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Commutative falling neutrosophic ideals in BCK-algebras

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Abstract: The notions of a commutative $(\in, \in)$-neutrosophic ideal and a commutative falling neutrosophic ideal are introduced, and several properties are investigated. Characterizations of a commutative $(\in, \in)$-neutrosophic ideal are obtained. Relations between commutative $(\in, \in)$-neutrosophic ideal and $(\in, \in)$-neutrosophic ideal are discussed. Conditions for an $(\in, \in)$-neutrosophic ideal to be a commutative $(\in, \in)$-neutrosophic ideal are established. Relations between commutative $(\in, \in)$-neutrosophic ideal, falling neutrosophic ideal and commutative falling neutrosophic ideal are considered. Conditions for a falling neutrosophic ideal to be commutative are provided.

Keywords: (commutative) $(\in, \in)$-neutrosophic ideal; neutrosophic random set; neutrosophic falling shadow; (commutative) falling neutrosophic ideal.

1 Introduction

Neutrosophic set (NS) developed by Smarandache [11, 12, 13] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part which is refered to the site http://fs.gallup.unm.edu/neutrosophy.htm. Jun, Borumand Saeid and Öztürk studied neutrosophic subalgebras/ideals in BCK/BCI-algebras based on neutrosophic points (see [1], [6] and [10]). Goodman [2] pointed out the equivalence of a fuzzy set and a class of random sets in the study of a unified treatment of uncertainty modeled by means of combining probability and fuzzy set theory. Wang and Sanchez [16] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. The mathematical structure of the theory of falling shadows is formulated in [17]. Tan et al. [14, 15] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Park [7] considered a fuzzy subalgebra and a fuzzy ideal as the falling shadow of the cloud of the subalgebra and ideal. Jun et al. [8] introduced the notion of neutrosophic random set and neutrosophic falling shadow. Using these notions, they introduced the concept of falling neutrosophic subalgebra and falling neutrosophic ideal in BCK/BCI-algebras, and investigated related properties. They discussed relations between falling neutrosophic subalgebra and falling neutrosophic ideal, and established a characterization of falling neutrosophic ideal.

In this paper, we introduce the concepts of a commutative $(\in, \in)$-neutrosophic ideal and a commutative falling neutrosophic ideal, and investigate several properties. We obtain characterizations of a commutative $(\in, \in)$-neutrosophic ideal, and discuss relations between a commutative $(\in, \in)$-neutrosophic ideal and an $(\in, \in)$-neutrosophic ideal. We provide conditions for an $(\in, \in)$-neutrosophic ideal to be a commutative $(\in, \in)$-neutrosophic ideal, and consider relations between a commutative $(\in, \in)$-neutrosophic ideal, a falling neutrosophic ideal and a commutative falling neutrosophic ideal. We give conditions for a falling neutrosophic ideal to be commutative.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [3] and [4]) and was extensively investigated by several researchers.

By a BCI*-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK*-algebra*. Any BCK/BCI-algebra X

satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\left(\begin{array}{l} x \le y \;\Rightarrow\; x * z \le y * z \\ x \le y \;\Rightarrow\; z * y \le z * x \end{array}\right), \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \le x * y) \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.5}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{2.6}$$

A subset I of a BCK-algebra X is called a *commutative ideal* of X if it satisfies (2.5) and

$$(x * y) * z \in I,\; z \in I \;\Rightarrow\; x * (y * (y * x)) \in I \tag{2.7}$$

for all $x, y, z \in X$.

Observe that every commutative ideal is an ideal, but the converse is not true (see [9]).

We refer the reader to the books [5, 9] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [12]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \rightarrow [0,1]$ is a truth membership function, $A_I : X \rightarrow [0,1]$ is an indeterminate membership function, and $A_F : X \rightarrow [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$T_\in(A;\alpha) := \{x \in X \mid A_T(x) \ge \alpha\},$$
$$I_\in(A;\beta) := \{x \in X \mid A_I(x) \ge \beta\},$$
$$F_\in(A;\gamma) := \{x \in X \mid A_F(x) \le \gamma\}.$$

We say $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are *neutrosophic* $\in$*-subsets*.

A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$*-neutrosophic subalgebra* of X (see [6]) if the following assertions are valid.

$$(\forall x, y \in X)\left(\begin{array}{l} x \in T_\in(A;\alpha_x),\; y \in T_\in(A;\alpha_y) \\ \qquad \Rightarrow\; x * y \in T_\in(A;\alpha_x \wedge \alpha_y), \\ x \in I_\in(A;\beta_x),\; y \in I_\in(A;\beta_y) \\ \qquad \Rightarrow\; x * y \in I_\in(A;\beta_x \wedge \beta_y), \\ x \in F_\in(A;\gamma_x),\; y \in F_\in(A;\gamma_y) \\ \qquad \Rightarrow\; x * y \in F_\in(A;\gamma_x \vee \gamma_y) \end{array}\right) \tag{2.8}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$*-neutrosophic ideal* of X (see [10]) if the following assertions are valid.

$$(\forall x \in X)\left(\begin{array}{l} x \in T_\in(A;\alpha_x) \;\Rightarrow\; 0 \in T_\in(A;\alpha_x) \\ x \in I_\in(A;\beta_x) \;\Rightarrow\; 0 \in I_\in(A;\beta_x) \\ x \in F_\in(A;\gamma_x) \;\Rightarrow\; 0 \in F_\in(A;\gamma_x) \end{array}\right) \tag{2.9}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} x * y \in T_\in(A;\alpha_x),\; y \in T_\in(A;\alpha_y) \\ \qquad \Rightarrow\; x \in T_\in(A;\alpha_x \wedge \alpha_y) \\ x * y \in I_\in(A;\beta_x),\; y \in I_\in(A;\beta_y) \\ \qquad \Rightarrow\; x \in I_\in(A;\beta_x \wedge \beta_y) \\ x * y \in F_\in(A;\gamma_x),\; y \in F_\in(A;\gamma_y) \\ \qquad \Rightarrow\; x \in F_\in(A;\gamma_x \vee \gamma_y) \end{array}\right) \tag{2.10}$$

for all $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

In what follows, let X and $\mathcal{P}(X)$ denote a BCK/BCI-algebra and the power set of X, respectively, unless otherwise specified.

For each $x \in X$ and $D \in \mathcal{P}(X)$, let

$$\bar{x} := \{C \in \mathcal{P}(X) \mid x \in C\}, \tag{2.11}$$

and

$$\bar{D} := \{\bar{x} \mid x \in D\}. \tag{2.12}$$

An ordered pair $(\mathcal{P}(X), \mathcal{B})$ is said to be a *hyper-measurable structure* on X if $\mathcal{B}$ is a σ-field in $\mathcal{P}(X)$ and $\bar{X} \subseteq \mathcal{B}$.

Given a probability space $(\Omega, \mathcal{A}, P)$ and a hyper-measurable structure $(\mathcal{P}(X), \mathcal{B})$ on X, a *neutrosophic random set* on X (see [8]) is defined to be a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which ξ_T, ξ_I and ξ_F are mappings from Ω to $\mathcal{P}(X)$ which are $\mathcal{A}$-$\mathcal{B}$ measurables, that is,

$$(\forall C \in \mathcal{B}) \left(\begin{array}{l} \xi_T^{-1}(C) = \{\omega_T \in \Omega \mid \xi_T(\omega_T) \in C\} \in \mathcal{A} \\ \xi_I^{-1}(C) = \{\omega_I \in \Omega \mid \xi_I(\omega_I) \in C\} \in \mathcal{A} \\ \xi_F^{-1}(C) = \{\omega_F \in \Omega \mid \xi_F(\omega_F) \in C\} \in \mathcal{A} \end{array} \right). \tag{2.13}$$

Given a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X, consider functions:

$$\begin{aligned} &\tilde{H}_T : X \to [0,1],\ x_T \mapsto P(\omega_T \mid x_T \in \xi_T(\omega_T)), \\ &\tilde{H}_I : X \to [0,1],\ x_I \mapsto P(\omega_I \mid x_I \in \xi_I(\omega_I)), \\ &\tilde{H}_F : X \to [0,1],\ x_F \mapsto 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)). \end{aligned}$$

Then $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a neutrosophic set on X, and we call it a *neutrosophic falling shadow* (see [8]) of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$, and $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *neutrosophic cloud* (see [8]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

For example, consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ where $\mathcal{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic set in X. Then a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\begin{aligned} &\xi_T : [0,1] \to \mathcal{P}(X), \alpha \mapsto T_\in(\tilde{H}; \alpha), \\ &\xi_I : [0,1] \to \mathcal{P}(X), \beta \mapsto I_\in(\tilde{H}; \beta), \\ &\xi_F : [0,1] \to \mathcal{P}(X), \gamma \mapsto F_\in(\tilde{H}; \gamma) \end{aligned}$$

is a neutrosophic random set and $\xi := (\xi_T, \xi_I, \xi_F)$ is a neutrosophic cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$. We will call $\xi := (\xi_T, \xi_I, \xi_F)$ defined above as the *neutrosophic cut-cloud* (see [8]) of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X. If $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are subalgebras (resp., ideals) of X for all $\omega_T, \omega_I, \omega_F \in \Omega$, then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *falling neutrosophic subalgebra* (resp., *falling neutrosophic ideal*) of X (see [8]).

3 Commutative $(\in, \in)$-neutrosophic ideals

Definition 3.1. A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK-algebra X is called a *commutative* $(\in, \in)$*-neutrosophic ideal* of X if it satisfies the condition (2.9) and

$$\begin{aligned} &(x*y)*z \in T_\in(A;\alpha_x),\ z \in T_\in(A;\alpha_y) \\ &\qquad \Rightarrow\ x*(y*(y*x)) \in T_\in(A;\alpha_x \wedge \alpha_y) \\ &(x*y)*z \in I_\in(A;\beta_x),\ z \in I_\in(A;\beta_y) \\ &\qquad \Rightarrow\ x*(y*(y*x)) \in I_\in(A;\beta_x \wedge \beta_y) \\ &(x*y)*z \in F_\in(A;\gamma_x),\ z \in F_\in(A;\gamma_y) \\ &\qquad \Rightarrow\ x*(y*(y*x)) \in F_\in(A;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

Example 3.2. Consider a set $X = \{0,1,2,3\}$ with the binary operation $*$ which is given in Table 1.

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then $(X;*,0)$ is a BCK-algebra (see [9]). Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 2

Table 2: Tabular representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.9	0.2
1	0.3	0.6	0.8
2	0.3	0.6	0.8
3	0.5	0.4	0.7

It is routine to verify that $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X.

Theorem 3.3. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK-algebra X, the following are equivalent.*

(1) *The non-empty $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$.*

(2) *$A = (A_T, A_I, A_F)$ satisfies the following assertions.*

$$(\forall x \in X) \left(\begin{array}{l} A_T(0) \geq A_T(x) \\ A_I(0) \geq A_I(x) \\ A_F(0) \leq A_F(x) \end{array} \right) \tag{3.2}$$

and for all $x, y, z \in X$,

$$\begin{aligned} &A_T(x*(y*(y*x))) \\ &\qquad \geq A_T((x*y)*z) \wedge A_T(z) \\ &A_I(x*(y*(y*x))) \\ &\qquad \geq A_I((x*y)*z) \wedge A_I(z) \\ &A_F(x*(y*(y*x))) \\ &\qquad \leq A_F((x*y)*z) \vee A_F(z) \end{aligned} \tag{3.3}$$

Proof. Assume that the non-empty $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. If $A_T(0) < A_T(a)$ for some $a \in X$, then $a \in T_\in(A; A_T(a))$ and $0 \notin T_\in(A; A_T(a))$. This is a contradiction, and so $A_T(0) \geq A_T(x)$ for all $x \in X$. Similarly,

$A_I(0) \geq A_I(x)$ for all $x \in X$. Suppose that $A_F(0) > A_F(a)$ for some $a \in X$. Then $a \in F_\in(A; A_F(a))$ and $0 \notin F_\in(A; A_F(a))$. This is a contradiction, and thus $A_F(0) \leq A_F(x)$ for all $x \in X$. Therefore (3.2) is valid. Assume that there exist $a, b, c \in X$ such that

$$A_T(a * (b * (b * a))) < A_T((a * b) * c) \wedge A_T(c).$$

Taking $\alpha := A_T((a * b) * c) \wedge A_T(c)$ implies that $(a * b) * c \in T_\in(A; \alpha)$ and $c \in T_\in(A; \alpha)$ but $a * (b * (b * a)) \notin T_\in(A; \alpha)$, which is a contradiction. Hence

$$A_T(x * (y * (y * x))) \geq A_T((x * y) * z) \wedge A_T(z)$$

for all $x, y, z \in X$. By the similar way, we can verify that

$$A_I(x * (y * (y * x))) \geq A_I((x * y) * z) \wedge A_I(z)$$

for all $x, y, z \in X$. Now suppose there are $x, y, z \in X$ such that

$$A_F(x * (y * (y * x))) > A_F((x * y) * z) \vee A_F(z) := \gamma.$$

Then $(x*y)*z \in F_\in(A; \gamma)$ and $z \in F_\in(A; \gamma)$ but $x*(y*(y*x)) \notin F_\in(A; \gamma)$, a contradiction. Thus

$$A_F(x * (y * (y * x))) \leq A_F((x * y) * z) \vee A_F(z)$$

for all $x, y, z \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying two conditions (3.2) and (3.3). Assume that $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are nonempty for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Let $x \in T_\in(A; \alpha)$, $a \in I_\in(A; \beta)$ and $u \in F_\in(A; \gamma)$ for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(a) \geq \beta$, and $A_F(0) \leq A_F(u) \leq \gamma$ by (3.2). It follows that $0 \in T_\in(A; \alpha)$, $0 \in I_\in(A; \beta)$ and $0 \in F_\in(A; \gamma)$. Let $a, b, c \in X$ be such that $(a * b) * c \in T_\in(A; \alpha)$ and $c \in T_\in(A; \alpha)$ for $\alpha \in (0, 1]$. Then

$$A_T(a * (b * (b * a))) \geq A_T((a * b) * c) \wedge A_T(c) \geq \alpha$$

by (3.3), and so $a * (b * (b * a)) \in T_\in(A; \alpha)$. If $(x * y) * z \in I_\in(A; \beta)$ and $z \in I_\in(A; \beta)$ for all $x, y, z \in X$ and $\beta \in (0, 1]$, then $A_I((x * y) * z) \geq \beta$ and $A_I(z) \geq \beta$. Hence the condition (3.3) implies that

$$A_I(x * (y * (y * x))) \geq A_I((x * y) * z) \wedge A_I(z) \geq \beta,$$

that is, $x * (y * (y * x)) \in I_\in(A; \beta)$. Finally, suppose that

$$(x * y) * z \in F_\in(A; \gamma) \text{ and } z \in F_\in(A; \gamma)$$

for all $x, y, z \in X$ and $\gamma \in (0, 1]$. Then $A_F((x * y) * z) \leq \gamma$ and $A_F(z) \leq \gamma$, which imply from the condition (3.3) that

$$A_F(x * (y * (y * x))) \leq A_F((x * y) * z) \vee A_F(z) \leq \gamma.$$

Hence $x * (y * (y * x)) \in F_\in(A; \gamma)$. Therefore the non-empty $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. □

Theorem 3.4. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X if and only if the non-empty neutrosophic $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.*

Proof. Let $A = (A_T, A_I, A_F)$ be a commutative $(\in, \in)$-neutrosophic ideal of X and assume that $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are nonempty for $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Then there exist $x, y, z \in X$ such that $x \in T_\in(A; \alpha)$, $y \in I_\in(A; \beta)$ and $z \in F_\in(A; \gamma)$. It follows from (2.9) that $0 \in T_\in(A; \alpha)$, $0 \in I_\in(A; \beta)$ and $0 \in F_\in(A; \gamma)$. Let $x, y, z, a, b, c, u, v, w \in X$ be such that

$$\begin{aligned} &(x * y) * z \in T_\in(A; \alpha),\ z \in T_\in(A; \alpha), \\ &(a * b) * c \in I_\in(A; \beta),\ c \in I_\in(A; \beta), \\ &(u * v) * w \in F_\in(A; \gamma),\ w \in F_\in(A; \gamma). \end{aligned}$$

Then

$$\begin{aligned} &x * (y * (y * x)) \in T_\in(A; \alpha \wedge \alpha) = T_\in(A; \alpha), \\ &a * (b * (b * a)) \in I_\in(A; \beta \wedge \beta) = I_\in(A; \beta), \\ &u * (v * (v * u)) \in F_\in(A; \gamma \vee \gamma) = F_\in(A; \gamma) \end{aligned}$$

by (2.10). Hence the non-empty neutrosophic $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X for which $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are nonempty and are commutative ideals of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$. Obviously, (2.9) is valid. Let $x, y, z \in X$ and $\alpha_x, \alpha_y \in (0, 1]$ be such that $(x * y) * z \in T_\in(A; \alpha_x)$ and $z \in T_\in(A; \alpha_y)$. Then $(x * y) * z \in T_\in(A; \alpha)$ and $z \in T_\in(A; \alpha)$ where $\alpha = \alpha_x \wedge \alpha_y$. Since $T_\in(A; \alpha)$ is a commutative ideal of X, it follows that

$$x * (y * (y * x)) \in T_\in(A; \alpha) = T_\in(A; \alpha_x \wedge \alpha_y).$$

Similarly, if $(x * y) * z \in I_\in(A; \beta_x)$ and $z \in I_\in(A; \beta_y)$ for all $x, y, z \in X$ and $\beta_x, \beta_y \in (0, 1]$, then

$$x * (y * (y * x)) \in I_\in(A; \beta_x \wedge \beta_y).$$

Now, suppose that $(x*y)*z \in F_\in(A; \gamma_x)$ and $z \in F_\in(A; \gamma_y)$ for all $x, y, z \in X$ and $\gamma_x, \gamma_y \in [0, 1)$. Then $(x * y) * z \in F_\in(A; \gamma)$ and $z \in F_\in(A; \gamma)$ where $\gamma = \gamma_x \vee \gamma_y$. Hence

$$x * (y * (y * x)) \in F_\in(A; \gamma) = F_\in(A; \gamma_x \vee \gamma_y)$$

since $F_\in(A; \gamma)$ is a commutative ideal of X. Therefore $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X. □

Corollary 3.5. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK-algebra X. Then $A = (A_T, A_I, A_F)$ is a commuta-*

tive $(\in, \in)$*-neutrosophic ideal of* X *if and only if it satisfies two conditions* (3.2) *and* (3.3).

Proposition 3.6. *Every commutative* $(\in, \in)$*-neutrosophic ideal* $A = (A_T, A_I, A_F)$ *of a* BCK*-algebra* X *satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{l} x * y \in T_\in(A;\alpha) \\ \quad \Rightarrow x * (y * (y * x)) \in T_\in(A;\alpha) \\ x * y \in I_\in(A;\beta) \\ \quad \Rightarrow x * (y * (y * x)) \in I_\in(A;\beta) \\ x * y \in F_\in(A;\gamma) \\ \quad \Rightarrow x * (y * (y * x)) \in F_\in(A;\gamma) \end{array}\right) \tag{3.4}$$

for all $\alpha, \beta \in (0, 1]$ *and* $\gamma \in [0, 1)$.

Proof. It is induced by taking $z = 0$ in (3.1). □

Theorem 3.7. *Every commutative* $(\in, \in)$*-neutrosophic ideal of a* BCK*-algebra* X *is an* $(\in, \in)$*-neutrosophic ideal of* X.

Proof. Let $A = (A_T, A_I, A_F)$ be a commutative $(\in, \in)$-neutrosophic ideal of a BCK-algebra X. Assume that

$$\begin{array}{l} x * y \in T_\in(A;\alpha_x),\ y \in T_\in(A;\alpha_y), \\ a * b \in I_\in(A;\beta_a),\ b \in I_\in(A;\beta_b), \\ c * d \in F_\in(A;\gamma_c),\ d \in F_\in(A;\gamma_d) \end{array}$$

for all $x, y, a, b, c, d \in X$. Using (2.1), we have

$$\begin{array}{l} (x * 0) * y = x * y \in T_\in(A;\alpha_x), \\ (a * 0) * b = a * b \in I_\in(A;\beta_a), \\ (c * 0) * d = c * d \in F_\in(A;\gamma_c). \end{array}$$

It follows from (3.1), (2.1) and (V) that

$$\begin{array}{l} x = x * 0 = x * (0 * (0 * x)) \in T_\in(A;\alpha_x \wedge \alpha_y), \\ a = a * 0 = a * (0 * (0 * a)) \in I_\in(A;\beta_a \wedge \beta_b), \\ c = c * 0 = c * (0 * (0 * c)) \in F_\in(A;\gamma_c \vee \gamma_d). \end{array}$$

Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. □

The converse of Theorem 3.7 is not true as seen in the following example.

Example 3.8. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 3

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Then $(X; *, 0)$ is a BCK-algebra (see [9]). Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by Table 4

Table 4: Tabular representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.66	0.77	0.27
1	0.55	0.45	0.37
2	0.33	0.66	0.47
3	0.33	0.45	0.67
4	0.33	0.45	0.67

Routine calculations show that $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. But it is not a commutative $(\in, \in)$-neutrosophic ideal of X since $(2 * 3) * 0 \in T_\in(A; 0.6)$ and $0 \in T_\in(A; 0.5)$ but $2 * (3 * (3 * 2)) \notin T_\in(A; 0.5 \wedge 0.6)$, $(1 * 3) * 2 \in I_\in(A; 0.55)$ and $2 \in I_\in(A; 0.63)$ but $1 * (3 * (3 * 1)) \notin I_\in(A; 0.55 \wedge 0.63)$, and/or $(2 * 3) * 0 \in F_\in(A; 0.43)$ and $0 \in F_\in(A; 0.39)$ but $2 * (3 * (3 * 2)) \notin F_\in(A; 0.43 \vee 0.39)$.

We provide conditions for an $(\in, \in)$-neutrosophic ideal to be a commutative $(\in, \in)$-neutrosophic ideal.

Theorem 3.9. *Let* $A = (A_T, A_I, A_F)$ *be an* $(\in, \in)$*-neutrosophic ideal of a* BCK*-algebra* X *in which the condition* (3.4) *is valid. Then* $A = (A_T, A_I, A_F)$ *is a commutative* $(\in, \in)$*-neutrosophic ideal of* X.

Proof. Let $A = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X and $x, y, z \in X$ be such that $(x * y) * z \in T_\in(A;\alpha_x)$ and $z \in T_\in(A;\alpha_y)$ for $\alpha_x, \alpha_y \in (0, 1]$. Then $x * y \in T_\in(A;\alpha_x \wedge \alpha_y)$ since $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X. It follows from (3.4) that $x * (y * (y * x)) \in T_\in(A;\alpha_x \wedge \alpha_y)$. Similarly, if $(x * y) * z \in I_\in(A;\beta_x)$ and $z \in I_\in(A;\beta_y)$, then $x * (y * (y * x)) \in I_\in(A;\beta_x \wedge \beta_y)$. Let $a, b, c \in X$ and $\gamma_a, \gamma_b \in [0, 1)$ be such that $(a * b) * c \in F_\in(A;\gamma_a)$ and $c \in F_\in(A;\gamma_a)$. Then $a * b \in F_\in(A;\gamma_a \vee \gamma_b)$, which implies from (3.4) that $a * (b * (b * a)) \in F_\in(A;\gamma_a \vee \gamma_b)$. Therefore $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X. □

Lemma 3.10. *Every* $(\in, \in)$*-neutrosophic ideal* $A = (A_T, A_I, A_F)$ *of a* BCK*-algebra* X *satisfies:*

$$\begin{array}{l} y, z \in T_\in(A;\alpha) \ \Rightarrow\ x \in T_\in(A;\alpha) \\ y, z \in I_\in(A;\beta) \ \Rightarrow\ x \in I_\in(A;\beta) \\ y, z \in F_\in(A;\gamma) \ \Rightarrow\ x \in F_\in(A;\gamma) \end{array} \tag{3.5}$$

for all $\alpha, \beta \in [0, 1)$, $\gamma \in (0, 1]$ *and* $x, y, z \in X$ *with* $x * y \leq z$.

Proof. For any $\alpha, \beta \in [0, 1)$, $\gamma \in (0, 1]$ and $x, y, z \in X$ with $x * y \leq z$, let $y, z \in T_\in(A;\alpha)$, $y, z \in I_\in(A;\beta)$ and $y, z \in F_\in(A;\gamma)$. Then

$$(x * y) * z = 0 \in T_\in(A;\alpha) \cap I_\in(A;\beta) \cap F_\in(A;\gamma)$$

by (2.9). It follows from (2.10) that

$$x * y \in T_\in(A; \alpha) \cap I_\in(A; \beta) \cap F_\in(A; \gamma)$$

and so that

$$x \in T_\in(A; \alpha) \cap I_\in(A; \beta) \cap F_\in(A; \gamma).$$

Thus (3.5) is valid. □

Theorem 3.11. *In a commutative BCK-algebra, every $(\in, \in)$-neutrosophic ideal is a commutative $(\in, \in)$-neutrosophic ideal.*

Proof. Let $A = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of a commutative BCK-algebra X. Let $x, y, z \in X$ be such that

$$(x * y) * z \in T_\in(A; \alpha_x) \cap I_\in(A; \beta_x) \cap F_\in(A; \gamma_x)$$

and

$$z \in T_\in(A; \alpha_y) \cap I_\in(A; \beta_y) \cap F_\in(A; \gamma_y)$$

for $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$. Note that

$$\begin{aligned}
&((x * (y * (y * x))) * ((x * y) * z)) * z \\
&= ((x * (y * (y * x))) * z) * ((x * y) * z) \\
&\leq (x * (y * (y * x))) * (x * y) \\
&= (x * (x * y)) * (y * (y * x)) \\
&= 0
\end{aligned}$$

by (2.3), (2.4) and (III), which implies that

$$(x * (y * (y * x))) * ((x * y) * z) \leq z.$$

It follows from Lemma 3.10 that

$$x * (y * (y * x)) \in T_\in(A; \alpha_x) \cap I_\in(A; \beta_x) \cap F_\in(A; \gamma_x).$$

Therefore $A = (A_T, A_I, A_F)$ is a commutative $(\in, \in)$-neutrosophic ideal of X. □

4 Commutative falling neutrosophic ideals

Definition 4.1. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on a BCK-algebra X. Then the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is called a *commutative falling neutrosophic ideal* of X if $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$.

Example 4.2. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 5

Table 5: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X; *, 0)$ is a BCK-algebra (see [9]). Consider $(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,3\} & \text{if } t \in [0, 0.25), \\ \{0,4\} & \text{if } t \in [0.25, 0.55), \\ \{0,1,2\} & \text{if } t \in [0.55, 0.85), \\ \{0,3,4\} & \text{if } t \in [0.85, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,1,2\} & \text{if } t \in [0, 0.45), \\ \{0,1,2,3\} & \text{if } t \in [0.45, 0.75), \\ \{0,1,2,4\} & \text{if } t \in [0.75, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.9, 1], \\ \{0,3\} & \text{if } t \in (0.7, 0.9], \\ \{0,4\} & \text{if } t \in (0.5, 0.7], \\ \{0,1,2,3\} & \text{if } t \in (0.3, 0.5], \\ X & \text{if } t \in [0, 0.3]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are commutative ideals of X for all $t \in [0, 1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a commutative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.3 & \text{if } x \in \{1,2\}, \\ 0.4 & \text{if } x = 3, \\ 0.45 & \text{if } x = 4, \end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x \in \{0,1,2\}, \\ 0.3 & \text{if } x = 3, \\ 0.25 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.5 & \text{if } x \in \{1,2,4\}, \\ 0.3 & \text{if } x = 3. \end{cases}$$

Given a probability space $(\Omega, \mathcal{A}, P)$, let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set

$\xi := (\xi_T, \xi_I, \xi_F)$. For $x \in X$, let

$$\begin{aligned}\Omega(x;\xi_T) &:= \{\omega_T \in \Omega \mid x \in \xi_T(\omega_T)\},\\ \Omega(x;\xi_I) &:= \{\omega_I \in \Omega \mid x \in \xi_I(\omega_I)\},\\ \Omega(x;\xi_F) &:= \{\omega_F \in \Omega \mid x \in \xi_F(\omega_F)\}.\end{aligned}$$

Then $\Omega(x;\xi_T), \Omega(x;\xi_I), \Omega(x;\xi_F) \in \mathcal{A}$ (see [8]).

Proposition 4.3. *Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of the neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a commutative falling neutrosophic ideal of X, then*

$$\begin{aligned}&\Omega((x*y)*z;\xi_T) \cap \Omega(z;\xi_T)\\ &\qquad \subseteq \Omega(x*(y*(y*x));\xi_T)\\ &\Omega((x*y)*z;\xi_I) \cap \Omega(z;\xi_I)\\ &\qquad \subseteq \Omega(x*(y*(y*x));\xi_I)\\ &\Omega((x*y)*z;\xi_F) \cap \Omega(z;\xi_F)\\ &\qquad \subseteq \Omega(x*(y*(y*x));\xi_F)\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\Omega(x*(y*(y*x));\xi_T) &\subseteq \Omega((x*y)*z;\xi_T)\\ \Omega(x*(y*(y*x));\xi_I) &\subseteq \Omega((x*y)*z;\xi_I)\\ \Omega(x*(y*(y*x));\xi_F) &\subseteq \Omega((x*y)*z;\xi_F)\end{aligned} \tag{4.2}$$

for all $x, y, z \in X$.

Proof. Let

$$\begin{aligned}\omega_T &\in \Omega((x*y)*z;\xi_T) \cap \Omega(z;\xi_T),\\ \omega_I &\in \Omega((x*y)*z;\xi_I) \cap \Omega(z;\xi_I),\\ \omega_F &\in \Omega((x*y)*z;\xi_F) \cap \Omega(z;\xi_F)\end{aligned}$$

for all $x, y, z \in X$. Then

$$\begin{aligned}&(x*y)*z \in \xi_T(\omega_T) \text{ and } z \in \xi_T(\omega_T),\\ &(x*y)*z \in \xi_I(\omega_I) \text{ and } z \in \xi_I(\omega_I),\\ &(x*y)*z \in \xi_F(\omega_F) \text{ and } z \in \xi_F(\omega_F).\end{aligned}$$

Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals of X, it follows from (2.7) that

$$x*(y*(y*x)) \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F)$$

and so that

$$\begin{aligned}\omega_T &\in \Omega(x*(y*(y*x));\xi_T),\\ \omega_I &\in \Omega(x*(y*(y*x));\xi_I),\\ \omega_F &\in \Omega(x*(y*(y*x));\xi_F).\end{aligned}$$

Hence (4.1) is valid. Now let

$$\begin{aligned}\omega_T &\in \Omega(x*(y*(y*x));\xi_T),\\ \omega_I &\in \Omega(x*(y*(y*x));\xi_I),\\ \omega_F &\in \Omega(x*(y*(y*x));\xi_F)\end{aligned}$$

for all $x, y, z \in X$. Then

$$x*(y*(y*x)) \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Note that

$$\begin{aligned}&((x*y)*z)*(x*(y*(y*x)))\\ &= ((x*y)*(x*(y*(y*x))))*z\\ &\leq ((y*(y*x))*y)*z = ((y*y)*(y*x))*z\\ &= (0*(y*x))*z = 0*z = 0,\end{aligned}$$

which yields

$$\begin{aligned}&((x*y)*z)*(x*(y*(y*x)))\\ &= 0 \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).\end{aligned}$$

Since $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals and hence ideals of X, it follows that

$$(x*y)*z \in \xi_T(\omega_T) \cap \xi_I(\omega_I) \cap \xi_F(\omega_F).$$

Hence

$$\begin{aligned}\omega_T &\in \Omega((x*y)*z;\xi_T),\\ \omega_I &\in \Omega((x*y)*z;\xi_I),\\ \omega_F &\in \Omega((x*y)*z;\xi_F).\end{aligned}$$

Therefore (4.2) is valid. □

Given a probability space $(\Omega, \mathcal{A}, P)$, let

$$\mathcal{F}(X) := \{f \mid f : \Omega \to X \text{ is a mapping}\}. \tag{4.3}$$

Define a binary operation $\circledast$ on $\mathcal{F}(X)$ as follows:

$$(\forall \omega \in \Omega)\ ((f \circledast g)(\omega) = f(\omega) * g(\omega)) \tag{4.4}$$

for all $f, g \in \mathcal{F}(X)$. Then $(\mathcal{F}(X); \circledast, \theta)$ is a BCK/BCI-algebra (see [7]) where θ is given as follows:

$$\theta : \Omega \to X,\ \omega \mapsto 0.$$

For any subset A of X and g_T, g_I, $g_F \in \mathcal{F}(X)$, consider the followings:

$$\begin{aligned}A_T^g &:= \{\omega_T \in \Omega \mid g_T(\omega_T) \in A\},\\ A_I^g &:= \{\omega_I \in \Omega \mid g_I(\omega_I) \in A\},\\ A_F^g &:= \{\omega_F \in \Omega \mid g_F(\omega_F) \in A\}\end{aligned}$$

and

$$\begin{aligned}&\xi_T : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_T \mapsto \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in A\},\\ &\xi_I : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_I \mapsto \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in A\},\\ &\xi_F : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \omega_F \mapsto \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in A\}.\end{aligned}$$

Then A_T^g, A_I^g, $A_F^g \in \mathcal{A}$ (see [8]).

Theorem 4.4. *If K is a commutative ideal of a BCK-algebra X, then*

$$\xi_T(\omega_T) = \{g_T \in \mathcal{F}(X) \mid g_T(\omega_T) \in K\},$$
$$\xi_I(\omega_I) = \{g_I \in \mathcal{F}(X) \mid g_I(\omega_I) \in K\},$$
$$\xi_F(\omega_F) = \{g_F \in \mathcal{F}(X) \mid g_F(\omega_F) \in K\}$$

are commutative ideals of $\mathcal{F}(X)$.

Proof. Assume that K is a commutative ideal of a BCK-algebra X. Since $\theta(\omega_T) = 0 \in K$, $\theta(\omega_I) = 0 \in K$ and $\theta(\omega_F) = 0 \in K$ for all $\omega_T, \omega_I, \omega_F \in \Omega$, we have $\theta \in \xi_T(\omega_T)$, $\theta \in \xi_I(\omega_I)$ and $\theta \in \xi_F(\omega_F)$. Let $f_T, g_T, h_T \in \mathcal{F}(X)$ be such that

$$(f_T \circledast g_T) \circledast h_T \in \xi_T(\omega_T) \text{ and } h_T \in \xi_T(\omega_T).$$

Then

$$(f_T(\omega_T) * g_T(\omega_T)) * h_T(\omega_T) = ((f_T \circledast g_T) \circledast h_T)(\omega_T) \in K$$

and $h_T(\omega_T) \in K$. Since K is a commutative ideal of X, it follows from (2.7) that

$$\begin{aligned}&(f_T \circledast (g_T \circledast (g_T \circledast f_T)))(\omega_T)\\ &= f_T(\omega_T) * (g_T(\omega_T) * (g_T(\omega_T) * f_T(\omega_T))) \in K,\end{aligned}$$

that is, $f_T \circledast (g_T \circledast (g_T \circledast f_T)) \in \xi_T(\omega_T)$. Hence $\xi_T(\omega_T)$ is a commutative ideal of $\mathcal{F}(X)$. Similarly, we can verify that $\xi_I(\omega_I)$ is a commutative ideal of $\mathcal{F}(X)$. Now, let $f_F, g_F, h_F \in \mathcal{F}(X)$ be such that $(f_F \circledast g_F) \circledast h_F \in \xi_F(\omega_F)$ and $h_F \in \xi_F(\omega_F)$. Then

$$\begin{aligned}&(f_F(\omega_F) * g_F(\omega_F)) * h_F(\omega_F)\\ &= ((f_F \circledast g_F) \circledast h_F)(\omega_F) \in K\end{aligned}$$

and $h_F(\omega_F) \in K$. Then

$$\begin{aligned}&(f_F \circledast (g_F \circledast (g_F \circledast f_F)))(\omega_F)\\ &= f_F(\omega_F) * (g_F(\omega_F) * (g_F(\omega_F) * f_F(\omega_F))) \in K,\end{aligned}$$

and so $f_F \circledast (g_F \circledast (g_F \circledast f_F)) \in \xi_F(\omega_F)$. Hence $\xi_F(\omega_F)$ is a commutative ideal of $\mathcal{F}(X)$. This completes the proof. $\square$

Theorem 4.5. *If we consider a probability space $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$, then every commutative $(\in, \in)$-neutrosophic ideal of a BCK-algebra is a commutative falling neutrosophic ideal.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a commutative $(\in, \in)$-neutrosophic ideal of X. Then $T_\in(\tilde{H};\alpha)$, $I_\in(\tilde{H};\beta)$ and $F_\in(\tilde{H};\gamma)$ are commutative ideals of X for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Hence a triple $\xi := (\xi_T, \xi_I, \xi_F)$ in which

$$\xi_T : [0,1] \to \mathcal{P}(X),\ \alpha \mapsto T_\in(\tilde{H};\alpha),$$
$$\xi_I : [0,1] \to \mathcal{P}(X),\ \beta \mapsto I_\in(\tilde{H};\beta),$$
$$\xi_F : [0,1] \to \mathcal{P}(X),\ \gamma \mapsto F_\in(\tilde{H};\gamma)$$

is a neutrosophic cut-cloud of $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a commutative falling neutrosophic ideal of X. $\square$

The converse of Theorem 4.5 is not true as seen in the following example.

Example 4.6. Consider a set $X = \{0,1,2,3,4\}$ with the binary operation $*$ which is given in Table 6

Table 6: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Then $(X;*,0)$ is a BCK-algebra (see [9]). Consider $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0,1\} & \text{if } t \in [0, 0.2),\\ \{0,2\} & \text{if } t \in [0.2, 0.55),\\ \{0,2,4\} & \text{if } t \in [0.55, 0.75),\\ \{0,1,2,3\} & \text{if } t \in [0.75, 1],\end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0,1\} & \text{if } t \in [0, 0.34),\\ \{0,4\} & \text{if } t \in [0.34, 0.66),\\ \{0,1,4\} & \text{if } t \in [0.66, 0.78),\\ X & \text{if } t \in [0.78, 1],\end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X),\quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.87, 1],\\ \{0,2\} & \text{if } t \in (0.76, 0.87],\\ \{0,4\} & \text{if } t \in (0.58, 0.76],\\ \{0,2,4\} & \text{if } t \in (0.33, 0.58],\\ X & \text{if } t \in [0, 0.33].\end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are commutative ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a commutative falling neutrosophic ideal of X, and it is given as follows:

$$\tilde{H}_T(x) = \begin{cases} 1 & \text{if } x = 0,\\ 0.45 & \text{if } x = 1,\\ 0.8 & \text{if } x = 2,\\ 0.25 & \text{if } x = 3,\\ 0.2 & \text{if } x = 4,\end{cases}$$

$$\tilde{H}_I(x) = \begin{cases} 1 & \text{if } x = 0,\\ 0.68 & \text{if } x = 1,\\ 0.22 & \text{if } x \in \{2,3\},\\ 0.66 & \text{if } x = 4,\end{cases}$$

and

$$\tilde{H}_F(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.67 & \text{if } x \in \{1,3\}, \\ 0.31 & \text{if } x = 2, \\ 0.24 & \text{if } x = 4. \end{cases}$$

But $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is not a commutative $(\in, \in)$-neutrosophic ideal of X since

$$(3 * 4) * 2 \in T_\in(\tilde{H}; 0.4) \text{ and } 2 \in T_\in(\tilde{H}; 0.6),$$

but $3 * (4 * (4 * 3)) = 3 \notin T_\in(\tilde{H}; 0.4)$.

We provide relations between a falling neutrosophic ideal and a commutative falling neutrosophic ideal .

Theorem 4.7. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a BCK-algebra. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a commutative falling neutrosophic ideal of X, then it is a falling neutrosophic ideal of X.*

Proof. Let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a commutative falling neutrosophic ideal of a BCK-algebra X. Then $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are commutative ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$. Thus $\xi_T(\omega_T)$, $\xi_I(\omega_I)$ and $\xi_F(\omega_F)$ are ideals of X for all $\omega_T, \omega_I, \omega_F \in \Omega$. Therefore $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X. □

The following example shows that the converse of Theorem 4.7 is not true in general.

Example 4.8. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 7

Table 7: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X; *, 0)$ is a BCK-algebra (see [9]). Consider $(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ and let $\xi := (\xi_T, \xi_I, \xi_F)$ be a neutrosophic random set on X which is given as follows:

$$\xi_T : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,3\} & \text{if } t \in [0, 0.27), \\ \{0,1,2,3\} & \text{if } t \in [0.27, 0.66), \\ \{0,1,2,4\} & \text{if } t \in [0.67, 1], \end{cases}$$

$$\xi_I : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0,3\} & \text{if } t \in [0, 0.35), \\ \{0,1,2,4\} & \text{if } t \in [0.35, 1], \end{cases}$$

and

$$\xi_F : [0,1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} \{0\} & \text{if } t \in (0.84, 1], \\ \{0,3\} & \text{if } t \in (0.76, 0.84], \\ \{0,1,2,4\} & \text{if } t \in (0.58, 0.76], \\ X & \text{if } t \in [0, 0.58]. \end{cases}$$

Then $\xi_T(t)$, $\xi_I(t)$ and $\xi_F(t)$ are ideals of X for all $t \in [0,1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ of $\xi := (\xi_T, \xi_I, \xi_F)$ is a falling neutrosophic ideal of X. But it is not a commutative falling neutrosophic ideal of X because if $\alpha \in [0, 0.27)$, $\beta \in [0, 0.35)$ and $\gamma \in (0.76, 0.84]$, then $\xi_T(\alpha) = \{0,3\}$, $\xi_I(\beta) = \{0,3\}$ and $\xi_F(\gamma) = \{0,3\}$ are not commutative ideals of X respectively.

Since every ideal is commutative in a commutative BCK-algebra, we have the following theorem.

Theorem 4.9. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on a commutative BCK-algebra. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is a commutative falling neutrosophic ideal of X.*

Corollary 4.10. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. For any BCK-algebra X which satisfies one of the following assertions*

$$(\forall x, y \in X)(x \leq y \;\Rightarrow\; x \leq y * (y * x)), \tag{4.5}$$

$$(\forall x, y \in X)(x \leq y \;\Rightarrow\; x = y * (y * x)), \tag{4.6}$$

$$(\forall x, y \in X)(x * (x * y) = y * (y * (x * (x * y)))), \tag{4.7}$$

$$(\forall x, y, z \in X)(x, y \leq z, z * y \leq z * x \;\Rightarrow\; x \leq y), \tag{4.8}$$

$$(\forall x, y, z \in X)(x \leq z, z * y \leq z * x \;\Rightarrow\; x \leq y), \tag{4.9}$$

let $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ be a neutrosophic falling shadow of a neutrosophic random set $\xi := (\xi_T, \xi_I, \xi_F)$ on X. If $\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)$ is a falling neutrosophic ideal of X, then it is a commutative falling neutrosophic ideal of X.

References

[1] A. Borumand Saeid and Y.B. Jun, Neutrosophic subalgebras of BCK/BCI-algebras based on neutrosophic points, Ann. Fuzzy Math. Inform. **14** (2017), no. 1, 87–97.

[2] I.R. Goodman, Fuzzy sets as equivalence classes of random sets, in "Recent Developments in Fuzzy Sets and Possibility Theory"(R. Yager, Ed.), Pergamon, New York 1982, pp. 327–343.

[3] K. Iséki, On BCI-algebras, Math. Seminar Notes **8** (1980), 125–130.

[4] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon. **23** (1978), 1–26.

[5] Y. Huang, *BCI-algebra*, Science Press, Beijing, 2006.

[6] Y.B. Jun, Neutrosophic subalgebras of several types in BCK/BCI-algebras, Ann. Fuzzy Math. Inform. **14** (2017), no. 1, 75–86.

[7] Y.B. Jun and C.H. Park, Falling shadows applied to subalgebras and ideals of BCK/BCI-algebras, Honam Math. J. **34** (2012), no. 2, 135–144.

[8] Y.B. Jun, F. Smarandache and H. Bordbar, Neutrosophic falling shadows applied to subalgebras and ideals in BCK/BCI-algebras, Ann. Fuzzy Math. Inform. (submitted).

[9] J. Meng and Y. B. Jun, *BCK-algebras*, Kyungmoonsa Co. Seoul, Korea 1994.

[10] M.A. Öztürk and Y.B. Jun, Neutrosophic ideals in BCK/BCI-algebras based on neutrosophic points, J. Inter. Math. Virtual Inst. **8** (2018), 1–17.

[11] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998. http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition online).

[12] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Reserch Press, Rehoboth, NM, 1999.

[13] F. Smarandache, Neutrosophic set-a generalization of the intuitionistic fuzzy set, Int. J. Pure Appl. Math. **24** (2005), no.3, 287–297.

[14] S.K. Tan, P.Z. Wang and E.S. Lee, Fuzzy set operations based on the theory of falling shadows, J. Math. Anal. Appl. **174** (1993), 242–255.

[15] S.K. Tan, P.Z. Wang and X.Z. Zhang, Fuzzy inference relation based on the theory of falling shadows, Fuzzy Sets and Systems **53** (1993), 179–188.

[16] P.Z. Wang and E. Sanchez, Treating a fuzzy subset as a projectable random set, in: "Fuzzy Information and Decision" (M. M. Gupta, E. Sanchez, Eds.), Pergamon, New York, 1982, pp. 212–219.

[17] P.Z. Wang, Fuzzy Sets and Falling Shadows of Random Sets, Beijing Normal Univ. Press, People's Republic of China, 1985. [In Chinese]

[18] Abdel-Basset, M., Mohamed, M., Smarandache, F., & Chang, V. (2018). Neutrosophic Association Rule Mining Algorithm for Big Data Analysis. *Symmetry*, *10*(4), 106.

[19] Abdel-Basset, M., & Mohamed, M. (2018). The Role of Single Valued Neutrosophic Sets and Rough Sets in Smart City: Imperfect and Incomplete Information Systems. Measurement. Volume 124, August 2018, Pages 47-55

[20] Abdel-Basset, M., Gunasekaran, M., Mohamed, M., & Smarandache, F. A novel method for solving the fully neutrosophic linear programming problems. *Neural Computing and Applications*, 1-11.

[21] Abdel-Basset, M., Manogaran, G., Gamal, A., & Smarandache, F. (2018). A hybrid approach of neutrosophic sets and DEMATEL method for developing supplier selection criteria. *Design Automation for Embedded Systems*, 1-22.

[22] Abdel-Basset, M., Mohamed, M., & Chang, V. (2018). NMCDA: A framework for evaluating cloud computing services. *Future Generation Computer Systems*, *86*, 12-29.

[23] Abdel-Basset, M., Mohamed, M., Zhou, Y., & Hezam, I. (2017). Multi-criteria group decision making based on neutrosophic analytic hierarchy process. *Journal of Intelligent & Fuzzy Systems*, *33*(6), 4055-4066.

[24] Abdel-Basset, M.; Mohamed, M.; Smarandache, F. An Extension of Neutrosophic AHP–SWOT Analysis for Strategic Planning and Decision-Making. *Symmetry* 2018, *10*, 116.

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