd(Obc), Acd, Oad\}$ has form 1. So 'Aab, Acd, Oad; so Obc' is valid. $\{MAac, LAcb, cd(MAab)\}$ has form 2. So '$MAac, LAcb$; so $MAab$' (Barbara LMM) is valid. $\{LAcb, Iac, cd(LIab)\}$ has form 11. So '$LAcb, Iac$; so $LIab$' (Darii LXL) is valid.

Notice that since 'E' occurs at most once in any of the forms, it follows that no valid syllogism, regardless of the number of premises, is such that 'E' occurs in two or more of its premises. A similar comment applies to occurrences of 'M'.

The following result is a corollary of theorem 6.4.

Theorem 6.5. (Smiley [1973], decision procedure for "AS-validity") Suppose "valid$_{AS}$" (assertoric syllogistic validity) is defined as in [1973]. Consider an inference in the assertoric syllogistic from premises $P_1, P_2, \ldots, P_n$ to conclusion C. This inference is valid$_{AS}$ if and only if $\{P_1, P_2, \ldots, P_n, cd(C)\}$ is a chain that has one of the following three forms:

1. Ax-y, Oxy (restriction of form 1 of theorem 6.4)
2. Az-x, Az-y, Exy (restriction of form 4 of theorem 6.4)
3. XAz-x, Au-y, Izu, Exy (or Eyx) (restriction of form 7 of theorem 6.4)

So, for example, given form 2 of the corollary both 'Aca, Acb; so Iab' (Darapti) and 'Aca, Eab; so Ocb' (Celaront) are valid.

On table 13 a syllogism is marked as valid by referring by number to the form listed in theorem 6.4 in virtue of which it is valid. So, for example, the first occurrence of '1' on the table indicates that Barbara XXX, XXM, XLX, XLM, LXX, LXM, LLX and LLM are valid in virtue of their relationship to $\{X/LAx - y, X/LOxy\}$. The 333 valid syllogisms marked on the table exactly match the 333 syllogisms that McCall accepts in his L-X-M system. See p. 46 of [McCall, 1963].

7   CONTINGENT SYLLOGISMS

A. N. Prior [1962, p. 188] gives a simple account of "the usual meaning of 'contingent'" in the following passage:

> In the *De Interpretatione* Aristotle remarks that the word 'possible' is ambiguous; we should sometimes say that 'It is possible that p' follows from 'It is necessary that p', but sometimes that it is inconsistent with it. In the former sense 'possible' means simply 'not impossible'; in the latter sense, 'neither impossible nor necessary'. It is for 'possible' in this second sense that the word 'contingent' is generally used. That is, 'It is contingent that p' means 'Both p and not-p are possible', KMpMNp [or $(Mp \wedge M\neg p)$]. Contingency in this sense stands between necessity and impossibility, but in quite a different way from that in which the simply factual stands between the

Table 13. Valid$_{AP}$ 2-premised syllogisms

	X/L	L	L	X	L	M	M	X	L	M
	X/L	L	X	L	M	L	X	M	M	L
	X/M	L	L	L	M	M	M	M	X	X
Barbara	1	3	3		2	2				
Celarent	7	9	9		8	11			8	11
Darii	7	11	11		9	8	8	9		
Ferio	7	8	8		9	11	11		9	
Cesare	7	9	9		8	11			8	11
Camestres	7	9		9	11	8			11	8
Festino	7	8	8		9	11	11		9	
Baroco	1	2			3	2			3	
Darapti	4	6	6	6	5	5	5	5		
Felapton	4	5	5		5	6	6		5	
Disamis	7	11		11	8	9	9	8		
Datisi	7	11	11		9	8	8	9		
Bocardo	1	2			2	3	3			
Ferison	7	8	8		9	11	11		9	
Bramantip	4	6		6	5	5	5	5		
Camenes	7	9		9	11	8			11	8
Dimaris	7	11		11	8	9	9	8		
Fresison	7	8	8		9	11	11		9	
Fesapo	4	5	5		5	6	6		5	
Barbari	4	6	6		5	5	5	5		
Celaront	4	5	5		5	6	6		5	6
Cesaro	4	5	5		5	6	6		5	6
Camestrop	4	5		5	6	5			6	5
Camenop	4	5		5	6	5			6	5
Total	8×24	24	15	8	24	24	16	7	15	8 = 333

necessary and the possible. It is not that necessity implies contingency, and contingency impossibility; rather we have three mutually exclusive alternatives which divide the field between them — either a proposition is necessary, or it is neither-necessary-nor-impossible (i.e. contingent), or it is impossible . . ..

On p. 190 of [1962] Prior introduces the symbol 'Q' and reads 'Qp' as 'It is contingent that p'. McCall adopts Prior's use of 'Q' to refer to Aristotle's contingency operator and Thom [1994, p. 91] refers to [McCall, 1963] to support his use of 'Q' in his discussions of contingency. In the discussion below, we shall also use 'Q'.[24]

---

[24]The following symbols are also found in the literature that formalizes contingency: 'E'_2' [Becker-Freyseng, 1933], '$\mathcal{T}$' [Łukasiewicz, 1957] and 'P()' [Smith, 1989]. Smith's P(Aab) is McCall's QAab, and Smith's PAab is McCall's MAab. Łukasiewicz used '$\mathcal{T}$' instead of 'Q' since earlier in his book he used 'Q' for 'is equivalent to'. McCall's Barbara LQM is Ross's [1949] $A^n A^c A^p$. Montgomery and Routley use ∇ for contingency in [1966] and [1968]. And Cresswell uses ∇ for contingency and $\triangle$ for non-contingency in [1988].

Thom makes the following remarks about contingency at the beginning of his article (p. 91):

> Aristotle's contingency syllogistic deals with the logic of derivations involving propositions that contain an expressed mode of contingency. The contingent is defined at I. 13, 32^a18-20, as that which is not necessary, but which being supposed does not result in anything impossible, i.e. as two-sided possibility.

Fitting Prior's remarks, the two sides of contingency (Q) are necessity and impossibility. The one side of possibility (M) is impossibility.

McCall in [1963] diminishes and extends the L-X-M calculus, formulating the Q-L-X-M calculus. We give the basis for it.

Primitive symbols

Use the primitive symbols for L-X-M together with

monadic operator Q

Formation rules

Use the formation rules for L-X-M, amending FR2' as follows.

FR2' If p is a categorical expression then $\neg p$ is a *categorical expression* and Lp and Qp are *wffs*.

Assertion axioms

Use A0-A4 from system $\mathrm{L}A$ and A5-A14 from system L-X-M. So A2 is IAA. Add the following axioms.

A15 (Barbara QQQ)	$\vdash$(QAbc →(QAab →QAac))
A16 (Darii QQQ)	$\vdash$(QAbc →(QIab →QIac))
A17 (QXQ-AAE, figure 1)	$\vdash$(QAbc →(Aab →QEac))
A18 (Darii QXQ)	$\vdash$(QAbc →(Iab →QIac))
A19 (Barbara XQM)	$\vdash$(Abc →(QAab →MAac))
A20 (Celarent XQM)	$\vdash$(Ebc →(QAab →MEac))
A21 (Ferio XQM)	$\vdash$(Ebc →(QIab →MOac))
A22 (complementary conversion, QE-QA)	$\vdash$(QEab →QAab)
A23 (complementary conversion, QI-QO)	$\vdash$(QIab →QOab)
A24 (complementary conversion, QO-QI)	$\vdash$(QOab →QIab)
A25 (QI conversion)	$\vdash$(QIab →QIba)
A26 (QE-ME subordination)	$\vdash$(QEab →MEab)
A27 (QI-MI subordination)	$\vdash$(QIab →MIab)

A28 (QO-MO subordination) $\vdash(QOab \to MOab)$

Assertion transformation rules

Use the assertion tranformation rules for L-X-M.

On p. 76 of [1963] McCall gives the following reason for changing A2 from $LIaa$ to Iaa.

> If we retain the axiom $LIaa$, we may, by means of the substitution $CKQAacLIaaIac$ $[((QAac \wedge LIaa) \to Iac))]$ of Darii QLX (proved below), derive the implication $CQAacIac$ $[(QAac \to Iac)]$, which is un-Aristotelian.

We shall present this reasoning systematically.

Proof.

1. $\vdash(Ebc \to (QAab \to MEac))$ (by A20)
2. $\vdash(QAab \to (Ebc \to MEac))$ (from 1 by AI)
3. $\vdash(QAab \to (LIac \to Ibc))$ (from 2 by RV and SE)
4. $\vdash(QAab \to (LIaa \to Iba))$ (from 2 by US)
5. $\vdash LIaa$ (by A2 for L-X-M)
6. $\vdash(QAab \to Iba)$ (from 4 and 5 by AI and MP)
7. $\vdash(QAab \to Iab)$ (from 6 by CW, given Con)
8. $\vdash(QAac \to Iac)$ (from 7 by US)   ∎

McCall devised his system Q-L-X-M so that it has this feature: $(QEab \to QEba)$ is not accepted. He wishes to reflect Aristotle's view that universally negative contingent propositions are not convertible.[25] McCall puts Aristotle's argument for the non-convertibility of such propositions as follows:

> ... in 36b35-37a3, Aristotle gives what is in essence the following argument. We know that $QAab$ implies $QEab$, and that $QEba$ implies $QAba$ [by complementary conversion]. Therefore if $QEab$ implied $QEba$, $QAab$ would imply $QAba$, which it does not. Hence $QEab$ is not convertible.

But, unfortunately, McCall's Q-L-X-M system is too strong. It forces us, for example, to accept $(QAbc \to (LAab \to LAde))$, which is clearly unAristotelian. It does not satisfy the chain condition mentioned above. After showing this, we shall lay out a system that is semantically consistent and maximizes Aristotelicity.

---

[25]On p. 198 of [1957] Łukasiewicz calls Aristotle's view a 'grave mistake'. Łukasiewicz says 'He [Aristotle] does not draw the right consequences from his definition of contingency, and denies the convertibility of universally-negative contingent propositions, though it is obviously admissible.' But, following McCall, one can attempt to formulate Aristotle's contingency syllogistic without, in effect, defining $QEab$ as $(\neg LEab \wedge \neg L\neg Eab)$.

7.1   Overlooked acceptances in the Q-L-X-M system

McCall claims that Barbara QLX is not a thesis in his Q-L-X-M system. See table 13 on p. 92 of [1963]. But this result is a corollary of the following theorem.

Theorem 7.1. $\vdash$(Q Abc $\to$(LAab $\to x$)), where x is any wff.

Proof.

1. $\vdash (Eca \to (QAbc \to MEba))$ (by A20 and US)
2. $\vdash (LAab \to LIba)$ (by Ap-sub-a)
3. $\vdash (MEba \to MOab)$ (from 2 by RV and SE)
4. $\vdash (Eca \to (QAbc \to MOab))$ (from 1 and 3 by CW)
5. $\vdash (QAbc \to (LAab \to Ica))$ (from 4 by AI, RV and SE)
6. $\vdash (LAab \to (Ica \to LIcb))$ (A7 and US)
7. $\vdash ((QAbc \to (LAab \to Ica)) \to ((LAab \to (Ica \to LIcb)) \to (QAbc \to (LAab \to LIcb))))$ (by AO)
8. $\vdash (QAbc \to (LAab \to LIcb))$ (from 5, 6 and 7 by MP)
9. $\vdash (QAbc \to QEbc)$ (by CC and US)
10. $\vdash (QEbc \to MEbc)$ (by A26 and US)
11. $\vdash (MEbc \to MEcb)$ (by Ap-con and US)
12. $\vdash (QAbc \to MEcb)$ (from 9, 10 and 11 by CW)
13. $\vdash (QAbc \to \neg LIcb)$ (from 12 by SE)
14. $\vdash ((QAbc \to (LAab \to LIcb)) \to ((QAbc \to \neg LIcb) \to (QAbc \to (LAab \to x))))$ (by A0)
15. $\vdash$(QAbc $\to$(LAab $\to x$)) (from 8, 13, and 14 by MP)   ∎

The following theorem provides additional evidence that McCall's Q-L-X-M system is too strong to be Aristotelian.

Theorem 7.2. $\vdash$(LAbc $\to$(QAab $\to x$)), where x is any sentence.

Proof.

1. $\vdash (Eac \to (QIba \to MObc))$ (by A21 and US)
2. $\vdash (QAab \to QIba)$ (by A18, US, A2, MP)
3. $\vdash (Eac \to (QAab \to MObc))$ (from 1 and 2 by AS)
4. $\vdash (LAbc \to (QAab \to Iac))$ (from 3 by RV and SE)
5. $\vdash (Iac \to Ica)$ (by Con)
6. $\vdash (LAbc \to (QAab \to Ica))$ (from 4 and 5 by CW)
7. $\vdash (QAab \to (Ica \to QIcb))$ (by A18)
8. $\vdash ((LAbc \to (QAab \to Ica)) \to ((QAab \to (Ica \to QIcb)) \to (LAbc \to (QAab \to QIcb))))$ (by AO)
9. $\vdash (LAbc \to (QAab \to QIcb))$ (from 6, 7 and 8 by MP)
10. $\vdash (QIcb \to QIbc)$ (by A25 and US)
11. $\vdash (QIbc \to QObc)$ (by A23 and US)

12. $\vdash (QObc \to MObc)$ (by A28 and US)

13. $\vdash (LAbc \to (QAab \to MObc))$ (from 9, 10, 11 and 12 by MP)

14. $\vdash (LAbc \to (QAab \to \neg LAbc))$ (from 13 by SE)

15. $\vdash ((LAbc \to (QAab \to \neg LAbc)) \to (LAbc \to (QAab \to x)))$ (by A0)

16. $\vdash (LAbc \to (QAab \to x))$ (from 14 and 15 by MP)   ∎

According to McCall's table 13 on p. 92 of [McCall, 1963], sentences representing Barbara QLX, Barbara LQX, Barbara LQQ, Baroco QXM and Bocardo XQM are not accepted in the Q-L-X-M system, though they correspond to inferences that Aristotle considered to be valid. But it is an immediate consequence of theorems 7.1 and 7.2 that the first three sentences are accepted. That the last two are accepted may be seen as follows:

1. $\vdash (LAbc \to (QAab \to Aac))$ (by theorem 7.2)

2. $\vdash (Oac \to (QAab \to MObc))$ (from 1 by RV and SE)

3. $\vdash (Obc \to (QAba \to MOac))$ (Bocardo XQM, from 2 by US)

4. $\vdash (QAbc \to (LAab \to Aac))$ (by theorem 7.1)

5. $\vdash (QAbc \to (Oac \to MOab))$ (from 4 by RV and SE)

6. $\vdash (QAcb \to (Oab \to MOac))$ (Baroco QXM, from 5 by US)

So McCall's claim on p. 93 of [1963] that Q-L-X-M has 85% Aristotelicity needs to be modified. Instead of 24 "non-Aristotelian moods" out of 154 moods marked on his table 13, there are 29 out of 154. So the Aristotelicity of the Q-L-X-M system is about 81%.

When determining the Aristotelicity of a system, McCall only uses figures 1, 2 and 3 and none of the "subaltern moods" such as Barbari. Given theorems 7.1 and 7.2, the following wffs are accepted in Q-L-X-M, though they are not marked as accepted on McCall's table 13: Bramantip QLQ, Camenes LQQ, Fesapo QLQ and Barbari LQQ.

In the following section we modify Q-L-X-M so that the resulting system, QLXM', does not have the unAristotelian features that result from theorems 7.1 and 7.2. Given the data – that Aristotle regarded Barbara LQM as invalid and Bocardo QLM as valid, for example – it is a virtue of the modified system that it does not have 100% Aristotelicity. Note that if $\vdash (QObc \to (LAba \to MOac))$ (Bocardo QLM) then $\vdash (LAac \to (QObc \to MOba))$ (Baroco LQM) by Reversal. In system QLXM' both Barbara LQM and Bocardo QLM are invalid. In contrast, in system Q-L-X-M both are valid.

8   QLXM'

To ensure that theorems 7.2 and 7.1 may not be proven in system QLXM' we exclude axioms A20 (Celarent XQM) and A21 (Ferio XQM). This decision is not difficult to make since, as McCall points out, Aristotle's proofs of Celarent XQM and Ferio XQM are flawed. McCall shows that one who endorses such reasoning, thinking that "what is impossible cannot follow from what is merely false, but not impossible", is committed to the absurd consequence that 'Some B are A; all C are A; so some C are A' is valid.

Only one other Q-L-X-M axiom is excluded to form QLXM': delete axiom A28, $\vdash(QOab \rightarrow MOab)$. This decision is a result of semantic considerations. For A28 to be validity preserving $QOab$ and $LAab$ must be semantically inconsistent. Since $LAab$ is true iff $^+(a) \subseteq n^+(b)$, we could make $QOab$ and $LAab$ contraries by fixing the semantics so that $QOab$ is true iff $^+(a) \circ q(b)$. But then we are forced to say that Bocardo QLQ, for example, is valid even though Aristotle considered it to be invalid. (Suppose that $V_{\mathcal{M}}(QOab) = t$ and $V_{\mathcal{M}}(LAac) = t$. Then $^+(a) \circ q(b)$ and $^+(a) \subseteq n^+(c)$. Then $n^+(c) \circ q(b)$. Then $^+(c) \circ q(b)$. Then $V_{\mathcal{M}}(QOcb) = t$.) Note, also, that if QOab is true iff $^+(a) \circ q(b)$ then we would want to ensure that $QIab$ is true iff $^+(a) \circ q(b)$ to guarantee the soundness of the complementary conversion principles that Aristotle clearly supported. But then we would be forced to say that Disamis QLQ is valid even though Aristotle considered it to be invalid. (Suppose that $V_{\mathcal{M}}(QIbc) = t$ and $V_{\mathcal{M}}(LAba) = t$. Then $^+(b) \circ q(c)$ and $^+(b) \subseteq n^+(a)$. Then $n^+(a) \circ q(c)$. Then $^+(a) \circ q(c)$. Then $V_{\mathcal{M}}(QIac) = t$.) Similar remarks may be made about Disamis QXQ, Datisi LQQ and Datisi XQQ.

Rather than fixing the conditions for the truth of $QOab$ as indicated above we may let $QOab$ be true iff either $QIab$ or $QIba$ is true.[26] To make A28 truth preserving we must ensure that if $LAab$ is true then both $QIab$ and $QIba$ are false. Such a position does not fit the sorts of examples Aristotle uses. Suppose, for example, that all things that are sleeping are necessarily men. It does not follow that it is not true that some men are contingently sleeping.

We avoid the above difficulties by deleting axiom A28 when defining QLXM'.

In this system, as in Q-L-X-M, there are no rejection axioms and no rejection rules.

Before giving a semantics for QLXM' we shall establish some immediate inferences that are conversions, subalternations or subordinations. With them we shall show the acceptance of various two-premised syllogisms indicated on table 15 by leaving a cell unmarked. After the semantics is given we shall show that sentences corresponding to the other cells, those in which numerals occur, are invalid. An occurrence of the "hat sign" in a cell in the table means the entry conflicts with Aristotle's judgments about validity as recorded on McCall's authoritative table 12 of [McCall, 1963].[27]

Theorem 8.1. (Ordinary Q-conversions, Q-con) i) $\vdash(QIab \rightarrow QIba)$; and ii) $\vdash(QOab \rightarrow QOba)$.

Proof. i) is A25. For ii) use A23, A24 and CW. ■

Theorem 8.2. (Contingency subalternations, Q-sub-a) i) $\vdash(QAab \rightarrow QIab)$; ii) $(QAab \rightarrow QOab)$; iii) $\vdash(QEab \rightarrow QIab)$; and iv) $\vdash(QEab \rightarrow QOab)$.

Proof. For i) use A18, AI, A2 and MP. For ii) use i), A23 and CW. For iii) use i), A22 and AS. For iv) use ii), A22 and AS. ■

---

[26]Thom evaluates $QOab$ in this way in [1993] and [1994].

[27]In the notes for table 12 McCall comments on tables in [Becker-Freyseng, 1933, p. 88] and [Ross, 1949, after p. 286].

Table 14. McCall's Table 12 and RV inconsistencies

	Q	Q	X	Q	L	Q	X	Q	L	Q	L
	Q	X	Q	L	Q	X	Q	L	Q	L	Q
	Q	Q	Q	Q	Q	M	M	X	X	M	M
Barbara	V	V		V		V^1	V			V^{13}	V
Celarent	V	V		V		V^2	V^7		V	V^{14}	V
Darii	V	V		V		V	V^8	4	12	V	V
Ferio	V	V		V		V^3	V^9		V	V^{15}	V
Cesare							V^{10}	3	V	15	V
Camestres						V^4		V	6	V	17
Festino							V^{11}	2	V	14	V
Baroco								1	5	13	16
Darapti	V	V	V			V	V			V	V
Felapton	V	V	V			V	V		V	V	V
Disamis	V		V		V	V	V	11	7	V	V
Datisi	V	V	V			V	V	10	9	V	V
Bocardo	V					V^5			V	V^{16}	V
Ferison	V	V	V			V^6	V^{12}		8	V^{17}	

McCall follows Ross's use of "complementary conversion" to refer to A22 to A24. On p. 298 of [Ross, 1949] Ross, in his discussion of 35a29-b1, identifies the following entailments, endorsed by Aristotle, as "complementary conversions":

'For all B, being A is contingent' [QAba] entails 'For all B, not being A is contingent' [QEba] and 'For some B, not being A is contingent' [QOba]. 'For all B, not being A is contingent' [QEba] entails 'For all B, being A is contingent' [QAba] and 'For some B, being A is contingent' [QIba]. 'For some B, being A is contingent' [QIba] entails 'For some B, not being A is contingent' [QOba]. 'For some B, not being A is contingent' [QOba] entails 'For some B, being A is contingent' [QIba].

Given the following theorem and US, Ross's six complementary conversions are asserted in QLXM'.

Theorem 8.3. (Complementary conversion, CC) i) $\vdash (QAab \to QEab)$; ii) $\vdash (QAab \to QOab)$; iii) $\vdash (QEab \to QAab)$; iv) $\vdash (QEab \to QIab)$; v) $\vdash (QIab \to QOab)$; and vi) $\vdash (QOab \to QIab)$.

Proof. For i) use A17, US, AI, A1 and MP. For ii) use Q-sub-a, A23 and CW. iii) is A22. For iv) use iii), Q-sub-a and CW. v) is A23. vi) is A24. ∎

Theorem 8.4. (Complementary conversions *per accidens*, CC(pa)) i) $\vdash (QAab \to QIba)$; ii) $\vdash (QAab \to QOba)$; iii) $\vdash (QEab \to QIba)$; and iv) $\vdash (QEab \to QOba)$.

Table 15. QLXM′ countermodels

	Q	Q	X/L	Q	X	Q	L	Q	L
	Q	X/L	Q	X	Q	L	Q	L	Q
	Q	Q	Q	M	M	X	X	M	M
Barbara			5ac			7ab	8ac		
Celarent			6ac		$\widehat{7ac}$				
Darii			5ac			7ab	8ac		
Ferio			6ac		$\widehat{7ac}$	5cb			
Cesare	7ac	9ca	6ac	5ca	$\widehat{7ac}$		$\frown$		$\frown$
Camestres	7ac	6ca	9ac	$\widehat{7ac}$	5ac		$\frown$		$\frown$
Festino	7ac	9ca	6ac		$\widehat{7ac}$	$\frown$		$\frown$	
Baroco	7ac	6ca	9ac	7ac	11bc		11bc	$\frown$	11bc
Darapti						7cb	7bc		
Felapton			9bc		$\widehat{8bc}$	1bc			
Disamis		5ca				7cb	7bc		
Datisi		5ca				7cb	7bc		
Bocardo		5ca	9bc	$\widehat{11ac}$	8bc	11ac		$\widehat{11ac}$	
Ferison			9bc		$\widehat{8bc}$	5cb	$\frown$		
Bramantip		5ca				8ca	7ba		
Camenes	10ac	6ca	7bc	7ac					
Dimaris		5ca				8ca	7ba		
Fresison	7ac	5ca	6bc		8bc				
Fesapo		5ca	6bc		8bc				
Barbari			5ac			7ab	8ac		
Celaront			6ac		7ac				
Cesaro	7ac	9ca	6ac		7ac				
Camestrop	7ac	6ca	9ac	7ac	1ab				
Camenop		6ca		7ac					

Proof. For i) use Q-sub-a, Q-con, US and CW. For ii) use i), A23, US and CW. For iii) use i), A22 and AS. For iv) use iii), A23, US and CW. ∎

Theorem 8.5. (Contingency subordinations, Q-sub-o) i) $\vdash (QAab \;\rightarrow\; MAab)$; ii) $\vdash (QEab \rightarrow MEab)$; and iii) $\vdash (QIab \rightarrow MIab)$.

Proof. For i) use A19, A1, US and MP. ii) is A26. iii) is A27. ∎

Uses of AS or CW in proofs of the following theorems involve only those immediate inferences that have been proven above. So, for example, in the proof that Celarent QQQ is asserted AS is used with Q-sub-a and US ($\vdash (QEbc \rightarrow QAbc)$) and CW is used with Q-sub-a and US ($\vdash (QAac \rightarrow QEac)$).

Theorem 8.6. (asserted QQQs) The non-numbered QQQ cells on table 15 correspond to asserted sentences.

Proof.

1. ⊢(QAbc →(QAab →QAac)) (Barbara QQQ, by A17)
2. ⊢(QEbc →(QAab →QEac)) (Celarent QQQ, from 1 by AS, US, CW)
3. ⊢(QAbc →(QIab →QIac)) (Darii QQQ, by A16)
4. ⊢(QEbc →(QIab →QOac)) (Ferio QQQ, from 3 by AS, US, CW)
5. ⊢(QAbc →(QAba →QIac)) (Darapti QQQ, from 3 by AS, US)
6. ⊢(QEbc →(QAba →QOac)) (Felapton QQQ, from 5 by AS, US, CW)
7. ⊢(QAbc →(QIba →QIac)) (Datisi QQQ, from 3 by AS, US)
8. ⊢(QEbc →(QIba →QOac)) (Ferison QQQ, from 7 by AS, US, CW)
9. ⊢(QIbc →(QAba →QIac)) (Disamis QQQ, from 7 by AI, CW, US)
10. ⊢(QObc →(QAba →QOac)) (Bocardo QQQ, from 9 by AS, US, CW)
11. ⊢(QIcb →(QAba →QIac)) (Dimaris QQQ, from 9 by AS, US)
12. ⊢(QAcb →(QAba →QIac)) (Bramantip QQQ, from 11 by AS, US)
13. ⊢(QEcb →(QAba →QOac)) (Fesapo QQQ, from 12 by AS, US, CW)
14. ⊢(QAbc →(QAab →QAac)) (Barbari QQQ, from 1 by CW, US)
15. ⊢(QEbc →(QAab →QEac)) (Celaront QQQ, from 2 by CW, US)
16. ⊢(QAcb →(QEba →QOac)) (Camenop QQQ, from 12 by AS, US, CW)   ∎

Theorem 8.7. (asserted QXQs and XQQs) The non-numbered QXQ and XQQ cells on table 15 correspond to asserted sentences.

Proof.

1. ⊢(QAbc →(Aab →QAac)) (Barbara QXQ, by A17, US, CW)
2. ⊢(QEbc →(Aab →QEac)) (Celarent QXQ, from 1 by AS, US, CW)
3. ⊢(QAbc →(Iab →QIac)) (Darii QXQ, by A18)
4. ⊢(QEbc →(Iab →QOac)) (Ferio QXQ, from 3 by AS, US, CW)
5. ⊢(QAbc →(Iba →QIac)) (Datisi QXQ, from 3 by AS, US)
6. ⊢(QEbc →(Iba →QOac)) (Ferison QXQ, from 5 by AS, US, CW)
7. ⊢(QAbc →(Aba →QIac)) (Darapti QXQ, from 5 by AS, US)
8. ⊢(QEbc →(Aba →QOac)) (Felapton QXQ, from 7 by AS, US, CW)
9. ⊢(QAbc →(Aab →QIac)) (Barbari QXQ, from 1 by US, CW)
10. ⊢(QEbc →(Aab →QOac)) (Celaront QXQ, from 2 by US, CW)
11. ⊢(Ibc →(QAba →QIac)) (Disamis XQQ, from 5 by AI, CW, US)
12. ⊢(Abc →(QAba →QIac)) (Darapti XQQ, from 11 by AS)
13. ⊢(Icb →(QAba →QIac)) (Dimaris XQQ, from 11 by AS)
14. ⊢(Acb →(QAba →QIac)) (Bramantip XQQ, from 13 by AS)
15. ⊢(Acb →(QEba →QOca)) (Camenop XQQ, from 14 by AS, CW)   ∎

Theorem 8.8. (asserted QLQs and LQQs) The non-numbered QLQ and LQQ cells on table 15 correspond to asserted sentences.

Proof. Use theorem 8.7 and Sub-o.   ∎

So, for example, $\vdash$(QAbc →(LAab →QAac)) since $\vdash$(QAbc →(Aab →QAac)) by the-
orem 8.7 and since $\vdash$(LAab →Aab) by Sub-o.

Theorem 8.9. (asserted QXMs and XQMs) The non-numbered QXM and XQM cells on
table 15 correspond to asserted sentences.

Proof. For non-numbered QXM and XQM cells referred to by names that do not end
with 'o' use theorem 8.7 wherever possible with Q-sub-o and CW. So Barbara QXM
is asserted since Barbara QXQ is assserted. And, by this reasoning, Celarent QXM,
Darii QXM, Darapti QXM, Datisi QXM, Barbari QXM, Darapti XQM, Disamis XQM,
Bramantip XQM, Camenes XQM, Dimaris XQM and Barbari XQM. For the remaining
non-numbered cells use asserted MXM syllogisms from table 13 wherever possibile with
Q-sub-o and AS. So, Ferio QXM is accepted since Ferio MXM is accepted. And, by
this reasoning, Festino QXM, Felapton QXM, Disamis QXM, Ferison QXM, Bramantip
QXM, Dimaris QXM, Fresison QXM, Fesapo QXM, Celaront QXM, Cesaro QXM, Darii
XQM, Datisi XQM and Barbari XQM. The only remaining non-numbered QXM and
XQM cells correspond to the axiom Barbara XQM (A21) and Camenop XQM, which is
deduced from Camenes XQM by CW given Ap-sub-a.   ∎

Theorem 8.10. (asserted QLXs and LQXs) The non-numbered QLX and LQX cells on
table 15 correspond to asserted sentences.

In the following proof the asterisks mark inconsistencies in the data as reported on
McCall's table 12 on pp. 84-85 of [1963].

Proof. Use theorem 8.9 with RV and SE. So i) Celarent QLX is asserted since Festino
QXM is asserted; ii) Celaront QLX (is asserted) since Cesaro QXM (is asserted); iii)* Ce-
sare QLX since Ferio QXM; iv) Camestres QLX since Darii QXM; v)* Festino QLX since
Celarent QXM; vi)* Baroco QLX since Barbara QXM; vii) Cesaro QLX since Celaront
QXM; viii) Camestrop QLX since Barbari QXM; ix) Camenes QLX since Dimaris XQM;
x) Fresison QLX since Camenes XQM; xi) Fesapo QLX since Camenop XQM; xii) Ca-
menop QLX since Bramantip XQM; xiii) Celarent LQX since Disamis XQM; xiv) Ferio
LQX since Datisi XQM; xv) Celaront LQX since Barbari XQM; xvi) Cesare LQX since
Datisi QXM; xvii)* Camestres LQX since Ferison QXM; xviii) Festino LQX since Dis-
amis QXM; xix) Cesaro LQX since Darapti QXM; xx) Camestrop LQX since Felapton
QXM; xxi) Felapton LQX since Barbari XQM; xxii) Bocardo LQX since Barbara XQM;
xxiii)* Ferison LQX since Darii XQM; xxiv) Camenes LQX since Fresison QXM; xxv)
Fresison LQX since Dimaris QXM; xxvi) Fesapo LQX since Bramantip QXM; and xxvii)
Camenop LQX since Fesapo QXM.   ∎

Theorem 8.11. The non-numbered QLM and LQM cells on table 15 correspond to as-
serted sentences.

Proof. For the QLMs use: i) results for the accepted QXM syllogisms stated in theorem
8.9, Sub-o and AS; or ii) results for the accepted QLX syllogisms stated in theorem 8.10,
Sub-o and CW. So, for example, $\vdash (QAbc \to (LAab \to MAac))$ (Barbara QLM is as-
serted) since $\vdash (QAbc \to (Aab \to MAac))$ and $\vdash (LAab \to Aab)$ given AS. $\vdash (QEcb \to$

$(LAab \rightarrow MEac))$ (Cesare QLM is asserted) since $\vdash(QEcb \rightarrow (LAab \rightarrow Eac))$ and
$\vdash(Eac \rightarrow MEac)$.

For the LQMs use results for the XQMs in theorem 8.9 or the LQXs in theorem 8.10
together with Sub-o, AS or CW. So, for example, $\vdash(LAbc \rightarrow (QIab \rightarrow MIac))$ (Darii
LQM is asserted) since $\vdash(LAbc \rightarrow (QIab \rightarrow Iac))$ and $\vdash(Iac \rightarrow MIac)$. $\vdash(LEbc \rightarrow$
$(QIab \rightarrow MOac))$ (Ferio LQM is asserted) since $\vdash(LEbc \rightarrow (QIab \rightarrow Oac))$ and
$\vdash(Oac \rightarrow MOac))$.   ∎

Theorem 8.12. The non-numbered QQMs on table 15 correspond to asserted sentences.

Proof. Obtain the assertion of Barbara QQM from the assertion of Barbara QQQ by using
CW with Q-sub-o. Use similar reasoning for Celarent, Darii, Barbari, Darapti, Disamis,
Datisi, Bramantip and Dimaris. We generate the remaining four QQMs as follows.

1. $\vdash(QEbc \rightarrow (QAab \rightarrow QEac))$ (Celarent QQQ)
2. $\vdash(QEac \rightarrow MEac)$ (by Q-sub-o)
3. $\vdash(MEac \rightarrow MOac)$ (by Ap-sub-a)
4. $\vdash(QEac \rightarrow MOac)$ (from 2 and 3 by CW)
5. $\vdash(QEbc \rightarrow (QAab \rightarrow MOac))$ (Celaront QQM, from 1 and 4 by CW)
6. $\vdash(QAab \rightarrow (QEbc \rightarrow QEac))$ (from 1 by AI)
7. $\vdash(MEac \rightarrow MEca)$ (by Ap-con)
8. $\vdash(QEac \rightarrow MEca)$ (from 2 and 7 by CW)
9. $\vdash(QAab \rightarrow (QEbc \rightarrow MEca))$ (from 6 and 8 by CW)
10. $\vdash(QAcb \rightarrow (QEba \rightarrow MEac))$ (Camenes QQM, from 9 by US)
11. $\vdash(MEac \rightarrow MOac)$ (Ap-sub-a and US)
12. $\vdash(QAcb \rightarrow (QEba \rightarrow MOac))$ (Camenop QQM, from 10 and 11 by CW)
13. $\vdash(QEcb \rightarrow (QAba \rightarrow MOac))$ (Fesapo QQM, from 12 by CC and AS)   ∎

8.1   Semantics for QLXM′

The semantics for QLXM′ is given by referring to Q-models.

Definition 8.13. (Q-model) $\mathcal{M}$ is a *Q-model* iff $\mathcal{M} = \langle W, n^+, q^+, n^-, q^- \rangle$, where W is
a non-empty set and n^+, q^+, n^-, and q^- are functions that map terms into subsets of W
and satisfy the following "base conditions":

BQ1 If f and g are any of the functions n^+, q^+, q^- or n^- and $f \neq g$, then, for every
term x, $f(x) \cap g(x) = \emptyset$; and for every x, $n^+(x) \cup q^+(x) \cup q^-(x) \cup n^-(x) = W$

BQ2 (For every x and y) if $^+(x) \subseteq n^-(y)$ then $^+(y) \subseteq n^-(z)$

BQ3 If $^+(y) \subseteq n^+(z)$ and $^+(x) \circ {}^+(y)$ then $n^+(x) \circ n^+(z)$

BQ4 If $^+(y) \subseteq n^-(z)$ and $^+(x) \circ {}^+(y)$ then $n^+(x) \circ n^-(z)$

BQ5 If $^+(z) \subseteq n^+(y)$ and $n^+(x) \circ n^-(y)$ then $n^+(x) \circ n^-(z)$

BQ6 If $^+(y) \subseteq q(z)$ and $^+(x) \subseteq q(y)$ then $^+(x) \subseteq q(z)$

BQ7 If $^+(y) \subseteq q(z)$ and $^+(x) \circ q(y)$ or $q(x) \circ {}^+(y)$ then $^+(x) \circ q(z)$ or $q(z) \circ {}^+(x)$

BQ8 If $^+(y) \subseteq +(z)$ and $^+(x) \subseteq q(y)$ then $n^+(x)$ does not overlap $n^-(z)$

Definition 8.14. (valuation relative to a Q-model) $V_{\mathcal{M}}$ is a *valuation relative to a Q-model* $\mathcal{M}$ iff it is is a valuation that satisfies the following "superstructural conditions":

S1 (For every x and y) $V_{\mathcal{M}}(Axy) = t$ iff $^+(x) \subseteq {}^+(y)$
S2 $V_{\mathcal{M}}(Ixy) = t$ iff $^+(x) \circ {}^+(y)$.
S3 $V_{\mathcal{M}}(LAxy) = t$ iff $^+(x) \subseteq n^+(y)$
S4 $V_{\mathcal{M}}(LIxy) = t$ iff $n^+(x) \circ n^+(y)$
S5 $V_{\mathcal{M}}(L\neg Axy) = t$ iff $n^+(x) \circ n^-(y)$
S6 $V_{\mathcal{M}}(L\neg Ixy) = t$ iff $^+(x) \subseteq n^-(y)$
S7 $V_{\mathcal{M}}(QAxy) = t$ iff $^+(x) \subseteq q(y)$
S8 $V_{\mathcal{M}}(QIxy) = t$ iff $^+(x) \circ q(y)$ or $q(x) \circ {}^+(y)$
S9 $V_{\mathcal{M}}(Q\neg Axy) = t$ iff $^+(x) \circ q(y)$ or $q(x) \circ {}^+(y)$
S10 $V_{\mathcal{M}}(Q\neg Ixy) = t$ iff $^+(x) \subseteq q(y)$

Definition 8.15. (Q-valid) $\models_Q \alpha$ (α is *Q-valid*) iff, for every Q-model $\mathcal{M}$, $V_{\mathcal{M}}(\alpha) = t$. α is *Q-invalid* ($\not\models_Q \alpha$) iff α is not Q-valid.

Theorem 8.16. (soundness) If α is an assertion in QLXM' then $\models_Q \alpha$.

Proof. We need to show that i) if $\vdash \alpha$ is an axiom of QLXM' then $\models_Q \alpha$; and ii) each assertion transformation rule of QLXM' preserves Q-validity. Some examples of the reasoning needed are given. For A1, $\models_Q Aaa$ since, for every Q-model $\mathcal{M}$, $V_{\mathcal{M}}(Aaa) = t$ since $^+(a) \subseteq {}^+(a)$. For A2, $\models_Q Iaa$ since, for every Q-model $\mathcal{M}$, $V_{\mathcal{M}}(Iaa) = t$ since $^+(a) \circ {}^+(a)$. For A5, suppose there is a Q-model $\mathcal{M}$ such that $V_{\mathcal{M}}(LAbc) = t$, $V_{\mathcal{M}}(Aab) = t$ and $V_{\mathcal{M}}(LAac) = f$. Then $^+(b) \subseteq n^+(c)$, $^+(a) \subseteq {}^+(b)$ and $^+(a) \not\subseteq n^+(c)$, which is impossible. So $\models_Q$ $(LAbc \rightarrow (Aab \rightarrow LAac))$. For A15, suppose there is a Q-model $\mathcal{M}$ such that $V_{\mathcal{M}}(QAbc) = t$, $V_{\mathcal{M}}(QAab) = t$ and $V_{\mathcal{M}}(QAac) = f$. Then $^+(b) \subseteq q(c)$, $^+(a) \subseteq q(b)$ and $^+(a) \not\subseteq q(c)$, which is impossible given **BQ6**. So $\models_Q$ $(QAbc \rightarrow (QAab \rightarrow QAac))$. For AR1 suppose $\models_Q$ $(\ldots x \ldots x \ldots)$ but $\not\models_Q$ $(\ldots y \ldots y \ldots)$, where $(\ldots y \ldots y \ldots)$ is the result of replacing every occurrence of term x in $(\ldots x \ldots x \ldots)$ with term y. Then, for some Q-model $\mathcal{M}$, $V_{\mathcal{M}}(\ldots y \ldots y \ldots) = f$, where $V_{\mathcal{M}}(y)$ is set S. Let $V_{\mathcal{M}}(x) = S$. Then $V_{\mathcal{M}}(\ldots x \ldots x \ldots) = f$. So it is impossible for AR1 not to preserve validity. For AR2 suppose a) $\models_Q (p \rightarrow q)$, b) $\models_Q p$ and c) $\not\models_Q q$. Then, for some Q-model $\mathcal{M}$, $V_{\mathcal{M}}(q) = f$, given c). Then, by a), $V_{\mathcal{M}}(p) = f$, which conflicts with b). So AR2 preserves validity. Reasoning for the other axioms and rules is straightforward and is omitted. ∎

Given the soundness of QLXM' every asserted sentence in Łukasiewicz's ŁA is Q-valid since ŁA is a fragment of QLXM'. All of the L-X-M syllogisms marked as asserted on table 5 are Q-valid since all of them are asserted in QLXM'. And, given the following theorem, all of the syllogisms marked as invalid on table 5 are Q-invalid.

Theorem 8.17. Models $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_4$ are Q-models.

Proof. By earlier arguments the four models satisfy conditions **BQ1** to **BQ5**. Consider $\mathcal{M}_1$. Suppose $^+(y) \subseteq q(z)$. Then $z = c$. So **BQ6** is trivially satisfied. For all x, $^+(x) \circ q(c)$. So **BQ7** is satisfied. For all z, $n^-(z) = \emptyset$. So **BQ8** is satisfied. Consider $\mathcal{M}_2$. Suppose $^+(y) \subseteq q(z)$. Then $y = a$ or $y = b$, and $z = c$. So **BQ6** is trivially satisfied. For all x, $^+(x) \circ q(c)$. So **BQ7** is satisfied. For all z, if $^+(c) \subseteq^+ (z)$ then $z = c$. Since $n^-(c) = \emptyset$, **BQ8** is satisfied. Consider $\mathcal{M}_3$. Suppose $^+(y) \subseteq q(z)$. Then $y = b$ or $y = c$, and $z = a$. So **BQ6** is trivially satisfied. For all x, $^+(a) \circ q(x)$. So **BQ7** is satisfied. For all z, if $^+(a) \subseteq^+ (z)$ then $z = a$. Since $n^-(a) = \emptyset$, **BQ8** is satisfied. Consider $\mathcal{M}_4$. For all y and z, if $^+(y) \not\subseteq q(z)$. So **BQ6, BQ7** and **BQ8** are trivially satisfied. ∎

Table 16. Q-model $\mathcal{M}_5$

	n^+	q^+	n^-	q^-
a	1	2	3	
b	3	1		2
c	1,3		2	

Table 16 expresses a model. **BQ1** and **BQ2**, here and below, require no comment. For every y and z, if $+(y) \subseteq n^+(z)$ then $z = c$. For every x, $n^+(x) \circ n^+(c)$. So **BQ3** is satisfied. For every y and z, $^+(y) \not\subseteq n^-(z)$. So **BQ4** is trivially satisfied. If $^+(y) \subseteq n^+(z)$ then $z = c$. For every x, y and z, if $z \subseteq n^+(y)$ then $n^+(x)$ does not overlap $n^-(y)$. So **BQ5** is satisfied. For every x and y, if $x \subseteq q(y)$ then $x = a$ and $y = b$. So **BQ6** is trivially satisfied. For all x, $x \circ q(b)$. So **BQ7** is satisfied. For all z, $^+(a)$ does not overlap $n^-(z)$. So **BQ8** is satisfied.

Given Q-model $\mathcal{M}_5$, $\not\models_Q (LAbc \to (QAab \to LAac))$. For, $V_{\mathcal{M}_5}(LAbc) = t$ since $^+(b) \subseteq n^+(c)$. $V_{\mathcal{M}_5}(QAab) = t$ since $^+(a) \subseteq q(b)$. And $V_{\mathcal{M}_5}(QAac) = f$ since $^+(a) \not\subseteq q(c)$. The occurrence of '4ac' in the Barbara/LQQ cell indicates Q-model $\mathcal{M}_5$ is a countermodel for Barbara LQQ, where 'a' is the minor term and 'c' is the major term. This method of listing minor and major terms will be followed below.

Table 17. Model $\mathcal{M}_6$

	n^+	q^+	n^-	q^-
a	1	2	3,4	
b	3	1	4	2
c	4		1,2,3	

Table 17 expresses a model. For every y and z, if $^+(y) \subseteq n^+(z)$ then $y = z = c$. For every x and y, if $^+(x) \circ {}^+(y)$ then $x = c$. Since $n^+(c) \circ n^+(c)$, **BQ3** is satisfied. For every y, if $^+(y) \subseteq n^-(a)$ then $y = b$ or $y = c$. For every x, if $x \circ {}^+(b)$ or $x \circ {}^+(c)$ then $n^+(x) \circ n^-(a)$. For every y, if $^+(y) \subseteq n^-(b)$ then $y = b$. For every x, if $x \circ {}^+(b)$

then $n^+(x) \circ n^-(b)$. For every y, if $^+(y) \subseteq n^-(c)$ then $y = a$ or $y = b$. For every x, if $x \circ {}^+(a)$ or $x \circ {}^+(b)$ then $n^+(x) \circ n^-(c)$. So **B4** is satisfied. For every y and z, if $^+(z) \subseteq n^+(y)$ then $n^-(y) \subseteq n^-(z)$ (that is, Thomason's **BT5** is satisfied).[28] So, **BQ5** is satisfied. For every x and y, if $^+(x) \subseteq q(y)$ then $x = a$ and $q = b$. So **BQ6** is trivially satisfied. For all w, if $^+(w) \circ q(a)$ or $q(w) \circ {}^+(a)$ then $w = a$ or $w = b$. Since $^+(a) \circ q(b)$ and $^+(b) \circ q(b)$, **BQ7** is satisfied. For every z, if $^+(b) \subseteq z$ then $z = b$. Since $^+(a)$ does not overlap $n^-(b)$, **BQ8** is satisfied.

   Given Q-model $\mathcal{M}_6$, $not \models_Q (LEac \rightarrow (QAab \rightarrow QObc))$. For, $V_{\mathcal{M}_6}(LEac) = t$ since $^+(a) \subseteq n^-(c)$. $V_{\mathcal{M}_6}(QAab) = t$ since $^+(a) \subseteq q(b)$. And $V_{\mathcal{M}_6}(QObc) = f$ since $^+(b)$ does not overlap $q(c)$ and $q(b)$ does not overlap $^+(c)$. The occurrence of '6bc' in the Felapton/LQQ cell indicates that Q-model $\mathcal{M}_6$ is a countermodel for Felapton LQQ, where 'b' is the minor term and 'c' is the major term.

Table 18. Model $\mathcal{M}_7$

	n^+	q^+	n^-	q^-
a	1		3	2
b	2	3		1
c	1		3	2

Table 18 expresses a Q-model. Since Thomason's **BT3** (if $^+(x) \circ {}^+(y)$ then $^+(x) \circ n^+(y)$) is satisfied, both **BQ3** and **BQ4** are satisfied. Since **BT5** is satisfied **BQ5** is satisfied. If $^+(x) \subseteq q(y)$ then $y = b$ and either $x = a$ or $x = c$. Then **BQ6** is trivially satisfied. If $^+(z) \circ q(a)$ or $q(z) \circ {}^+(a)$ and if $^+(z) \circ q(c)$ or $q(z) \circ {}^+(c)$ then $^+(z) \circ {}^+(b)$. So **BQ7** is satisfied. If $^+(b) \subseteq {}^+(z)$ then $z = b$. Since $n^-(b) = \emptyset$, **BQ8** is satisfied.

   Use Q-model $\mathcal{M}_8$ to show that Barbari LQX and others are invalid.

Table 19. Model $\mathcal{M}_8$

	n^+	q^+	n^-	q^-
a	1	2	3	4
b	4	3		1,2
c	3,4		2	1

Table 19 expresses a Q-model. Suppose $^+(y) \subseteq n^+(z)$. Then $y = b$ or $y = c$, and $z = c$. Since $n^+(b) \circ n^+(c)$ and $n^+(c) \circ n^+(c)$, **BQ3** is satisfied. Since there is no x such that $n^+(x) \circ n^-(c)$, **BQ5** is satisfied. Since, for every x and y, $x \not\subseteq y$, **BQ4** is

---

[28] As noted above, **BQ5** is a weaker condition than **BT5**. Replacing **BQ5** with **BT5** in the definition of a Q-model, forming a Q' model, yields this highly unAristotelian result: $\models_{Q'} (QAab \rightarrow Eab)$. For, suppose that for some Q'-model $\mathcal{M}$, $V_{\mathcal{M}}(Eab) = f$. Then $V_{\mathcal{M}}(Iab) = t$. By S2 and **BT5**, $^+(a) \circ n^+(b)$. By S9, $V_{\mathcal{M}}(QAab) = f$.

trivially satisfied. Suppose $^+(y) \subseteq q(z)$. Then $y = a$ and $z = b$. Then **BQ6** is trivially satisfied. Since, for every x, $^+(x) \circ q(b)$ or $q(x) \circ {}^+(b)$, **BQ7** is satisfied. If $^+(b) \subseteq {}^+(z)$ then $z = b$ or $z = c$. Since, for all x, $n^+(x)$ does not overlap $n^-(b)$ and $n^+(x)$ does not overlap $n^-(c)$, **BQ8** is satisfied.

Table 20. Model $\mathcal{M}_9$

	n^+	q^+	n^-	q^-
a	1	2	3,4	
b	3,4	1		2
c	3		1,2,4	

Table 20 expresses a Q-model. If $^+(y) \subseteq n^+(z)$ then $y = c$ and either $z = c$ or $z = b$. If $^+(x) \circ {}^+(c)$ then $x = b$ or $x = c$. Since $n^+(x) \circ n^+(z)$, **BQ3** is satisfied. For every y, if $^+(y) \subseteq n^-(a)$ then $y = c$. For every x, if $x \circ {}^+(c)$ then $n^+(x) \circ n^-(a)$. There are no y such that $y \subseteq n^-(b)$. For every y, if $^+(y) \subseteq n^-(c)$ then $y = a$. For every x, if $x \circ {}^+(a)$ then $n^+(x) \circ n^-(c)$. So **BQ4** is satisfied. If $^+(z) \subseteq n^+(y)$ then $z = c$ and either $y = b$ or $y = c$. $n^+(x)$ does not overlap $n^-(b)$. If $n^+(x) \circ n^-(c)$ then $x = a$ or $x = b$. Since $n^+(a) \circ n^-(c)$ and $n^+(b) \circ n^-(c)$, **BQ5** is satisfied. For every x and y, if $x \subseteq q(y)$ then $x = a$ and $y = b$. So **BQ6** is trivially satisfied. For all z, if $^+(z) \circ {}^+(a)$ then $z = a$ or $z = b$. Since $^+(a) \circ q(b)$ and $^+(b) \circ q(b)$, **BQ7** is satisfied. For all z, if $^+(b) \subseteq {}^+(z)$ then $b = z$. Since $n^-(z) = \emptyset$, **BQ8** is satisfied.

Table 21. Model $\mathcal{M}_{10}$

	n^+	q^+	n^-	q^-
a	1	2		3,4
b	3	2	1	4
c	4	2	1,3	

Table 21 expresses a Q-model. For every x and y, $^+(x) \not\subseteq n^+(y)$. So **BQ3** and **BQ5** are satisfied. For all x and y, $^+(x) \not\subseteq n^-(y)$. So **BQ4** is satisfied. If $^+(x) \subseteq q(y)$ and $^+(y) \subseteq q(z)$ then $x = a$ and $z = c$. So **BQ6** is satisfied. For every x and y, $^+(x) \circ q(y)$ or $q(x) \circ {}^+(y)$. So **BQ7** is satisfied. If $^+(x) \subseteq q(y)$ then $x = b$ or $x = c$. For all z, $n^+(b)$ does not overlap $n^-(z)$ and $n^+(c)$ does not overlap $n^-(z)$. So **BQ8** is satisfied.

Table 22 expresses a Q-model. Since **BT3** is satisfied, **BQ3** and **BQ4** are satisfied. Since **BT5** is satisfied, **BQ5** is satisfied. For every x and y, $x \not\subseteq y$. So **BQ6**, **BQ7** and **BQ8** are trivially satisfied.

Table 22. Model $\mathcal{M}_{11}$

	n^+	q^+	n^-	q^-
a	1,2			
b	1	2		
c	1,2			

8.2 Q-valid moods needed for completeness

Aristotle did not discuss any moods with possiblity, as opposed to contingency, premises (or antecedents). But, given the semantics proposed for QLXM′ we must recognize the Q-validity of some moods in which an M-wff is a premise (or an antecedent). In particular Darii QMQ is Q-valid. So, to move in the direction of obtaining completeness results for QLXM′ we shall amend the system by making Darii QMQ axiom 29 (A29).

Theorem 8.18. (soundness of amended QLXM′) Suppose QLXM′ is amended by making the assertion of Darii QMQ, $\vdash (QAbc \to (MIab \to QIac))$ be an axiom. Leave everything else unchanged. Then the resulting system is sound.

Proof. Suppose $\mathcal{M}$ is a Q-model, $V_{\mathcal{M}}(QAbc) = t$ and $V_{\mathcal{M}}(MIab) = t$. Given the definition of a Q-model, at least one of these three conditions is met: i) $^+(a) \circ {}^+(b)$, ii) $^+(a) \circ q(b)$ or iii) $^+(a) \subseteq n^-(b)$. If i)is met then $^+(a) \circ q(c)$ and thus $V_{\mathcal{M}}(QIac) = t$. If ii) is met then $^+(a) \circ q(c)$ or $q(a) \circ {}^+(c)$ and thus $V_{\mathcal{M}}(QIac) = t$. If iii) is met then $V_{\mathcal{M}}(MIab) = t$ and $V_{\mathcal{M}}(MIab) = f$. Given this absurdity $V_{\mathcal{M}}(QIac) = t$. So $\models_Q (QAbc \to (MIab \to QIac))$. ∎

Assertions that are Q-valid correspond to unmarked cells on table 23. The marks in cells indicate how countermodels may be found for the Q-invalid syllogisms the table refers to.

For each unmarked cell we shall show how the indicated syllogism is asserted in the system.

Theorem 8.19. (asserted QMQs and MQQs) The non-numbered QMQ and MQQ cells on table 23 correspond to asserted wffs.

Proof.

1. $\vdash (QAbc \to (MIab \to QIac))$ (Darii QMQ, A29)
2. $\vdash (QEbc \to (MIab \to QOac))$ (Ferio QMQ, from 1 by CC, AS, CW)
3. $\vdash (QAbc \to (MIba \to QIac))$ (Datisi QMQ, from 1 by Ap-con, AS)
4. $\vdash (QAbc \to (MAba \to QIac))$ (Darapti QMQ, from 3 by Ap-con, AS)
5. $\vdash (QEbc \to (MIba \to QOac))$ (Ferison QMQ, from 3 by CC, AS, CW)
6. $\vdash (QEbc \to (MAba \to QOac))$ (Felapton QMQ, from 4 by CC, AS, CW)
7. $\vdash (QAbc \to (MAab \to QIac))$ (Barbari QMQ, from 1 by Ap-sub-a, AS)

Table 23. Additional Q-syllogisms

		QMQ	MQQ	QMM	MQM	QLL	LQL	QQM
Figure 1	Barbara	12ac	5ac	12ac	15ca	7ab	8ac	
	Celarent	12ac	6ac	13ac	7ac	7ab		
	Darii		5ac		15ca	7ab	14bc	
	Ferio		6ac	14ac	14ac	7ab	15ba	14ac
Figure 2	Cesare	9ca	6ac	7ac	7ac	14ab		7ac
	Camestres	6ca	9ac	7ac	7ac		14ba	7ac
	Festino	9ca	6ac	7ac	7ac	13ab	15ba	7ac
	Baroco	6ca	9ac	7ac	7ac	12ab	14ba	7ac
Figure 3	Darapti					7cb	7bc	
	Felapton		9bc	14ac	14ac	7cb	15bc	14ac
	Disamis	5ca		15ac		7cb	7bc	
	Datisi		5ca		15ca	7cb	14bc	
	Bocardo	5ca	9bc	14ac	8bc	7cb	15ba	14ac
	Ferison		9bc	14ac	14ac	7cb	15ba	14ac
Figure 4	Bramantip	5ca		15ac		8ca	7ba	
	Camenes	6ca	7bc	7ac	13ca		7ba	
	Dimaris	5ca		15ac		7cb	7ba	
	Fresison	5ca	6bc	7ac	7ac	13ab	15ba	7ca
	Fesapo	5ca	6bc	7ac	8bc	16cb	15ba	
Subalterns	Barbari		5ac		15ac	7ab	14bc	
	Celaront		6ac	13ac	7ac	7ab		
	Cesaro	9ca	6ac	7ac	7ac	13ab		7ac
	Camestrop	6ca	9ac	7ac	7ac		14ba	7ac
	Camenop	6ca		7ac	16ac		7ba	

8. $\vdash (QEbc \rightarrow (MAab \rightarrow QOac))$ (Celaront QMQ, from 7 by CC, AS, CW)

9. $\vdash (MIbc \rightarrow (QAba \rightarrow QIac))$ (Disamis MQQ, from 3 by AI, Q-con, CW)

10. $\vdash (MAbc \rightarrow (QAba \rightarrow QIac))$ (Darapti MQQ, from 3 by Ap-sub-a, AS)

11. $\vdash (MIcb \rightarrow (QAba \rightarrow QIac))$ (Dimaris MQQ, from 1 by AI, Q-con, CW, US)

12. $\vdash (MAcb \rightarrow (QAba \rightarrow QIac))$ (Bramantip MQQ, from 11 by Ap-sub-a, AS)

13. $\vdash (MAcb \rightarrow (QEba \rightarrow QOac))$ (Camenop MQQ, from 12 by CC, AS, CW)

■

Theorem 8.20. (asserted QMMs and MQMs) The non-numbered QMM and MQM cells on table 23 correspond to asserted sentences.

Proof. Use theorem 8.19 and Q-sub-o.   ■

Theorem 8.21. (asserted QLLs and LQLs) The non-numbered QLL and LQL cells on table 23 correspond to asserted wffs.

Proof. Use theorem 8.20 and RV. ∎

Theorem 8.22. (asserted QQMs) The non-numbered QQM cells on table 23 correspond to asserted wffs.

Proof. Use theorem 8.6, Q-sub-o and CW for cells other than Camenes, Fesapo, Celaront and Camenop QQM. For them use the following reasoning.

1. $\vdash (QEbc \to (QAab \to MEac))$ (Celarent QQM)
2. $\vdash (QEbc \to (QAab \to MOac))$ (Celaront QQM, from 1 by Ap-sub-a, CW)
3. $\vdash (QAcb \to (QAba \to MEac))$ (Camenes QQM, from 1 by AI, Ap-con, CW, US)
4. $\vdash (QAcb \to (QAba \to MOac))$ (Camenop QQM, from 3 by Ap-sub-a, CW)
5. $\vdash (QEcb \to (QAba \to MOac))$ (Fesapo QQM, from 4 by CC, AS) ∎

Table 24. Model $\mathcal{M}_{12}$

	n^+	q^+	n^-	q^-
a	1	2	3	4
b	3	4	2	1
c	2	3	1	4

Table 24 expresses a Q-model. For every x and y, $^+(x) \not\subseteq n^+(y)$. So **BQ3** and **BQ5** are trivially satisfied. For every x and y, $^+(x) \not\subseteq n^-(y)$. So **BQ4** is trivially satisfied. Suppose $^+(y) \subseteq q(z)$. Then $y = b$ and $z = c$. So **BQ6** is trivially satisfied. For every x, $^+(x) \circ q(b)$ or $q(x) \circ {}^+(b)$. So **BQ7** is satisfied. If $^+(c) \subseteq z$ then $z = c$. Since $n^+(b)$ does not overlap $n^-(c)$, **BQ8** is satisfied.

Table 25. Model $\mathcal{M}_{13}$

	n^+	q^+	n^-	q^-
a	1	2	3	4
b	4	3	2	1
c	1,2	3		4

Table 25 expresses a Q-model. Suppose $^+(y) \subseteq n^+(z)$. Then $y = a$ and $z = c$. If $^+(x) \circ {}^+(a)$ then $x = a$ or $x = c$. Since $n^+(a) \circ n^+(c)$, **BQ3** is satisfied. Since $n^-(c) = \emptyset$, **BQ5**is trivially satisfied. For every x and y, $^+(x) \not\subseteq n^-(y)$. So **BQ4** is trivially satisfied. Suppose $^+(y) \subseteq q(z)$. Then $y = b$ and $z = c$. So **BQ6** is trivially satisfied. For all x, $^+(c) \circ q(x)$. So **BQ7** is satisfied. Since $n^-(z) = \emptyset$, **BQ8** is satisfied.

Table 26 expresses a Q-model. For every x and y, if $^+(x) \subseteq n^+(y)$ then $x = a$ and $y = c$. If $^+(x) \circ {}^+(a)$ then $n^+(x) \circ n^+(c)$. So **BQ3** is satisfied. For every x and y,

Table 26. Model $\mathcal{M}_{14}$

	n^+	q^+	n^-	q^-
a	1	2		3,4
b	3	4	1	2
c	1,2	4		3

$^+(x) \not\subseteq n^-(y)$. So **BQ4** is satisfied. Since $n^-(c) = \emptyset$, **BQ5** is satisfied. For every x and y, if $^+(x) \subseteq q(y)$ then $y = a$ or $y = c$. So **BQ6** is trivially satisfied. For every z, $^z \circ q(a)$ or $q(z) \circ {}^+(a)$. And for every z, $^z \circ q(c)$ or $q(z) \circ {}^+(c)$. So **BQ7** is satisfied. For every z, if $^+(a) \subseteq {}^+(z)$ then $z = a$. And for every z, if $^+(c) \subseteq {}^+(z)$ then $z = c$. Since $n^-(a) = \emptyset$ and $n^-(a) = \emptyset$, **BQ8** is satisfied.

Table 27. Model $\mathcal{M}_{15}$

	n^+	q^+	n^-	q^-
a	1		2,3	4
b	4	3		1,2
c	2		1,4	3

Table 27 expresses a Q-model. Since **BT3** is satisfied, both **BQ3** and **BQ4** are satisfied. Since **BT5** is satisfied **BQ5** is satisfied. Suppose $^+(y) \subseteq q(z)$. Then $z = b$. So **BQ6** is trivially satisfied. For all x, $x \circ q(b)$. So **BQ7** is satisfied. If $^+(b) \subseteq {}^+(x)$ then $x = b$. Since $n^-(b) = \emptyset$, **BQ8** is satisfied.

Table 28. Model $\mathcal{M}_{16}$

	n^+	q^+	n^-	q^-
a	1	2		3
b	2	3		1
c	1,2			3

Table 28 expresses a Q-model. Suppose $^+(y) \subseteq n^+(z)$. Then $z = c$. Since, for all x, $n^+(x) \circ n^+(c)$, **BQ3** is satisfied. Since $n^-(c) = \emptyset$, **BQ5** is satisfied. Since, for all x and y, $^+(x) \not\subseteq n^-(y)$, **BQ4** is trivially satisfied. Suppose $^+(y) \subseteq q(z)$. Then $y = b$ and $z = a$. So **BQ6** is trivially satisfied. For all x, $x \circ q(a)$. So **BQ7** is satisfied. For all x, $n^-(x) = \emptyset$. So **BQ8** is trivially satisfied.

Note that the acceptance of all but two of the QLM and LQM moods is generated from acceptances involving the MLM and LMM moods.

9   THE ARISTOTELICITY OF QLXM'

Of the 154 first, second or third figure syllogisms referred to on table 15 there are exactly thirteen that are Q-valid but invalid for Aristotle. And there are exactly nine that are Q-invalid but are valid for Aristotle. So the Aristotelicity of QLXM' system is about 86%. Of the twenty-two discrepancies seventeen are due to mistakes involving the use of Reversal. These mistakes are marked on table 14 by using pairs of numbers from 1 to 17. So, for example, on this table both Barbara QXM and Baroco QLX are marked with '1', indicating that by Reversal both should be valid or both should be invalid. But Aristotle regarded only the former as valid. Both Ferison QLM and Camestres LQM are marked with '17', indicating that by Reversal both should be valid or both should be invalid. Aristotle regarded only the former as valid.

So, there are five remaining discrepancies to account for. i) Darapti XQQ: Aristotle could have used Disamis XQQ to show its validity. ii) Darapti LQQ: Aristotle could have used Darapti XQQ to show its validity. iii) Festino QXM: As noted by McCall in [1963, p. 93], Aristotle could have used Festino MXM to show its validity. Given Reversal, Festino MXM is valid in virtue of Disamis XLL. iv) Celarent QLX: Aristotle could have used Reversal and Festino QXM to show it is valid. v) Felapton XQM: Aristotle properly regarded it as valid since he regarded Ferio XQM as valid. Given our interest in developing a formal system that would not have the unAristotelian results, noted in theorem 7.2, which are present in McCall's Q-L-X-M system, we chose to regard Ferio XQM as Q-invalid.

10   TALLY OF THE TWO-PREMISED Q-VALID SYLLOGISMS

The 333 syllogisms marked on Table 13 are the Q-valid apodeictic two-premised syllogisms in which no contingent wff is a premise or a conclusion. Table 15 and table 23 refer to some of the Q-valid 2-premised syllogisms that involve contingent wffs. To count all of them we need to take account of complementary conversions. Note, for example, that AEA QQQ-figure 1 (that is $(QAbc \rightarrow (QEab \rightarrow QAac))$) is Q-valid by complementary conversion since Barbara QQQ is Q-valid.[29] In this section we shall count all of the 2-premised syllogisms that are Q-valid.

When counting the valid moods we shall use '[A]' to mean that the premise or conclusion indicated may be either an A or an E wff. Similarly we shall use '[I]' to mean the premise or the conclusion indicated may be either an I or an O formula. So, by saying that QQQ [A][A][A] in figure 1 is Q-valid, we are claiming the validity of eight figure 1 QQQ syllogisms: QQQ AAA (AAE, AEA, AEE, EAA, EAE, EEA, and EEE). By saying that QXQ [A]I[I] in figure 1 is Q-valid, we are claiming the Q-validity of four figure 1 QXQ syllogisms: QXQ AII (AIO, EII, and EIO).

---

[29] See Ross's table in [1949, facing p. 286] for references to this as well as several other syllogisms that may be validated by using complementary conversion.

Q-valid QQQs (64):
   Figure 1: [A][A][A], [A][I][I], [A][A][I]
   Figure 3: [A][A][I], [I][A][I], [A][I][I]
   Figure 4: [A][A][I], [I][A][I]

Q-valid QXQs and QLQs (40):
   Figure 1: [A]A[A], [A]I[I], [A]A[I],
   Figure 3: [A]A[I], [A]I[I]

Q-valid XQQs and LQQs (32):
   Figure 3: A[A][I], I[A][I]
   Figure 4: A[A][I], I[A][I]

Q-valid QXMs (34):
   Figure 1: [A]AA, [A]AE, [A]II, [A]IO, [A]AI, [A]AO
   Figure 2: [A]IO, [A]AO
   Figure 3: [A]AI, [A]AO, [I]AI, [A]II, [A]IO
   Figure 4: [A]AI, [I]AI, [A]IO, [A]AO

Q-valid XQMs (20):
   Figure 1: A[A]A, A[I]I, A[A]I
   Figure 3: A[A]I, I[A]I, A[I]I,
   Figure 4: A[A]I, A[A]E, I[A]I, A[A]O

Q-valid QLXs (24):
   Figure 1: [A]AE, [A]AO
   Figure 2: [A]AE, [A]EE, [A]IO, [A]OO, [A]AO, [A]EO
   Figure 4: [A]EE, [A]IO, [A]AO, [A]EO

Q-valid LQXs (30):
   Figure 1: E[A]E, E[I]O, E[A]O
   Figure 2: E[A]E, A[A]E, E[I]O, E[A]O, A[A]O
   Figure 3: E[A]O, O[A]O, E[I]O
   Figure 4: A[A]E, E[I]O, E[A]O, A[A]O

Q-valid QLMs (46):
   Figure 1: [A]AA, [A]AE, [A]II, [A]IO, [A]AI, [A]AO
   Figure 2: [A]AE, [A]EE, [A]IO, [A]OO, {A]AO, [A]EO
   Figure 3: [A]AI, [A]AO, [I]AI, [A]II, [A]IO
   Figure 4: [A]AI, [A]EE, [I]AI, [A]IO, [A]AO, [A]EO

Q-valid LQMs (46):
   Figure 1: A[A]A, E[A]E, A[I]I, E[I]O, A[A]I, E[A]O
   Figure 2: E[A]E, A[A]E, E[I]O, E[A]O, A[A]O

Figure 3: A[A]I, E[A]O, I[A]I, A[I]I, O[A]O, E[I]O
Figure 4: A[A]I, A[A]E, I[A]I, E[I]O, E[A]O, A[A]O

Q-valid QMQs (16):
    Figure 1: [A]I[I], [A]A[I]
    Figure 3: [A]A[I], [A]I[I]

Q-valid MQQs (16):
    Figure 3: A[A][I], I[A][I]
    Figure 4: A[A][I], I[A][I]

Q-valid QMMs (8):
    Figure 1: [A]II, [A]AI
    Figure 3: [A]AI, [A]II

Q-valid MQMs (8):
    Figure 3: A[A]I, I[A]I
    Figure 4: A[A]I, I[A]I

Q-valid QLLs (8):
    Figure 2: [A]EE, [A]EO
    Figure 4: [A]EE, [A]EO

Q-valid LQLs (8):
    Figure 1: E[A]E, E[A]O
    Figure 2: E[A]E, E[A]O

Q-valid QQMs (48):
    Figure 1: [A][A]A, [A][A]E, [A][I]I, [A][A]I, [A][A]O
    Figure 3: [A][A]I, [I][A]I, [A][I]I
    Figure 4: [A][A]I, [A][A]E, [I][A]I, [A][A]O

There are $333 + 64 + 40 + 32 + 34 + 20 + 24 + 30 + 46 + 46 + 16 + 16 + 8 + 8 + 8 + 8 + 48$ (that is, 781) Q-valid 2-premised syllogisms found in thirty five "general moods": LLX, LLM, LXX, LXM, XLX, XLM, XXX, XXM, LLL, LXL, XLL, LMM, MLM, MXM, XMM, LMX, MLX, QQQ, QXQ, QLQ, XQQ, LQQ, QXM, XQM, QLX, LQX, QLM, LQM, QMQ, MQQ, QMM, MQM, QLL, LQL, and QQM.

11 EXTENSIONS

The most natural extension of the above work on QLXM' would be to develop a Smiley-type decision procedure for validity for the n-premised syllogisms, for $n \geq 2$, where these syllogisms meet the chain condition. Though Smiley's decision procedure for the assertoric syllogistic pairs inconsistent sets of wffs with syllogisms construed as inferences,

the pairing could also be between sets of wffs and syllogisms constructed as implications. The decision procedure would list Q-inconsistent sets such as $\{P_1 A x_1 x_2, P_2 A x_3 x_4, \ldots,$ $P_n A x_{2n-1} x_{2n}, \widetilde{Q} A x_1 x_{2n}\}$, where: i) each P_i, for $1 \leq i < n$, is X, L or Q; ii) P_n is Q; and iii) $\widetilde{Q}$ (the negation of Q) is a new quantifier. Given the decision procedure it would follow that $(QAab \rightarrow (Abc \rightarrow (LAcd \rightarrow (QAde \rightarrow QAae))))$, for example, is Q-valid.

Though it is argued above that QLXM′ is more Aristotelian than McCall's Q-L-X-M there are several other systems that could be developed to bring coherence into Aristotle's discussions of modalities. For example, consider Barbara XQM. McCall points out that Aristotle's defense of its validity is flawed, but McCall chooses to take it as an axiom in his Q-L-X-M. It is also an axiom in QLXM′. Dropping this axiom would mean that the semantics for the weaker system would be simpler.

BIBLIOGRAPHY

[Anderson and Belnap, 1975 and 1992] Alan Ross Anderson and Nuel D. Belnap, Jr., editors. *Entailment: The Logic of Relevance and Necessity*, volume I and volume II. Princeton University Press, Princeton, 1975 and 1992.

[Becker-Freyseng, 1933] Albrecht Becker-Freyseng. *Die Aristotelische Theorie der Möglichkeitsschlüsse: eine logischphilologische Untersuchung der Kapitel 13-22 von Aristoteles Analytica priora I.* Junker and Dünnhaupt, Berlin, 1933.

[Bell *et al.*, 2001] John L. Bell, David DeVidi, and Graham Solomon. *Logical Options: an Introduction to Classical and Alternative Logics*. Broadview, Peterborough, 2001.

[Bocheński, 1963] I. M. Bocheński. *Ancient Formal Logic*. North-Holland, Amsterdam, 1963.

[Corcoran, 1972] John Corcoran. The completeness of an ancient logic. *Journal of Symbolic Logic*, 37:696–702, 1972.

[Cresswell, 1988] M. J. Cresswell. Necessity and contingency. *Studia Logica*, 47:145–149, 1988.

[Fitting and Mendelsohn, 1998] Melvin Fitting and Richard L. Mendelsohn. *First-order Modal Logic*. Kluwer, Dordrecht, 1998.

[Frege, 1960] Gottlob Frege. Negation. In Peter Geach and Max Black, editors, *Translations from the Philosophical Writings of Gottlob Frege*, pages 117–135. Blackwell, Oxford, second edition, 1960.

[Geach, 1964] P. T. Geach. Review of McCall, 1963. *Ratio*, 6:200–206, 1964.

[Geach, 1972] P. T. Geach. *Logic Matters*. Blackwell, Oxford, 1972.

[Girle, 2000] Rod Girle. *Modal Logics and Philosophy*. McGill-Queen's University Press, Montreal, 2000.

[Hughes and Cresswell, 1968] G. E. Hughes and M. J. Cresswell. *An Introduction to Modal Logic*. Methuen, London, 1968.

[Hughes and Cresswell, 1996] G. E. Hughes and M. J. Cresswell. *A New Introduction to Modal Logic*. Routledge, London, 1996.

[Johnson, 1989] Fred Johnson. Models for modal syllogisms. *Notre Dame Journal of Formal Logic*, 30:271–284, 1989.

[Johnson, 1991] Fred Johnson. Three-membered domains for Aristotle's syllogistic. *Studia Logica*, 14:181–187, 1991.

[Johnson, 1993] Fred Johnson. Modal ecthesis. *History and Philosophy of Logic*, 14:171–182, 1993.

[Johnson, 1994] Fred Johnson. Syllogisms with fractional quantifiers. *Journal of Philosophical Logic*, 23:401–422, 1994.

[Johnson, 1995] Fred Johnson. Apodeictic syllogisms: Deductions and decision procedures. *History and Philosophy of Logic*, 16:1–18, 1995.

[Johnson, 1997] Fred Johnson. Extended Gergonne syllogisms. *Journal of Philosophical Logic*, 23:553–567, 1997.

[Johnson, 1999a] Fred Johnson. Parry syllogisms. *Notre Dame Journal of Formal Logic*, 40:414–419, 1999.

[Johnson, 1999b] Fred Johnson. Rejection and truth-value gaps. *Notre Dame Journal of Formal Logic*, 40:574–577, 1999.

[Kneale and Kneale, 1962] William Kneale and Martha Kneale. *The Development of Logic*. Clarendon Press, Oxford, 1962.

[Lear, 1980] Jonathan Lear. *Aristotle and Logical Theory*. Cambridge University Press, Cambridge, 1980.

[Łukasiewicz, 1957] Jan Łukasiewicz. *Aristotle's Syllogistic: From the Standpoint of Modern Formal Logic*. Clarendon Press, Oxford, second edition, 1957.

[McCall, 1963] Storrs McCall. *Aristotle's Modal Syllogisms*. Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1963.

[Meredith, 1953] C. A. Meredith. The figures and models of the n-termed Aristotelian syllogisms. *Dominican Studies*, 6:42–47, 1953.

[Montgomery and Routley, 1966] H. A. Montgomery and F. R. Routley. Contingency and non-contingency bases for normal modal logics. *Logique et Analyse*, 35-36:318–328, 1966.

[Nortmann, 1990] Ulrich Nortmann. Über die Stärke der Aristotelischen Modallogik. *Erkenntnis*, 32:61–82, 1990.

[Patterson, 1989] Richard Patterson. The case of the two Barbaras: basic approaches to Aristotle's modal logic. *Oxford Studies in Ancient Philosophy*, 7:1–40, 1989.

[Patterson, 1990] Richard Patterson. Conversion principles and the basis of Aristotle's modal logic. *History and Philosophy of Logic*, 11:151–172, 1990.

[Prior, 1962] A. N. Prior. *Formal Logic*. Clarendon Press, Oxford, second edition, 1962.

[Rescher, 1963] Nicholas Rescher. Aristotle's theory of modal syllogisms and its interpretation. In *The Critical Approach: Essays in honor of Karl Popper*. Free Press of Glencoe, New York, 1963.

[Ross, 1949] W. D. Ross. *Aristotle's Prior and Posterior Analytics*. Clarendon Press, Oxford, 1949.

[Routley and Montgomery, 1968] R. Routley and H. Montgomery. On systems containing Aristotle's thesis. *Journal of Symbolic Logic*, 33:82–96, 1968.

[Rumfitt, 1997] Ian Rumfitt. The categoricity problem and truth-value gaps. *Analysis*, 57:223–235, 1997.

[Smiley, 1961] Timothy Smiley. On Łukasiewicz's Ł-modal system. *Notre Dame Journal of Formal Logic*, 2:149–153, 1961.

[Smiley, 1973] Timothy Smiley. What is a syllogism? *Journal of Philosophical Logic*, 2:136–154, 1973.

[Smiley, 1994] Timothy Smiley. Aristotle's completeness proof. *Ancient Philosophy*, 14:25–38, 1994.

[Smiley, 1996] Timothy Smiley. Rejection. *Analysis*, 56:1–9, 1996.

[Smith, 1982] Robin Smith. What is Aristotelian ecthesis? *History and Philosophy of Logic*, 3:113–127, 1982.

[Smith, 1989] Robin Smith. *Aristotle, Prior Analytics*. Hackett, Indianapolis, 1989.

[Sugihara, 1957a] T. Sugihara. Necessity and possiblity in Aristotelian syllogistic. *Memoirs of the Faculty of Liberal Arts, Fukui University*, 6:75–87, 1957.

[Sugihara, 1957b] T. Sugihara. Necessity and possiblity in Aristotelian syllogistic. *Memoirs of the Faculty of Liberal Arts, Fukui University*, 7:15–22, 1957.

[Thom, 1991] Paul Thom. The two Barbaras. *History and Philosophy of Logic*, 12:135–149, 1991.

[Thom, 1993] Paul Thom. Apodeictic ecthesis. *Notre Dame Journal of Formal Logic*, 34:193–208, 1993.

[Thom, 1994] Paul Thom. Interpreting Aristotle's contingency-syllogistic. In C. C. W. Taylor, editor, *Oxford Studies in Ancient Philosophy*, volume 12, pages 91–109. Clarendon Press, Oxford, 1994.

[Thom, 1996] Paul Thom. *The Logic of Essentialism: An Interpretation of Aristotle's Modal Syllogistic*, volume 43 of *The New Synthese Historical Library*. Kluwer, Dordrecht, 1996.

[Thomason, 1993] S. K. Thomason. Semantic analysis of the modal syllogistic. *Journal of Philosophical Logic*, 22:111–128, 1993.

[Thomason, 1997] S. K. Thomason. Relational models for the modal syllogistic. *Journal of Philosophical Logic*, 26:129–141, 1997.

[van Rijen, 1989] Jeroen van Rijen. *Aspects of Aristotle's Logic of Modalities*. Reidel, Dordrecht, 1989.

[von Wright, 1951] G. H. von Wright. *An Essay in Modal Logic*. North-Holland, Amsterdam, 1951.