Philosophy Study, ISSN 2159-5313
November 2012, Vol. 2, No. 11, 773-793

Carnap's Thought on Inductive Logic

Yusuke Kaneko
Meiji University

Although we often see references to Carnap's inductive logic even in modern literatures, seemingly its confusing style has long obstructed its correct understanding. So instead of Carnap, in this paper, I devote myself to its necessary and sufficient commentary. In the beginning part (Sections 2-5), I explain why Carnap began the study of inductive logic and how he related it with our thought on probability (Sections 2-4). Therein, I trace Carnap's thought back to Wittgenstein's Tractatus as well (Section 5). In the succeeding sections, I attempt the simplest exhibition of Carnap's earlier system, where his original thought was thoroughly provided. For this purpose, minor concepts to which researchers have not paid attention are highlighted, for example, *$\mathfrak{m}$-function* (Section 8), *in-correlation* (Section 10), *C-correlate* (Section 10), *statistical distribution* (Section 12), and *fitting sequence* (Section 17). The climax of this paper is the proof of theorem (56). Through this theorem, we will be able to overview Carnap's whole system.

Keywords: inductive logic, confirmation, probability, the earlier system in *Foundations*, λ-system

1. Introduction

In his later theory called λ-system, Carnap provided the simplest way to calculate the probability in inductive inference.

(0) $\mathfrak{c}_\lambda\,(\mathrm{M}\,(\mathrm{a}_{s+1}), i) = \dfrac{s_M + \frac{w_M}{\kappa}\lambda}{s+\lambda}$ (Carnap 1952, 33)

Here, "$\mathfrak{c}_\lambda$" is a $\mathfrak{c}$-function (cf. (31) below). "M" is a molecular predicate (cf. (13) below). "*i*" is an individual distribution (cf. (41) below). "λ" is a parameter weighting the logical factor $\frac{w_M}{\kappa}$ (Section 19). "s_M" is the number of individual constants of which M is predicated. "w_M" is the width of M (cf. (18) below). "κ" is the number of Q-predicates (cf. (17) below). "s" stands for the number of individuals observed until then; thus "a_{s+1}" for the individual observed next.

After its publication, most of the researchers began to study this system alone. It had, however, an undesirable outcome as well: they forgot the basic idea underlying Carnap's inductive logic.

It is well known that Carnap published his inductive logic mainly in two books: *The Logical Foundations of Probability* (1950, 1st ed.) and *The Continuum of Inductive Methods* (1952). We may call the one published in *Foundations*[1] "the earlier system," and the other published in *Continuum* "the later system" or simply "λ-system."

Formula (0) was provided in the later system. However, the main thought of inductive logic is, without a doubt, stated exclusively in the earlier system. So by adopting Formula (0), we may say, researchers have

Yusuke Kaneko, Ph.D., The Faculty of Commerce, Meiji University, Japan; main research fields: Analytic Philosophy (presently focusing on Logical Positivism, especially Carnap), Ethics (Kant, Utilitarianism, Business Ethics, Biomedical Ethics), Traditional Theoretical Philosophy (Kant, Hume, Descartes).

forgotten the central ideas of Carnap's inductive logic.

The aim of this paper is nothing but getting back these lost ideas. Although my commentary is sometimes a bit lengthy, I am sure that through it, readers will be led to the kernel of inductive logic.

2. From Confirmation to Probability

Carnap's inductive logic can be put in the context of his wider program, *the logic of science*, which was already stated in *Syntax* (1937, xiii). As early as in *Testability*, Carnap found inductive logic necessary for this program and mentioned its key concept, confirmation, as well (1936-37, 87). Confirmation is the concept to characterize the relationship between observed evidence and theoretical hypotheses. Carnap grouped it under *inductive inference*. Thus, inductive logic was, first of all, a theory to clarify the confirmation concept (1962, 1f.).

The procedure to clarify a concept was called *explication* by Carnap (1962, 3f.). Explication goes through three stages (Carnap 1962, 12). First, a concept is originally used as a classificatory one. Second, the concept is refined into a comparative one. Third, it is explicated into a quantitative one (Carnap 1962, 8f.; 1966, 51f.).

Confirmation was no exception. Corresponding to the three stages, three concepts of confirmation were considered.

(1) The classificatory concept of confirmation (Carnap 1962, 21f.):
- (i) h is confirmed by e.
- (ii) $\mathfrak{C}(h, e)$.

(2) The comparative concept of confirmation (Carnap 1962, 22f.):
- (i) h is more strongly confirmed by e than h' by e'.
- (ii) $\mathfrak{MC}(h, e, h', e')$.

(3) The quantitative concept of confirmation (Carnap 1962, 23):
- (i) The degree of confirmation of h with respect to e is q.
- (ii) $\mathfrak{c}\,(h, e) = q$.

Here, (ii) is a mere abbreviation of (i).

$\mathfrak{MC}$ and $\mathfrak{C}$ are predicates, but $\mathfrak{c}$ is a functor. This difference is important: confirmation becomes a function in its third stage, which is regarded as probability function "P" today.[2] In this manner, Carnap proceeded from the explication of confirmation to the probability function. He called such a confirmation concept $\mathfrak{c}$-functions (1962, 293f.). And in this context, probability was named *probability$_1$* (Carnap 1962, 25).

3. Probability$_1$ and Probability$_2$

This is how, in his course of argument, Carnap took up the probability concept as well. Yet, this treatment seemingly oversimplified the concept.

Carnap's distinction between probability$_1$ and probability$_2$ might become the answer (1962, 23f.).[3] Probability$_1$ is a *degree of confirmation*, which is, as just explained, the quantitative concept of confirmation (Carnap 1962, 25), whereas probability$_2$ is a *relative frequency in the long run* (Carnap 1962, 25).

In ordinary talks, we usually think about probability$_2$. Suppose, e.g., we talk about the batting average of a batter in a baseball game, saying, "The probability of the batter making a hit this time is 0.3." This talk is thought to be about probability$_2$; indeed, it matches Reichenbach's *straight rule* quite well (Reichenbach is a typical supporter of probability$_2$).

Carnap objected to this analysis for the reason that even in the case, the probability we have in mind is not probability$_2$ but an *estimate* of relative frequency (1962, 168); the estimate of relative frequency is not probability$_2$ but *the probability$_1$-weighted mean*, a kind of probability$_1$ (1962, 169, 525); that is why our mindset is only directed to probability$_1$.[4]

Reflecting on history, Carnap further said: probability had been originally considered as probability$_1$ (1962, 182-83), and probability$_2$ had been derived from probability$_1$ because of the thinkers' inadvertent ways of thought (cf. Section 14 below).

In this way, Carnap reduced probability$_2$ to probability$_1$. His standpoint was not dualism but monism in spite of the dichotomy.

4. Relativity and the Basic Idea

This is Carnap's stance to the probability concept. Probability is, for him, probability$_1$, in other words, a degree of confirmation.

He was interested especially in the relative character of probability. This was because his concept, probability$_1$, was the expression of the relationship between a hypothesis and evidence. Carnap named the character *relativity* (1962, 31f.). The relativity is identified with what we call *conditional probability* today. In this way, Carnap centered conditional probability in his system. This choice is instructive, considering that the axiom systems for conditional probability and those for unconditional probability easily coexist in modern probability theories.

But how did Carnap formulate the relativity? This question leads us to a deeper part of inductive logic. Let us trace Carnap's statements.

> (4) Suppose now that X is interested in a certain hypothesis h; he wants to obtain a judgment about h on the basis of his knowledge. For this purpose, he examines the relation between the range of e and that of h (Carnap 1962, 298).

Here, Carnap has possible world semantics in mind; "range" is a set of possible worlds where the sentence h or e holds, though Carnap called possible worlds "state-descriptions" (cf. (19) below).[5] The range of e is written $\mathfrak{R}(e)$, and that of h $\mathfrak{R}(h)$. The rest of this passage is as follows:

> (5) Suppose the whole knowledge which an observer X has gained … is expressed by the sentence e.… If he finds that $\mathfrak{R}(e)\subset\mathfrak{R}(h)$, then (since the actual world in which X has observed e will be, through the inclusion, the world in which X observes h, too) h must likewise be true (Carnap 1962, 298).[6]

Here, Carnap has deductive logic in mind: the logical implication $e\rightarrow h$ (Carnap 1962, 84). However, what if the whole inclusion does not hold?

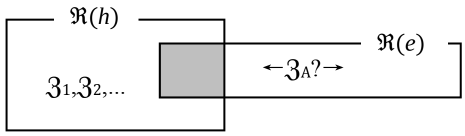
> (6) If, however, he finds that *only a part of* the range of e is contained in that of h, … then he cannot find *certainty* concerning the truth of h.… He knows the actual state—or the actual world, hereafter written $\mathfrak{Z}_A$—is described by one of the state-descriptions in $\mathfrak{R}(e)$.… If now $\mathfrak{Z}_A$ belonged to $\mathfrak{R}(e)\cap\mathfrak{R}(h)$, then h would be true; otherwise, h would be false (Carnap 1962, 84).[7]

However, the problem is: to X, the pinpoint place of the actual world $\mathfrak{Z}_A$ in $\mathfrak{R}(e)$ is not known. Thus, the size of each range must be counted.

> (7) Therefore, the larger the part of $\mathfrak{R}(e)$ overlapping with $\mathfrak{R}(h)$ is in relation to the whole of $\mathfrak{R}(e)$, in other words, the more of those possibilities which are still left open by e are such that h would hold in them, the more reason has X, who knows e, for expecting h to be true (Carnap 1962, 84).

Carnap depicted this idea as follows:

(8)



The shaded area is $\mathfrak{R}(e) \cap \mathfrak{R}(h)$. (Carnap 1962, 297)

We do not know where the actual world $\mathfrak{Z}_A$ is in $\mathfrak{R}(e)$. However, the larger $\mathfrak{R}(e) \cap \mathfrak{R}(h)$ is, the more is the possibility of $\mathfrak{Z}_A$ being the world in which *h* holds. This is the basic idea called *partial inclusion*. (Of course, it is different from the relation of the proper subset in set theory.)

Here let us recall: if $\mathfrak{R}(e) \subset \mathfrak{R}(h)$, the logical implication $e \rightarrow h$ holds (see (5) above). After this, the relation of *e* and *h* can be named *partial logical implication* when partial inclusion holds between $\mathfrak{R}(e)$ and $\mathfrak{R}(h)$. This is why Carnap called the relationship of confirmation, probability, or inductive inference "partial logical implication" (1962, 31).

5. Wittgenstein as Background

This is the basic idea of inductive logic. It is the theory that handles the interface of ranges, to which possible worlds belong. Yet there still remain questions. How can we formally deal with ranges? How can we measure them? What about their contents? What on earth are the possible worlds in Carnap's system?

To answer these questions, we may ascribe inductive logic to another thought; in my opinion, Wittgenstein's *Tractatus* is justly counted as such.[8]

Theses from 5.15 to 5.156 in *Tractatus* famously deal with probability. Carnap's interest in them was also remarkable (1962, 299); he took his idea of range from Wittgenstein's *Spielraum* in 4.463 (1942, 96-97, 107; 1962, 83).

The characteristic of Wittgenstein's idea is making use of *truth-tables*. This is stated from 4.25. Carnap mostly followed it, constructing his system in such a manner that its theorems are provable from truth-tables (e.g., see notes 16, 17, and 18).

In this context, we may specify the following passage as the origin of inductive logic.

(9) Regarding the holding and not holding of n states of affairs, there are $\sum_{k=0}^{n} {}_{n}C_{k}$ combinations (Wittgenstein 1918, 4.27).

This passage is conspicuous in *Tractatus*, because it tries to solve the problem of the world, exclusively using a mathematical method.

We may relate this passage with inductive logic as follows. First, using the *Binomial Theorem*, we obtain $2^n = \sum_{k=0}^{n} {}_{n}C_{k}$ (Carnap 1962, 152).[9] Second, counting the number of possible worlds in inductive logic, we find it to be 2^n. (This anticipates the later arguments; see (11) and the following, plus (20) below.) From these two facts, it can be said that "combinations" in (9) correspond to the possible worlds in inductive logic. These "combinations" are customarily listed as, e.g., "1100" on a truth-table. (We can check them on table (12) below.)

Now that the relationship between Wittgenstein's passage (= 9) and Carnap's thought (on possible worlds) is clarified, we may approach the picture of inductive logic in terms of *Tractatus*, especially by means of its truth-tables.

But before rushing into this analysis, we must remove one obstacle between these two thinkers, that is, their discrepancy in the understanding of possible worlds, which is especially concerned with their units. Let us take up "states of affairs" in (9) as such units. States of affairs are customarily listed on the headings of the left-most columns in a truth-table, for example, as "$P_1(a_1)$" (see table (12) below). Wittgenstein regarded them as *elementary sentences* (1918, 4.25). Elementary sentences are the complete picture of the external world, so they are fixed, not changeable. On the contrary, Carnap did not regard "$P_1(a_1)$" or the like as such. For him, they are merely *atomic sentences*, that is, the products of an artificial language, so they are changeable.

What results from this difference? It follows that possible worlds are absolute and objective for Wittgenstein, whereas they are arbitrary and subjective for Carnap. Later, in this very respect, Carnap decisively broke with *Tractatus*, as Friedman pointed out (Friedman 1999, 179, 183; see also Carnap 1936-37, 58).

It is true that we may ascribe Carnap's thought to Wittgenstein's *Tractatus* fundamentally. But Carnap's picture is more flexible. We are allowed to *design* our view of the world, because in Carnap's system, the possible worlds are simply derived from our *artificial* languages.[10]

6. Inductive Logic as Meta-theory

In this way, we may ascribe Carnap's thought to Wittgenstein's *Tractatus*. However, there was also a discrepancy between them, according to which, within Carnap's system, we can imagine possible worlds as we like. Carnap also emphasized this point, in the construction of inductive logic, he fully took the importance of artificial languages into account (Carnap 1962, 54).

Besides this relationship, we may also refer to the one between Carnap and Tarski. That is, for the designed language, Carnap provided Tarski's semantical analysis (Carnap 1942, vif.). The language we design is an *object language*. For this language, probability theory and inductive logic are provided as its *meta-theory* (Carnap 1966, 29-39). Therefore, probability functions or $\mathfrak{c}$-functions were thought to be *semantical functions* (Carnap 1962, 164, 283, 522).

In this way, inductive logic was grouped under semantics. So the preceding sentences, such as (1), (2), and (3), were regarded as those in a *meta-language*, and many expressions, such as "*h*," "*e*," "*i*," "*j*," "$\mathfrak{S}$," "$\mathfrak{Z}$," "$\mathfrak{Str}$," etc., were assorted into *meta-arguments* (Carnap 1962, 55f.).

By reference to this semantical analysis, we find that Carnap became the first person[11] to answer the naïve question, "To what do we assign probability?". His answer was "a name of a sentence" (Carnap 1962, 279f.). The argument of a probability function is *a name of a sentence*.

7. Definitions of Terminology

Now that Carnap's philosophical backgrounds were revealed, we may go into its technical phases. Let us, first, define possible worlds formally; after that, we get back to the previous question "How can we measure ranges?" (cf. Section 5).

According to the preceding arguments, we may artificially design a language in inductive logic. As an example, let us take the following language.

(10) $\mathfrak{L}_2^2$: Const. = $\{a_1, a_2\}$ Pred. = $\{P_1, P_2\}$

Here, "Const." is the abbreviation of "individual constant," and "Pred." of "primitive predicate." Note that in inductive logic, Const. means the *population* (Carnap 1962, 207, 493f.). Predicates are confined to monadic predicates.[12] Carnap called a language having π predicates and N individual constants "$\mathfrak{L}_N^\pi$" (1962, 123). $\mathfrak{L}_N^\pi$ has also logical symbols of the first-order predicate logic.

About this language, the following obviously holds:

(11) There are $N \times \pi$ atomic sentences in $\mathfrak{L}_N^{\pi}$. (Carnap 1962, 121)

This "$N \times \pi$" is equal to "n" in (9). But according to (9), there are $\sum_{k=0}^{n} {}_{n}C_{k}$ ($= 2^n$) combinations of the atomic sentences regarding holding and not holding. How can we count it?

As for this question, compiling a truth-table in $\mathfrak{L}_2^2$ will help:

(12)

$P_1(a_1)$	$P_2(a_1)$	$P_1(a_2)$	$P_2(a_2)$			$P_1(a_1)\rightarrow P_2(a_1)$	$P_1(a_2)\rightarrow P_2(a_2)$
1	1	1	1	$\mathfrak{Z}_1$	$Q_1(a_1)\wedge Q_1(a_2)$	1	1
1	1	1	0	$\mathfrak{Z}_2$	$Q_1(a_1)\wedge Q_2(a_2)$	1	0
1	1	0	1	$\mathfrak{Z}_3$	$Q_1(a_1)\wedge Q_3(a_2)$	1	1
1	1	0	0	$\mathfrak{Z}_4$	$Q_1(a_1)\wedge Q_4(a_2)$	1	1
1	0	1	1	$\mathfrak{Z}_5$	$Q_2(a_1)\wedge Q_1(a_2)$	0	1
1	0	1	0	$\mathfrak{Z}_6$	$Q_2(a_1)\wedge Q_2(a_2)$	0	0
1	0	0	1	$\mathfrak{Z}_7$	$Q_2(a_1)\wedge Q_3(a_2)$	0	1
1	0	0	0	$\mathfrak{Z}_8$	$Q_2(a_1)\wedge Q_4(a_2)$	0	1
0	1	1	1	$\mathfrak{Z}_9$	$Q_3(a_1)\wedge Q_1(a_2)$	1	1
0	1	1	0	$\mathfrak{Z}_{10}$	$Q_3(a_1)\wedge Q_2(a_2)$	1	0
0	1	0	1	$\mathfrak{Z}_{11}$	$Q_3(a_1)\wedge Q_3(a_2)$	1	1
0	1	0	0	$\mathfrak{Z}_{12}$	$Q_3(a_1)\wedge Q_4(a_2)$	1	1
0	0	1	1	$\mathfrak{Z}_{13}$	$Q_4(a_1)\wedge Q_1(a_2)$	1	1
0	0	1	0	$\mathfrak{Z}_{14}$	$Q_4(a_1)\wedge Q_2(a_2)$	1	0
0	0	0	1	$\mathfrak{Z}_{15}$	$Q_4(a_1)\wedge Q_3(a_2)$	1	1
0	0	0	0	$\mathfrak{Z}_{16}$	$Q_4(a_1)\wedge Q_4(a_2)$	1	1

There are 16 lines marked as, e.g., "1100" in the four left-most columns. They express the combinations of atomic sentences regarding holding and not holding. The combinations are calculated as a repeated permutation that takes four units out of two different things (1, 0). Thus, $2^4 = 16$. In general, $2^{N\times\pi} = 2^n$.

In this way, we can count the number of combinations of atomic sentences; and those combinations express possible worlds, as we saw earlier (Section 5). They are marked in the fifth column as "$\mathfrak{Z}_1$," for example. In terms of semantics of propositional logic, they are regarded as truth assignments, too.

However, in (12), a possible world is formed as a conjunction of atomic sentences, e.g., $\mathfrak{Z}_7$ as $P_1(a_1)\wedge\neg P_2(a_1)\wedge\neg P_1(a_2)\wedge P_2(a_2)$. This formulation is somewhat complicated. Thus, we introduce another formulation of possible worlds using *Q-predicates*, instead:

(13) Only for abbreviation, we write, e.g., "$P_1\rightarrow P_2(a_1)$" instead of "$P_1(a_1)\rightarrow P_2(a_1)$" and we call such an expression a *molecular predicate expression*. Further, we write, e.g., "$M(a_1)$" instead of "$P_1\rightarrow P_2(a_1)$" and call such an expression a *molecular predicate* (Carnap 1962, 104-05).

(14) If molecular predicates $M_1, \ldots, M_p$ fulfill the following conditions, then they are called *forming a division* (Carnap 1962, 107-08).

① $\vDash$[13] $\forall x[M_1(x)\vee \ldots \vee M_p(x)]$ (exhaustiveness)

② For any M_i, M_j ($1 \leq i, j \leq p$), $\vDash \forall x\neg[M_i(x)\wedge M_j(x)]$ (exclusiveness)

③ For no M_i ($1 \leq i \leq p$), $\vDash\neg\exists x M_i(x)$ (M_i is not logically empty).

(15) The molecular predicates introduced with the following definition are called *Q-predicates*.

$$\forall x[Q_i(x)\leftarrow\rightarrow(\neg)P_1(x)\wedge \ldots \wedge(\neg)P_\pi(x)]$$ (Carnap 1962, 125)

Here, "P_1" to "P_π" are all the primitive monadic predicates in $\mathfrak{L}_N^{\pi}$. "$(\neg)$" stands for either affirmation or negation. It is obvious that the Q-predicates form a division (Carnap 1962, 126). Note that Q-predicates belong not to meta-languages but to object languages. In $\mathfrak{L}_2{}^2$, all Q-predicates are as follows:

(16) $\forall x[Q_1(x)\longleftrightarrow P_1(x)\land P_2(x)]$
$\forall x[Q_2(x)\longleftrightarrow P_1(x)\land\neg P_2(x)]$
$\forall x[Q_3(x)\longleftrightarrow\neg P_1(x)\land P_2(x)]$
$\forall x[Q_4(x)\longleftrightarrow\neg P_1(x)\land\neg P_2(x)]$

In general, the following holds:

(17) There are 2^{π} Q-predicates in $\mathfrak{L}_N{}^{\pi}$. (Carnap 1962, 126)[14]

Carnap wrote "κ" instead of "2^{π}" (Carnap 1962, 126).

Q-predicates play two important roles in inductive logic. One is the following:

(18) Any formula $\mathfrak{M}(x)$[15] in $\mathfrak{L}_N{}^{\pi}$ is expressed with a disjunction of Q-predicates in the following way:
$\ulcorner\forall x[\mathfrak{M}(x)\longleftrightarrow Q_{i1}(x)\lor Q_{i2}(x)\lor \ldots \lor Q_{iw}(x)]\urcorner$ (Carnap 1962, 126)[16]

Here, the number "w" is called width (Carnap 1962, 127).

The other role is that we can define state-descriptions with Q-predicates, this is what we intended:

(19) The conjunctions introduced, as follows, by predicating one Q-predicate of each individual constant in $\mathfrak{L}_N{}^{\pi}$ are called state-descriptions.
$\mathfrak{Z}_i = \ulcorner Q_{i1}(a_1)\land Q_{i2}(a_2)\land \ldots \land Q_{ic}(a_N)\urcorner$ (Carnap 1962, 72)

Here, Q_{i1}-Q_{ic} is one of Q_1-Q_{κ} in $\mathfrak{L}_N{}^{\pi}$. (The second subscript "c" in "$Q_{ic}(a_N)$" shows the number of Q-predicates appearing in the world $\mathfrak{Z}_i$.) This is the other formulation of possible worlds we intended. Additionally, let me mention the following theorem as a counterpart to *Tractatus* 4.27 (= 9):

(20) There are $2^{N\times\pi}$ state-descriptions in $\mathfrak{L}_N{}^{\pi}$. (Carnap 1962, 121)[17]

Carnap wrote "ζ" instead of "$2^{N\times\pi}$" (Carnap 1962, 121). It is obvious that state-descriptions form a division with the same meaning as (14) (Carnap 1962, 94). Their important role in inductive logic is as follows:

(21) Any sentence $\mathfrak{S}$[18] in $\mathfrak{L}_N{}^{\pi}$ can be expressed with a disjunction of state-descriptions as follows:
$\ulcorner\mathfrak{S}\longleftrightarrow\mathfrak{Z}_{i1}\lor\mathfrak{Z}_{i2}\lor \ldots \lor\mathfrak{Z}_{im}\urcorner$ (Carnap 1962, 94)[19]

The second subscript "m" in "$\mathfrak{Z}_{im}$" is an arbitrary number.

In this way, we can define possible worlds formally in inductive logic. Then, based on it, let us try to answer the question asked from the beginning of Section 5.

(22) For any $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$, and $\mathfrak{Z}_i$ in $\mathfrak{L}_N{}^{\pi}$, we adopt the following recursive definition (Carnap 1962, 78-79):
$\mathfrak{S}_1$ *holds in* $\mathfrak{Z}_i$ $\Longleftrightarrow$ ① If $\mathfrak{S}_1$ is an atomic sentence, then $\mathfrak{S}_1$ is a conjunct of $\mathfrak{Z}_i$.
② If $\mathfrak{S}_1$ is $\ulcorner\neg\mathfrak{S}_2\urcorner$, then $\mathfrak{S}_2$ does not hold in $\mathfrak{Z}_i$.
③ If $\mathfrak{S}_1$ is $\ulcorner\mathfrak{S}_2\lor\mathfrak{S}_3\urcorner$, then $\mathfrak{S}_2$ or $\mathfrak{S}_3$ holds in $\mathfrak{Z}_i$.
④ If $\mathfrak{S}_1$ is $\ulcorner\forall x\mathfrak{M}(x)\urcorner$, then all of (a_1), … $\mathfrak{M}(a_N)$ hold in $\mathfrak{Z}_i$.

Those who found state-descriptions correspondent to truth assignments (cf. (12) above) find this (22) correspondent to the usual definition of truth (cf. Carnap 1962, 68-69, 78).

From this recursive definition, we can obtain a strict definition of the range:

(23) For any sentence $\mathfrak{S}$ in $\mathfrak{L}_N{}^{\pi}$, its *range* is defined as follows:

$\mathfrak{R}(\mathfrak{S}) = \{\mathfrak{Z}_i \mid \mathfrak{S} \text{ holds in } \mathfrak{Z}_i\}$ (Carnap 1962, 79)

8. The Problem of Measure

With these definitions, the preceding arguments are followed up. Based on it, then, let us tackle the issue "How can we measure ranges?" (Section 5).

One might think that it suffices to identify the measure of a range with its cardinality—card($\mathfrak{R}(\mathfrak{S})$). This is much the same idea as the measure that Carnap called $\mathfrak{m}^\dagger$ (1962, 564-65).[20] However, if we adopt $\mathfrak{c}$-functions based on $\mathfrak{m}^\dagger$ or cardinality, inductive logic disregarding empirical data is constructed, that is, with respect to relevant evidence, $\mathfrak{c}^\dagger(h, e_1) = \mathfrak{c}^\dagger(h, e_1 \wedge e_2) = \mathfrak{c}^\dagger(h, e_1 \wedge e_2 \wedge e_3) = \cdots$.[21] Therefore, Carnap had to turn it down, and sought for the better measure for his logic.

At the beginning, he assumed a set of all measures, calling it *$\mathfrak{m}$-functions* (1962, 293f.). His strategy was, then, choosing better $\mathfrak{m}$-functions from this great number of candidates. However, in this course of thought, he seems to have faced a fatal difficulty: it is impossible to measure a range itself directly; in other words, we cannot give any values to $\mathfrak{m}(\mathfrak{R}(\mathfrak{S}))$. We must recognize this fact first.

Take another look at (23), according to which a range is merely a set of state-descriptions. So the following equation may be expected.

(24) $\mathfrak{m}(\mathfrak{R}(\mathfrak{S})) = \mathfrak{m}(\{\mathfrak{Z}_{i1}, \dots, \mathfrak{Z}_{in}\})$ (Provable)

If the following (25) held in addition to this (24), we could proceed further smoothly:[22]

(25) $\mathfrak{m}(\{\mathfrak{Z}_{i1}, \dots, \mathfrak{Z}_{in}\}) = \mathfrak{m}(\mathfrak{Z}_{i1} \vee \dots \vee \mathfrak{Z}_{in})$ (Unprovable)

But this does not hold. For, certainly, the following holds:

(26) $\mathfrak{R}(\mathfrak{Z}_{i1} \vee \dots \vee \mathfrak{Z}_{in}) = \{\mathfrak{Z}_{i1}, \dots, \mathfrak{Z}_{in}\}$ (Provable)

However, between the sentence $\mathfrak{Z}_{i1} \vee \dots \vee \mathfrak{Z}_{in}$ and the set $\{\mathfrak{Z}_{i1}, \dots, \mathfrak{Z}_{in}\}$, we cannot prove any identity. In other words, $\{\mathfrak{Z}_{i1}, \dots, \mathfrak{Z}_{in}\}$ can be *the range of* $\mathfrak{Z}_{i1} \vee \dots \vee \mathfrak{Z}_{in}$ but cannot be $\mathfrak{Z}_{i1} \vee \dots \vee \mathfrak{Z}_{in}$ *itself*.

If we find a sentence with which $\{\mathfrak{Z}_{i1}, \dots, \mathfrak{Z}_{in}\}$ is identified, we can proceed further than (24), but we cannot find it.

On trial, let us take up the following statement of Carnap's as a clue:

(27) A set of sentences is logically equivalent to the conjunction of its elements. (Carnap 1962, 86)

This statement is acceptable, considering a set of sentences working in $\{\mathfrak{S}_1, \mathfrak{S}_2\} \vDash \mathfrak{S}_3$, for example. Thus, the following identity may be expected:

(28) $\mathfrak{m}(\mathfrak{R}(\mathfrak{S})) = \mathfrak{m}(\{\mathfrak{Z}_{i1}, \dots, \mathfrak{Z}_{in}\}) = \mathfrak{m}(\mathfrak{Z}_{i1} \wedge \dots \wedge \mathfrak{Z}_{in})$ (Provable, but nonsense)

But this idea does not hold, either. For state-descriptions are exclusive with each other (cf. (19) above), so that $\mathfrak{Z}_{i1} \wedge \dots \wedge \mathfrak{Z}_{in}$ is contradictory.

After all, we must give up measuring a range itself directly. In other words, we cannot afford any values to $\mathfrak{m}(\mathfrak{R}(\mathfrak{S}))$.

This is why an alternative method is sought for. As such, we may focus on the state-descriptions inside a range. That is, measuring not the range itself but the state-descriptions inside it—this could be the alternative:

(29) For any sentence $\mathfrak{S}$ in $\mathfrak{L}_N^\pi$, $\mathfrak{m}(\mathfrak{S}) = \sum_{j=1}^{n} \mathfrak{m}(\mathfrak{Z}_{ij})$ for $\mathfrak{Z}_{ij} \in \mathfrak{R}(\mathfrak{S})$. (Carnap 1962, 295)

This idea is naturally derived from the preceding theorem (21) as well.

Anyway, this alternative finally gives us the answer to our original question "How can we measure a range?". According to it, the answer is: "By measuring the state-descriptions *inside* the range." So we must not forget: in (29), "$\mathfrak{m}(\mathfrak{S})$" plays a role of "$\mathfrak{m}(\mathfrak{R}(\mathfrak{S}))$." This is the method Carnap finally adopted.

9. The Explication of Conditional Probability

Now we got to the starting point of inductive logic. Together with (29), by adopting the following rule, we can axiomatize inductive logic.

(30) For any sentence $\mathfrak{S}$ in $\mathfrak{L}_N{}^{\pi}$, if $\mathfrak{S}$ is logically false, then $\mathfrak{m}(\mathfrak{S}) = 0$. (Carnap 1962, 295)

In a great number of $\mathfrak{m}$-functions, those fulfilling (29) and (30) are to form a subset of *regular* members. Accordingly, Carnap called them *regular $\mathfrak{m}$-functions* (1962, 295). But, why "regular"? The answer is easy. It is because such $\mathfrak{m}$-functions fulfill the orthodox axioms of probability. That is, those axioms are proved from (29) and (30).[23]

On the other hand, we must carefully treat the fact that $\mathfrak{m}$-functions now play the role of *unconditional probability*. It appears to conflict with another fact that Carnap preferred conditional probability in his system (cf. Section 4). To reconcile it with the other, we must say: $\mathfrak{m}$-functions were merely a minor point in his course of thought. Indeed, Carnap admitted it with this formula:

(31) $\mathfrak{c}(h, e) = \frac{\mathfrak{m}(h \wedge e)}{\mathfrak{m}(e)}$ (Carnap 1962, 295)

$\mathfrak{M}$-functions are no more than parts of a $\mathfrak{c}$-function. Let me further explain it from the viewpoint of Section 4.

Firstly, we can prove that $\mathfrak{R}(h) \cap \mathfrak{R}(e) = \mathfrak{R}(h \wedge e)$ (Carnap 1962, 79). According to Section 8, the size of $\mathfrak{R}(h \wedge e)$ is measured as $\mathfrak{m}(h \wedge e)$, and that of $\mathfrak{R}(e)$ as $\mathfrak{m}(e)$. Therefore, in (31), $\mathfrak{m}(h \wedge e)$ means the size of $\mathfrak{R}(h \wedge e)$, and $\mathfrak{m}(e)$ the size of $\mathfrak{R}(e)$.

In Section 4, we saw that the larger $\mathfrak{R}(h) \cap \mathfrak{R}(e)$ is, the greater is the possibility that *h* becomes true (on the assumption that the size of $\mathfrak{R}(e)$ is not changed). This was the case when X thought about the probability of *h* with regard to *e*. Furthermore, we can say that the larger $\mathfrak{R}(e)$ is, the *lower* is the possibility that *h* becomes true (on the assumption that the size of $\mathfrak{R}(h) \cap \mathfrak{R}(e)$ is not changed). In short, the size of $\mathfrak{R}(h \wedge e)$ is in direct proportion to the probability of *h* with regard to *e*, but the size of $\mathfrak{R}(e)$ is in inverse proportion to it. This is the meaning of (31).

10. Symmetrical Functions

This is how Carnap explicated the concept of conditional probability (1962, 298). Here, let us recall that conditional probability was the same as what Carnap intended to clarify in his system: probability$_1$ or a degree of confirmation (Sections 2-4). Then, have we already reached his goal?

Not yet. Recall the confirmation Carnap pursued was the quantitative concept, namely $\mathfrak{c}$-function (Section 2). So he still wondered how concrete values are assigned to $\mathfrak{c}$-functions. And to solve the problem, he narrowed down the great many $\mathfrak{m}$-functions from regular $\mathfrak{m}$-functions further to *symmetrical $\mathfrak{m}$-functions*. Let us trace his argument below.

(32) The function C that substitutes for individual constants appearing in a sentence $\mathfrak{S}$ other ones in accordance with a particular rule is called an *in-correlation* (Carnap 1962, 109).

"$\mathfrak{In}$" of $\mathfrak{in}$-correlation is a mere abbreviation of "individual constant" (Carnap 1962, 55-56). Carnap symbolized "a particular rule" in this definition by $\begin{pmatrix} a_1 & a_2 \\ \downarrow & \downarrow \\ a_2 & a_1 \end{pmatrix}$,[24] for example. Following this rule, $P_1(a_1)$ in $\mathfrak{L}_2^2$ is altered to $P_1(a_2)$. The latter is expressed as $C(P_1(a_1))$ and called a C-correlate of $P_1(a_1)$ (Carnap 1962, 55-56). As in this example, it is possible that individual constants in $\mathfrak{S}$ do *not* appear in $C(\mathfrak{S})$. On the contrary, if all individual constants in $\mathfrak{S}$ also appear in $C(\mathfrak{S})$, that is, $C(\mathfrak{S})$ is different from $\mathfrak{S}$ only in the order of individual constants, we call $C(\mathfrak{S})$ *isomorphic* to $\mathfrak{S}$ (Carnap 1962, 55-56). Obviously, any C-correlate of a state-description is isomorphic to the state-description, that is, $C(\mathfrak{Z}_i)$ is isomorphic to $\mathfrak{Z}_i$. For example, considering the rule above in $\mathfrak{L}_2^2$ and (12), $C(\mathfrak{Z}_5) = \mathfrak{Z}_2$ is isomorphic to $\mathfrak{Z}_5$. Symmetrical $\mathfrak{m}$-functions have to do with such state-descriptions:

(33) Those $\mathfrak{m}$-functions which assign the same value to all isomorphic state-descriptions are called *symmetrical $\mathfrak{m}$-functions* (Carnap 1962, 485).

In (12), if $\mathfrak{m}$ is symmetrical, $\mathfrak{m}(\mathfrak{Z}_2) = \mathfrak{m}(\mathfrak{Z}_5), \mathfrak{m}(\mathfrak{Z}_3) = \mathfrak{m}(\mathfrak{Z}_9)$, and so on.

Now then, how many isomorphic state-descriptions are there for one state-description? The answer is as follows:

(34) Suppose concerning a state-description $\mathfrak{Z}_i$ in $\mathfrak{L}_N^\pi$, the number of individual constants which Q_1 is predicated of is N_1, and likewise, N_2 for Q_2, …, N_κ for Q_κ. Then, the number of state-descriptions isomorphic to $\mathfrak{Z}_i$ is $\frac{N!}{N_1! \times N_2! \times \ldots \times N_\kappa!}$. (Carnap 1962, 140)[25]

I write "ξ" instead of "$\frac{N!}{N_1! \times N_2! \times \ldots \times N_\kappa!}$".

11. Statistics in Inductive Logic

A $\mathfrak{c}$-function was defined in (31); additionally, if $\mathfrak{m}$ is a regular $\mathfrak{m}$-function there, the $\mathfrak{c}$-function defined from it is also called *regular* (Carnap 1962, 295). Furthermore, if $\mathfrak{m}$ is a symmetrical $\mathfrak{m}$-function, it is called a *symmetrical $\mathfrak{c}$-function* (Carnap 1962, 486).

We have already seen that regular $\mathfrak{c}$-functions were devised for probabilistic logic, such as Kolmogoroff's system (Section 9). In contrast, the symmetrical $\mathfrak{c}$-functions just mentioned were devised for statistics including the *Probability Integral* Φ(z) for *normal distribution*, *Bernoulli's Limit Theorem*, and so on. It is a bit surprising that Carnap included even these famous statistical laws as theorems of his system.[26]

Unfortunately, it is impossible to explain all the details here, but we can abstract its main parts. (Note that "p," "s_1," or the like below express cardinal numbers. See note 12.)

(35) In $\mathfrak{L}_N^\pi$, consider p molecular predicates forming a division. Let i be an individual distribution (cf. (41) below) stating that over the first sample $K_1 = \{a_1, \ldots, a_s\}$, M_1 is predicated of s_1 individual constants, and likewise, s_2 for M_2, …, s_p for M_p. Similarly i', over the second sample $K_2 = \{a_{s+1}, \ldots, a_{s+s'}\}$, s'_1 is for M_1, s'_2 for M_2, …, s'_p for M_p. Let j be a statistical distribution (cf. (43) below) corresponding to i, and j' to i'. Since $i \wedge i'$ is also an individual distribution, we can form a statistical distribution J corresponding to $i \wedge i'$. (J is to be a statement over s + s′ individual constants.) Then, with a symmetrical $\mathfrak{c}$-function, the following holds:

$$\mathfrak{c}(j, J) = \frac{{}_{s_1+s'_1}C_{s_1} \times {}_{s_2+s'_2}C_{s_2} \times \cdots \times {}_{s_p+s'_p}C_{s_p}}{{}_{s+s'}C_s} \quad \text{(Carnap 1962, 491)}$$

This is a key theorem for symmetrical functions; but we put off its proof until succeeding sections (Sections 12-13). Before that, we see the relationship of this theorem with statistical investigation.

In (35), let $K_1 \cup K_2$ be equal to Const., i.e., the population, and let N_1 be the number of individual constants which M_1 is predicated of in Const., and likewise, N_2 for M_2, …, N_p for M_p; hence, $s + s' = N$, $s_1 + s'_1 = N_1$, …, $s_p + s'_p = N_p$. Then, the following holds as a corollary of (35):

(36) Consider $\mathfrak{L}_N^1$: Const. = $\{a_1, \ldots, a_N\}$, Pred. = $\{P_1\}$, $\forall x[Q_1(x) \leftarrow\rightarrow P_1(x)]$, $\forall x[Q_2(x) \leftarrow\rightarrow \neg P_1(x)]$. As evidence, let *J* be a statistical distribution stating that over the population Const., the absolute frequency of Q_1 is N_1, and that of Q_2 is N_2. As a hypothesis, let *j* be a statistical distribution stating that over the sample $K_1 = \{a_1, \ldots, a_s\}$, the absolute frequency of Q_1 is s_1, and that of Q_2 is s_2. Then, with a symmetrical $\mathfrak{c}$-function, the following holds:

$$\mathfrak{c}(j, J) = \frac{{}_{N_1}C_{s_1} \times {}_{N_2}C_{s_2}}{{}_{N}C_{S}} \qquad \text{(cf. Carnap 1962, 495)}$$

We may start our statistical investigation from this (36). After that, as the number of N, S, N_1, N_2, s_1, or s_2 become larger, we may proceed to use the *Binomial Law*:[27]

(37) $$\mathfrak{c}(j, J) \cong {}_{s}C_{s_1} \times \left(\frac{N_1}{N}\right)^{s_1} \times \left(\frac{N_2}{N}\right)^{s_2} \qquad \text{(Carnap 1962, 499)}^{28}$$

Furthermore, when the number becomes much larger, we may use the *Normal Law*:

(38) $$\mathfrak{c}(j, J) \cong \frac{1}{\sigma \times \sqrt{2\pi}} \times e^{-\frac{\delta^2}{2\sigma^2}} = \frac{1}{\sigma} \times \phi\left(\frac{\delta}{\sigma}\right) \qquad \text{(Carnap 1962, 504)}^{29}$$

Here, $\sigma = \sqrt{s \times \left(\frac{S_1}{N}\right) \times \left(\frac{S_2}{N}\right)}$, i.e., the standard deviation; π is the ratio of the circumference; $\delta = s_1 - s \times \left(\frac{S_1}{N}\right)$, i.e., the deviation; e is the base of natural logarithms; $\phi(x) = \frac{1}{\sigma \times \sqrt{2\pi}} \times e^{-\frac{x^2}{2}}$, i.e., the *Normal Function* (Carnap 1962, 153).

These theorems are provable in inductive logic, as stated above. Carnap never disregarded statistical procedures.

12. The Form of Evidence and Hypotheses

Now then, let us get back to theorem (35). To prove it, however, we need some more terminology. Let me complement it.

(39) The disjunction that connects all isomorphic state-descriptions in the following way is called a *structure description*:

$$\mathfrak{Str}_i = \ulcorner \mathfrak{Z}_{i1} \vee \mathfrak{Z}_{i2} \vee \ldots \vee \mathfrak{Z}_{i\xi} \urcorner \qquad \text{(Carnap 1962, 116)}$$

Isomorphic state-descriptions have individual constants—all individual constants in $\mathfrak{L}_N^\pi$—in common (see (32) and the following). Thus, all the isomorphic state-descriptions connected in one disjunction are thought to express a *statistical datum* focusing only on properties (Q-predicates) and abstracting data on individuals. This is the meaning of structure-descriptions.

With regard to table (12) in $\mathfrak{L}_2^2$, 16 state-descriptions are reduced to 10 structure-descriptions: $\mathfrak{Str}_1 = \mathfrak{Z}_1$, $\mathfrak{Str}_2 = \mathfrak{Z}_2 \vee \mathfrak{Z}_5$, $\mathfrak{Str}_3 = \mathfrak{Z}_3 \vee \mathfrak{Z}_9$, $\mathfrak{Str}_4 = \mathfrak{Z}_4 \vee \mathfrak{Z}_{13}$, $\mathfrak{Str}_5 = \mathfrak{Z}_6$, $\mathfrak{Str}_6 = \mathfrak{Z}_7 \vee \mathfrak{Z}_{10}$, $\mathfrak{Str}_7 = \mathfrak{Z}_8 \vee \mathfrak{Z}_{14}$, $\mathfrak{Str}_8 = \mathfrak{Z}_{11}$, $\mathfrak{Str}_9 = \mathfrak{Z}_{12} \vee \mathfrak{Z}_{15}$, $\mathfrak{Str}_{10} = \mathfrak{Z}_{16}$. In general, the following holds about the number of structure-descriptions:

(40) There are ${}_{\kappa+N-1}C_N$ structure-descriptions in $\mathfrak{L}_N^\pi$. (Carnap 1962, 138)[30]

Carnap wrote "τ" instead of "${}_{\kappa+N-1}C_N$" (Carnap 1962, 121).

(41) The conjunction stating over s individual constants and p molecular predicates forming a division *which*

predicate is predicated of *which individual constant* is called an individual distribution:

$$i_k = \ulcorner M_{k1}(a_{j1}) \wedge M_{k2}(a_{j2}) \wedge \ldots \wedge M_{ks}(a_{js}) \urcorner$$ (Carnap 1962, 111)

"M_{k1}" to "M_{ks}" is one of p molecular predicates. "s" is an arbitrary number. Note that "M_{ki}" ($1 \leq I \leq s$) is not confined to a Q-predicate, and individual distributions do not have to provide a datum on all individual constants in contrast with state-descriptions.

For example, suppose that "M_1" and "M_2" are defined as follows: $\forall x[M_1(x) \longleftrightarrow \{P_1(x) \rightarrow P_2(x)\}]$, $\forall x[M_2(x) \longleftrightarrow \{P_1(x) \wedge \neg P_2(x)\}]$ in $\mathfrak{L}_3{}^2$, where Const. = $\{a_1, a_2, a_3\}$ and Pred. = $\{P_1, P_2\}$. Then, $M_2(a_1) \wedge M_1(a_2)$ is an individual distribution.

The following is analogous to (34):

(42) Let *i* be an individual distribution in $\mathfrak{L}_N{}^\pi$ stating that over s individual constants, M_1 is predicated of s_1 individual constants, and likewise, s_2 for M_2, …, s_p for M_p. Then, the number of individual distributions isomorphic to *i* is $\frac{s!}{s_1! \times s_2! \times \ldots \times s_p!}$. (Carnap 1962, 158)

I write "ς" instead of "$\frac{s!}{s_1! \times s_2! \times \ldots \times s_p!}$".

(43) The disjunction connecting all isomorphic individual distributions in the following way is called a *statistical distribution*.

$$j_k = \ulcorner i_{k1} \vee i_{k2} \vee \ldots \vee i_{k\varsigma} \urcorner$$ (Carnap 1962, 111)

For example, in $\mathfrak{L}_3{}^2$ above, $\{M_1(a_1) \wedge M_2(a_2)\} \vee \{M_2(a_1) \wedge M_1(a_2)\}$ is a statistical distribution.

Statistical distributions state the absolute frequency or relative frequency of each M_i ($1 \leq I \leq p$). As individual distributions correspond to state-descriptions, so statistical distributions correspond to structure-distributions (Carnap 1962, 115).

We adopt individual-distributions and statistical-distributions as the form of hypotheses and evidence.

13. The Proof of (35)

Let us then prove theorem (35). The subsequent (44) to (51) are lemmas for the proof; it is premised that $\mathfrak{m}$ is a symmetrical $\mathfrak{m}$-function and i, j, j', and J are the same as (35).

(44) Let $\mathfrak{S}_i$ be a sentence in $\mathfrak{L}_N{}^\pi$ and $\mathfrak{R}(\mathfrak{S}_i) = \{\mathfrak{Z}_{i1}, \ldots, \mathfrak{Z}_{in}\}$. Then, the range of $C(\mathfrak{S}_i)$ is constructed from elements in $\mathfrak{R}(\mathfrak{S}_i)$, i.e., $\mathfrak{R}(C(\mathfrak{S}_i)) = \{C(\mathfrak{Z}_{i1}), \ldots, C(\mathfrak{Z}_{in})\}$. (Carnap 1962, 110)[31]

(45) For any sentence $\mathfrak{S}_i$, $\mathfrak{m}(\mathfrak{S}_i) = \mathfrak{m}(C(\mathfrak{S}_i))$. (Carnap 1962, 488)[32]

(46) Let *i* be an individual distribution and *j* a statistical distribution corresponding to *i*. Then,

$$\mathfrak{m}(j) = \frac{s!}{s_1! \times s_2! \times \ldots \times s_p!} \times \mathfrak{m}(i)$$ (Carnap 1962, 490)[33]

(47) $\vDash j \wedge j' \rightarrow J$ (Carnap 1962, 491)[34]

(48) $\vDash j \wedge J \rightarrow j'$ (Carnap 1962, 491)[35]

(49) $\vDash j \wedge J \longleftrightarrow j \wedge j'$ (Carnap 1962, 491)[36]

(50) Let $\mathfrak{S}_k$ be a sentence that has no individual constants in common with *j*; then,

$$\mathfrak{m}(\mathfrak{S}_k \wedge j) = \frac{s!}{s_1! \times s_2! \times \ldots \times s_p!} \times \mathfrak{m}(\mathfrak{S}_k \wedge i)$$ (Carnap 1962, 490)[37]

(51) $\mathfrak{m}(j \wedge j') = \frac{s!}{s_1! \times s_2! \times \ldots \times s_p!} \times \frac{s'!}{s'_1! \times s'_2! \times \ldots \times s'_p!} \times \mathfrak{m}(i \wedge i')$ (Carnap 1962, 491).[38]

From these lemmas, (35) is proved as follows:

(52) $\mathfrak{c}(j, J) = \frac{\mathfrak{m}(j \wedge J)}{\mathfrak{m}(J)}$ from (31)

$$= \frac{\mathfrak{m}(j \wedge j')}{\mathfrak{m}(J)} \qquad \text{from (49)}$$

$$= \frac{\frac{s!}{s_1! \times s_2! \times \ldots \times s_p!} \times \frac{s'!}{s'_1! \times s'_2! \times \ldots \times s'_p!} \times \mathfrak{m}(i \wedge i')}{\frac{(s+s')!}{(s_1+s'_1)! \times (s_2+s'_2)! \times \ldots \times (s_p+s'_p)!} \times \mathfrak{m}(i \wedge i')} \qquad \text{from (51) and (46)}$$

$$= \frac{\frac{(s_1+s'_1)!}{s_1! \times s'_1!} \times \frac{(s_2+s'_2)!}{s_2! \times s'_2!} \times \ldots \times \frac{(s_p+s'_p)!}{s_p! \times s'_p!}}{\frac{(s+s')!}{s! \times s'!}} = \frac{{}_{s_1+s'_1}C_{s_1} \times {}_{s_2+s'_2}C_{s_2} \times \cdots \times {}_{s_p+s'_p}C_{s_p}}{{}_{s+s'}C_s} \quad \blacksquare$$

14. From Direct Inference to Predictive Inference

Theorem (35) provides inductive logic with a concrete value for the first time (Carnap 1962, 492). Therefore, Carnap seems to have attained his aim here. But this is not the case. Take another look at (35). Therein, not only the information of relative frequency over a sample (here I regard K_2 as the sample) but also that over a population (here I regard $K_1 \cup K_2$ as the population) is provided in advance. This is strange, because in most cases where we make an inductive inference, we do not have information all over the population; our information is restricted to a sample, a part of the population. We must make an inference from one sample (e.g., K_2 in (35)) to the other (e.g., K_1 in (35)) within this restriction. Nevertheless, the information of both samples is provided in (35) in advance.

Carnap noticed this strangeness. So, naming such inferences as (35) *direct inferences*, he strictly distinguished them from customary inductive inferences (1962, 207). The customary inductive inferences go from one sample to the other within restricted information. Carnap called such inferences *predictive inferences* (1962, 207). It is this kind of inferences that inductive logic had to deal with.

15. The First Solution

For the reason just mentioned, (35) could not be the final solution for Carnap. It is a method only for direct inferences; but our inductive inferences are usually predictive inferences.

The target of inductive logic must be predictive inferences. We may define them formally as the inferences from the first sample $K_1 = \{a_1, \ldots, a_s\}$ to the second sample $K_2 = \{a_{s+1}, \ldots, a_{s+s'}\}$, where K_1 and K_2 are subsets of the population Const. (In this definition, symbols are used in a different way from those in theorem (35). But it does not influence the following arguments.)

Here, imagine this scenario. In $\mathfrak{L}_2^2$ above, we research the properties expressed by M_1 and M_2. Their definitions are: $\forall x[M_1(x) \longleftrightarrow Q_1(x) \vee Q_2(x)]$, i.e., P_1, and $\forall x[M_2(x) \longleftrightarrow Q_3(x) \vee Q_4(x)]$, i.e., P_2 respectively. Again, we have already observed, as the first sample, a_1 being M_1. In this situation, what is the probability of a_2 also being M_1?

This is a simple, good example to consider a predictive inference formally. Since we are supposed to have no information about population, the preceding method exhibited in Section 11 is not available now. Thus, we must construct a new method. This is possible, developing the basic idea mentioned in Section 4.

The basic idea was concerned with ranges. We have already seen that ranges are measured by (29). But, following (29) alone, an $\mathfrak{m}$-function (i.e., a regular $\mathfrak{m}$-function) could not yield any concrete values.

In contrast, if only a measure, i.e., the value of $\mathfrak{m}(\mathfrak{Z}_i)$, is decided for all state-descriptions, we are able to give concrete values to any sentence. This is the breakthrough we are seeking for. So how we decide the value of $\mathfrak{m}(\mathfrak{Z}_i)$—this is the most important problem.

One might think it suffices to give equal values to each state-description; this is much the same idea as

Laplace's *principle of indifference*. But we have already turned it down (Section 8). As $\mathfrak{m}^{\dagger}$, Carnap also denied it (1962, 564).[39] In the end, the measure he adopted was the following $\mathfrak{m}^*$:

(53) For any $\mathfrak{Z}_i$ in $\mathfrak{L}_N^{\pi}$, $\mathfrak{m}^*(\mathfrak{Z}_i) = \frac{1}{\tau} \times \frac{1}{\xi}$. (Carnap 1962, 563)

The idea behind this formula is as follows.[40] The fault of $\mathfrak{m}^{\dagger}$ is that it cannot reflect the influence of empirical evidence (cf. Section 8). To reflect it, then, what should be done? Suppose that we observed regularized samples, e.g., $M_1(a_1) \wedge M_1(a_2)$. If we will utilize this precious experience, we must prefer the hypothesis describing the more regularized world, e.g., $\mathfrak{Z}_I = M_1(a_1) \wedge M_1(a_2) \wedge M_1(a_3)$ to the one describing the less regularized world, e.g., $\mathfrak{Z}_{II} = M_1(a_1) \wedge M_1(a_2) \wedge M_2(a_3)$. (Here, for simplicity, we allow the notation of state-descriptions by molecular predicates.) If only we introduce this way of thinking, the better function will be provided.

Generally speaking, structure-descriptions to which more regularized worlds belong are composed of fewer disjuncts. For example, compare $\mathfrak{Str}_1$ with $\mathfrak{Str}_2$ in $\mathfrak{L}_2^2$ (see (39) and the following). The world that belongs to $\mathfrak{Str}_1$, i.e., $\mathfrak{Z}_1$, is more regularized than the worlds to $\mathfrak{Str}_2$, i.e., $\mathfrak{Z}_2$ and $\mathfrak{Z}_5$. To be concrete, $\mathfrak{Z}_1$ in $\mathfrak{Str}_1$ is described with only one Q-predicate, i.e., Q_1; on the contrary, $\mathfrak{Z}_2$ and $\mathfrak{Z}_5$ in $\mathfrak{Str}_2$ are described with two Q-predicates, i.e., Q_1 and Q_2. On account of this, $\mathfrak{Str}_1$ is composed of fewer disjuncts (= $\mathfrak{Z}_1$) than $\mathfrak{Str}_2$ (= $\mathfrak{Z}_2$, $\mathfrak{Z}_5$).

This fact is understandable from (34). Focus on the second subscript "ξ" of the last disjunct in (39). ξ is the number stated in (34). It is obvious from (34) that the state-description in which fewer Q-predicates appear (so, which is "regularized") has fewer state-descriptions isomorphic to itself. Therefore, the structure-description composed of regularized state-descriptions has fewer disjuncts.

In accordance with this fact, firstly, we assign equal values to each structure-description, and secondly, make the assigned value divided among disjuncts of each structure-description:

(54)
$$
\begin{array}{lllll}
1 & \rightarrow & \mathfrak{Str}_1 = \frac{1}{10} & \longrightarrow & \mathfrak{Z}_1 = \frac{1}{10} \\
 & \searrow & \mathfrak{Str}_2 = \frac{1}{10} & \longrightarrow & \mathfrak{Z}_2 = \frac{1}{5} \\
 & & \vdots & \searrow & \mathfrak{Z}_5 = \frac{1}{5} \\
 & \searrow & \mathfrak{Str}_{10} = \frac{1}{10} & &
\end{array}
$$
Example in (12)

In this way, more regularized worlds (state-descriptions) come to obtain higher values; in consequence, the hypotheses including more regularized worlds will obtain higher values. (Recall that an hypothesis is restated as a disjunction of state-descriptions. See (21) again.)

This is the idea behind (53). And it gives the answer to our original question "What is the probability of a_2 also being M_1?".

(55)
$$c^*(M_1(a_2), M_1(a_1)) = \frac{\mathfrak{m}^*(M_1(a_2) \wedge M_1(a_1))}{\mathfrak{m}^*(M_1(a_1))} \quad \text{from (31)}$$
$$= \frac{\mathfrak{m}^*(\mathfrak{Z}_1 \vee \mathfrak{Z}_5 \vee \mathfrak{Z}_2 \vee \mathfrak{Z}_6)}{\mathfrak{m}^*(\mathfrak{Z}_1 \vee \mathfrak{Z}_2 \vee \mathfrak{Z}_3 \vee \mathfrak{Z}_4 \vee \mathfrak{Z}_5 \vee \mathfrak{Z}_6 \vee \mathfrak{Z}_7 \vee \mathfrak{Z}_8)} \quad \text{from (21) and (12)}$$
$$= \frac{\mathfrak{m}^*(\mathfrak{Z}_1)+\mathfrak{m}^*(\mathfrak{Z}_5)+\mathfrak{m}^*(\mathfrak{Z}_2)+\mathfrak{m}^*(\mathfrak{Z}_6)}{\mathfrak{m}^*(\mathfrak{Z}_1)+\mathfrak{m}^*(\mathfrak{Z}_2)+\mathfrak{m}^*(\mathfrak{Z}_3)+\mathfrak{m}^*(\mathfrak{Z}_4)+\mathfrak{m}^*(\mathfrak{Z}_5)+\mathfrak{m}^*(\mathfrak{Z}_6)+\mathfrak{m}^*(\mathfrak{Z}_7)+\mathfrak{m}^*(\mathfrak{Z}_8)} \quad \text{from exclusiveness (Carnap 1962, 307)}$$
$$= \frac{3}{5} \quad \text{from (53)}$$

This value is a little higher than the value given before observation: $c^*(M_1(a_1), t) = \frac{1}{2}$. ("t" means tautology, which means "We have not observed anything yet." See Carnap 1962, 307f.)

16. The Second Solution

This is how Carnap created the function suitable for predictive inferences. However, on the other hand, he noticed its limitation.

As the number of N or that of π in $\mathfrak{L}_N^{\pi}$ becomes larger, the number of state-descriptions and that of structure-descriptions become extravagantly large (Carnap 1962, 139). Again, to make matters worse, when using this method, we must take great pains to find, for each sentence, the state-descriptions the disjunction of which is equivalent to the sentence. For example, see the second step of (55). There, we had to look into table (12) thoroughly to find, for the sentence $M_1(a_1)$ in the denominator, the state-descriptions the disjunction of which is equivalent to it. (Recall that $M_1(a_1)$ is equivalent $Q_1(a_1) \vee Q_2(a_1)$, as stated in the second paragraph of Section 15. And see theorem (21) and note 19. Of course, the same is true of the numerator, i.e., $M_1(a_2) \wedge M_1(a_1) \longleftrightarrow \{Q_1(a_2) \vee Q_2(a_2)\} \wedge \{Q_1(a_1) \vee Q_2(a_1)\} \longleftrightarrow \{Q_1(a_1) \wedge Q_1(a_2)\} \vee \{Q_2(a_1) \wedge Q_1(a_2)\} \vee \{Q_1(a_1) \wedge Q_2(a_2)\} \vee \{Q_2(a_1) \wedge Q_2(a_2)\}$.) But this is a vital step to the present method.

Carnap noticed this difficulty and complexity (1962, 566). Thus, he looked for an alternative method successively (1962, 117). As a result, what he provided was the following:

(56) In $\mathfrak{L}_N^{\pi}$, consider p molecular predicates forming a division. As evidence, let *i* be an individual distribution stating that over the first sample $K_1 = \{a_1, \ldots, a_s\}$, M_1 is predicated of s_1 individual constants, and likewise, s_2 for M_2, …, s_p for M_p; let *j* be a statistical distribution corresponding to *i*. As a hypothesis, let *j'* be a statistical distribution stating that over the second sample $K_2 = \{a_{s+1}, \ldots, a_{s+s'}\}$, the absolute frequency of M_1 is s'_1, and likewise, s'_2 for M_2, …, s'_p for M_p. Then, the following holds:

$$\mathfrak{c}^*(j',i) = \mathfrak{c}^*(j',j) = \frac{\prod_{i=1}^{p} {}_{w_i+(s_i+s'_i)-1}C_{s'_i}}{{}_{\kappa+(s+s')-1}C_{s'}} \qquad \text{(Carnap 1962, 568)}$$

We may say: this is *the heart of* Carnap's inductive logic.

17. Fitting Sequence

Now we are at the heart of inductive logic. Theorem (56) provides us the most general method to calculate predictive inferences; furthermore, it becomes a connective point with the later system.

For its proof, first of all, we must recognize the most important feature of $\mathfrak{m}^*$.

(57) Consider the following two similar languages:
$\mathfrak{L}_N^{\pi}$: Const. = $\{a_1, \ldots, a_N\}$, Pred. = $\{P_1, \ldots, P_\pi\}$.
$\mathfrak{L}_{N+1}^{\pi}$: Const. = $\{a_1, \ldots, a_N, a_{N+1}\}$, Pred. = $\{P_1, \ldots, P_\pi\}$.
Let ${}_N\mathfrak{m}$ be the $\mathfrak{m}$-function for $\mathfrak{L}_N^{\pi}$, and ${}_{N+1}\mathfrak{m}$ for $\mathfrak{L}_{N+1}^{\pi}$.
Let ${}_N\mathfrak{Z}_j$ be a state-description in $\mathfrak{L}_N^{\pi}$, and ${}_{N+1}\mathfrak{Z}_j$ in $\mathfrak{L}_{N+1}^{\pi}$. Then,

$${}_N\mathfrak{m}^*({}_N\mathfrak{Z}_j) = {}_{N+1}\mathfrak{m}^*({}_N\mathfrak{Z}_j).$$

Let us prove this theorem, using the subsequent four paragraphs.

First, consider the measure of ${}_N\mathfrak{Z}_j$ in $\mathfrak{L}_{N+1}^{\pi}$, i.e., ${}_{N+1}\mathfrak{m}^*({}_N\mathfrak{Z}_j)$. Here, note that ${}_N\mathfrak{Z}_j$ is *not* a state-description in $\mathfrak{L}_{N+1}^{\pi}$ but a mere sentence in $\mathfrak{L}_{N+1}^{\pi}$. In accordance with (29), we can regard $\sum_{i=1}^{n} {}_{N+1}\mathfrak{m}^*({}_{N+1}\mathfrak{Z}_{ji})$ for ${}_{N+1}\mathfrak{Z}_{ji} \in \mathfrak{R}({}_N\mathfrak{Z}_j)$ as ${}_{N+1}\mathfrak{m}^*({}_N\mathfrak{Z}_j)$. But, what is the number of the superscript "n" of Σ? The answer is given in terms of (21):

(58)
$$\begin{aligned} {}_N\mathfrak{Z}_j &\longleftrightarrow {}_N\mathfrak{Z}_j \wedge \{Q_1(a_{N+1}) \vee Q_2(a_{N+1}) \vee \ldots \vee Q_\kappa(a_{N+1})\} && \text{from the exhaustiveness of Q-Predicates} \\ &\longleftrightarrow ({}_N\mathfrak{Z}_j \wedge Q_1(a_{N+1})) \vee ({}_N\mathfrak{Z}_j \wedge Q_2(a_{N+1})) \vee \ldots \vee ({}_N\mathfrak{Z}_j \wedge Q_\kappa(a_{N+1})) \\ &\longleftrightarrow {}_{N+1}\mathfrak{Z}_{j1} \vee {}_{N+1}\mathfrak{Z}_{j2} \vee \ldots \vee {}_{N+1}\mathfrak{Z}_{j\kappa}. \end{aligned}$$

Hence, n = κ.

Second, from (53), we obtain the measure of ${}_N\mathfrak{Z}_j$ in $\mathfrak{L}_N^{\pi}$:

(59) $${}_N\mathfrak{m}^*({}_N\mathfrak{Z}_j) = \frac{1}{{}_{\kappa+N-1}C_N} \times \frac{1}{\frac{N!}{N_1! \times N_2! \times \cdots \times N_\kappa!}} = \frac{N! \times (\kappa-1)!}{(\kappa+N-1)!} \times \frac{N_1! \times N_2! \times \cdots \times N_\kappa!}{N!}$$ See also (40) and (34).

Third, we obtain the concrete value of ${}_{N+1}\mathfrak{m}^*({}_N\mathfrak{Z}_j)$ in the following way:

(60) $${}_{N+1}\mathfrak{m}^*({}_N\mathfrak{Z}_j)$$
$$= {}_{N+1}\mathfrak{m}^*({}_{N+1}\mathfrak{Z}_{j1}) + {}_{N+1}\mathfrak{m}^*({}_{N+1}\mathfrak{Z}_{j2}) + \cdots + {}_{N+1}\mathfrak{m}^*({}_{N+1}\mathfrak{Z}_{j\kappa})$$ (See (58) and its preceding paragraph.)
$$= \frac{1}{{}_{\kappa+(N+1)-1}C_{N+1}} \times \frac{1}{\frac{(N+1)!}{(N_1+1)! \times N_2! \times \cdots \times N_\kappa!}} + \frac{1}{{}_{\kappa+(N+1)-1}C_{N+1}} \times \frac{1}{\frac{(N+1)!}{N_1! \times (N_2+1)! \times \cdots \times N_\kappa!}} + \cdots + \frac{1}{{}_{\kappa+(N+1)-1}C_{N+1}} \times \frac{1}{\frac{(N+1)!}{N_1! \times N_2! \times \cdots \times (N_\kappa+1)!}}$$ from(53)
$$= \frac{(\kappa-1)! \times (N+1)!}{(\kappa+N)!} \times \frac{N_1! \times N_2! \times \cdots \times N_\kappa!}{(N+1)!} \times (N+\kappa)$$ (Recall that $N_1 + N_2 + \cdots + N_\kappa = N$. cf. (34).)
$$= \frac{(\kappa-1)! \times N!}{(\kappa+N-1)!} \times \frac{N_1! \times N_2! \times \cdots \times N_\kappa!}{N!}$$

From (59) and (60), we realize that (57) holds. ■

In this proof, it is easily seen that the following also holds:

(61) $${}_N\mathfrak{m}({}_N\mathfrak{Z}_j) = \sum_{i=1}^{\kappa} {}_{N+1}\mathfrak{m}({}_{N+1}\mathfrak{Z}_{ji}),$$ where ${}_N\mathfrak{Z}_j$ is a subconjunction[41] of ${}_{N+1}\mathfrak{Z}_{ji}$.

Carnap called the $\mathfrak{m}$-functions fulfilling this condition a *fitting sequence* (1962, 309f.); thus, ${}_N\mathfrak{m}^*$ and ${}_{N+1}\mathfrak{m}^*$ form a fitting sequence.

Now, from (57), we can also obtain the following:

(62) If $\mathfrak{S}_i$ is a non-general sentence, i.e., neither universal nor existential, then,
$${}_N\mathfrak{m}^*(\mathfrak{S}_i) = {}_{N+1}\mathfrak{m}^*(\mathfrak{S}_i).$$ (Carnap 1962, 310)[42]

From this, by mathematical induction, we obtain the following:

(63) If $\mathfrak{S}_i$ is a non-general sentence, ${}_N\mathfrak{m}^*(\mathfrak{S}_i) = {}_{N+m}\mathfrak{m}^*(\mathfrak{S}_i)$. (Carnap 1962, 311).

"m" is an arbitrary number.

18. The Proof of (56)

Let us continue the proof of (56). Besides the theorem just proved (= 63), we need some more lemmas.

(64) If $\mathfrak{m}$ is a symmetrical function and the hypothesis has no individual constant in common with its evidence, it makes no difference whether the evidence is an individual distribution or a statistical distribution. (Carnap 1962, 490)[43]

(65) Let *J* be a statistical distribution stating that over a population, i.e., N individual constantsand p molecular predicates, i.e., $M_1, \ldots, M_p$, M_1 is predicated of N_1 individual constants, and likewise, N_2 for M_2, …, N_p for M_p. Then, *J* is also regarded as something like a structure-description. (Note that *J* is described with molecular predicates while structure-descriptions are described with Q-predicates. See (39), (19), (43), and (41) again.) On *J* regarded as such, we can find $\prod_{i=1}^{p} {}_{w_i+N_i-1}C_{N_i}$ patterns of structure-description. In other words:
$$J \longleftrightarrow \mathfrak{Str}_{J1} \vee \ldots \vee \mathfrak{Str}_{Jm}, \text{ where } m = \prod_{i=1}^{p} {}_{w_i+N_i-1}C_{N_i}.$$ [44]

(66) For any $\mathfrak{Str}_i$ in $\mathfrak{L}_N^{\pi}$, $\mathfrak{m}^*(\mathfrak{Str}_i) = \frac{1}{{}_{N+\kappa-1}C_{\kappa-1}}$. (Carnap 1962, 563).[45]

Then, let us return to (56). For its proof, we use the notation in (57) again. We begin with the left-most side of (56):

(67) ${}_N\mathfrak{c}^*(j', i) = {}_N\mathfrak{c}^*(j', j)$ from (64)

$$= \frac{{}_{N}\mathfrak{m}^{*}\ (j \wedge j')}{{}_{N}\mathfrak{m}^{*}\ (j)} \qquad \text{from (31)}$$

$$= \frac{\frac{s'!}{s'_1! \times \ldots \times s'_p!} \times {}_{N}\mathfrak{m}^{*}\ (i \wedge i')}{{}_{N}\mathfrak{m}^{*}\ (i)} \qquad \text{from (51) and (46)} \ \cdots ①.$$

In this last ①, we can regard $i \wedge i'$ as well as i as an individual distribution, and further, form a statistical distribution corresponding to $i \wedge i$, which we call J_1. In addition, we can also form a statistical distribution corresponding to i, calling it J_2. Let us resume the proof.

$$(68) \qquad ① = \frac{\frac{s'!}{s'_1! \times \ldots \times s'_p!} \times \frac{1}{\frac{(s+s')!}{(s_1+s'_1)! \times \ldots \times (s_p+s'_p)!}} \times {}_{N}\mathfrak{m}^{*}\ (J_1)}{\frac{1}{\frac{s!}{s_1! \times \ldots \times s_p!}} \times {}_{N}\mathfrak{m}^{*}\ (J_2)} \cdots ② \qquad \text{from (46)}$$

Here, by applying (63), we *upgrade* statistical distributions J_1 and J_2 to structure-descriptions in order to use (66) afterward:

$$(69) \qquad ② = \frac{\frac{s'!}{s'_1! \times \ldots \times s'_p!} \times \frac{1}{\frac{(s+s')!}{(s_1+s'_1)! \times \ldots \times (s_p+s'_p)!}} \times {}_{s+s'}\mathfrak{m}^{*}\ (J_1)}{\frac{1}{\frac{s!}{s_1! \times \ldots \times s_p!}} \times {}_{s}\mathfrak{m}^{*}\ (J_2)}$$

$$= \frac{\frac{s'!}{s'_1! \times \ldots \times s'_p!} \times \frac{1}{\frac{(s+s')!}{(s_1+s'_1)! \times \ldots \times (s_p+s'_p)!}} \times \frac{1}{{}_{\kappa+(s+s')-1}C_{s+s'}} \times \prod_{i=1}^{p} {}_{w_i+(s_i+s'_i)-1}C_{(s_i+s'_i)}}{\frac{1}{\frac{s!}{s_1! \times \ldots \times s_p!}} \times \frac{1}{{}_{\kappa+s-1}C_{s}} \times \prod_{i=1}^{p} {}_{w_i+s_i-1}C_{s_i}} \qquad \text{from (65) and (66)}$$

$$= \frac{\frac{1}{s'_1! \times \ldots \times s'_p!} \times \prod_{i=1}^{p} \frac{(w_i-1)!}{(w_i+s_i-1)!} \times \prod_{i=1}^{p} \frac{(w_i+(s_i+s'_i)-1)!}{(w_i-1)!}}{\frac{1}{s'} \times \frac{(\kappa+(s+s')-1)!}{(\kappa-1)!} \times \frac{(\kappa-1)!}{(\kappa+s-1)!}}$$

$$= \frac{\prod_{i=1}^{p} {}_{w_i+(s_i+s'_i)-1}C_{s'_i}}{{}_{\kappa+(s+s')-1}C_{s'}} \quad \blacksquare$$

19. Conclusion

This is how the most important theorem (56) is proved. To my surprise, Carnap did not prove it in his books. Yet, as we saw before (Section 14), this theorem alone provides a method for customary inductive inferences. So its proof must be the climax of Carnap's inductive logic. Why did Carnap omit it, then?

One reason is, presumably, that he gave up publishing a book that would complement his earlier system (Carnap 1962, xiii). As already stated in Section 1, Carnap published inductive logic mainly in two books: *Foundations* and *Continuum*. When writing *Foundations*, he seems to have planned to publish a pair of books named *Probability and Induction*.[46] *Foundations* was intended to be the first volume of the pair (Carnap 1962, viif.). This plan was still in place when *Continuum* was published (e.g., Carnap 1952, iii). However, when the second edition of *Foundations* was published in 1962, he suddenly gave up this plan (1962, xiii). This is why the second volume never appeared.

The disappearance of Vol. II is not settled merely as a change of Carnap's plan; it meant the disappearance of the parts that Carnap had postponed until Vol. II. In *Foundations*, especially, Carnap said many times that the concrete development of inductive logic was postponed until Vol. II (e.g., Carnap 1952, 45; 1962, 23; note 55). Therefore, it is not too much to say that the disappearance of Vol. II meant the disappearance of major parts of Carnap's inductive logic as well.

Among them was the proof of (56). But as stated above, it is the heart of Carnap's inductive logic. Its importance is not confined to that it is the sole formula for predictive inferences. It is nothing but the connective point between the earlier system and the later system. This is known by letting *j'* be a singular sentence of a molecular predicate in (56). Then, we obtain the following corollary:

(70) $c^*(M(a_{s+1}), i) = c^*(M(a_{s+1}), j) = \frac{s_M + w_M}{s + \kappa}$ (Carnap 1962, 568)

This shows how firmly the earlier system is connected with the later system. We know it from the similarity between (70) and (0). Furthermore, by letting $\lambda = \kappa$, we actually obtain (70) from (0).

But we should rather derive (0) from (70) conversely. Let us call $\frac{s_M}{s}$ in (70), the *empirical factor*, and $\frac{w_M}{\kappa}$, the *logical factor*, following Carnap (1952, 27f.; 1962, 568), and then, consider *weighting* these two factors in the form of a *weighted mean*:

(71) $\frac{W_1\frac{s_M}{s} + W_2\frac{w_M}{\kappa}}{W_1 + W_2}$ (cf. Carnap 1952, 27f.)

Here, "W_1" is a weight for $\frac{s_M}{s}$, and "W_2" for $\frac{w_M}{\kappa}$.

In this formula, if only we fix W_1 as "s" (Carnap 1952, 27) and write "λ" instead of W_2—this is merely for showing that it was the weight for the logical factor (Carnap 1952, 28), we obtain (0). And along this line, we see the order: (56)→(70)→(71)→(0). Herein, again, we ascertain that (56) certainly underlies all the predictions.

This insight leads to the argument we made in Section 5, according to which Carnap's inductive logic has a property of artificiality, in other words, *subjectivity*. This picture was attributed to the earlier system alone, because Carnap's possible world semantics was stated only in it.

Carnap's view of the world is considered to be designed subjectively; this is because it reflects our design of languages. But this view is eliminated, or gets lost, in λ-system. It is true that parameter λ took over such a property; but it is concerning numerical values alone. What we want to know is why such and such a numerical value is yielded. We need the answer not mathematically but philosophically. And for this interest, the earlier system is indispensable. So putting the formula in the earlier system, (56), at the bottom is necessary.

From this insight, I have so far drawn many philosophical insights in other papers (Kaneko 2007; 2009; 2012a; 2012b). This article constitutes their theoretical background in a strictly formal way.

Index for Symbols

("f." means "and the following")

n	: (9)	w	: (18) f.	s_p	: (35)
Const.	: (10) f.	$\mathfrak{Z}$	: (19)	τ	: (40)f.
Pred.	: (10) f.	ζ	: (20) f.	*i*	: (41)
π	: (10) f.	$\mathfrak{R}$	: (23)	ς	: (42)f.
N	: (10) f.	C	: (32)	*j*	: (43)
$\mathfrak{L}_N^{\pi}$	: (10) f.	ξ	: (34) f.	*J*	: (35)
M	: (13)	K_1	: (35)	${}_N\mathfrak{m}$	: (57)
Q	: (15)	K_2	: (35)	${}_N\mathfrak{Z}$	: (57)
κ	: (17) f.	s_p	: (35)		

Notes

1. As for the abbreviations, see References below.
2. Carnap ascribed the notation "P(*h*/*e*)" to Keynes and "P(*h*|*e*)" to Jeffreys (Carnap 1962, 280-81).
3. To my surprise, Carnap abandoned these names too easily later (e.g., Carnap 1966, 19-39).
4. The probabibity$_1$-weighted mean is the same as *mathematical expectation*. For further explanation, see Carnap, 1952, 19;

1962, 525.

5. Strictly speaking, we must distinguish his idea from possible world semantics like Kripke's because, by refraining from such terminology as "possible worlds," Carnap tried to avoid committing himself in superfluous metaphysical controversies (1962, 71).

6. The omission and the parentheses are by Kaneko.

7. Some parts of this quotation were altered by Kaneko.

8. Of course, we can find many origins for inductive logic. But even so, its philosophical background is *Tractatus*. I relate this idea with Carnap's philosophical lineage as a logical positivist.

9. In Binomial Theorem $(a + b)^n = \sum_{k=0}^{n} {}_nC_k\, a^{n-k}b^k$, let a = 1 and b = 1. ■

10. I already elaborated on this aspect in many articles (Kaneko 2007; 2009; 2011; 2012). The present article is considered to be a final, theoretical foundation of those thoughts.

11. Strictly speaking, Carnap referred to Mazurkiewics and Hosiasson (Carnap 1962, 281).

12. This is a very controversial constraint (Carnap 1962, 122-24), which I have dealt with in Kaneko 2011; 2012.

13. "⊨" means "logically true" though Carnap wrote "⊢" (1962, 83).

14. *Proof*. Consider the repeated permutation that takes π units out of two things (affirmation or negation). ■

15. Carnap called a formula a *matrix* and used "$\mathfrak{M}$" for its meta-argument (1962, 55). So I must mention them with $\ulcorner\mathfrak{M}(x)\urcorner$ using Quine's quasi-quotes since variables do not belong to meta-language. However, for legibility, I often omit them.

16. The proof is realized from truth-tables. For example, from the right-most columns, the right-most columns in (12) and definition (16), we can see that $P_1(a_1)\rightarrow P_2(a_1)$ is equivalent to $Q_1(a_1)\lor Q_3(a_1)\lor Q_4(a_1)$, and $P_1(a_2)\rightarrow P_2(a_2)$ to $Q_1(a_2)\lor Q_3(a_2)\lor Q_4(a_2)$. (Note that we regard $P_1(a_1)\rightarrow P_2(a_1)$ and $P_1(a_2)\rightarrow P_2(a_2)$ as equivalent to $\forall x[P_1(x)\rightarrow P_2(x)]$. See (22)-④ below.) ■

Also, we can find a more practical way to find the disjunction (cf. Kaneko 2009, 60). Stating it as to $P_1(x)\rightarrow P_2(x)$:

① Transform $P_1(x)\rightarrow P_2(x)$ into $\neg P_1(x)\lor P_2(x)$ (disjunctive normal form).

② Substitute $Q_3(x)\lor Q_4(x)$ for $\neg P_1(x)$, and $Q_1(x)\lor Q_3(x)$ for $P_2(x)$.

③ Connect them: $Q_1(x)\lor Q_3(x)\lor Q_4(x)$.

17. Various proofs are possible. As already stated (cf. under (12)), one is a proof using a truth-table. Otherwise, consider the repeated permutation that takes N units (a_1-a_N) out of 2^π things (Q_1-Q_κ). $(2^\pi)^N = 2^{N\times\pi}$. ■

18. Since Carnap adopted the substitutional definition for universal and existential sentences (cf. (22)-④), we can regard $\mathfrak{S}$ as all kinds of sentences.

19. Like (18), the proof of this theorem is also realized from truth tables, e.g., (12). ■

Also, we can find more practical way to find the disjunction (cf. Kaneko 2009, 61). Stating it as to $P_1(a_1)\rightarrow P_2(a_1)$:

① Transform $P_1(a_1)\rightarrow P_2(a_1)$ into $\neg P_1(a_1)\lor P_2(a_1)$ (disjunctive normal form).

② Substitute $Q_3(a_1)\lor Q_4(a_1)$ for $\neg P_1(a_1)$, and $Q_1(a_1)\lor Q_3(a_1)$ for $P_2(a_1)$ (cf. definition 16).

③ Substitute $\mathfrak{Z}_9\lor\mathfrak{Z}_{10}\lor\mathfrak{Z}_{11}\lor\mathfrak{Z}_{12}$ for $Q_3(a_1)$, $\mathfrak{Z}_{13}\lor\mathfrak{Z}_{14}\lor\mathfrak{Z}_{15}\lor\mathfrak{Z}_{16}$ for $Q_4(a_1)$, and $\mathfrak{Z}_1\lor\mathfrak{Z}_2\lor\mathfrak{Z}_3\lor\mathfrak{Z}_4$ for $Q_1(a_1)$.

④ Connect them: $\mathfrak{Z}_1\lor\mathfrak{Z}_2\lor\mathfrak{Z}_3\lor\mathfrak{Z}_4\lor\mathfrak{Z}_9\lor\mathfrak{Z}_{10}\lor\mathfrak{Z}_{11}\lor\mathfrak{Z}_{12}\lor\mathfrak{Z}_{13}\lor\mathfrak{Z}_{14}\lor\mathfrak{Z}_{15}\lor\mathfrak{Z}_{16}$.

20. Strictly speaking, $\frac{\mathrm{card}(\mathfrak{R}(\mathfrak{S}))}{\zeta} = \mathfrak{m}^{\dagger}(\mathfrak{S})$.

21. In detail, see Kaneko 2009, 62f.

22. For if (25) held, we could easily admit the following definition:

$$\mathfrak{m}\big(\mathfrak{R}(\mathfrak{S})\big) = \sum_{j=1}^{n} \mathfrak{m}(\mathfrak{Z}_{ij}) \text{ for } \mathfrak{Z}_{ij} \in \mathfrak{R}(\mathfrak{S}).$$

However, the fact that we cannot adopt this definition compels us to adopt the alternative definition (29) below. By the way, in the subsequent argument, I often anticipate well-known principles for $\mathfrak{m}$-functions (Carnap 1962, 306f.).

23. See Carnap, 1962, 306. Strictly speaking, (29) and (30) are obtained as the extension of the following axioms for state-descriptions (Carnap 1962, 295):

(i) For any $\mathfrak{Z}_i, \mathfrak{m}(\mathfrak{Z}_i) > 0 \quad (1 \le i \le \zeta)$

(ii) $\sum_{i=1}^{\zeta} \mathfrak{m}(\mathfrak{Z}_i) = 1$

Although more terminological definitions must be added, inductive logic is mainly axiomatized from these (i), (ii), (29), and (30). I express this view in order to clarify where the core of inductive logic is.

Here I want to touch on the history of probability theory as well. It is said that the axiomatization of probability theory was made by Kolmogoroff in the 20th century (e.g., Swinburne 2002, 5). Although Carnap knew it, his evaluation of Kolmogoroff is low; he thought that Kolmogoroff had made the axiomatization only for probability$_2$ (1962, 343). Carnap's study of the axiomatization was thoroughgoing; he examined almost all of the contemporary axiomatic systems of probability and showed that his regular $\mathfrak{m}$- and $\mathfrak{c}$-functions fulfilled them (1962, 337f.).

24. Arrows were not written by Carnap.

25. *Proof*. Consider the permutation including the same $N_1, \ldots, N_p$ things (Q-predicates). ■

26. Strictly speaking, the mathematical theory of approximation is required for the proof. See **T96-1.c** for Bernoulli's Limit Theorem, **T96-1.b.(3)** for the Probability Integral (Carnap 1962, 505).

27. As for the difference among values of (36), (37), and (38), see Carnap, 1962, 502-10.

28. (37) is derived as a theorem of symmetrical functions although the knowledge of approximation is also required.

29. In statistics, the Normal Law is derived as the limit of the Binomial Law.

30. Consider the repeated combination that takes N units out of κ things (Q-predicates). ■

31. *Proof by mathematical induction.*

① Suppose $\mathfrak{S}_i$ is an atomic sentence. Then take up one of state-descriptions $\mathfrak{Z}_{ij}$ in $\mathfrak{R}(\mathfrak{S}_i)$. Atomic sentences directly appear in state-descriptions (cf. (12)), so in order to decide a state-description belonging to $\mathfrak{R}(C(\mathfrak{S}_i))$, it suffices to form $C(\mathfrak{Z}_{ij})$ simply. For example, in (12), consider $P_1(a_1)$, $\mathfrak{Z}_2$ belonging to $\mathfrak{R}(P_1(a_1))$ and C mentioned under (32). We can simply regard $C(\mathfrak{Z}_2) = \mathfrak{Z}_5$ as a member of $\mathfrak{R}(C(P_1(a_1))) = \mathfrak{R}(P_1(a_2))$.

② Suppose $\mathfrak{S}_i$ is $\neg\mathfrak{S}_k$. As a hypothesis of mathematical induction, we assume that (44) holds for $\mathfrak{S}_k$; that is, when $\mathfrak{R}(\mathfrak{S}_k) = \{\mathfrak{Z}_{k1}, \ldots, \mathfrak{Z}_{km}\}$, $\mathfrak{R}(C(\mathfrak{S}_k)) = \{C(\mathfrak{Z}_{k1}), \ldots, C(\mathfrak{Z}_{km})\}$. In addition, let $V_{\mathfrak{Z}}$ be a set of all state-descriptions; hence $V_{\mathfrak{Z}} - \mathfrak{R}(\mathfrak{S}_k) = \{\mathfrak{Z}_{km+1}, \ldots, \mathfrak{Z}_{k\zeta}\}$. Then, using **T18-1.e** (Carnap 1962, 79), $\mathfrak{R}(\neg\mathfrak{S}_k) = V_{\mathfrak{Z}} - \mathfrak{R}(\mathfrak{S}_k) = \{\mathfrak{Z}_{km+1}, \ldots, \mathfrak{Z}_{k\zeta}\}$. We can easily see that $C(\neg\mathfrak{S}_k) = \neg C(\mathfrak{S}_k)$. Thus, $\mathfrak{R}(C(\neg\mathfrak{S}_k)) = \mathfrak{R}(\neg C(\mathfrak{S}_k)) = V_{\mathfrak{Z}} - \mathfrak{R}(C(\mathfrak{S}_k)) = V_{\mathfrak{Z}} - \{C(\mathfrak{Z}_{k1}), \ldots, C(\mathfrak{Z}_{km})\} = \{C(\mathfrak{Z}_{km+1}), \ldots, C(\mathfrak{Z}_{k\zeta})\}$. Other connectives are proved in the same way. ■

32. From (29), $\mathfrak{m}(\mathfrak{S}_i) = \sum_{j=1}^{n} \mathfrak{m}(\mathfrak{Z}_{ij})$ for $\mathfrak{Z}_{ij} \in \mathfrak{R}(\mathfrak{S}_i)$. From (44), $\mathfrak{m}(C(\mathfrak{S}_i)) = \sum_{j=1}^{n} \mathfrak{m}(C(\mathfrak{Z}_{ij}))$ for $C(\mathfrak{Z}_{ij}) \in \mathfrak{R}(C(\mathfrak{S}_i))$. From the explanation under (32), $\mathfrak{m}(\mathfrak{Z}_{ij}) = \mathfrak{m}(C(\mathfrak{Z}_{ij}))$. Therefore, $\mathfrak{m}(\mathfrak{S}_i) = \mathfrak{m}(C(\mathfrak{S}_i))$. ■

33. Let j be $i_1 \vee i_2 \vee \ldots \vee i_\varsigma$, where i_1 is i. Since $i_1, i_2, \ldots, i_\varsigma$ are isomorphic, they can be C-correlates with each other (cf. (32)). Thus, using $C_1, \ldots, C_{\varsigma-1}$, we can write them as $i_1, C_1(i_1), \ldots, C_{\varsigma-1}(i_1)$.

From (45), $\mathfrak{m}(i_1) = \mathfrak{m}(C_1(i_1)) = \cdots = \mathfrak{m}(C_{\varsigma-1}(i_1))$, and from (42), $\mathfrak{m}(i_1) + \mathfrak{m}(C_1(i_1)) + \cdots + \mathfrak{m}(C_{\varsigma-1}(i_1)) = \frac{s!}{s_1! \times s_2! \times \ldots \times s_p!} \times \mathfrak{m}(i_1)$. ■

34. From (43), j states that over s individual constants, the absolute frequency of M_1 is s_1, and likewise, s_2 for M_2, …, s_p for M_p. Similarly j', over s′ individual constants, s'_1 for M_1, …, s'_p for M_p. These two assertions entails J: over s+s′ individual constants, $s_1+s'_1$ for M_1, …, $s_p+s'_p$ for M_p. ■

35. Consider in the same way as in note 34. ■

36. From (47) and (48). ■

37. Let j be $i_1 \vee i_2 \vee \ldots \vee i_\varsigma$, where i_1 is i. Then,

$$\mathfrak{m}(\mathfrak{S}_k \wedge j) = \mathfrak{m}(\mathfrak{S}_k \wedge (i_1 \vee i_2 \vee \ldots \vee i_\varsigma)) = \mathfrak{m}((\mathfrak{S}_k \wedge i_1) \vee (\mathfrak{S}_k \wedge i_2) \vee \ldots \vee (\mathfrak{S}_k \wedge i_\varsigma))$$

$= \mathfrak{m}(\mathfrak{S}_k \wedge i_1) + \mathfrak{m}(\mathfrak{S}_k \wedge i_2) + \cdots + \mathfrak{m}(\mathfrak{S}_k \wedge i_\varsigma)$ from the exclusiveness of i_1-i_ς (M_{ki} forms a division. See (41).)

$= \mathfrak{m}(\mathfrak{S}_k \wedge i_1) + \mathfrak{m}(C_1(\mathfrak{S}_k \wedge i_1)) + \cdots + \mathfrak{m}(C_{\varsigma-1}(\mathfrak{S}_k \wedge i_1))$ (As for "C_1," … "$C_{\varsigma-1}$," see note 33.)

$= \frac{s!}{s_1! \times s_2! \times \ldots \times s_p!} \times \mathfrak{m}(\mathfrak{S}_k \wedge i)$ from (45) and (42). ■

38. From (50). ■

39. I add to this that Carnap thought of not only Laplace but also Wittgenstein as the theorists who adopted $\mathfrak{m}^\dagger$ (Carnap 1962, 565; Wittgenstein 1918, 5.15). In this respect, we know, Carnap had to alter the picture of *Tractatus* for the completion of his system (cf. Section 5).

40. I learned this from Uchii, 1972, 410-11.

41. Consider two sentences, one of which includes every conjunct of the other, e.g., $\mathfrak{S}_I = \ulcorner\mathfrak{S}_1 \wedge \mathfrak{S}_2 \wedge \mathfrak{S}_3\urcorner$ and $\mathfrak{S}_{II} = \ulcorner\mathfrak{S}_1 \wedge \mathfrak{S}_2\urcorner$, in which case we call the latter a *subconjunction* of the former (Carnap 1962, 81).

42. *Proof.* ${}_N\mathfrak{m}(\mathfrak{S}_i) = {}_N\mathfrak{m}({}_N\mathfrak{Z}_{i1} \vee \ldots \vee {}_N\mathfrak{Z}_{im})$ from (21)

$= {}_N\mathfrak{m}({}_N\mathfrak{Z}_{i1}) + \ldots + {}_N\mathfrak{m}({}_N\mathfrak{Z}_{im})$ from the exclusiveness of state-descriptions

$= {}_{N+1}\mathfrak{m}({}_N\mathfrak{Z}_{i1}) + \ldots + {}_{N+1}\mathfrak{m}({}_N\mathfrak{Z}_{im})$ from (57)

$= {}_{N+1}({}_N\mathfrak{Z}_{i1} \vee \ldots \vee {}_N\mathfrak{Z}_{im}) = {}_{N+1}(\mathfrak{S}_i)$. ■

43. Under the same premise with (56), let h be the hypotheses that do not have any individual constants in common with i or j. Then, $\mathfrak{c}(h, j) = \frac{\mathfrak{m}(h \wedge j)}{\mathfrak{m}(j)} = \frac{\mathfrak{m}(h \wedge i)}{\mathfrak{m}(i)} = \mathfrak{c}(h, i)$, from (50) and (46). ■

44. For example, J is as follows:

$$(M_2(a_1) \wedge M_1(a_2) \wedge \ldots \wedge M_p(a_N)) \vee (M_1(a_1) \wedge M_p(a_2) \wedge \ldots \wedge M_2(a_N)) \vee \ldots \vee (M_p(a_1) \wedge M_1(a_2) \wedge \ldots \wedge M_2(a_N))$$ (cf. (43) and (41)).

Here, in each individual distribution as disjunct, $N_1, N_2, \ldots$, and N_p are the same. Now, focusing on $M_i(a_j)$ ($1 \leq i \leq p$, $1 \leq j \leq N$), we can find it equivalent to $Q_{i1}(a_j) \vee \ldots \vee Q_{iw_i}(a_j)$ from (18). Thus, in one place occupied with $M_i(a_j)$, w_i kinds of Q-predicates can appear. That is, $M_i(a_j)$ can be $Q_{i1}(a_j)$ or … or $Q_{iw_i}(a_j)$. From this point of view, informally, we can represent J in the following way:

$$J = \{\underset{\substack{\uparrow \\ w_1 \text{ kinds of Q-predicates}}}{\underline{M_1}} \times N_1, \ldots, \underset{\substack{\uparrow \\ w_p \text{ kinds of Q-predicates}}}{\underline{M_p}} \times N_p\}$$

Here, per M_i ($1 \leq i \leq p$), there are ${}_{w_i+N_i-1}C_{N_i}$ patterns of appearances of Q-predicates. This is derived as the repeated combination that takes N_i units out of w_i things (Q-predicates). Therefore, if we paraphrase J into a structure-description with Q-predicates, ${}_{w_1+N_1-1}C_{N_1} \times \ldots \times {}_{w_p+N_p-1}C_{N_p}$ patterns are possible as a whole. Thus, (65) holds. ■

45. From (53). ■

46. There is a book with a similar title: *Inductive Logik und Wahrscheinlichkeit*. But this book is no more than the summary and translation of *Foundations* and *Continuum* by Stegmüller (Carnap 1959, Vorwort).

Works Cited

Carnap, Rudolf. *The Continuum of Inductive Methods*. (*Continuum*). Chicago: Chicago UP, 1952.

---. *Inductive Logik und Wahrscheinlichkeit*. Wien: Springer, 1959.

---. *Introduction to Semantics*. Cambridge, Mass.: Harvard UP, 1942.

---. *The Logical Foundations of Probability*. (*Foundations*). 2nd ed. Chicago: Chicago UP, 1962.

---. *Logical Syntax of Language*. (*Syntax*). Trans. Amethe Smeaton. London: Routledge, 1937.

---. *Philosophical Foundations of Physics*. Ed. Martin Gardner. London: Basic Books, 1966.

---. *Testability and Meaning. Reading in the Philosophy of Science*. (*Testability*). New York: Appleton-Century-Crofts, 1936-37.

Friedman, Michael. *Reconsidering Logical Positisvism*. Cambridge: Cambridge UP, 1999.

Kaneko, Yusuke. "Belief as Probability." (In Japanese). *Philosophical Studies* XXV (2007): 107-20.

---. "Belief in Causation: One Application of Carnap's Inductive Logic." *Academic Research International* 3.1 (2012a): 11-22.

---. "The Confirmation of Singular Causal Statement by Carnap's Inductive Logic." *The Logica Year Book 2011*. London: College Publications, 2012b. 77-95.

---. *The Phases of Ethical Judgements: From a Motivistic Point of View*. (Japanese, A doctoral dissertation submitted to the University of Tokyo). 2009.

Swinburne, Richard. "Introduction." *Bayes's Theorem*. Oxford: Oxford UP, 2002. 1-20.

Uchii, Soshichi. "Inductive Logic and Probability." (In Japanese). *Tetsugaku Kenkyuu* 523 (1972): 403-41.

Wittgenstein, Ludwig. *Tractatus Logico-Philosophicus*. (*Tractatus*). Suhrkamp Taschenbuch Wissenschaft, Nr. 501, 1918.