SPECIAL SUBSET LINGUISTIC TOPOLOGICAL SPACES

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PREFACE

In this book, authors, for the first time, introduce the new notion of special subset linguistic topological spaces using linguistic square matrices.

This book is organized into three chapters. Chapter One supplies the reader with the concept of ling set, ling variable, ling continuum, etc. Specific basic linguistic algebraic structures, like linguistic semigroup linguistic monoid, are introduced. Also, algebraic structures to linguistic square matrices are defined and described with examples. For the first time, non-commutative linguistic topological spaces are introduced.

The notion of linguistic special subset doubly non-commutative topological spaces of linguistic topological spaces in this book.

This book gives examples and problems for the reader to familiarize themselves with these concepts.

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Chapter One

BASIC CONCEPTS ON LINGUISTIC SETS AND SUBSETS OF LINGUISTIC SETS

In this chapter, we define the concept of linguistic (ling) sets and describe a few of its properties.

We mainly want to make this book a self-contained one. Further, we only give the definition of a ling sets associated with the ling variable in a very different way. We, as defined by Zadeh [41-3] or Zimmerman, [44-6] do not consider a ling variable as 5-tuple with a fuzzy membership value which plays a major role in the ling variables and the related sets.

We will first give some examples of ling variables and their related ling sets.

Example 1.1. Consider the ling variable age of people. Let S be the ling set associated with this ling variable. S = [youngest, oldest] where youngest is the age at the birth of the child and oldest is the time of death of that person in case of one person is associated and is 100 is it is the ling set associated with people in general.

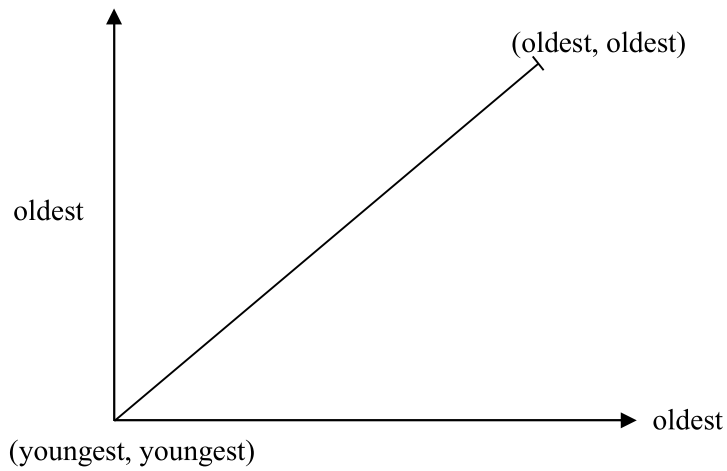
The following observations are important.

1. S = [youngest, oldest] is in the increasing order, that is youngest ≤ … ≤ middle age ≤ … ≤ old ≤ … ≤ very old ≤ … ≤ oldest

That is we see the age steadily increases with time.

As time goes the age increases it cannot be constant at some time; so it is a steadily increasing one with time.

Ling. Graph



If we take the vertical and horizontal axis as youngest to oldest we get a steadily increasing curve.

It is important to note the ling set S and this case is a ling continuum and in fact a totally ordered set.

Recall we say a set S to be totally ordered if we have the following conditions to be true.

Definition 1.1 *[12] A non-empty set S is said to be totally ordered set under the order ≤ (≥) if for every pair of elements x, y ∈ S we have either x ≤ y (y ≥ x) (or y ≤ x i.e. x ≥ y). We will be using ≤ for we say it is in increasing order.*

Thus we say (S,≤) is a totally ordered set under the ordering ≤.

For more refer [12].

We will illustrate first the totally ordered set by some examples.

Example 1.2. Let Z^+ be the set of positive integers. $\{Z^+, \leq\}$ is a totally ordered set as

$$1 \leq 2 \leq 3 \leq \ldots \leq n \leq n + 1 \leq \ldots \leq \infty.$$

We use ≤ but to be in proper notation we should write it as

$$1 < 2 < 3 < \ldots < n < n +1 < \ldots < \infty.$$

However, by default of notation we accept ≤ for if $2 \in Z^+$ we say $2 \leq 2$ (as we cannot say $2 < 2$). Another abbreviation we use in this book is linguistic set is replaced by ling set.

Further the order of Z^+ is infinite.

Now we give yet another example of a totally ordered set of finite order.

Example 1.3. Let S = {1, $\sqrt{2}$, 5.3, –7, –9, –18, 9.9, 0.9, 7.9, 12.8, –19.9, 99, –999} be a finite order set o(S) = order of

S = |S| = 13. We see there is a total order defined on S.

For $-999 \leq -19.9 \leq -18 \leq -9 \leq -7 \leq 0.9 \leq 1 \leq \sqrt{2} \leq 5.3 \leq 7.9 \leq 9.9 \leq 12.8 \leq \leq 99$.

We now give an example of a non empty set which is not totally ordered.

Example 1.4. Let S = {a, b, c, d} be a non empty set of 4 concepts. P(S) be the power set of S.

P(S) = {ϕ, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {c, b, d} and S = {a, b, c, d}} be the power set of S.

We see P(S) is not a totally ordered set for consider {a} and {c, b, d} ∈ P(S) we cannot order them by containment relation similarly {a} and {b, c} ∈ P(S), are not ordered by containment relation ⊆. Thus P(S) is not a totally ordered set.

Now we recall the definition of partially ordered set; for more refer [12].

Definition 1.2. *Let S be a non empty set. S is said to be a partially ordered set if there exist a distinct pair of elements in S which are comparable under an ordering ≤ or ⊆ containment relation.*

For more refer[12].

We will illustrate this situation by some example.

Example 1.5. Let S = {a, b, c} be a set. P(S) be the power set of S. P(S) = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}.

Clearly, we have proved P(S) is not a totally ordered set, however {a}, {a, c} ∈ P(S) is such that {a} ⊆ {a, c}. Thus {P(S), ⊆} is only a partially ordered set under the containment relation. We see P(S) is a finite partially ordered set which is not totally ordered set. New we give yet another example of a partially ordered set.

Example 1.6. Let S be an infinite ordered set say S = N natural integers; P(S) be power set of S. P(S) is an infinite ordered partially ordered set under the set containment relation. Clearly {P(S), ⊆} is not a totally ordered set as if {3, 4} and {7, 8, 9} ∈ P(S) we cannot order them under containment relation. Consider {5, 6, 9} and {19, 5, 6, 9, 18, 27} ∈ P(S).

We see {5, 6, 9} ⊆ {5, 6, 9, 18, 19, 27}; so P(S) is only a partially ordered set and not a totally ordered set of infinite cardinality.

There can be sets which are neither partially ordered nor is totally ordered.

We will prove this by some examples.

Example 1.7. Consider the set of all 4 elements taken from the six elements {a, b, c, d, e, f} = S.

Let M = {{a, b, c, d}, {a, b, c, e}, {a, b, c, f}, {a, b, d, e}, {a, c, e, d}, {a, b, d, f}, {a, b, f, e}, {a, c, e, f}, {a, c, d, f}, {b, c, d, e}, {b, c, d, f}, {b, c, f, e}, {b, d, e, f}, {a, d, e, f}, {c, e, d, f}} $\subseteq$ P(S), the power set of S.

Clearly none of the elements in M are comparable. Thus M is neither a totally ordered set nor a partially ordered set.

Now having seen examples of partially order sets, totally ordered set and unordered set we proceed onto describe ling sets which are partially ordered, totally ordered and unordered by examples.

Example 1.8. Consider the ling variable V colour of the eyes of people internationally. Let S be the linguistic set associated with V.

Clearly S = {yellow, green, blue, brown, black, dark brown, light brown}.

We see no two colours in the set are comparable so S is an unordered ling set can be partially ordered if we used qualifying norm of light or dark shades of a same colour.

We give an example of a totally ordered ling set [24-29].

Example 1.9. Let us consider the ling variable V weight of people.

Let S be the ling set associated with this ling variable V.

S = [lowest, highest], the lowest weight corresponds to weight in general of a born baby which is the least ling element of S and highest weight corresponds to the maximum weight of a person that can be reached in general (of course we do not include the extreme case of either highest weight or lowest weight.

Now we see S = [lowest, highest] is a totally ordered set and is infact a ling continuum.

Thus the interval is an increasing continuum.

We have lowest; ≤ … ≤ low ≤ just low ≤ … ≤ medium weight ≤ … ≤ very high ≤ … ≤ highest.

Hence, S is a totally ordered increasing set. It is not like the age which is also a totally ordered increasing continuum because we see the age is the person increase with time however one cannot say weight of a person is not increase with time for a person in his childhood may have low weight then in the middle age they attain the maximum and highest around the age of 40 - 55 years and then they gradually may decrease in weight and may be the lowest at the very old age and so on.

This is the difference between the two ling variables age of people and weight of people.

Both the linguistic sets associated with these two ling variables age and weight are totally ordered ling sets.

Infact both these ling sets are of infinite ordered.

Now we provide yet another example of a ling variable whose associated ling set is a finite totally ordered set.

Example 1.10. Let us consider the ling variable performance of students in a classroom. The number of students whom the teacher has to assess is only 20. The ling set associated with this ling variable of 20 students falls in the following ling set S = {good, bad, very bad, very good, just good, fair, just fair, very very bad}.

We see the ling set S is a totally ordered set which is expressed in the following

very very bad ≤ very bad ≤ bad ≤ just fair ≤ fair ≤ just good ≤ good ≤ very good.

Clearly (S, ≤) is a totally ordered set of finite order.

Having seen examples of both finite and infinite totally ordered ling set, we proceed onto describe partially ordered ling sets.

We have already provided example of unordered ling sets, for more refer [24-29].

Now we proceed onto give examples of ling sets which are partially ordered ling sets.

Example 1.11. Consider the ling variable customers rating of a product from an industry. Let B be the ling set associated with this ling variable.

B = {good, bad, very worst, just medium, medium, just good}. Clearly B is a totally ordered set.

Consider P(B) the ling power set of the ling set B. P(B) = {ϕ, {just good}, {medium}, {just medium}, {good}, {bad}, {worst}, {just good, good}, {just good, medium}, {just good, just medium}, {just good, bad}, {just good, worst}, {good, bad}, {good, medium}, {good, just medium}, {good, worst}, {medium, just medium}, {medium, bad}, {medium, worst}, {just, medium, bad}, {just medium, worst}, {just good, medium, just medium}, {just good, medium, bad}, …, {good, bad, worst}, {just good, medium, just medium, good}, …, {worst, bad, good, just medium}, {just good, medium, bad, just medium, good}, …, {just medium, medium, good, bad, worst}, B}.

Clearly P(B) is not a totally ordered set for consider the subsets {good, bad}, {worst} $\in$ P(B). They cannot be compared, hence P(B) is not a totally ordered set. Now consider the subsets {bad, good}, {bad, good, medium, just good} $\in$ P(B).

We see under the containment of subsets relation $\subseteq$; {bad, good} $\subseteq$ {bad, good} $\subseteq$ {bad, good, medium, just good} so P(B) is a partially ordered ling set.

Infact {P(B), $\subseteq$} is a finite order partially ordered ling set.

We will now provide an example of a partially ordered linguistic set of infinite cardinality.

Example 1.12. Consider the ling variable age of people. The ling set related with this ling variable age is a ling continuum S = [youngest, oldest] which is of infinite cardinality.

Clearly P(S) the ling power set of P(S) is of infinite cardinality as S $\subseteq$ P(S).

Now to show P(S) is a partially ordered ling set and not a totally ordered ling set it is enough if we can prove P(S) has a distinct pair of subsets which are not comparable or which do not satisfy the containment relation of subsets which will prove {P(S), ⊆} is not a totally ordered ling set.

To show {P(S), ⊆} is a partially ordered ling set it is enough if we show the existence of a distinct pair of ling subsets of P(S) which are comparable or subsets which satisfy the containment relation.

Consider the distinct ling subsets {old, young, just old} and {old, young, just old, very old, very young, middle aged} ∈ P(S).

We see {old, young, just old} ⊆ {old, young, just old, very old, very young, middle aged}.

Hence {P(S), ⊆} is a partially ordered ling set as there exists a pair of distinct ling subsets of P(S) which satisfy containment relation.

Thus {P(S), ⊆} is a partially ordered ling set which is not a totally ordered ling set and is of infinite cardinality.

Now we define operations on ling sets. For our aim of this book is defining ling topological spaces.

So we need to have operations both on ling sets and ling subsets of a ling set S.

We will provide examples of them.

Example 1.13. Consider the ling variable performance aspects of 25 workers employed in a factory. Let S be the ling set associated with this variable. The expert find that the 20 of these workers fall under 6 ling terms / words given by

S = {good, bad, very good, fair, just bad, very bad}.

Now suppose we wish to define a max operation on S.

We want to prove {S, max} is a ling semigroup under max operation, infact a commutative one.

The table of {S, max} is as follows.

max	good	bad	fair	very bad	very good	just bad
good	good	good	good	good	very good	good
bad	good	bad	fair	bad	very good	just bad
fair	good	fair	fair	fair	very good	fair
very bad	good	bad	fair	very bad	very good	just bad
very good	very good	very good	very god	very good	very good	very good
just bad	good	just bad	fair	just bad	very good	just bad

Table 1.1

The total order enjoyed by this ling set S is as follows.

very bad ≤ bad ≤ just bad ≤ fair ≤ good ≤ very good.

Clearly {S, max} is a ling commutative semigroup of order 6. Infact the ling term very bad under max is such that max{very bad, s} = s for all s ∈ S.

Thus we can say the ling term very bad acts as the ling identity for the ling set S under max operation.

Infact {S, max} is a ling commutative monoid of finite order.

Now we define the min operation on the ling set S by the following table.

min	good	bad	fair	very bad	very good	just bad
good	good	bad	fair	very bad	good	just bad
bad	bad	bad	bad	very bad	bad	bad
fair	fair	bad	fair	very bad	fair	just bad
very bad	very bad	very bad	very bad	very bad	very bad	very bad
very good	good	bad	fair	very bad	very good	just bad
just bad	just bad	bad	just bad	very bad	just bad	just bad

Table 1.2

From the table 1.2 it is clear {S, min} is closed under min operation and infact min operation is both associative and commutative, hence {S, min} is a commutative semigroup of finite order.

We see for very good $\in$ S is such that min{very good, s} = s for all s $\in$ S. Thus very good acts as the ling identity for the ling semigroup {S, min}.

Hence {S, min} is a commutative monoid of finite order,.

The following observations are vital.

i) For any totally ordered ling set S, finite or infinite we have a greatest element and a least element.

ii) The least element serves as the ling identity of S under the max operation.

For max{least element, s} = s for all s $\in$ S.

iii) On similar lines the greatest element of S acts as the ling identity for min operation.

For min {greatest element, s} = s for all s $\in$ S. Hence the claim.

We see if the ling set S is not a totally ordered set then {S, min} and {S, max} are not even closed.

We will illustrate this situation by some examples.

Example 1.14. Let S = {good, bad, fair, tall, short, fat, thin} be a ling set we try to define max operation on S.

max{good, bad} = good,

max{good, fat} is undefined (ud),

max{tall, short} = tall   max{short, bad} = ud

and max{tall, thin} = ud.

Thus {S, max} is not even a closed under max operation so the algebraic structure of S under max cannot be defined.

So it is mandatory for one to define max operation on any ling set S we need to have S to be a totally ordered set.

Even if S is a partially ordered set then also S cannot be closed under the max operation.

Now we test if S is closed at least under the min operation.

We see

min{good, bad} = bad,   min {good, fair} = fair,

min{good, tall} is undefined = ud,

min{good, short} = ud,   min{good, fat} = ud,

min{good, thin} = ud,   min{bad, fair} = bad

min{bad, tall} = ud,   min{bad, short} = ud and so on.

Thus S is not even closed under min operation.

Clearly S is only a partially ordered set and not a totally ordered set.

Hence for min operation to be defined on S it is mandatory that S must be a totally ordered set.

Now we proceed onto define two other operations on S; viz '∪' - union of ling terms and '∩' - intersection of ling terms in S. S is taken as in example 1.13; where S = {good, bad, fair, very bad, very good, just bad}.

We see S is a totally ordered set.

very bad ≤ bad ≤ just bad ≤ fair ≤ good ≤ very good.

We define {very bad ∩ bad}

= very bad (as very bad ≤ bad).

Similarly {fair ∩ good} = fair (as fair ≤ good) and so on.

Now we give the ling table of {S, ∩}.

∩	good	bad	fair	very bad	very good	just bad
good	good	bad	fair	very bad	good	just bad
bad	bad	bad	bad	very bad	bad	bad
fair	fair	bad	fair	very bad	fair	just bad
very bad	very bad	very bad	very bad	very bad	very bad	very bad
very good	good	bad	fair	very bad	very good	just bad
just bad	just bad	bad	just bad	very bad	just bad	just bad

Table 1.3

Consider the table 1.2 of {S, min} and table 1.3 {S, ∩} we see both are identical. Thus we can say in case of totally ordered ling sets both the operation ∩ and min are identical that is

$$\min\{s_1, s_2\} = s_1 \cap s_2 \qquad \ldots I$$

(for all $s_1, s_2 \in S$)

I is true so we cannot distinguish them infact the greatest element of S also acts as the ling identity of {S, ∩}. Now we define '∪' operation on the ling set S.

good ∪ bad = good for bad ≤ good.

good ∪ very good = very good as good ≤ very good

good ∪ very bad = good as very bad ≤ good

fair ∪ just bed = fair as just bad ≤ just fair.

Keeping these laws in mind we define table of {S, ∪} in the following:

∪	good	bad	fair	very bad	very good	just bad
good	good	good	good	good	very good	good
bad	good	bad	fair	bad	very good	just bad
fair	good	fair	fair	fair	very good	fair
very bad	good	bad	fair	very bad	very good	just bad
very good	very good	very good	very good	very good	very good	very good
just bad	good	just bad	fair	just bad	very good	just bad

Table 1.4

We observe the table 1.1 of {S, max} with table 1.4 of {S, ∪} are both identical.

Hence we conclude in case of totally ordered ling set {S, max} and {S, ∪} are identical.

Infact we have max {x, y} = x ∪ y …II

From I and II we can easily conclude {S, max} and {S, ∪} are same on totally ordered ling sets.

Now we show however this is not true in case of ling power set of a set for more refer [24-28].

We will illustrate this situation by an example.

Example 1.15. Let S = {good, bad, fair} be a totally ordered ling set associated with the ling variable performance of a student in the classroom.

Let P(S) = {ϕ, {good}, {bad}, {fair}, {good, bad}, {good, fair}, {bad, fair}, {good, bad, fair} = S}

be the ling power set of P(S).

We first define {P(S), ∩} where ∩ is just the set theoretic intersection used in the classical sense.

The table 1.5 of {P(S), ∩} is as follows.

∩	ϕ	{good}	{bad}	{fair}	{good, bad}	{good, fair}	{fair, bad}	S
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
{good}	ϕ	{good}	ϕ	ϕ	{good}	{good}	ϕ	{good}
{bad}	ϕ	ϕ	{bad}	ϕ	{bad}	ϕ	{bad}	{bad}
{fair}	ϕ	ϕ	ϕ	{fair}	ϕ	{fair}	{fair}	{fair]
{good, bad}	ϕ	{good}	{bad}	ϕ	{good, bad}	{good}	{bad}	{good, bad}
{good, fair}	ϕ	{good}	ϕ	{fair}	{good}	{good, fair}	{fair}	{good, fair}
{bad, fair}	ϕ	ϕ	{bad}	{fair}	{bad}	{fair}	{bad, fair}	{bad, fair}
S	ϕ	{good}	{bad}	{fair}	{good, bad}	{good, fair}	{bad, fair}	S

Table 1.5

From the above table we see '∩' is a closed operation on P(S). Infact '∩' and P(S) is both associative and commutative.

Further we see S acts as the linguistic identity of this ling power set P(S).

We see S ∩ {A} = {A} for all {A} ∈ P(S).

Thus P(S), ∩} is a commutative semigroup of finite order with S as its ling identity.

Now we define the operation ∪ on P(S) and it is given by the following table.

∪	ϕ	{good}	{bad}	{fair}	{good, bad}	{good, fair}	{good, fair}	S
ϕ	ϕ	{good}	{bad}	{fair}	{good, bad}	{good, fair}	{bad, fair}	S
{good}	{good}	{good}	{good, bad}	{good, fair}	{good, bad}	{good, fair}	S	S
{bad}	{bad}	{bad, good}	{bad}	{bad, fair}	{good, bad}	S	{bad, fair}	S
{fair}	{fair}	{good, fair}	{bad, fair}	{fair]	S	{good, fair}	{bad, fair}	S
{good, bad}	{good, bad}	{good, bad]	{good, bad}	S	{good, bad}	S	S	S
{good, fair}	{good, fair}	{good, fair}	S	{good, fair}	S	{good, fair}	S	S
{bad, fair}	{bad, fair}	S	{bad, fair}	{bad, fair]	S	S	{bad, fair}	S
S	S	S	S	S	S	S	S	S

Table 1.6

Thus {P(S), ∪} is a commutative ling semigroup.

We see for very ling subset

A ∈ P(S), A ∪ S = S and A ∪ {ϕ} = A.

Thus {ϕ} acts as the ling identity in case of '∪' operation on the ling power set P(S) of S. {P(S), ∪} is a ling commutative monoid of order $2^{|S|}$.

We have provided just one example of this situations.

We just state a few interesting results related with them.

Theorem 1.1. *Every totally ordered set (ling set) is always a partial ordered set (ling set).*

Proof is left as an exercise to the reader.

Theorem 1.2. *A partially ordered set (ling set) in general is not a totally ordered set (ling set).*

Proof is left as an exercise to the reader.

Theorem 1.3. *Every subset (ling subset) of totally ordered set (ling set) is again a totally ordered set (ling set).*

Proof is left as an exercise for the reader.

Theorem 1.4. *A proper subset (ling subset) of a partially ordered sets (ling set) can be*

i) *Unordered sub set (ling subset) of S*
ii) *Partially ordered subset (ling subset) of S*
iii) *Totally ordered subset (ling subset) of S*

We just give an illustration to this effect by an example.

Example 1.16. Let S = {a, b, c, d} be a set (ling set; that is a, b, c, d can be some concept or can also be ling terms) P(S) denote the powerset (ling powerset of S).

P(S) = {{ϕ}, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d} , {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, S} be the power set of S or ling power set of S.

Now (P(S), ⊆) (⊆ subset containment relation) is a partially ordered set (ling partially ordered set).

Take the subset

R = {{a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {d, c}} ⊆ P(S).

Clearly R is an unordered set (ling subset) of P(S).

Now take

V = {{a}, {a, b}, {a, b, c}, {a, b, c, d} = S} ⊆ P(S).

We see {V, ⊆} is a totally ordered subset (ling subset) of {P(S), ⊆}.

Let W = {{a}, {a, b}, {b, c}, {b}, {a, b}} ⊆ P(S) be a subset (ling subset) of P(S) {W, ⊆} is only a partially ordered set (ling set) as {a} and {b, c} cannot be ordered by containment relation.

However {a} ⊆ {a, b}, {b} ⊆ {b, c} and {c} ⊆ {b, c}.

Hence {W, ⊆} is only a partially ordered subset (ling subset of P(S)) of P(S).

So the theorem can be proved for P(S) any set (ling set) with n terms; S = {a_1, a_2, …, a_n}.

|P(S)| = 2^n. {P(S), ⊆} is only a partially ordered set (ling set) under containment relation.

Consider all the subsets of order two in P(S);

$B = \{\{a_1, a_2\}, \{a_1, a_3\}, \ldots, \{a_1, a_n\}, \{a_2, a_3\}, \ldots, \{a_2, a_n\}, \ldots., \{a_{n-1}, a_n\}\} \subseteq P(S)$ is an unordered set.

Consider the subset collection

$C = \{\{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_3, a_4\}, \{a_1, a_2, a_3, a_4, a_5\}, \{a_1, a_2, a_3, a_4, a_5, a_6, \ldots, \{a_1, a_2, \ldots, a_{n-1}\}, S\} \subseteq P(S)$ is clearly a totally ordered set (ling set) as

$\{a_1\} \subseteq \{a_1, a_2\} \subseteq \{a_1, a_2, a_3\} \subseteq \{a_1, a_2, a_3, a_4\} \subseteq \{a_1, a_2, a_3, a_4, a_5\} \subseteq \ldots \subseteq \{a_1, a_2, \ldots, a_{n-1}\} \subseteq \{a_1, a_2, \ldots, a_n\} = S.$

Consider the collection of subsets (ling subsets) D given by

$$D = \{\{a_1\}, \{a, a_2\}, \{a_3, a_4, a_5\}, \{a_4, a_5\}, \{a_6, a_7, a_8, a_9, a_1\}\} \subseteq P(S).$$

D is only a partially ordered set (ling set) and not a totally ordered set (ling set).

We see the pair of elements $\{a_1\}$ and $\{a_3, a_4, a_5\}$ are not comparable or does not satisfy the containment relation.

Similarly the pair of subsets $\{a_1, a_2\}$ and $\{a_1, a_6, a_7, a_8, a_9\}$ are not compatible with the containment relation. $\{a_4, a_5\}$ and $\{a_2, a_1\}$ are not compatible under containment relation.

However $\{a_1\} \subseteq \{a_1, a_2\}$ $\{a_1\} \subseteq \{a_6, a_7, a_8, a_9, a_1\}$, $\{a_4, a_5\} \subseteq \{a_3, a_4, a_5\}$ are pairs of subsets which are compatible under the containment relation.

Thus D is only a partially ordered set (ling set) and not totally ordered set (ling set).

Having seen some of the properties of partially ordered set (ling set), totally ordered set (ling set) and undered set (ling set) we now proceed onto study other prospered enjoyed by these sets (ling sets).

We leave it for the reader to prove the following results.

Theorem 1.5. *Let S be a totally ordered ling set {S, min} is a ling commutative monoid with the greatest element of S as the ling identity.*

Theorem 1.6. *Let S be a totally ling set S. {S, max} is a ling commutative monoid with the least ling element as its ling identity.*

Theorem 1.7. *{S, $\cup$} be the ling monoid where S is a totally ordered ling set which is identical with {S, max}.*

Theorem 1.8. *{S, $\cap$} is the ling monoid identical with {S, min} where S is a totally ordered ling set.*

Theorem 1.9. *S be a ling set P(S) be the ling power set of the ling set S {P(S), $\cap$} is a commutative monoid with the ling set S as its ling identity.*

Theorem 1.10. *Let S be a ling set P(S) be the ling power set of S. {P(S), $\cup$} is a commutative monoid with the ling identity {ϕ}.*

Proof of these two theorems are left as an exercise.

However P(S) under min or max is very differently defined. We describe how min and max operations are defined on P(S) by an example.

Example 1.17: Let S = {a, b, c, d} be a totally ling set (set). P(S) be the ling power set of S; we have a ≤ b ≤ c ≤ d.

P(S) = {ϕ, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, S = {a, b, c, d}} be the ling power set of the ling set S.

Consider A = {a, b, c} and B = {b, c, d} ∈ P(S).

min {A, B} = min{{a, b, c}, {b, c, d}} = {min {a, b}, min{a, c}, min{a, d}, min{b, d}, min{b, c}, min {b, d}, min{c, b}, min{c, c}, min {c, d}} = {a, b, c}.

We see A ∩ B = {b, c} and A ∪ B = {a, b, c, d}

max{A, B} = max {{a, b, c}, {b, c, d}} = {max{a, b}, max{a, c}, max{a, d}, max{b, d}, max{b, c}, min {b, d} max{c, d}, max{c, c}, max{c, d}} = {b, c, d}.

However A ∪ B = {a, b, c} ∪ {b, c, d} = {a, b, c, d}.

Clearly A ∪ B ≠ min {A, B} ≠ A ∩ B
≠ max{A, B}.

A ∩ B ≠ A ∪ B ≠ min {A, B}
≠ max{A, B}.

max {A, B} ≠ A ∪ B ≠ A ∩ B
≠ min {A, B}.

and min {A, B} ≠ A ∪ B ≠ A ∩ B
≠ max {A, B}.

Thus unlike ling sets where min {a, b} = a ∩ b

and max {a, b} = a ∪ b we see in case of ling power sets all the four operations are distinct.

We will illustrate in case of S = {a, b, c} by an example.

Example 1.18. Let S = {a, b, c} be a totally ordered ling set. P(S) be the ling power set of S.

P(S) = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}

S = {a, b, c}} be the ling power set of S. o(P(S)) = 8.

We will define the operations max and min P(S).

We will illustrate this by the following tables. Table 1.7 gives the operations max on P(S).

a < b < c

max	{ϕ}	{a}	{b}	{c}	{a, b}	{a, c}	{b, c}	{a,b,c}
ϕ	{ϕ}	{a}	{b}	{c}	{a,b}	{a, c}	{b, c}	{a,b,c}
{a}	{a}	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}
{b}	{b}	{b}	{b}	{c}	{b}	{b,c}	{b,c}	{b,c}
{c}	{c}	{c}	{c}	{c}	{c}	{c}	{c}	{c}
{a,b}	{a,b}	{a,b}	{b}	{c}	{a,b}	{a,b,c}	{b,c}	{a,b,c}
{a,c}	{a,c}	{a,c}	{b,c}	{c}	{a,c,b}	{a,c}	{b,c}	{a,b,c}
{c,b}	{c,b}	{c,b}	{b,c}	{c}	{b,c}	{b,c}	{b,c}	{b,c}
{a,b,c}	{a,b,c}	{a,b,c}	{b,c}	{c}	{a,b,c}	{a,b,c}	{b,c}	{a,b,c}

Table 1.7

We see for max on P(S) the least element ϕ acts as the ling identity.

The {P(S), max} is a commutative ling monoid of order 8.

Now we give the table of min operation on P(S)

where a ≤ b ≤ c

min	ϕ	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}
ϕ	{ϕ}	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
{a}	ϕ	{a}	{a}	{a}	{a}	{a}	{a}	{a}
{b}	ϕ	{a}	{b}	{b}	{a,b}	{a,b}	{b}	{a,b}
{c}	ϕ	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{b,c,a}
{a,b}	ϕ	{a}	{a,b}	{a,b}	{a,b}	{a,b}	{a,b}	{a,b}
{a,c}	ϕ	{a}	{a,b}	{a,c}	{a,b}	{a,c}	{a,b,c}	{a,b,c}
{b,c}	ϕ	{a}	{b}	{b,c}	{a,b}	{a,b,c}	{b,c}	{b,a,c}
{a,b,c}	ϕ	{a}	{a,b}	{a,b,c}	{a,b}	{a,b,c}	{a.b,c}	{a,b,c}

Table 1.8

Clearly min operation on P(S) is not the same as max operation P(S).

Both {P(S), min} and {P(S), max} are very different.

Now we give the tables for {P(S), ∩} and {P(S), ∪} and show that all the four operation is different and distinct resulting a ling commutative monoid.

We will table them to show they have distinct linguistic identities.

Now the table for {P(S), ∩} is as follows.

∩	{ϕ}	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
{a}	ϕ	{a}	ϕ	ϕ	{a}	{a}	ϕ	{a}
{b}	ϕ	ϕ	{b}	ϕ	{b}	ϕ	{b}	{b}
{c}	ϕ	ϕ	ϕ	{c}	ϕ	{c}	{c}	{c}
{a,b}	ϕ	{a}	{b}	{ϕ}	{a,b}	{a]	{b}	{a,b}
{a,c}	ϕ	{a}	ϕ	{c}	{a}	{a,c}	{c}	{a,c}
{b,c}	ϕ	ϕ	{b}	{c}	{b}	{c}	{b,c}	{b,c}
{a,b,c}	ϕ	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}

Table 1.9

Now the table for {P(S), ∪} is given in the following:

∪	ϕ	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}
ϕ	ϕ	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}
{a}	{a}	{a}	{a.b}	{a,c}	{a,b}	{a,c}	{a,b,c}	{a,b,c}
{b}	{b}	{a,b}	{b}	{b,c}	{a,b}	{a,b,c}	{b,c}	{a,b,c}
{c}	{c}	{a,c}	{b,c}	{c}	{a,b,c}	{a, c}	{b, c}	{a,b,c}
{a,b}	{a,b}	{a,b}	{a,b}	{a,b,c}	{a,b}	{a,b,c}	{a,b,c}	{a,b,c}
{a,c}	{a,c}	{a,c}	{a,b,c}	{a,c}	{a,b,c}	{a,c}	{a,b,c}	{a,b,c}
{b,c}	{b,c}	{a,b,c}	{b,c}	{b,c}	{a,b,c}	{a,b,c}	{b, c}	{a,b,c}
{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}

Table 1.10

We tabulate these 4 ling monoids using P(S) under the four operations ∪, ∩ max and min.

S. No.	P(S)	Identity	How they are different
1.	{P(S), ∪}	{ϕ}	{a} ∪ {b} = {a, b}
2.	{P(S), min}	{a, b, c}	min{{a}, {b}} = {a}
3.	{P(S), ∩}	{a, b, c}	{a} ∩ {b} = {ϕ}
4.	{P(S), max}	{ϕ}	max {{a}, {b}} = {b}

Table 1.11

For same of a pair of elements the four operations yield 4 different values evident from the last column of the table.

Hence the claim.

These concepts will be used while building ling topological spaces.

However if the underlying ling set is not a totally ordered set we can define only the two operation ∪ and ∩.

We will recall the basic properties about substructures of these monoids [29, 34-36].

Theorem 1.11. *Let S be a ling finite set P(S) be the ling power set of S. {P(S), ∩} is a ling commutative monoid. All proper subsets of P(S) in general need not in general be submonoids or subsemigroups.*

Proof: Let S = $\{a_1, a_2, \ldots, a_n\}$ be the ling set and P(S) the ling power set of S.

Consider the subset P = {{a}, {a_2}, … {a_n}} all singleton subsets of P(S).

We see {a_i} ∩ {a_j} = {ϕ}; i ≠ j and ϕ ∉ P so {P, ∩} is not a submonoid as the very closure axiom is flouted.

Hence every proper subsets collection of P(S) need not in general be a ling submonoid or even a ling - subsemigroup.

Theorem 1.12. *Let S be a ling finite set. P(S) be the ling power set of S. If W is a collection of subsets from P(S). {P(S), ∪} is a ling commutative monoid then in general {W, ∪} is not submonoid or even a subsemigroup.*

Proof. Given S = {a_1, a_2, …, a_n} be a ling finite set. P(S) be the ling power set of S.

W = {{a_1}, {a_2}, …, {a_n}} ⊆ P(S) is not even clsed under the ∪ operation.

Consider {a_1}, {a_2} ∈ W; we see {a_1} ∪ {a_2} = {a_1, a_2} ∉ W. Hence {W, ∪} is not even closed under the union operation. Thus the claim.

Theorem 1.13. *Let S be a totally ordered finite ling set, P(S) be the ling power set of S, {P(S), max}. Every proper subset of P(S) in general is not a ling monoid or a ling subsemigroup. Infact the proper ling subset of P(S) need not in general be even closed under the max operation or min operation.*

Proof. Let S = {a_1, a_2, …, a_n} be a totally ordered ling set

$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_{n-1} \leq a_n$ be the total order enjoyed by S. P(S) be the ling power set of S. {P(S), max} is a ling monoid which is commutative.

Consider W = {{a_1, a_2} and {a_3, a_4}} be two ling subset which comprises P(S). max{{a_1, a_5}, {a_3, a_4}} {a_3, a_4, a_5} $\notin$ W as $a_1 \leq a_3 \leq a_4 \leq a_5$.

Thus {W, max} is not even closed under the max operation.

Take the same W; consider {W, min}

min{{a_1, a_5}, {a_3, a_4}} = {min {a_1, a_5}, min {a_1, a_4}, min {a_5, a_3}, min{a_5, a_4}} (given $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq \ldots \leq a_{n-1} \leq a_n$).

So; min{{a_1, a_5}, {a_3, a_4}} = {a_1, a_1, a_3, a_4}

= {a_1, a_3, a_4} $\notin$ W.

Hence {W, min} is not even closed under the min operation.

We define ling subsemigroup or ling submonoid of a ling commutative monoid.

Definition 1.3. *Let S be a ling set and P(S) the ling power set of S. {P(S), ∪} be a ling monoid. A subset collection of P(S) say W is said to be a ling submonoid if and only if {W, ∪} is a ling monoid or in short W itself is a ling monoid.*

We will provide some examples of this situation.

Example 1.19. Let S = {a, b, c, d} be a ling set P(S) be the ling power set of S. P(S) = {ϕ{a}, {b}, {c}, {d}, {a, c} {a, b}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, S} be the ling power set of the set S.

{P(S), ∪} is a ling monoid

V = {{a}, ϕ, {b}, {a, b}} ⊆ P(S) is a ling submonoid of P(S) under ∪; W is given by the following table.

∪	ϕ	{a}	{b}	{a, b}
ϕ	ϕ	[a}	{b}	{a, b}
{a}	{a}	{a}	{a, b}	{a, b}
{b}	{b}	{a, b}	{b}	{a, b}
{a, b}	{a, b}	{a, b}	{a, b}	{a, b}

Table 1.12

{W, ∪} is a ling submonoid.

Consider V = {{a}, {a, b}, {a, c}, {a, b, c}} ⊆ P(S).

We obtain the table of V under ∪ to find out {V, ∪} is a ling submonoid or not.

∪	{a}	{a,c}	{a,b}	{a, b, c}
{a}	{a}	{a, c}	{a, b}	{a, b, c}
{a, c}	{a, c}	{a, c}	{a, b, c}	{a, b, c}
{a, b}	{a, b}	{a, b, c}	{a, b}	{a, b, c}
{a, b, c}	{a, b, c}	{a, b, c}	{a, b, c}	{a, b, c}

Table 1.13

{V, ∪} is a ling submonoid of {P(S), ∪}.

For {W, ∪} to be a submonoid {ϕ} is a ling identity.

For the ling submonoid {V, ∪} we see {a} is the ling identity of {V, ∪} the ling submonid of {P(S), ∪}.

We see for {P(S), ∪} the ling identity is {ϕ}. Similarly for the ling subset {W, ∪}, which is a ling submonoid of {P(S), ∪} has ϕ to be a ling identity so also for the ling submonoid {W, ∪}, ϕ is the ling identity.

Hence a very natural question arises will every ling submonoid of {P(S), ∪} has ϕ to be ling identity.

To this question we give the answer as in general ϕ need not be always the ling identity of all ling submonoids. {V, ∪} ⊆ {P(S), ∪} is a ling submonoids {V, ∪} ⊆ {P(S), ∪}is a ling submonoid of {P(S), ∪} but ϕ∉V; only {a} ∈ V is the ling identity of the ling submonoid of {P(S), ∪}.

Consider the ling subset P = {{a, b}, {b, c}, {c, a}, {a, b, c}} ⊆ P(S). In {P, ∪} a ling submonoid of P(S).

We first find the table of P under the ling operation ∪ in the following:

∪	{a, b}	{b, c}	{c, a}	{a, b, c}
{a, b}	{a, b}	{a, b, c}	{a, b, c}	{a, b, c}
{b, c}	{a, b, c}	{b, c}	{a, b, c}	{a, b, c}
{c, a}	{a, b, c}	{a, b, c}	{c, a}	{a, b, c}
{a, b, c}	{a, b, c}	{a, b, c}	{a, b, c}	{a, b, c}

Table 1.14

We see {P, ∪} is only a ling subsemigroup of {P(S), ∪}. Clearly {P, ∪} has no ling identity.

Consider the ling subset

T = {{a}, {b}, {a, c}, {c}} ⊆ P(S).

We find the table of T under ∩.

∪	{a}	{b}	{c}	{a, c}
{a}	{a}	{a, b}	{a, c}	{a, c}
{b}	{a, b}	{b}	{b, c}	{a, b, c}
{c}	{a, c}	{b, c}	{c}	{a, c}
{a, c}	{a, c}	{a, c, b}	{a, c}	{a, c}

Table 1.15

We see T under ∩ is not even a closed ling subset hence {T, ∪} is not a ling submonoid or a ling subsemigroup.

Thus we see all the 3 ling subsets of P(S) behave very differently under the ∪ operation.

Now if we consider P(S), P(S) under ∩ is a ling monoid with {a, b, c} as its ling identity under ∩.

Now take W = {ϕ, {a}, {b}, {a, b}} ⊆ P(S).

We build the table of {W, ∩} in the following.

$\cap$	ϕ	{a}	{b}	{a,b}
ϕ	ϕ	ϕ	ϕ	ϕ
{a}	ϕ	{a}	ϕ	{a}
{b}	ϕ	ϕ	{b}	{b}
{a,b}	ϕ	{a}	{b}	{a,b}

Table 1.16

We see {a, b} $\in$ W is the ling identity of W under $\cap$ where {W, $\cap$} is the ling submonoid of {P(S), $\cap$}.

Consider P = {{a, b}, {b, c}, {c, a}, {a, b, c}} $\subseteq$ P(S).

We give the table {P, $\cap$} in the following.

$\cap$	{a, b}	{b, c}	{c, a}	{a, b, c}
{a, b}	{a, b}	{b}	{a}	{a, b}
{b, c}	{b}	{b, c}	{c}	{b, c}
{c, a}	{a}	{c}	{c, a}	{c, a}
{a, b, c}	{a, b}	{b, c}	{c, a}	{a, b, c}

Table 1.17

Clearly {P, $\cap$} is not even closed under $\cap$; so is just a ling subset of P(S).

Thus we have seen like the classical algebraic structure group in the case of ling monoids the ling identity of the ling submonoids are not always the same.

To this effect we have given examples.

Also a ling monoid can have a ling subsemigroup which is not a ling submonoid of only a ling subsemigroup.

We have given examples to this effect also.

We will show this is the case when we use the operator max or min on ling powerset P(S) of a ling set S which is a totally ordered set.

We give min and max operations on the ling subsets W, T and P so that it is easy for comparison of them.

Given T = {{a}, {b}, {a, c}, {c}} ⊆ P(S).

We have already proved T is not a ling subsemigroup or ling submonoid under the union operator.

max	{a}	{b}	{a, c}	{c}
{a}	{a, b}	{b}	{a, c}	{c}
{b}	{b}	{b}	{b, c}	{c}
{a,c}	{a, c}	{b, c}	{a, c}	{c}
{c}	{c}	{c}	{c}	{c}

Table 1.18

We will be using the rule $a \leq b \leq c$

We see in the table {b, c} is not in T so not even closed under max operation.

Hence T is not a ling subsemigroup or ling submonoid

Now consider the operator min of the ling subset T of P(S).

We will describe it by the following table

min	{a}	{b}	{a,c}	{c}
{a}	{a}	{a}	{a}	{a}
{b}	{a}	{b}	{a, b}	{b}
{a, c}	{a}	{b,c}	{a, c}	{c}
{c}	{a}	{b}	{c}	{c}

Table 1.19

We see {T, min} is not a ling subsemigroup or a ling submonoid as T is not even closed under min operation for {a, b} $\notin$ T but {a, b} min{{b}, {a, c}} = {min {b, a}, min{b, c}} = {a, b} $\notin$ T. (we have a ≤ b ≤ c)

Thus {T, min} and {T, max} are not even closed under the operations min and max respectively.

Now consider the ling subset W of P(S);

W = {ϕ, {a}, {b}, {a, b}}.

We find how max operator functions on W

max	ϕ	{a}	{b}	{a, b}
ϕ	ϕ	{a}	{b}	{a, b}
{a}	{a}	{a}	{b}	{a, b}
{b}	{b}	{b}	{b}	{b}
{a, b}	{a, b}	{a, b}	{b}	{a, b}

Table 1.20

We see {W, max} is a ling submonoid of P(S).

Clearly the ling identity of W and P(S) are the same viz ϕ. We know the least element of P(S) is the ling identity element of max operation in P(S) and W are the same then the ling submonoid W will have the same ling identi0ty as that of P(S).

Now we find the table of {W, min}

min	ϕ	{a}	{b}	{a,b}
ϕ	ϕ	ϕ	ϕ	ϕ
{a}	ϕ	{a}	{a}	{a}
{b}	ϕ	{a}	{b}	{a, b}
{a,b}	ϕ	{a}	{a,b}	{a, b}

Table 1.21

We see {b} $\in$ W acts as the ling identity under max. However for P(S) the ling identity with respect to min is S = {a, b, c}.

So the ling submonoid W of P(S) has a different ling identity from that of P(S).

We consider {V, max} and represent in the following table V = {{a}, {a, c}, {a, b}, {a, b, c}} $\subseteq$ P(S). $a \leq b \leq c$.

max	{a}	{a, c}	{a, b}	{a, b, c}
{a}	{a}	{a, c}	{a, b}	{a, b, c}
{a, c}	{a, c}	{a, c}	{a, b, c}	{a, b, c}
{a, b}	{a, b}	{a, b, c}	{a, b}	{a, b, c}
{a, b, c}	{a, b, c}	{a, b, c}	{a, b, c}	{a, b, c}

Table 1.22

We see the least element is {a} in V. So the ling identity of V is {a} under max operation. However the ling identity for max operation in P(S) is {ϕ}.

Hence for the ling submonoid {V, max} ⊆ {P(S), max} we see the linguistic identity of V different from the ling identity of P(S) under max operation.

Now we give the table of {V, min} in the following.

min	{a}	{a, b}	{a, c}	{a, b, c}
{a}	{a}	{a}	{a}	{a}
{a, b}	{a}	{a, b}	{a, b}	{a, b}
{a, c}	{a}	{a, b}	{a, c}	{a, c, b}
{a, b, c}	{a}	{a, b}	{a, c, b}	{a, b, c}

Table 1.23

Hence we list out the following observations.

1. If P(S) be the ling power set of S. ϕ in P(S) is the least element and S to be the greatest element of P(S).
2. If min(and max) operations are to be defined on the ling power set P(S) it is mandatory that the ling set S must be a totally ordered set. For otherwise min or max operations on P(S) cannot be defined.

3. {P(S), min}, {P(S), max}, {P(S), $\cup$}, and {P(S), $\cap$} are the four distinct ling monoids under the respective operations mentioned.

4. A proper ling subset M from P(S) can be just a ling subset closure and may not be compatible with any of the four operations $\cup$, $\cap$, min or max.

5. Even if {M, min}, {M, max}, {M, $\cup$} and {M, $\cap$} happen to be closed under their respective operations still {M, min}, {M, max}, {M, $\cup$} and {M, $\cap$} may not be ling submonoid it may be only a ling subsemigroup.

6. Even if {M, min}, {M, max}, {M, $\cup$} and {M, $\cap$} are ling submonoids of {P(S), min}, {P(S), max}, {P(S), $\cup$} and {P(S), $\cap$} respectively still they may have a different ling identity from that of the ling identity mentioned in {P(S), min}, {P(S), max}, {P(S), $\cup$} and {P(S), $\cap$}.

To this effect we have shown examples.

However to make these notions more understandable we will describe some examples to this effect.

Example 1.20. Consider the totally ordered ling set

S = {a, b, c, d}; P(S) be the ling power set of S.

P(S) = {ϕ, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, {a, b, c, d} = S} be the ling power set of S.

$o(P(S)) = 2^4 = 16$. S is totally ordered $a \leq b \leq c \leq d$.

Now {P(S), ∪} is a commutative ling monoid with {ϕ} is its ling identity.

For A ∈ P(S) we have A ∪ {ϕ} = A for every A ∈ P(S).

We see {P(S), ∩} is a commutative ling monoid with S = {a, b, c, d} as its ling identity.

Thus for A ∈ P(S); A ∩ S = A for all A ∈ P(S).

Consider {P(S), max}; this is again a ling monoid with ϕ as its ling identity.

For max {ϕ, A} = A for every A ∈ P(S).

The following observation is mandatory.

We have both {P(S), max} and {P(S), ∪} have the same ling identity viz ϕ. However both the operations ∪ and max are distinct.

For if we take A = {a, b, d} and B = {b, c, d} ∈ P(S).

We find A ∩ B = {a, b, d} ∩ {b, c, d} = {b, d} …I

A ∪ B = {a, b, d} ∪ {b, c, d} = {a, b, c, d} …II

min{A, B} = min{{a, b, d}, {b, c, d}}

= {min{a, b}, min{a, c}, min{a, d}, min{b, b}, min{b, c}, min{b, d}, min{d, b}, min{d, c}, min{d, d}}

= {a, b, c, d} …III

(using $a \leq b \leq c \leq d$)

max{A, B} = max{{a, b, d}, {b, c,d}}

= {max{a, b}, max{a, c}, max{a, d}, max{b, c}, max{b, d}, max{b, b}, max{d, b}, max{d, c}, max[d, d}}

= {b, c, d} …IV

(using $a \leq b \leq c \leq d$).

We see II and III are identical however I, II and IV are distinct or I, III and IV are distinct or different.

Now take C = {a, d} and D = {b, c} $\in$ P(S)

$D \cap C$ = {a, d} $\cap$ {b, c} = ϕ …I

$D \cup C$ = {a, b} $\cup$ {b, c} = {a, b, c, d} …II

min{D, C} = min{{a, d}, {b, c}} = {min{a, b}, min{a, c}

min {d, b}, min{d, c}} = {a, b, c}; ($a \leq b \leq c \leq d$) …III

Consider max{C, D} = max {{a, d}, {b, c}}

= {max{a, b}, max{a, c}, max{d, b}, max{d, c}}

= {b, c, d}; ($a \leq b \leq c \leq d$) …IV

Thus we see I, II, III and IV are distinct hence {P(S), $\cap$}, {P(S), $\cup$}, {P(S), min} and {P(S), max} are four different ling monoids. Hence our claim.

Now we show how they work on ling subsets of P(S).

Let M = {{a, b}, {b, c}, {c, d}, {d, a}, {b, d}} ∈ P(S).

We give the table of {M, ∪} in the following.

∪	{a, b}	{b, c}	{c, d}	{d, a}	{b, d}
{a, b}	{a, b}	{a, b, c}	{a,b,c, d}	{a, b, d}	{a, b, d}
{b, c}	{a, b, c}	{b, c}	{b,c,d}	{a,b,c,d}	{b, c, d}
{c, d}	{a,b,c,d}	{b, c, d}	{c,d}	{a, c, d}	{b, c, d}
{d, a}	{a, b, d}	{a,b,c,d}	{a,c,d}	{d, a}	{a, b, d}
{b, d}	{a, b, d}	{b, c, d}	{b,c,d}	{a, b, d}	{b, d}

Table 1.24

We see {M, ∪} is not even closed under the operation ∪. For {a, b, c}, {b, c, d}, {a, b, c, d}, {a, c, d}, {a, b, d} ∉ M.

So closure axiom is not true in case the operator ∪.

Now consider {M, ∩} the operation ∩ on M.

We give the table associated with {M, ∩} in the following.

∩	{a, b}	{b, c}	{c, d}	{d, a}	{b, d}
{a, b}	{a, b}	{b}	{ϕ}	{a}	{b}
{b, c}	{b}	{b, c}	{c}	{ϕ}	{b}
{c, d}	{ϕ}	{c}	{c, d}	{d}	{d}
{d, a}	{a}	{ϕ}	{d}	{a, d}	{d}
{b, d}	{b}	{b}	{d}	{d}	{b, d}

Table 1.25

{M, ∩} not even closed under the operation ∩.

For {b}, ϕ, {a}, {c}, {d} ∉ M.

So both {M, ∪} and {M, ∩} are even closed under the respective operations.

Now we give the table {M, max}; a ≤ b ≤ c ≤ d.

max	{a, b}	{b, c}	{c, d}	{d, a}	{b, d}
{a, b}	{a, b}	{b, c}	{c, d}	{a, d, b}	{b, d}
{b, c}	{b, c}	{b, c}	{c, d}	{b, c, d}	{b, c, d}
{c, d}	{c, d}	{c, d}	{c, d}	{c, d}	{c, d}
{a, d}	{a, d, b}	{b, c, d}	{c, d}	{a, d}	{b, d}
{b, d}	{b, d}	{b, c, d}	{c, d}	{b, d}	{b, d}

Table 1.26

We see {M, max} is not even closed under the operator max {a, b, d}, {b, c, d} ∉ M. This is clear from the table .

Now consider the table for {M, min}

min	{a, b}	{b, c}	{c, d}	{d, a}	{b, d}
{a, b}	{a, b}	{a, b}	{a, b}	{a, b}	{a, b}
{b, c}	{a, b}	{b, c}	{b, c}	{b, a, c}	{b, c}
{c, d}	{a, b}	{c, b }	{c, d}	{d, a, c}	{b, d, c}
{d, a}	{a, b}	{a, b, c}	{d, a, c}	{d, a}	{d, b, a}
{b, d}	{a, b}	{b, c}	{b, d, c}	{d, a, b}	{b, d}

Table 1.27

We see {M, min} is not closed under the operation min, for {d, c, a}, {b, d, c}, {a, d, b}, {a, b, c} ∉ M.

We see all the four operations on M are distinct and none of them is closed on M.

However we will consider a ling subset

R = {{a}, {a, b}, {a, b, c}, {b, c}, {c, a}} ⊆ P(S).

Now the table for {R; ∪} is given

∪	{a}	{a, b}	{b, c}	{c, a]	{a, b, c}
{a}	{a}	{a, b}	{a, b, c}	{c, a}	{a, b, c}
{a, b}	{a, b}	{a, b}	{a, b, c}	{a, b, c}	{a, b, c}
{b, c}	{a, b, c}	{a, b, c}	{b, c}	{a, b, c}	{a, b, c}
{c, a}	{c, a}	{a, b, c}	{a, b, c}	{c, a}	{a, b, c}
{a, b, c}	{a, b, c}	{a, b, c}	{a, b, c}	{a, b, c}	{a, b, c}

Table 1.28

We see {R, ∪} is a ling subsemigroup as ∪ is a closed operation on R.

We see {R, ∪} does not contain any ling identity in R.

Consider the table of {R, ∩}.

∩	{a}	{a, b}	{b, c}	{a, c}	{a, b, c}
{a}	{a}	{a}	{ϕ}	{a}	{a}
{a, b}	{a}	{a, b}	{b}	{a}	{a, b}
{b, c}	ϕ	{b}	{b, c}	{c}	{b, c}
{a, c}	{a}	{a}	{c}	{a, c}	{a, c}
{a, b, c}	{a}	{a, b}	{b, c}	{a, c}	{a, b, c}

Table 1.29

R under the ling operation ∩ is not closed. Hence {R, ∩} is not a ling subsemigroup or a ling submonoid as under the operator ∩ the closure axiom is flouted.

Now we give the table {R, min} by the following table.

min	{a}	{a, b}	{b, c}	{a, c]	{a, b, c}
{a}	{a}	{a}	{a}	{a}	{a}
{a, b}	{a}	{a, b}	{a, b}	{a, b}	{a, b}
{b, c}	{a}	{a, b}	{b, c}	{a, b, c}	{b, c, a}
{a, c}	{a}	{a, b}	{a, c, b}	{a, c}	{a, c, b}
{a, b, c}	{a}	{a, b}	{a, b, c}	{a, c, b}	{a, c, b}

Table 1.30

We see R under min is closed. Thus {R, min} is a ling subsemigroup which is not a ling submonoid as it has no ling identify in it (R).

Now we give the table for {R, max}

max	{a}	{a, b}	{b, c}	{a, c}	{a, b, c}
{a}	{a}	{a, b}	{b, c}	{a, c}	{a, b, c}
{a,b}	{a,b}	{a, b}	{b, c}	{a, b, c}	{a, b, c}
{b, c}	{b, c}	{b, c}	{b, c}	{b, c}	{b, c}
{a, c}	{a, c}	{a, b, c}	{b, c}	{a, c}	{a, b, c}
{a, b, c}	{a, b, c}	{a, b, c}	{b, c}	{a, b, c}	{a, b, c}

Table 1.31

We see max is a closed operation on R. {R, max} is infact a ling submonoid with {a} as its ling identity; however for P(S), ϕ is the ling identity under max operation.

Now we table the four structures of R.

S. No.		**Not closed under the operation**	**Ling subsemi group**	**Ling submonoid**
1.	{R, $\cup$}		ling subsemigroup	
2.	{R, $\cap$}	not closed under the operation $\cap$		
3.	{R,min}		ling subsemigroup	
4.	{R, max}	-	-	a ling sub monoid with {a} as ling identity

Table 1.32

We see the four ling operations behave in four different ways.

Consider the ling subsets

B = {{a}, {b}, {c}, {d}, {a, b, c}, {b, d}} ⊆ P(S).

Now we find all the four operations on B . The table for {B, ∪} is as follows.

∪	{a}	{b}	{c}	{d}	{b, d}	{a,b,c}
{a}	{a}	{a, b}	{a, c}	{a, d}	{a, b, d}	{a,b,c}
{b}	{a, b}	{b}	{b, c}	{b, d}	{b,d}	{a,b,c}
{c}	{a, c}	{b, c}	{c}	{c,d}	{c, b, d}	{a,b,c}
{d}	{a,d}	{b,d}	{c,d}	{d}	{b,d}	{a,b,c,d}
{b,d}	{a,b,d}	{b,d}	{b,c,d}	{b,d}	{b,d}	{a,b,c,d}
{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c,d}	{a,b,c,d}	{a,b,c}

Table 1.33

Clearly B is not closed under the ∪ operation.

Now we find the table {B, ∩}.

∩	{a}	{b}	{c}	{d}	{b,d}	{a,b,c}
{a}	{a}	ϕ	ϕ	ϕ	ϕ	{a}
{b}	ϕ	{b}	ϕ	ϕ	{b}	{b}
{c}	ϕ	ϕ	{c}	ϕ	ϕ	{c}
{d}	ϕ	ϕ	ϕ	{d}	{d}	{ϕ}
{b,d}	ϕ	{b}	{ϕ}	{d}	{b,d}	{b}
{a,b,c}	{a}	{b}	{c}	{ϕ}	{b}	{a,b,c}

Table 1.34

Clearly the operation ∩ is not closed for the ling subsets B ⊆ P(S). Now we find the table for max on B

a ≤ b ≤ c ≤ d.

max	{a}	{b}	{c}	{d}	{b,d}	{a,b,c}
{a}	{a}	{b}	{c}	{d}	{b,d}	{a,b,c}
{b}	{b}	{b}	{c}	{d}	{b,d}	{b,c}
{c}	{c}	{c}	{c}	{d}	{c,d}	{c}
{d}	{d}	{d}	{d}	{d}	{d}	{d}
{b,d}	{b,d}	{b,d}	{c,d}	{d}	{b,d}	{b,c,d}
{a,b,c}	{a,b,c}	{b,c}	{c}	{d}	{b,c,d}	{a,b,c}

Table 1.35

Clearly max operation is not closed for B.

We give the table for {B, min};

min	{a}	{b}	{c}	{d}	{b,d}	{a,b,c}
{a}	{a}	{a}	{a}	{a}	{a}	{a}
{b}	{a}	{b}	{b}	{b}	{b}	{a,b}
{c}	{a}	{b}	{c}	{c}	{b,c}	{a,b,c}
{d}	{a}	{b}	{c}	{d}	{b,d}	{a,b,c}
{b,d}	{a}	{b}	{b,c}	{b,d}	{b,d}	{a,b,c}
{a,b,c}	{a}	{a,b}	{a,b,c}	{a,b,c}	{a,b,c}	{a,b,c}

Table 1.36

{b,c}, {a, b} $\notin$ B. So B is not even closed under min operation.

Now we table the four operations on B.

S. No.		Not closed operation	Ling sub semigroup	Ling sub monoid
1.	{B, $\cup$}	Not closed	-	-
2.	{B, $\cap$}	Not closed	-	-
3.	{B, min}	Not closed	-	-
4.	{B, max}	Not closed	-	-

Table 1.37

Thus B remains to be a ling subset collection of P(B) which is not closed under all the four operations.

We have seen all the cases which proves the observations we have listed.

Now we would next define the new notion of ling semirings and ling semifields.

To achieve this we need to recall the definition of semirings and semifields.

Definition 1.4. *Let S be a non empty set. S is defined to be a semiring under the operations + and × if the following conditions are satisfied.*

1. *For all x,y $\in$ S we have x + y $\in$ S (closure axiom).*
2. *For all x, y, z $\in$ S we have*

$x + (y + z) = (x + y) + z$ (associativity law).

3. For all $x, y \in S$ we have $x + y = y + x$ commutative law.

4. There exists a unique $0 \in S$ such that $0 + x = x + 0 = x$ for all $x \in S$; 0 is called the additive identity of S.

5. {S, +} is a commutative monoid.

6. There is defined a binary operation $\times$ on S, such that for $x, y \in S$ we have $x \times y \in S$ (closure axiom)

7. For all $x, y, z \in S$ we have $x \times (y \times z) = (x \times y) \times z$ (associative law).

8. For all $x, y \in S$ if $x \times y = y \times x$ (commutative law) we define {S, $\times$} as a commutative semigroup under $\times$.

9. We define {S, +, $\times$} as a semiring if there exists $1 \in S$ such that $x \times 1 = 1 \times x = x$ for all $x \in S$; then we define {S, $\times$} as the commutative semigroup under the $\times$; if $1 \in S$ we define {S, $\times$} to be ling monoid.

Now if we have the following operation to be true that is

$$x \times (y + z) = x \times y + x \times z$$

The distributive law is true.

Then {S, +, ×} is defined as the semiring.

For more about semirings please refer [31-37].

We will first provide some examples of them.

Example 1.21. Let S = $Z^o = Z^+ \cup \{0\}$ be the set of positive integers with zero. {S, +} is a monoid of infinite order.

For if 5, 9 ∈ {S, +} we know 5 + 9 = 14 ∈ S.

Thus {S, +} a monoid and its identity is zero; {S, ×} is a monoid using Z^+, under the product.

Thus {S, +, ×} is a semiring of infinite order.

In case $R^o = R^+ \cup \{0\}$ is again a semiring of infinite order.

$Q^o = Q^+ \cup \{0\}$ is also a semiring of infinite order.

We now proceed onto give examples of semirings.

Example 1.22. $Q^o[x] = \{\sum_{i=1}^{\infty} a_i x^i \mid a_i \in Q^o = Q^+ \cup \{0\}$; x is an indeterminate} be the semiring of polynomials with coefficients from the positive rationals together with 0.

It is easily verified that for any p(x) ∈ $Q^o[x]$, we cannot find a negative q(x) ∈ $Q^o[x]$ for p(x) such that

$q(x) + p(x) = 0(x)$ where $0(x) = 0x^n + 0x^{n-1} + \ldots + 0x + 0$

the zero polynomial.

Since in $Q^o = Q^+ \cup \{0\}$ we do not have for any

$$n \in Q^o, (-n) \notin Q^+$$

such that $n + (-n) = 0$ as Q^+ is only the collection of positive rationals.

We know $\{Q^o = Q^+ \cup \{0\}, +, \times\}$ is a semiring hence $Q^o[x]$ is also a semiring under + and × [31-37].

Example 1.23. Let $S = \{R^o[x] / R^+ \cup \{0\} = R^o$; x an indeterminate} be the semiring of polynomials {S, +, ×} is a semiring.

Further we see this S has no inverses with respect + and ×.

Now all the examples provided of semirings are of infinite cardinality or to be more appropriate technically all of them are of characteristic zero.

Now we will provide examples of finite ordered semiring.

We just recall the definition of lattice to make this book a self contained one.

We define the notion of relation on two non-empty sets A and B.

Definition 1.5. *Let A and B two non empty sets. A relation R from A to B is a subset of A × B. Relations from A to B are called relations on A for short if (a,b) ∈ R then we write aRb and say that 'a' is in relation R to b. Also if a is not in relation R to be and write a* $a \not{R} b$.

A may have some of the following properties.

R is reflexive if for all a in A we have aRa.

R is symmetric if for a and b in A aRb implies bRa.

R is antisymmetric if for all a and b in A aRb and bRa imply a = b.

R is transitive if for a, b, c in A aRb and bRc imply aRc.

A relation R on a set A is called a partial order (relation) if R is reflexive, antisymmetric and transitive.

In this case we say (A, R) is a partially ordered set or a poset.

A partial ordered is denoted by $\leq$ or $\subseteq$.

We say a partial order on A is called a total order if for every pair a, b $\in$ A either a $\leq$ or b $\leq$ a. {A, $\leq$} is called a chain or a totally ordered set.

We provide examples of a partially ordered set and a totally ordered set.

Example 1.24. Let S = {a, b, c, d} be a set of four elements. The powerset P(S) of S is given by

S = {ϕ, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, {a, b, c, d}}. |S| = 2^4 = 16.

The containment is only a partial order on P(S) as {b, c} and {c, d} $\in$ P(S) {b, c} $\not\subset$ {c, d} however we have {a, b}, {a, b, d} ∈ P(S) such that {a, b} $\not\subset$ {a, b, d}.

Thus {P(S), ⊆} is only a partial order set.

Example 1.25. Let S = {1, 9, –9, 6, 8, 12, –18} be a nonempty set S is a totally ordered set under '≤' for

$$-18 \leq -9 \leq 1 \leq 6 \leq 8 \leq 9 \leq 12.$$

Hence the claim.

We can represent the two examples diagrammatically.

The diagrammatic representation is defined as the Hassee diagrams.

The Hassee diagram for the totally ordered set S is as follows.

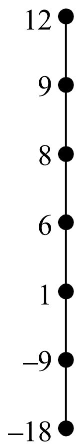


Figure 1.1

The Hasse diagram for P(S) given in example 1.24 is as follows.

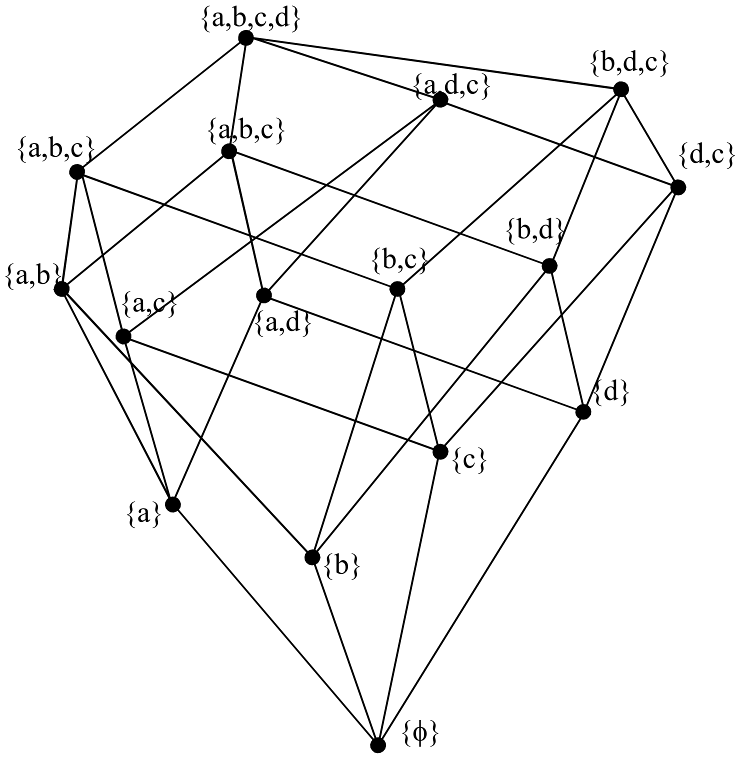


Figure 1.2

Now we proceed onto define the notion of upper bound, lower bound, infimum and supremum of a partially ordered set or poset in the following.

Definition 1.6. *Let {S, ≤} be a partially ordered set and P ⊆ S.*

i) *x ∈ S is called an upper bound of P if and only if for all p ∈ P; p ≤ x.*

ii) *x ∈ S is called a lower bound of P if and only if for all p ∈ P, x ≤ p.*

iii) *The greatest amongst the lower bounds, whenever it exists is called the infimum of P and is denoted by inf P.*

iv) *The least upper bound of P whenever it exists is called the supremum of P and is denoted by sup B.*

We now recall the definition of a semilattice.

Definition 1.7. *Let (L, ≤) be a partially ordered set (L, ≤) is called a semilattice order if for every pair if elements, a, b ∈ L the sup(a, b) exists (or equivalently inf (a, b) exists).*

A partially ordered set (poset) (L, ≤) is called a lattice ordered if for every pair of elements, a, b ∈ both inf (x, y) and sup (x, y) exists.

We will first provide examples of semilattices.

Example 1.26. Consider A the subset of P(S) where

S = {a, b, c}.

Here A = {{a, b, c} = S, {a, b} {a, c} and {b, c}}. The semilattice associated with A is as follows;

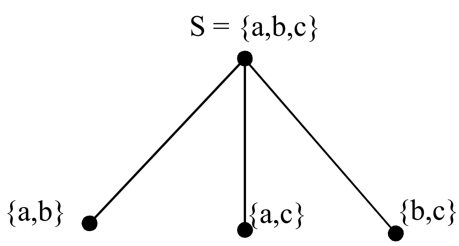


Figure 1.3

Here this semilattice has operation sup defined on it and inf operation is not defined on it.

Example 1.27. Consider the subset B of the power set P(S) of S where S = {a, b, c, d}.

The subset B $\subseteq$ P(S) has the following elements B = {ϕ,{a}, {b}, {c}, {d}}.

The semilattice of B which has only inf defined on it is given by

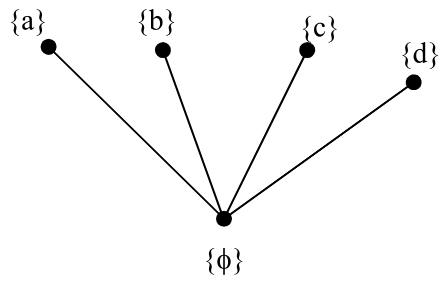


Figure 1.4

We can have several such semilattices.

Now we can technically define semilattice.

Definition 1.8. *Let {L, ≤} be a poset. We say {L, ≤} is a semilattice under sup if the following statements are equiv*alent.

i) $x \leq y$

ii) *sup {x, y} = y.*

We say {L, ≤} is a semilattice under inf if the following statements are equivalent.

i) $x \leq y$

ii) $inf\,(x,y) = x$.

(Now to this effect we have already provided examples).

Now we define equivalent of a lattice as follows.

Definition 1.9

i) Every ordered set is lattice ordered.

ii) In a lattice ordered set (L, ≤) the following statements are equivalent for all x, y ∈ L

a. $x \leq y$

b. $sup\,\{x, y\} = x$ *and*

c. $inf\,\{x, y\} = x$.

We will provide one or two examples of the same.

Example 1.28. Let L be a lattice given by the following figure.

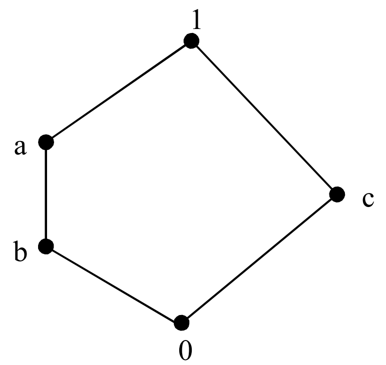


Figure 1.5

This lattice will be called as a pentagon lattice.

We see L = {a, b, c, 0, 1} the '≤' order is as follows

0 ≤ b ≤ a ≤ 1 and 0 ≤ c ≤ 1.

a and b are comparable with c. This is also evident from the Hasse diagram given in figure.

Example 1.29. Let S = {0, 1, a, b, c} be a set with the partial order ≤ as follows

0 ≤ a ≤ 1, 0 ≤ b ≤ 1 and 0 ≤ c ≤ 1.

Clearly a, b and c are not comparable.

We have the following Hasse diagram for the S.

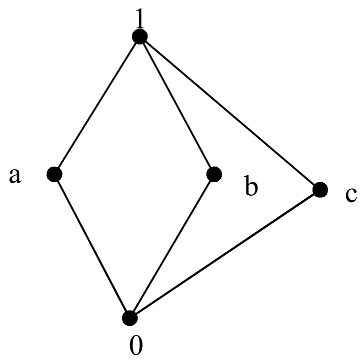


Figure 1.6

This lattice will be addressed as a diamond lattice.

If we remove from the set S the element 1 then we have S \ {1} = P = {0, a, b, c} such that

0 ≤ a, 0 ≤ b and 0 ≤ c

The resultant can only yield a semi lattice under inf only. The Hasse diagram is given by the following figure.

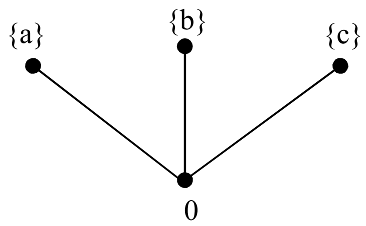


Figure 1.7

For only if {a, b} = 0 = inf {b, c} = inf {a, c}.

However sup{a, b}, sup{b, c} and sup{a, c} is not defined.

Now if we remove from the set S the element 0. Then we have S \ {0} = B = {1, a, b, c}.

We have a ≤ 1, b ≤ 1 and c ≤ 1.

For this B we can only define sup for

sup{a, b} = sup{b, c} = sup{a, c} = 1.

So it a semilattice under sup.

We do not have inf{a, b}, inf{b, c} and inf{a, c} to be defined.

The Hasse diagram for B is as follows.

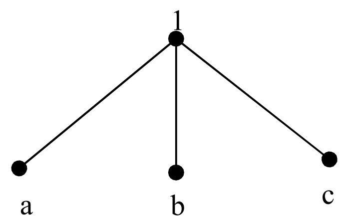


Figure 1.8

Thus B is a semilattice under the sup operation.

Next we define the notion of a line lattice.

We saw all the two lattices and semilattice given in figure are only partially ordered set. They are not totally ordered set.

If we have a lattice whose set under consideration is a totally ordered set then we have a special name for then we call it either as a line lattice or more mathematically a chain lattice.

Definition 1.10. *Let {L, ≤} be a lattice. If '≤' is a total order on L and L is a lattice order we call L a chain lattice. Thus we have a chain lattice L if for every pair x, y ∈ L we have either x ≤ y or y ≤ x.*

We now give some examples of chain lattices.

Example 1.30. Consider l = [x, y] any closed interval on the real line R, [x, y] under total order is a chain lattice.

Example 1.31. [0, ∞] is also a chain lattice of infinite order.

Example 1.32. [–∞, 1] is a chain lattice of infinite order.

Example 1.33. {0, 1} = L the two element set 0 ≤ 1 is a chain lattice having the following Hasse diagram.

Figure 1.9

This lattice will be denoted by C_2.

Now we give the algebraic definition of semilattice and lattice.

Definition 1.11. *An algebraic semilattice {L, $\cup$} (under sup) is a nonempty set L with a binary operation $\cup$ (join) (also known as sum or union) which satisfy the following conditions.*

i) $x \cup y = y \cup x$ *(commutative law)*
ii) $x \cup (y \cup z) = (x \cup y) \cup z$ *(associative law)*
iii) $x \cup (x \cap y) = x$ *(absorption law).*

(This condition (iii)) yields an additional condition such that

iv) $x \cup x = x$ *(idempotent law)*

An algebraic semilattice (L, $\leq$) L a nonempty set with a binary operation meet '$\cap$' (also called as intersection or product) which satisfy the following conditions for all x, y, z $\in$ L.

i) $x \cap y = y \cap x$ *(commutative law)*
ii) $(x \cap y) \cap z = x \cap (y \cap z)$ *(associative law)*
iii) $x \cap (x \cup y) = x$ *(absorption law)*

This law leads to additional conclusion $x \cap x = x$

iv) $x \cap x = x$ *(idempotent law).*

In view of all these we will now give the definition of an algebraic lattice.

Definition 1.12. *An algebraic lattice {L, $\cap$, $\cup$} is a nonempty set L with two binary operations $\cup$ and $\cap$ ($\cup$ (join) and $\cap$ (meet)) (also called as union or sum and intersection or product) which satisfy the following conditions for all*

$$x, y, z \in L.$$

i) $x \cap y = y \cap x$; $x \cup y = y \cup x$ *(commutative law)*

ii) $x \cap (y \cap z) = (x \cap y) \cap z$,

$(x \cup y) \cup z = x \cup (y \cup z)$ *(associative law)*

iii) $x \cap (x \cup y) = x \quad x \cup (x \cap y) = x$ *(absorption law)*

Thus (iii) condition leads to a new law viz.

$x \cap x = x$ *and* $x \cup x = x$ *(idempotent law).*

The relation between lattice ordered sets and algebraic lattice is as follows.

Let (L, $\leq$) be a lattice ordered set.

If we define $x \cap y = \inf (x, y)$ and $x \cup y = \sup (x,y)$

Then {L $\cup$, $\cap$} is an algebraic lattice.

If on the other hand {L, $\cup$, $\cap$} is an algebraic lattice if we define $x \leq y$ if and only if $x \cap y = x$ (or $x \leq y$) if and only if ($x \cup y = y$) then (L, $\leq$) is a lattice ordered set.

Now we give yet another example of a algebraic lattice.

Example 1.34. Let S = {a, b, c} be a set and P(S) be the power set of S.

P(S) = {ϕ, {a}, {b}, {c}, {a, c}, {a, b}, {b, c}, {a, b, c} = S} be the power set and order of P(S) = 2^3 = 8.

The semilattice under ∪ or sup is given below

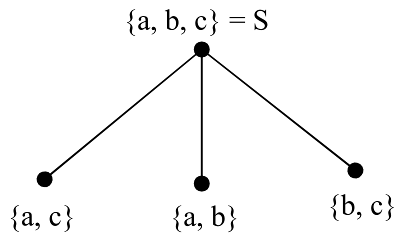


Figure 1.10

The semilattice under ∩ or inf is given below.

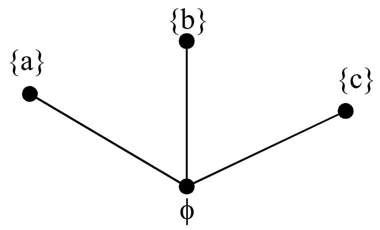


Figure 1.11

The algebraic lattice with ∪ and ∩ is given below.

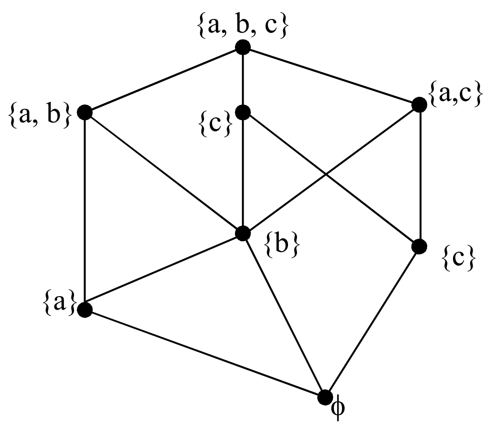


Figure 1.12

Now we define to notion of a sublattice of a lattice.

Definition 1.13. *Let L be a lattice.*

S be a non-empty subset of L if S is lattice with respect to the restriction of $\cup$ and $\cap$ of L onto S then S is a sublattice of L.

We will give examples of sublattice of a lattice L.

Example 1.35. Let L be a lattice given by the following figure 1.13. S be a sublattice of L which is also given in figure 1.14.

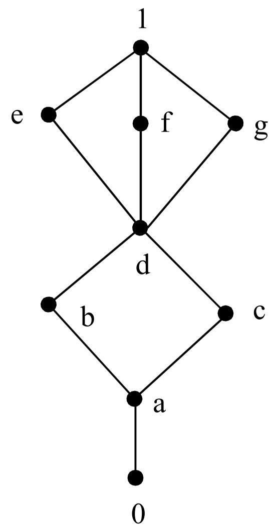


Figure 1.13

Consider the subset S of L given by {1, e, f, d, b, c, a, 0}.

The figure associated with S is as follows.

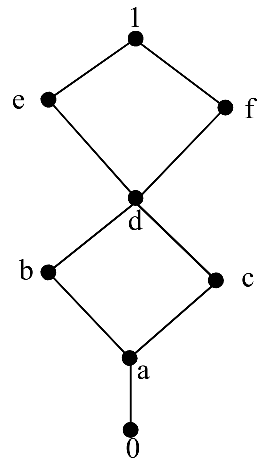


Figure 1.14

Take S \ {d} = P then P does not form a sublattice. Take M = {b a d c} $\subseteq$ L; M is a sublattice given by the following figure.

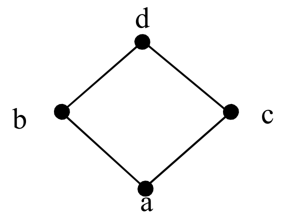


Figure 1.15

Let N = {e, f, g, d, 1} $\subseteq$ L; the sublattice associated with N is given by the following figure.

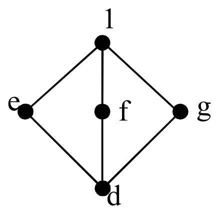


Figure 1.16

We give definition of modular lattice and distributive lattice in the following.

Definition 1.14. *Let L be a lattice L is called or defined as a distributive lattice if for all x, y, z $\in$ L; the following equations hold good;*

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z) \text{ or}$$

$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$$

called the distributivity equations.

We give examples of distributive lattices.

Example 1.36. Let L be a lattice given by the following figure 1.17.

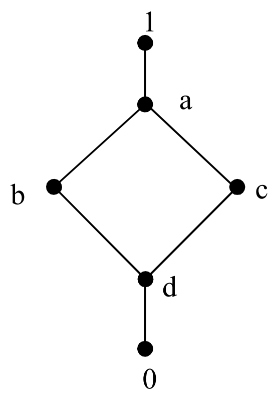


Figure 1.17

To prove a $\cup$ (b $\cap$ d) = (a $\cup$ b) $\cap$ (a $\cup$ d).

Consider a $\cup$ (b $\cap$ d) = a $\cup$ d = a

Now (a $\cup$ b) $\cap$ (a $\cup$ d) = a $\cap$ a = a.

Hence distributive law is true for the triplet.

Consider x ∩ (y ∪ z) = (x ∩ y) ∪ (x ∩ z)

x ∩ (y ∪ z) = a ∩ (b ∪ d) = a ∩ b = b

(a ∩ b) ∪ (a ∩ d) = b ∪ d = b.

Consider {b, d, c}

To prove b ∪ (d ∩ c) = (b ∪ d) ∩ (b ∪ c)

b ∪ (d ∩ c) = b ∪ d = b.

Now

(b ∪ d) ∩ (b ∪ c) = b ∩ a ∩ b.

b ∩ (c ∪ d) = (b ∩ c) ∪ (b ∪ d)

Consider

b ∩ (c ∪ d) = b ∩ c = d.

and (b ∩ c) ∪ (b ∩ d) = d ∪ d = d.

Thus b ∩ (c ∪ d) = (b ∩ c) ∪ (b ∩ d) = d.

Consider the lattice L given by the following figure 1.18.

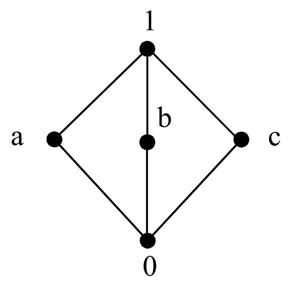


Figure 1.18

Take a, b, c $\in$ L

$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$

$a \cup (b \cap c) = a \cup 0 = a$ ... I

$(a \cup b) \cap (a \cup c) = 1 \cap 1 = 1$ …II

$(a \cup b) \cap (a \cup c) = 1 \cap 1 = 1$

Equation I and II are not equal in L so figure 1.18 is not a distributive lattice.

$(a \cap b) \cup (b \cap c) \cup (c \cap a) = 0 \cup 0 \cup 0 = 0.$ …I

$(a \cup b) \cap (b \cap c) \cap (c \cup a) = 1 \cap 1 \cap 1 = 1.$

So distributive law is not true in case of this diamond lattice. Thus we can conclude the diamond lattice is not a distributive lattice.

Consider the pentagon lattice L = {0, b, a, 1, c}

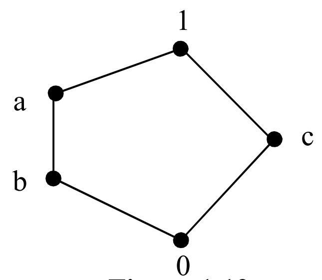


Figure 1.19

Is a pentagon lattice distributive or not?

Consider a, b, c $\in$ L.

To prove $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ if the pentagon lattice is distributive.

Consider $a \cap (b \cup c) = a \cap 1 = a$ …I

$(a \cap b) \cup (a \cup c) = b \cup 0 = b$ …II

Clearly I and II are distinct, hence the pentagon lattice is not a distributive lattice.

Now we recall the definition of modular lattices.

Definition 1.15. *A lattice L is called modular if for all*

$a, b, c \in L$, $a \leq c$ imply $a \cup (b \cap c) = (a \cup b) \cap c$.

We can prove the diamond lattice L is a modular lattice, but just we have proved that a diamond lattice is not distributive.

Consider the modular lattice 1, a, b (we cannot take the triple a, b, c as all the three are not comparable so modular law cannot be implemented.

To prove for a, b, 1 with $a \leq c = 1$, the identity

$x \cup (y \cap z) = (x \cup y) \cap z$; $x \leq z$ …I

Consider $a = x$, $b = y$ and $z = 1$;

$x \cup (y \cap z) = a \cup (b \cap 1)$ $(a = x, b = y, z = 1)$

$= a \cup b = 1$ …I

$(x \cup y) \cap z = (a \cup b) \cap 1 = 1 \cap 1 = 1$ … II

I and II are identical hence modular law is true.

We can take (a, b, 0); $x = 0$ $y = b$ $z = a$

$x \cup (y \cap z) = 0 \cup (b \cap a) = 0$ …I

$(x \cup y) \cap z = (0 \cup b) \cap a = b \cap a = 0$ …II

I and II identical hence modular law is true.

Thus the diamond lattice is modular however it is not a distributive lattice.

To test if the pentagon lattice is modular.

Consider the pentagon lattice.

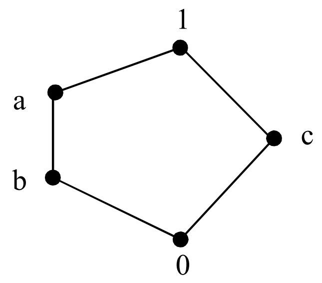


Figure 1.20

To prove $x \cup (y \cap z) = (x \cup y) \cap z; \quad x \leq z$

Take $x = a$, $y = b$ and $z = 1$; $a \cup (b \cap 1) = a \cup b = 1$ … I

$(x \cup y) \cup z = (a \cup b) \cap z = a \cap 1 = a$ …II

I and II are not the same so the modular law does not hold good for the triplet $(a, b, 1;\ a \leq 1)$ hence the pentagon lattice is not modular.

Now a chain lattice is always distributive. For elements of a chain lattice is a totally ordered set; they satisfy both modular and distributive lattice.

All distributive lattices are modular but in general a modular lattice is non distributive.

A pentagon lattice is both non distributive and non modular.

A diamond lattice is distributive but is a modular lattice. The lattice given in the following figure is a distributive lattice.

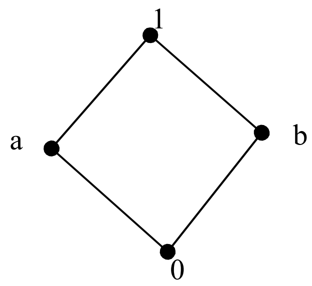


Figure 1.21

Now we define complemented lattice [12, 22].

Definition 1.16. *A lattice L with 0 and 1 is called complemented if for each $x \in L$ there is at least one $y \in L$ such that*

$x \cap y = 0$ and $x \cup y = 1$; y is called the complement of L.

For the figure 1.21 given, a is the complement of b as

$a \cup b = 1$ and $a \cap b = 0$.

Further the complement is unique.

Consider the following lattice $L = \{1, a_1, a_2, \ldots, a_5, 0\}$

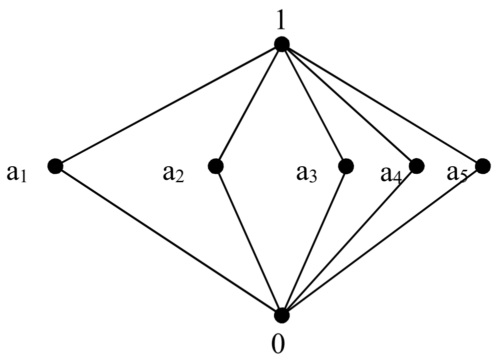


Figure 1.22

The complement of a_1 can be a_2, a_3, a_4, a_5

for $a_i \cap a_j = 0$ and $a_i \cup a_j = 1, i \neq j$.

We see 0 is the complement of 1 or 1 is the complement of 0

as $0 \cup 1 = 1$ and $0 \cap 1 = 0$.

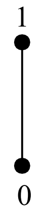


Figure 1.23

Thus C_2 or the chain lattice is such that it has only two element.

Consider the lattice L = {0, 1, a, b} given by the following figure.

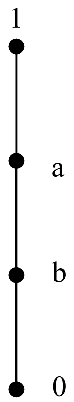


Figure 1.24

a has no complement and b has no complement.

Thus we can have lattices such that the elements other than 0 and 1 have no complement.

The class of line or chain lattices falls under the class. For in a chain lattice L other than 0 and 1 no other element has a complement.

The chain lattice L given in the following figure 1.25, no element other than 0 and 1 have complements.

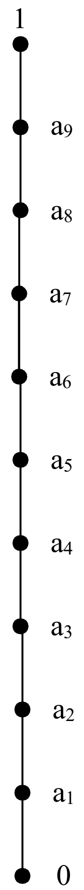


Figure 1.25

$0 \cup 1 = 1, \; 0 \cap 1 = 0.$

But $\quad a_i \cap a_j = a_i \qquad$ if $i \leq j$

$a_i \cup a_j = a_j \qquad$ since $i \leq j$ for $1 \leq i, j \leq 9$

hence L has no complement for its elements other than 0 and 1.

Recall the definition of Boolean algebra [12, 22].

Definition 1.17. *A complemented distributive lattice is called or defined as a Boolean algebra. Distributivity in a Boolean algebra guarantees the uniqueness of complements.*

The Hasse diagram of the smallest Boolean algebra is C_2.

Figure 1.26

The complements are unique for $1 \cup 0 = 1$ and $0 \cap 1 = 0$.

Now consider

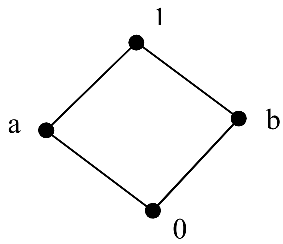


Figure 1.27

a is the unique complement of b and vice versa.

0 is the complement of 1 and vice versa

$a \cup b = 1$ and $a \cap b = 0$

$0 \cup 1 = 1$ and $1 \cap 0 = 0$.

Consider the direct product of the lattice C_2 with itself.

$C_2 \times C_2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$.

Now this $C_2 \times C_2$ can be given a lattice representation which is as follows.

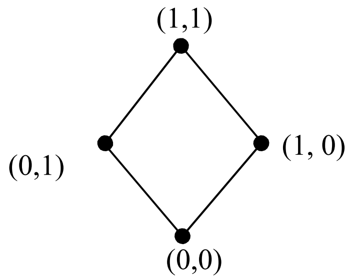


Figure 1.28

Clearly $(0, 0) \cup (1, 1) = (1, 1)$ and

$(0, 0) \cap (1, 1) = (0, 0)$.

So (1, 1) is the complement of (0, 0) and vice versa.

Consider $(1, 0) = (0, 1) \in L = C_2 \times C_2$.

$(1, 0) \cup (0, 1) = (1, 1)$ and $(1, 0) \cap (0, 1) = (0, 0)$

Complement of (0, 1) is (1, 0) and vice versa.

So the lattice got by $C_2 \times C_2$ is a distributive lattice and is a Boolean algebra by the very definition of Boolean algebra.

Consider the power set of two element {a, b} = S.

P(S) = {ϕ,{a}, {b}, {a, b}}.

The lattice representation of P(S) is as follows.

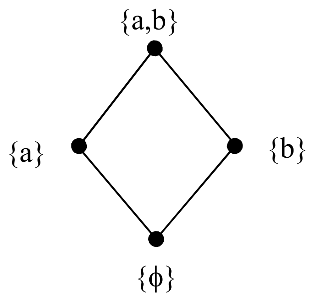


Figure 1.29

This lattice is identical or isomorphic with the Boolean algebra got by $C_2 \times C_2$ and P(S)

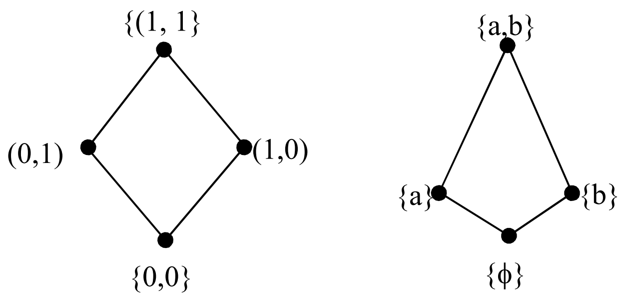


Figure 1.30

So by matching

{0, 0} ↔ {ϕ}, {{0, 1)} ↔ {a},

{(1,0)} ↔ {b}, {(1,1)} ↔ {a, b}.

Now consider the triple product of C_2;

$L = C_2 \times C_2 \times C_2 = \{(0, 0, 0), (1, 0, 0), (0, 1, 0)\ (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$. $|L| = 8$.

The Hasse diagram for L is as follows.

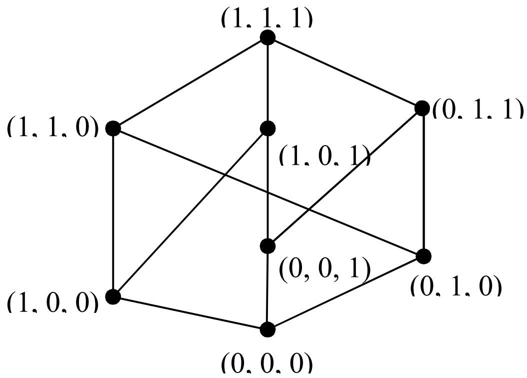


Figure 1.31

Now consider the power set P(S) of S = {a, b, c}.

We have

P(S) = {ϕ, {a}, {b}, {c}, {a, c}, {a, b}, {b, c} = S = {a, b, c}} is the power set of order $8 = 2^3$. The lattice associated with P(S) is given by the following figure.

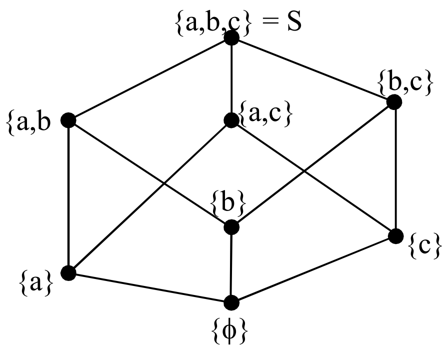


Figure 1.32

We see $C_2 \times C_2 \times C_2$ is also a lattice infact a Boolean algebra of order 8 and P(S) is again a Boolean algebra of order 8 which is identical or isomorphic to each other.

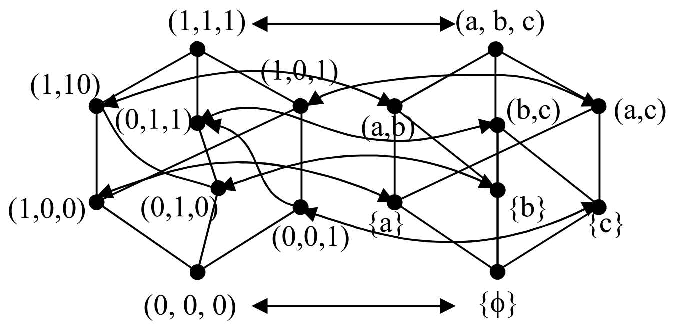


Figure 1.33

{(1, 1, 1)} ↔ {a, b, c}, (1, 1,0) ↔ {a, b}

(0, 1, 1) ↔ {b, c}, (1, 0, 1) ↔ {a, c}

{(0, 0, 0) ↔ ϕ, (1, 0, 0) ↔ {a}

(0, 1, 0) ↔ {b}, (0, 0, 1) ↔ {c}}

In view of this we have the following result.

Theorem 1.14. *Let L =* $\underbrace{C_2 \times \ldots \times C_2}_{n\text{-times}}$ *be a Boolean algebra of order* 2^n. *Let K = P(S), the power set of S =* $\{a_1, \ldots, a_n\}$.

|P(S)| = 2^n. *K = P(S) is again a Boolean algebra of order* 2^n.

We have the two Boolean algebras are isomorphic or identical.

Proof is left as an exercise to the reader.

Boolean algebras are also semirings of finite order.

Thus we have given semirings of finite order.

We suggest a few problems to the reader so that by working with them the reader will become familiar with the basic concepts of ling variables, ling sets, partially ordered ling sets and totally ordered sets and lings sets; lattices, semirings Boolean lattice or Boolean algebra and so on which are described in this chapter.

SUGGESTED PROBLEMS

1. Give an example of a ling variable which has a ling continuum associated with it.

2. Give an example of a ling variable whose associate ling set is an unordered finite asset.

3. Illustrate by an example of a ling variable which ling set is continuous totally ordered by not increasing through out the set.

4. Can we say a ling variable whose ling set is totally ordered set increases till the end?

5. Can we say all ling variables which have a ling continuum is always a time dependent one?

   Justify your claim.

6. Obtain any other special feature associated with ling variables and their ling sets.

7. Does there exist ling variables which is time dependent still their ling set is a ling continuum?

8. Find a ling variables whose ling set repeats the same form of occurrence.

9. Compare and contrast the ling variables.

   a) The weather report of a month.

   b) The speed of a car on road.

   The ling set associated with them

10. For the ling variable age of persons find the related ling set S.

    i) Is {S, min} a semigroup?

    ii) Is {S, max} a semigroup?

    iii) Can we define {S, $\cup$} and {S, $\cap$} on S?

    iv) What are the special features enjoyed by {S, max}?

11. For the ling variable, weight of people find the related ling set B.

    a) Does B dependent on time?

    b) Can we say B is a continuous set?

    c) Does B increase with time?

    d) Can we define {B, max} and {B, min} on B?

e) Is {B, max} the same as {B, ∪}? Justify your claim!

f) Is {B, min} the same as {B, ∩}? Substantiate your answer.

g) Can the graph of B be constant at an interval on the ling axis?

h) Compare this ling set B with the ling set S of age.

12. Let the ling variable be the colour of the eyes of people internationally. Let C be the ling set associated this ling variable.

a) Is C is a finite or infinite set?

b) Can C be a partially ordered set or a totally ordered set or neither? Substantiate your claim.

c) Does {C, min} a ling semigroup?

d) What is the largest cardinality of C?

13. Let V be ling variable associated with the yield of paddy plants. Let D be the ling set associated with the ling variable V.

i) Is D a totally ordered set?

ii) Can we say D is at least a partially ordered set?

iii) Is D of infinite cardinality or finite cardinality?

iv) Compare this D with the ling set C given in problem 12.

v) Compare and contrast this D with the ling set B given in problem 10. How do they differ from each other?

vi) Compare this ling set D with the ling sets given in problem 9.

vii) Is D time dependent or time independent?

14. Let V be a ling variable and S the associated ling set for the ling variable V. P(S) be the ling power set of S.

i) What is the ling algebraic structure enjoyed by $\{P(S), \cap\}$?

ii) Describe the ling algebraic structure enjoyed by $\{P(S), \min\}$.

iii) Does the $\{P(S), \cap\}$ and $\{P(S), \min\}$ identical or distinct? Substantiate your claim.

iv) What is the ling algebraic structure enjoyed by $\{P(S), \max\}$?

v) Prove $\{P(S), \cup\}$ is a ling monoid!

vi) What is the ling identity of $(P(S), \cup\}$?

vii) Is ling algebraic structure of $\{P(S), \cup\}$ the same as $\{P(S), \max\}$?

viii) Can we prove $\{P(S), \cup, \cap\}$ is a ling semiring?

ix) Find the linguistic semilattice $\{P(S), \cup\}$. Is it under inf or sup?

x) Can we prove $\{P(S), \max, \min\}$ is a ling semiring?

xi) Is {P(S), ∩} a ling semilattice under inf?

xii) Is the ling lattice {P(S), ∪, ∩} a distributive ling lattice?

xiii) Prove {P(S), ∪, ∩} is a Boolean algebra.

xiv) Prove every ling element or ling subset in P(S) is ling complemented.

xv) Will every ling sublattice of {P(S), ∪, ∩} be a ling Boolean sublattice? Justify your claim.

15. Let S = $\{s_1, s_2, s_3, s_4\}$ be a ling set. P(S) its ling power set.

a) Study questions (i) to (xv) of problem 14 for this ling power set.

b) Is P(S) a totally ordered ling set?

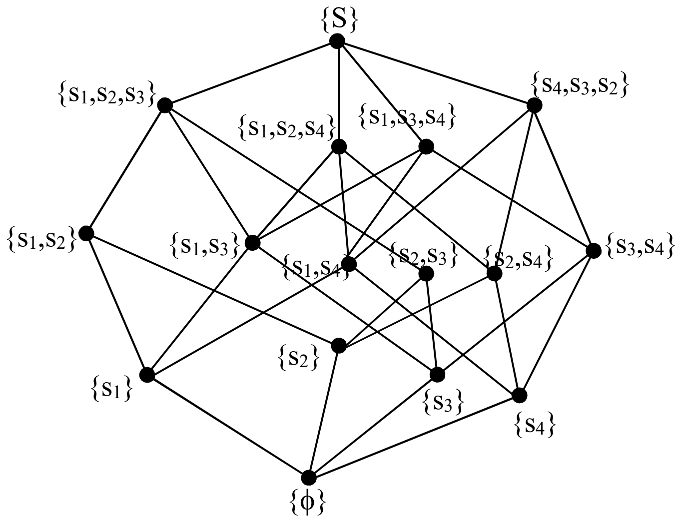


Figure 1.34

i) Prove every element in P(S) has a unique ling complement.

ii) Find the ling complement of

a) $\{s_1, s_4\}$

b) $\{s_3\}$

c) $\{s_1, s_3, s_2\}$ and

d) $\{\phi\}$.

iii) Is the ling subsets A = $\{s, \{s_1, s_2, s_3\}, \{s_2, s_2\}, \{s_1\}\} \subseteq$ P(S) a ling Boolean subalgebra of P(S)? Justify your claim!

iv) Is B = $\{\{s_1, s_2\}, \{s_1\}, \{s_2\}, \{\phi\}\} \subseteq$ P(S) a ling Boolean subalgebra of P(S)? Prove!

v) Is D = $\{\{s_2, s_3, s_4\}, \{s_2, s_3\}, \{s_2, s_4\}, \{s_3, s_4\}, \{s_4\}, \{s_2\}, \{s_3\}, \phi\} \subseteq$ P(S) a ling Boolean subalgebra of P(S)?

vi) What is the ling algebraic structure enjoyed by X = $\{\{s\}, \{s_1, s_2, s_4\}, \{s_1, s_2, s_3\}, \{s_1, s_3, s_4\}, \{s_2, s_3, s_4\}\} \subseteq$ P(S) (a) ling lattice? (b) ling semilattice (c) ling Boolean subalgebra. Justify your claim!

vii) Is $\{\{s_1, s_2, s_4\}, \{s_2, s_3, s_4\}, \{s_2\}, \{s_3\}, \{s_4\}\} \subseteq$ P(S) a ling sublattice of P(S)?

16. a. Give an example of a ling semilattice order of 10 under sup.

b. Give an example of a ling semilattice of order 10 under inf.

17. Compare the ling semilattices obtained in problems 16 (a) and (b).

18. Give an example of a ling modular lattice that is not distributive.

19. Is this ling lattice given in the following figure distributive?

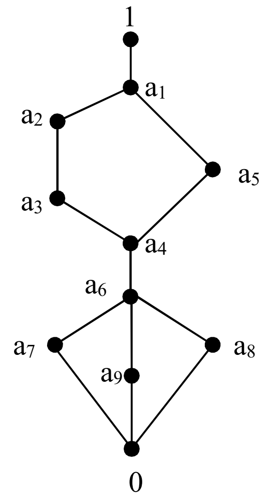


Figure 1.35

Does L satisfy modular law?

b) Is this ling sublattice modular or distributive? Prove your claim.

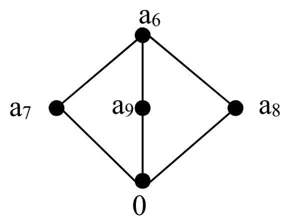


Figure 1.36

c) Is this ling sublattice modular or distributive or neither? Prove your claim.

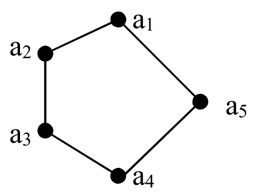


Figure 1.37

d)

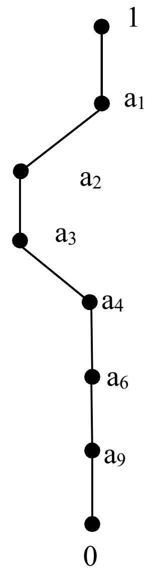


Figure 1.38

i) Is the ling sublattice given by F distributive or modular or both?

ii) Can we say F is a ling chain lattice?

iii) Prove or disprove F is the longest maximal ling chain of L!

iv) How many such maximal ling chains L has?

v) Obtain any other interesting property enjoyed by L.

20. Define ling complements of a ling lattice.

21. Can a ling chain lattice have complements? Justify your claim.

22. Find the ling complements of the following ling lattice L.

L =

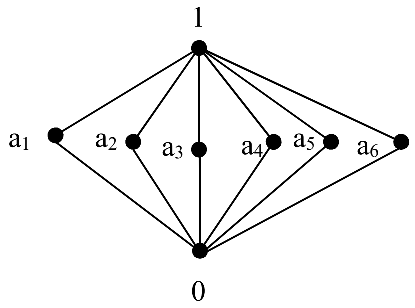


Figure 1.39

a) Is the ling complement of any $a_i \in L$ unique? $(1 \leq i \leq 6)$ Justify your claim.

b) Can we say L satisfies distributive identity?

c) Will L satisfy the modular identity?

d) Can we say L is not a Boolean algebra? Prove It!

23.

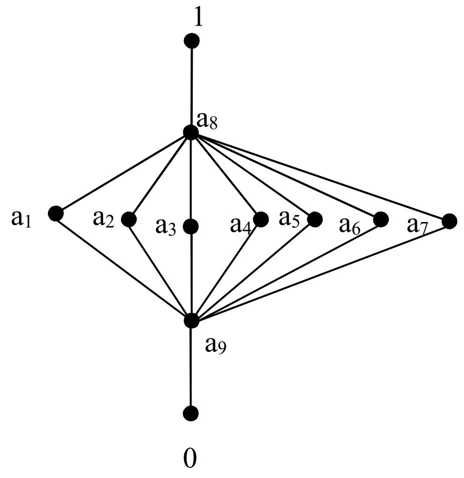


Figure 1.40

L_1 is a ling lattice.

a) Is L_1 a ling complemented lattice?

b) How is the L_1 different from the L given in problem (22)?

24. Prove C = $\underbrace{C_2 \times C_2 \times C_2 \times C_2 \times C_2}_{5-\text{times}}$ is a ling Boolean algebra L of order 2^5.

25. Prove P(S) a ling power set of S; |S| = 5 is a ling Boolean algebra B of order 2^5.

26. Prove L is isomorphic B as ling Boolean algebras.

27. Is the given ling lattice a semiring? Justify your claim.

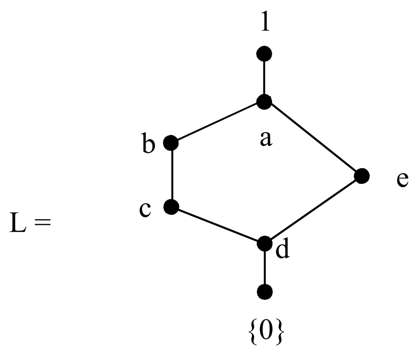


Figure 1.41

28. Is the given line lattice L given by the following figure a semiring. Prove your claim.

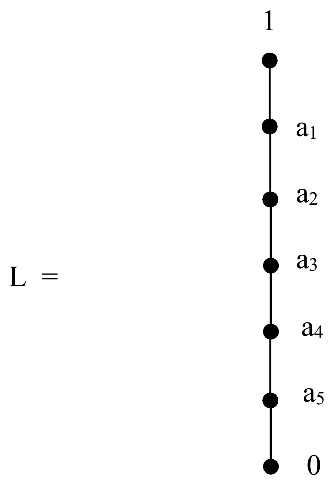


Figure 1.42

29. Is the lattice P given in the following figure a semiring?

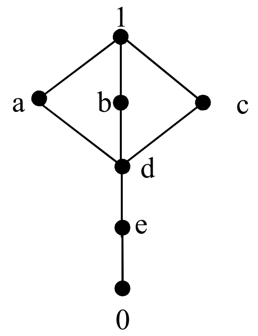


Figure 1.43

Justify your claim.

30. Let L be a ling lattice B given by the following figure will B be a semiring?

B =

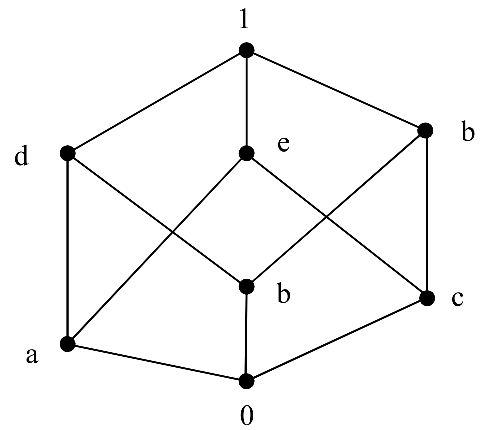


Figure 1.44

Justify your claim.

31. Let S = $Z^o[x]$ be the polynomial ring $Z^o = Z^+ \cup \{0\}$. Prove {S, +, ×} is a semiring of infinite order.

32. Is P = $Z^o \times Z^o \times Z^o = \{(x, y, z\} / x, y, z \in Z^o\}$ a semiring of infinite order under componentwise addition and product?

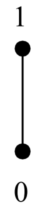
33. If C_2 =  is a line lattice. Prove or

Figure 1.45

disprove B = $C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2$ is a semiring.

34. Can we say for any set S, P(S) its power set is a semiring under ∪ and ∩?

35. Define a sublattice of a lattice L.

    a) Can we say for the modular lattice L all its sublattices will be modular

    b) Prove or disprove this in case

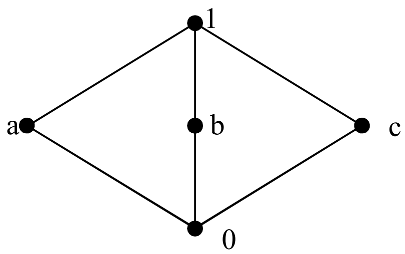


Figure 1.46

36. Find all sublattices of order greater of the lattice L given in figure.

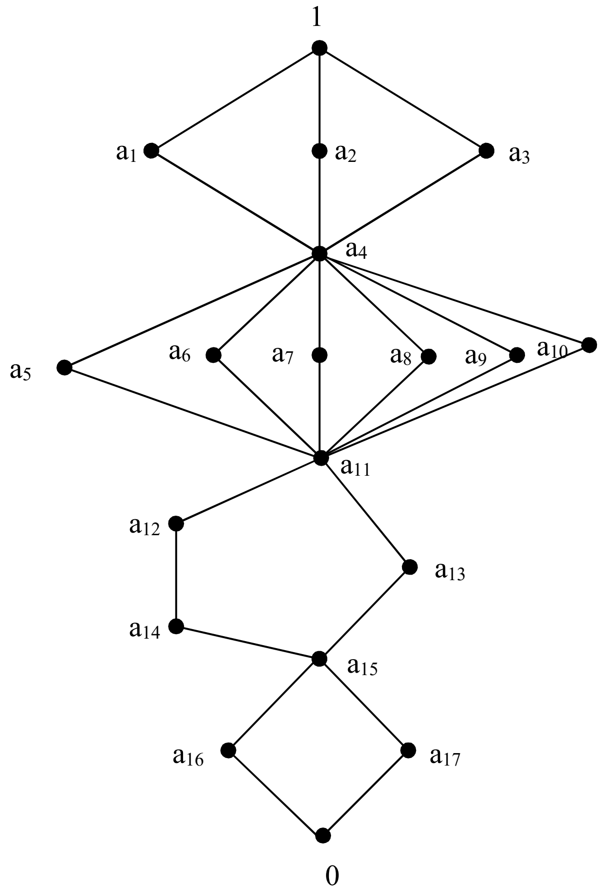


Figure 1.47

a) Does L contain modular sublattice?

b) How many modular sub lattices does L contain?

c) Is L modular?

d) Prove L has a sublattice which is not modular.

e) How many sublattices of L are Boolean algebra of order 4?

f) Can L have a sublattice of order 8 which is a Boolean algebra?

g) How many maximal chains does L contain?

h) What is the length of the maximal chain?

i) Is P =

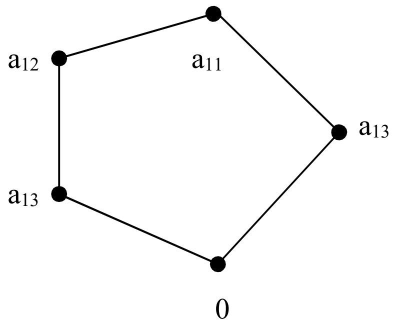


Figure 1.48

a modular sublattice of L.

j) Does L contain elements which are complements (of course other than 0 and 1).

37. Let L be a lattice given by the following example.

L =

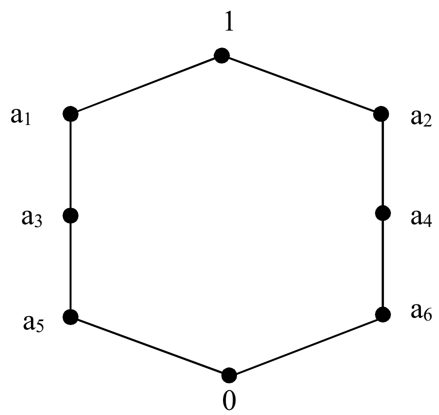


Figure 1.49

a) Does L have sublattices which are modular?

b) Is L a distributive lattice?

c) Prove or disprove L has only 2 maximal chains.

d) Can L has complements other than (0 and 1)?

e) What is the complement of a_6?

f) What is complement of a_1?

g) Can the complements of L be unique?

38. Obtain any other special properties associated with ling lattices.

39. Is the lattice L given in problem 37 distributive?

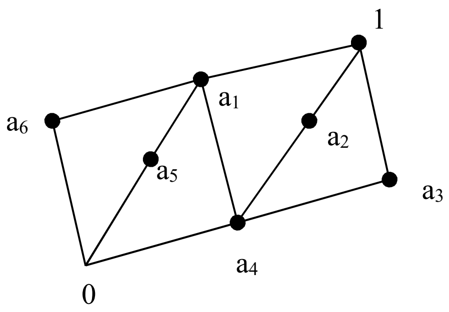


Figure 1.50

i) Is it a modular lattice?

ii) Can L have sublattices which are distributive?

iii) Find the complement of a_6 (does it exist).

iv) Find the complement of a_2.

40. Let S = {good, bad, best, worst, very good, fair, just bad, just good, just fair, very bad} be the ling set associated with the ling variable performance of students in a classroom.

a) Prove {S, min, max} is a ling semiring.

b) If P(S) be ling powerset of S disprove or prove {P(S), min, max} = M is a ling Boolean algebra.

c) Prove {P(S), ∪, ∩} is a ling semiring.

d) Is every Boolean algebra a semiring?

41. Can every semiring be a Boolean algebra? Justify your claim.

42. Is every chain lattice a semiring?

43. Prove or disprove all semirings are not chain lattices.

44. Enumerate all interesting properties about

    i) ling semiring

    ii) ling lattices and

    iii) ling chain lattices.

Chapter Two

LINGUISTIC TOPOLOGICAL SPACES AND THEIR PROPERTIES

In this chapter, we for the first time define, describe and develop the new notion of linguistic topological spaces using subsets of a linguistic set associated with the linguistic variable.

We recall the definition of topological spaces from [16].

Definition 2.1. *Let X be a non empty set. A class T of subsets of X is called a topology on X if it satisfies the following two conditions.*

i) *The union of every class of sets in T is a set in T.*

ii) *The intersection of every finite class of sets in T is in T.*

A topology on X is thus is a class of subsets of X which is closed under the operations of arbitrary unions and finite intersections.

Thus a topological space consists of two objects: a non empty set X and a topology T on X. The sets in the class T are called the open sets of the topological space (X, T) and the elements of X are called points.

We provide examples of them.

Example 2.1. Let X be a non empty set. Let the topology τ be the collections of all subsets of X including X and ϕ.

Thus (X, τ) is a topological space called the discrete topological space.

The following example gives the extreme form of the topological space described above.

Example 2.2. Let X be a non empty set, T = {X, ϕ} be the subsets of X. X is a topological space which is opposite extreme case of the example 2.1.

Example 2.3. Let X = {(0, ∞)} be the continuum of reals.

T = {collection of all open intervals of (0, ∞)}. We see {X, T) is a topological space of infinite order.

If x = (5, 20) and y = (3, 26) $\in$ T then

x $\cap$ y = (5, 28) $\cap$ (3, 26) = (5, 26) and

x $\cup$ y = (5, 28) $\cup$ (3, 26) = (3, 28).

This is the way operations are performed on the subsets of (0, ∞); that is on P((0, ∞)).

We proceed onto give one more example.

Example 2.4. Let X = {a, b, c, d} be a set of order four.

T = {ϕ, {a}, {a, b}, {a, c}, {a, d}, {a, b, c}, {a, c, d}, {a, b, d}, {a, b, c, d} = X}. {X, T} is a topological space of order 9.

Now having seen the concept of topological spaces we now proceed onto define ling topological spaces associated with a ling variable V.

Example 2.5. Let us consider the interval; I = (0, 1). The subintervals of I denoted by B = {(a, b) / 0 < a, b < 1} be the collection of all open subintervals of I.

We see (B, T) is a topological space of open intervals, where if x = (0.3, 0.6) and y = (0.75, 0.9) in B then

$x \cap y = (0.3, 0.6) \cap (0.25, 0.9) = (0.3, 0.6) \in B$

and $x \cup y = (0.3, 0.6) \cup (0.25, 0.9) = (0.25, 0.9) \in B$

Hence our claim.

Now before we proceed onto define subspaces of topological spaces. We define the new concept of linguistic topological spaces. Just before we define ling topological spaces we proceed onto provide a few examples of them.

Example 2.6. Let V be a linguistic variable associated with height of people.

Let S = {very tall, just tall, tall, short, very short, shortest, just short, medium, just medium} be the ling set associated with the ling variable V. P(S); be the powerset of S. S and ϕ be elements in P(S) τ = P(S) under $\cup$ and $\cap$ is closed.

Thus (S, τ) is a topological space defined as the linguistic topological space of finite order.

Example 2.7. Let V be the ling variable associated with the age of people. Let S = [youngest, oldest] be the linguistic continuum.

We can have three types of ling topological spaces.

If τ_1 = {collection of all subsets of S including S and ϕ} then (S, $\tau_1(\cup, \cap)$) is a ling topological space as for all A, B $\in \tau_1$ both $A \cap B$ and $A \cup B \in \tau$, infact finite intersection of ling subsets and union of ling subsets from τ_1 is in τ_1 referred as τ_1 ($\cup$, $\cap$).

Next consider

τ_2 = {$[\ell_1, \ell_2]$ / ℓ_1, $\ell_2 \in$ [youngest, oldest] be the collection of all closed ling intervals of [youngest, oldest]}.

We define operations on τ_2 as follows.

If A = [a_1, a_2] and B = [b_1, b_2] $\in \tau_2$

where a_1, a_2, b_1, $b_2 \in$ [youngest, oldest] = S.

Clearly S is a totally ordered set.

Further min{a_1, a_2} = a_1, max{a_1, a_2) = a_2,

min {b_1, b_2} = b_1 and max{b_1, b_2} = b_2.

Suppose min{a_1, b_1} = a_1 and min{a_2,b_2} = b_2 then

$\min \{A, B\} = \min\{[a_1, a_2], [b_1, b_2]\}$

$= [\min \{a_1, b_1\}, \min\{a_2, b_2\}]$

$= [a_1, b_2] \in \tau_2.$

$\max \{a_1, b_1\} = b_1, \max \{a_2, b_2\} = a_2.$

Thus $\max\{A, B\} = \max \{[a_1, a_2], [b_1, b_2]\}$

$= [\max \{a_1, b_1\}, \max\{a_2, b_2\}] = [b_1, a_2\} \in \tau_2$

Hence $\{S, \tau_2 (\min, \max)\}$ is a ling topological space under min, max operations.

Let τ_3 = {collection of all open intervals of S = [youngest, oldest]} = {(a, b) / a, b $\in$ S = [youngest, oldest]}.

Let A = (a_1, a_2) and B = (b_1, b_2) be two ling open intervals of τ_3, where $a_1, a_2, b_1, b_2 \in$ S = [youngest, oldest].

We know $\min\{a_1, a_2\} = a_1, \max\{a_1, a_2) = a_2$.

$\min\{b_1, b_2\} = b_1$ and $\max\{b_1, b_2\} = b_2, \min\{a_1, b_1\} = b_1,$

$\max\{b_2, b_2\} = b_2, \max\{a_1, b_1\} = a_1$ and $\min\{a_2, b_2\} = a_2.$

Now $\min\{A, B\} = \min\{(a_1, a_2), (b_1, b_2)\}$

$= (\min\{a_1, b_1\}, \min\{a_2, b_2\}) = (b_1, a_2) \in \tau_3.$

$\max \{A, B\} = \max\{(a_1, a_2), (b_1, b_2)\}$

$= (\max \{a, b\}, \max\{a_2, b_2\}) = (a_1, b_2) \in \tau_3.$

Thus {S, τ_3 (min,max)} is a ling topological space of open ling intervals.

All the three ling topological spaces are different and distinct.

Next we provide yet another example of ling topological space.

Example 2.8. Let V be the ling variable associated with the colour of the eyes of internationals. The ling set S associated with V is given by

S = {black, blue, green, amber, brown, light brown, dark brown}. Let P(S) be the power set of S.

τ = P(S) is closed under both the operations $\cup$ and $\cap$.

Infact (S, τ) is a ling topological space of finite order.

Definition 2.2. *Let V be a ling variable. S be a ling set associated with the ling variable V. Suppose τ_ℓ be the collection of ling subsets of S such that if A, B $\in \tau_\ell$ then A $\cup$ B and A $\cap$ B are in τ_ℓ and further both S and ϕ are in τ_ℓ.*

We define the pair (S, τ_ℓ) as a ling topological space.

We have illustrated these by several examples.

However if instead of the ling set S and τ_ℓ just the collection of ling subsets satisfying conditions if we take S to be a ling continuum that is the ling variable V leads that to the ling set which is a ling continuum then we can define open or closed ling intervals on these ling continuum and on these class of open or closed ling intervals τ_1 or τ_2 respectively of the ling continuum we can define max and min operations so that the ling continuum S together with τ_1 or τ_2 forms ling topological spaces denoted by $\{S, \tau_1 \text{ (min, max)}\}$ and $\{S, \tau_2 \text{ (min, max)}\}$.

However we wish to record the following facts.

i) We in case of ling topological spaces get ling topological space only depending on the ling variable V under consideration.

Thus there are several ling topological spaces depending on the underlying ling variable.

ii) All the ling variables in general will not yield a ling continuum, as the ling set associated with it. Only some of the ling sets will be a ling continuum, some can be just finite ling sets S where the ling set S cannot be even a partially ordered set, where as some of the ling sets can be just partially ordered sets and some of them can be totally ordered sets.

So all ling variables will not yield ling continuum as the ling set associated with it.

However we have several ling topological spaces only depending on the ling variable under study.

Now if the ling variable results in a ling set which is a ling continuum then we see we can build three different types of ling topological spaces. However with just ling sets which are not ling continuum; we have defined and given examples of only one type of ling topological spaces using class of subset of the ling set under consideration.

Now having seen examples of just topological spaces and ling topological spaces we proceed onto define the notion of subspaces or to be more specific topological subspaces and describe them by examples.

Definition 2.3. *Let (X, τ) be a classical topological space. Y be a non empty subset of X. We define Y is a topological subspace of X if Y itself under the topology τ of X is a topological space, that is (Y, τ) is a topological space.*

We however will give a modified form of this definition once we succeed in defining the notion of metric on ling spaces and so on.

Now as an immediate examples we have the following.

Example 2.9. Let S = {a, b, c, d, e, f} be a set.

P(S) = {ϕ, {a}, {b}, {c}, …, {f}, {a, b}, {a, c}, …, {e, f}, {a, b, c}, {a, b, d}, …, {d, e, f}, {a, b, c, d}, {a, b, c, e}, …, {c, d, e, f}, {a, b, c, d, e}, {a, b, c, d, f}, …, {b, c, d, e, f}, {a, b, c, d, e, f} = S} = τ

be the class of subsets of S. (S, τ) is a topological space under the topology of sets of finite intersection and arbitrary unions of subsets are in τ.

Suppose τ′ = {{ϕ}, {a}, {a, b}, {a, c}, {a, b, c}, {a, b, d}, {a, b, c, d}. {a, b, c, e}, {a, b, c, d, e}, {a, b, c, d, f}, S = {a, b, c, d, e, f}} ⊆ τ

is such that (S, τ′) is a topological subspace of (S, τ).

Infact we can have several such topological subspaces of the topological space (S, τ). However as we have not defined a metric or distance so far on ling sets we give a crude form of definition of ling topological subspaces as one given earlier in 2.34. We provide examples of them.

Example 2.10. Let V be the ling variable associated with performance aspects of workers in a company. S be the ling set associated with V given by

S = {good, bad, very good, very bad, just bad, very fair, fair, just fair}.

Clearly o(S) = 8 and infact S is a totally ordered set; τ_ℓ = P(S) be the collection of all ling subsets of S. We see {S, τ_ℓ } is a ling topological space.

Consider the subsets collection

τ'_ℓ = {ϕ, {good}, {bad}, {fair}, {good, bad}, {good, fair}, {fair, bad}, {good, bad, fair}}.

We see {S, τ'_ℓ } is a ling topological subspace of (S, τ).

We provide yet another example of the same.

Example 2.11. Let V be the ling variable associated with the weight of people. Let S = [lowest weight, highest weight] be the ling continuum associated with the ling variable V.

Let T = {(a, b) / a, b ∈ S be the collection of all ling open intervals of S. T is closed under the operations min and max.

Thus {S, T(min, max)} is a ling topological space of infinite order.

Consider P = [medium weight, high weight] ⊆ S = [lowest weight, highest weight].

Let T′ = {(a, b) / a, b ∈ P} ⊆ T.

(P, $T'_{(\min,\max)}$ ⊆ (S, T(min, max)) is a ling topological subspace of S.

Now having seen examples of ling topological subspaces we now proceed onto define special type of ling topological spaces which we call as special subset ling topological spaces.

We will illustrate this situation by some examples.

Example 2.12. Let V be the ling variable which studies the performance aspects of students in a class room.

Let S = {fair, good, best, just good, bad, very fair, just fair, very bad worst}

be the ling set associated with the ling variable V.

Let τ = P(S) = {collection of all subsets of S including S and the empty set ϕ}; powerset of S.

For A, B ∈ P(S) we define min{A, B} and max{A, B} as follows.

Suppose A = {good, best, just, good, bad, fair} and

B = {very bad, fair, good} ∈ P(S).

We find min{A, B} = min{{good, best, just good, bad, fair}, {fair, good, very bad}}

= {min{good, good}, min{good, fair}, min{good, very bad}, min{best, good}, min{best, fair}, min{best, very bad}, min{just good, good}, min{just good, fair}, min{just good, very bad}, min{bad, good}, min{bad, fair}, min{bad, very bad}, min{fair, very bad}, min{fair, fair}, min{fair, very bad}}

= {good, fair, very bad, just good, bad} ..I

Clearly min{A, B} ∈ P(S) max{A, B} = max{good, best, just good, bad, fair}, {very bad, fair, good}}

= {max{good, very bad}, max {good, fair}, mad{good, good}, max{best, very bad}, max{best, fair}, max{best, good}, max{just good, good}, max{just good, fair}, max{just good, bad}, max{bad, very bad}, max{bad, fair}, max{good, bad}, max{fair, very bad}, max{fair, fair}, max{fair, good}}

= {good, best, just good, bad, fair} …II

We see max{A, B} ∈ P(S).

We see {P(S), max, min} is a special subset ling topological space.

We see these special ling topological are different from the ling topological spaces.

It is pertinent to keep on record that even in case of classical topological spaces if we define special topological spaces they are different. This is established by the following examples.

Example 2.13. Let S = {2, 4, 6, 20, 15, 18, 25, 14, 9, 19} be a set of integers.

Let P(S) \ {ϕ} be the power set of S in which the empty set ϕ is deleted from P(S).

Let X and Y be two subsets of P(S) given by

X = {4, 20, 9, 18, 14} and Y = {25, 18, 15, 9, 19, 2}.

We find min{X, Y} and max{X, Y} min {X, Y}

= {4, 2, 18, 15, 9, 19, 14} $\in$ P(S) \ ϕ ..I

max{X, Y} = {25, 18, 15, 9, 19, 4, 20, 14} $\in$ P(S) \ ϕ. ..II

We see max{X, Y} and min{X, Y} are different. Further {P(S) \ ϕ, max min} is a special subset topological space of different type.

However {S, τ} the classical topological space is different from the special subset topological space as

$X \cap Y$ = {18, 9} and …III

$X \cup Y$ = {4, 20, 9, 18, 14, 25, 15, 2, 19} …IV

Clearly I and II are different from III and IV.

Suppose if we take S = Z or Z^+ or $Z^+ \cup \{0\}$, clearly {S, $\times$} and {S, +} are semigroups and S is closed under $\times$ and + respectively.

Then we can built yet another special type of topological spaces which cannot be built using ling sets or subsets of along set.

However for the sake of completeness we make a mention of them by an example.

Example 2.14. Let S = $\{Z^+ \cup \{0\}\}$ be the set of positive integers P(S) be the collection of all subsets of S including S and ϕ.

We define $\times$ and + on P(S) as follows.

Let A = {3, 5, 9, 2, 20, 18, 1} and B = {0, 1, 6, 10, 15, 7, 8, 9}

be two proper subsets of P(S).

A × B = {3, 5, 9, 2, 20, 18, 1} × {0, 1, 6, 10, 15, 7, 8, 9}

= {0, 3, 18, 30, 45, 21, 24, 27, 5, 50, 75, 35, 40, 9, 43, 90, 135, 63, 72, 81, 2, 12, 20, 14, 16, 120, 160, 200, 140, 180, 108, 270, 126, 144, 162, 1, 6, 10, 15, 7, 8} …I

Now A + B = {3, 5, 9, 2, 20, 18, 1} + {0, 1, 6, 10, 15, 7, 8, 9}

= {3, 4, 9, 13, 18, 10, 11, 12, 5, 6, 15, 20, 14, 19, 24, 16, 17, 2, 8, 21, 26, 30, 35, 27, 28, 29, 33, 25, 1, 7} …II

Clearly I and II are distinct.

We see {P(S) \ {ϕ}, +} is a closed set; it can only be a semigroup with {0} as its additive identity. As + operation is performed we see {0} = {ϕ}.

Now {P(S) \ {0}, ×} is only a semigroup of infinite order which is commutative. {1} acts as its multiplicative identity.

Thus both {P(S) \ {ϕ}, +} and {P(S) \ {ϕ}, ×} are commutative monoids of infinite order.

Thus we see under these operations + and × P(S) \ {ϕ} enjoys a commutative monoid structure. Infact what we need is if the set P(S) \ {ϕ}; (S = $Z^+ \cup$ {0}) is closed under these two operations it is sufficient for us.

Now we can say $\{S, P(S) \setminus \{\phi\}, +, \times\}$ enjoys a special type of topological structure which we choose to call as special subset topological spaces.

We call it special as the two operation + over × are not distributive in general.

For if A, B, C ∈ P(S) then A × (B + C) = A × B + A × C may not hold good.

Take A = {3, 5}, B = {4, 1} and C = {9, 10} ∈ P(S) \ {ϕ}.

We find out if A × (B + C) = A × B + A × C.

Consider A × (B + C) = {3, 5} × ({4, 1} + {9, 10})

= {3, 5} × ({13, 10, 14, 11})

= {39, 30, 42, 33, 65, 50, 70, 55} …I

Clearly o(A × (B + c)) = 8.

Consider A × B + A × C

= {3, 5} × {4, 1} + {3, 5} × {9, 10}.

= {12, 3, 5, 20} + {27, 30, 45, 50}.

= {39, 30, 32, 47, 42, 33, 35, 50, 57, 48, 65,

62, 53, 55, 70} …II

o(A × B + A × C) = 15.

However A × (B + C) ≠ A × B + A × C.

Hence the + operation on P(S) \ {ϕ} does not distributive over ×.

So only we define the topological space with + and × defined over P(S) \ {ϕ} as a special subset topological space of infinite order.

We have several such type of special subset topological spaces which are illustrated by the following examples.

Example 2.15. Let Z be the set of integers P(Z) be the power set of Z. Take P(Z) \ {ϕ}; {P(Z) \ {ϕ}, +} and {P(Z) \ {0}, ×} are infinite commutative monoids with identities {0} and {1}. However it can be easily verified + does not distributive over ×.

Thus P(Z) \ {0} gives way to the special subset topological space of infinite order.

Example 2.16. Let Q be set of rationals P(Q) be the power set of Q {P(Q) \ {ϕ}, ×, +} is a special subset topological space of infinite order.

Example 2.17. Let R be the reals P(R) the powerset of R. {P(R) \ {ϕ, +, ×} is a special subset topological space of infinite order.

Infact we have {P(Z) \ {ϕ}, +, ×} ⊆ {P(Q) \ {ϕ}, +, ×}

⊆ {P(R) \ {ϕ}, +, ×} ⊆ {P(ℂ) \ {ϕ}, +, ×}

are 4 special subset topological spaces such that they form a chain of special subset topological spaces.

In fact we can say {P(Z) \ {ϕ}, +, ×} is a special subset sub topological space (topological subspace) of {P(Q) \ {ϕ}, +, ×}, {P(R) \ {ϕ}, +, ×} and {P(ℂ) \ {ϕ}, +, ×} what do we term as special subset topological subspaces, we define it in a crude way.

Definition 2.4. *Let {P(S) +, ×} be a special subset topological space. We say a proper subset V of P(S) is a special subset topological subspace of V if {V, +, ×} itself under the operations of P(S) is a special subset topological space of P(S).*

For a given special subset topological space we may have several subset special topological subspaces.

We will illustrate this situation by some examples.

Example 2.18. Let $\{P(\mathbb{C}) \setminus \{\phi\}, +, \times\}$ be a special subset topological space. We see $\{P(Z^+ \cup \{0\}) \setminus \{\phi\}, +, \times\}$, $\{P(Z) \setminus \{\phi\}, +, \times\}$, $\{P(Q) \setminus \{\phi\}, +, \times\}$, $\{P(R) \setminus \{\phi\}, +, \times\}$, $\{P(Q^+ \cup \{0\}) \setminus \{\phi\}, +, \times\}$, $\{P(R^+ \cup \{0\} \setminus \{\phi\}, +, \times\}\}$ are some of the special subset topological subspaces of $\{P(\mathbb{C}) \setminus \{\phi\}, +, \times\}$.

All these subset special topological spaces so far given by us are only of infinite order.

Now we proceed onto describe some special subset topological spaces of finite order.

We will illustrate this situation by some examples.

Example 2.19. Let Z_3 be the ring of modulo integer mod3.

$P(Z_3) = \{\{\phi\}, \{0\}m\ \{1\}, \{2\}, \{1, 2\}, \{0, 2\}, \{1, 0\}, \{1, 2, 0\}$. $\{P(Z_3) \setminus \{\phi\}, +, \times\}$ is a special subset topological space of order 7.

Example 2.20. Let $Z_6 = \{1, 2, 3, 4, 5, 0\}$ be the set of modulo integers mod 3. $\{P(Z_6) \setminus \{\phi\}, +, \times\}$ is a special subset topological space of order 63.

Consider

$A = \{2, 4, 0\}$, $B = \{1, 3, 5, 2\}$ and $C = \{5, 4\} \in P(Z_6) \setminus \{\phi\}$.

We will check if the distributive law is true in case of this A, B, C.

Consider $A \times (B + C) = \{2, 4, 0\} \times (\{1, 3, 5, 2\} + \{4, 5\})$

$= \{2, 4, 0\} \times \{5, 1, 3, 0, 2, 4\} = \{4, 2, 0\}$ …I

$A \times B + A \times C = \{2, 4, 0\} \times \{1, 3, 5, 2\} + \{2, 4, 0\} \times \{5, 4\}$

$= \{2, 0, 4\} + \{4, 2, 0\} = \{4, 2, 0\}$ …II

So I and II are equal so the distributive law is true for this triple.

Let $A = \{1, 2, 5, 3\}$, $B = \{1, 3, 4\}$ and

$C = \{5, 3, 0\} \in P(Z_6) \setminus \{\phi\}$.

$A \times (B + C) = \{1, 2, 5, 3\} \times (\{1, 3, 4\} + \{5, 3, 0\})$

$= \{1, 2, 5, 3\} \times \{0, 4, 1, 2, 3\}$

$= \{0, 4, 1, 2, 3, 5\}$ …I

$A \times B + A \times C = \{1, 2, 5, 3\} \times \{1, 3, 4\} +$

$\{1, 2, 5, 3\} \times \{5, 3, 0\} = \{1, 2, 5, 3, 0, 4\}$

$+ \{5, 4, 2, 3, 0\} = \{1, 2, 5, 3, 0, 4\}$ …II

I and II are identical.

We leave it for the reader prove or disprove the validity of the distributive law in the case of {P(Z_n) \ {0}, +, ×}, the special subset topological space of finite order.

Having seen examples of both special subset topological spaces of finite and infinite order we see we have infinitely many subset special topological spaces of finite order.

We see all the special subset topological spaces which we constructed are both associative and commutative; however we say they may or may not satisfy the distributive laws.

We provide one or two examples of non commutative special subset topological spaces.

Example 2.21. Let M = {collection of all 2 × 2 matrices with entries from Z or Q or $\mathbb{C}$ or R or Z_m; $2 \leq m < \infty$} be the set under consideration.

Let P(M) be the power set of M, {P(M) \ S {ϕ}, +} be the commutative monoid of infinite or finite order depending on the fact where the entries are taken from {P(M) \ {ϕ}, ×} is a non commutative monoid of infinite order.

For {P(M) \ {ϕ}, +} the subset {$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$} acts as the additive identity where as for {P(M) \ {ϕ}, ×} the subset {$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$} acts as the multiplicative identity.

Thus {P(M) \ {ϕ}, +, ×} is a special type of subset topological spaces or in short special subset topological spaces.

For more about special subset topological spaces refer [WBV + FS].

Since our aim is to build special subset ling topological spaces and show how this same concept exist in case of classical spaces. As + and × cannot be defined we are forced to define only max and min for ∪ and ∩ will yield only linguistic topological spaces which will not be special subset topological spaces.

Now translate this to the case of special subset linguistic topological spaces.

We first illustrate this situation by some examples.

Example 2.22. Let us consider the ling variable V age of people. Suppose S = [youngest, oldest] be the ling continuum associated with the ling variable V.

Let M $\left\{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} / a_i \in S = [\text{youngest, oldest}];\ 1 \leq i \leq 4\right\}$

be the collection of ling matrices of 2 × 2 order.

Clearly {M, min, max} is a ling topological space of infinite order as cardinality of M is infinite.

Now consider P(M) the power set of M.

We see {{P(M) \ {ϕ}, min max} is a special subset topological space of 2 × 2 ling matrices.

However for the set P(M) if are define min, max min operations we see {P(M) \ {ϕ}, min, max min} is a non commutative special subset ling topological space.

We first show that for the same M defined in example {M, min, max min} is a non commutative ling topological space of matrices.

Let $A = \begin{bmatrix} \text{young} & \text{old} \\ \text{just old} & \text{middle age} \end{bmatrix}$ and

$B = \begin{bmatrix} \text{middle age} & \text{young} \\ \text{old} & \text{old} \end{bmatrix} \in M.$

We define

$$\max\{\min\{A, B\}\} = \max\{\min \begin{bmatrix} \text{young} & \text{old} \\ \text{just old} & \text{middle age} \end{bmatrix}, \begin{bmatrix} \text{middle age} & \text{young} \\ \text{old} & \text{old} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \begin{matrix} \max\{\min\{\text{young, middle age}\}, \\ \min\{\text{old, old}\}\} \end{matrix} & \begin{matrix} \max\{\min\{\text{young, young}\}, \\ \min\{\text{old, old}\} \end{matrix} \\ \begin{matrix} \max\{\min\{\text{just old, middle age}\} \\ \min\{\text{old, middle age}\} \end{matrix} & \begin{matrix} \max\{\min\{\text{just old, young}\}, \\ \min\{\text{middle age, old}\}\} \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \max\{\text{young}, \text{old}\} & \max\{\text{young}, \text{old}\} \\ \max\{\text{middle age}, & \max\{\text{young}, \\ \text{middle age}\} & \text{middle age}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{old} & \text{old} \\ \text{middle age} & \text{middle age} \end{bmatrix} \quad \ldots \text{I}$$

Consider max{min{B, A}}

$$= \max\{\min \begin{bmatrix} \text{middle age} & \text{young} \\ \text{old} & \text{old} \end{bmatrix} \begin{bmatrix} \text{young} & \text{old} \\ \text{just old} & \text{middle age} \end{bmatrix}\}$$

$$= \begin{bmatrix} \max\{\min\{\text{middle age}, \text{young}\}, & \max\{\min\{\text{middle age}, \text{old}\} \\ \min\{\text{young}, \text{just old}\}\} & \min\{\text{middle age}, \text{middle age}\}\} \\ \max\{\min\{\text{old}, \text{young}\}, \min & \max\{\min\{\text{old}, \text{old}\}, \\ \{\text{old just}, \text{old}\}\} & \min\{\text{old}, \text{middle age}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{young} & \text{middle age} \\ \text{just old} & \text{old} \end{bmatrix} \quad \ldots \text{II}$$

Clearly I and II are distinct.

Hence max{min{A, B}} ≠ max{min{B, A}}.

Thus {M, min, maxmin} is a linguistic topological space which is non commutative.

We now we prove {M, min min max} is also a non commutative linguistic topological space of infinite order.

Consider min{max{A, B}}

$$= \min\ \{\max \begin{bmatrix} \text{young} & \text{old} \\ \text{just old} & \text{middle age} \end{bmatrix}, \begin{bmatrix} \text{middle age} & \text{young} \\ \text{old} & \text{old} \end{bmatrix}\}$$

$$= \begin{bmatrix} \min\{\max\{\text{young}, \text{middle age}\}, & \min\{\max\{\text{young}, \text{young}\}, \\ \max\{\text{old}, \text{old}\}\} & \max\{\text{old}, \text{old}\}\} \\ \min\{\max\{\text{just old}, \text{middleage}\}, & \min\{\max\{\text{just old}, \text{young}\} \\ \max\{\text{middle age}, \text{old}\} & \max\{\text{middle age}, \text{old}\}\} \end{bmatrix}$$

$$= \begin{bmatrix} \min\{\text{middle age}, \text{old}\} & \min\{\text{young}, \text{old}\} \\ \min\{\text{just old}, \text{old}\} & \min\{\text{just old}, \text{old}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{middle age} & \text{young} \\ \text{just old} & \text{just old} \end{bmatrix} \qquad \ldots \text{I}$$

min{max{B,A}

$$= \min\{\max \begin{bmatrix} \text{middle age} & \text{young} \\ \text{old} & \text{old} \end{bmatrix}, \begin{bmatrix} \text{young} & \text{old} \\ \text{just old} & \text{middle age} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \min\{\max\{\text{middle age}, \text{young}\} & \min\{\max\{\text{middle age}, \text{old}\} \\ \max\{\text{old}, \text{just old}\} & \max\{\text{young}, \text{middle age}\}\} \\ \min\{\max\{\text{old}, \text{young}\}, & \min\{\max\{\text{old}, \text{old}\}, \max \\ \max\{\text{old}, \text{just old}\}\} & \{\text{old}, \text{middle age}\} \end{bmatrix}$$

$$= \begin{bmatrix} \min\{\text{middle age}, & \min\{\text{old}, \\ \text{old}\} & \text{middle age}\} \\ \min\{\text{old}, \text{old}\} & \min\{\text{old}, \text{old}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{middle age} & \text{middle age} \\ \text{old} & \text{old} \end{bmatrix} \qquad \ldots \text{II}$$

Clearly I and II are not equal, hence {M, min, min max} is a non commutative linguistic topological space of 2 × 2 matrices.

Now having seen examples of non commutative linguistic topological spaces we proceed onto develop and describe special subset of non commutative linguistic topological spaces by examples both of finite and infinite order.

At the outset we first provide examples of non commutative linguistic topological spaces of finite order.

Example 2.23. Consider the linguistic variable V performance aspects of 10 students in the classroom.

The linguistic set associated with the linguistic variable V be

S = {good, fair, bad, very bad, best, very good, just good, just fair}.

Clearly S is a totally ordered set and the total order is given by very bad < bad < just fair < fair < just good < good < very good < best.

Now let M = {collection of all 2 × 2 linguistic matrices with entries from S}

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / \ a, b, c, d \in S \right\}.$$

Now {M, min} is a ling semigroup of finite order which is commutative.

For if A = $\begin{pmatrix} \text{good} & \text{best} \\ \text{bad} & \text{fair} \end{pmatrix}$ and B = $\begin{pmatrix} \text{bad} & \text{best} \\ \text{very good} & \text{very bad} \end{pmatrix}$ ∈ M

min {A, B} = min $\{\begin{pmatrix} \text{good} & \text{best} \\ \text{bad} & \text{fair} \end{pmatrix}, \begin{pmatrix} \text{bad} & \text{best} \\ \text{very good} & \text{very bad} \end{pmatrix}\}$

$= \begin{pmatrix} \min\{\text{good, bad}\} & \min\{\text{best, best}\} \\ \min\{\text{bad, very good}\} & \min\{\text{very bad, fair}\} \end{pmatrix}$

$= \begin{pmatrix} \text{bad} & \text{best} \\ \text{bad} & \text{very bad} \end{pmatrix}$. Clearly min{A,B}

= min{B,A} as we know min{s_1, s_2} = min{s_2, s_1} as every term or component in this linguistic matrix is commutative hence the fact min{A, B} = min{B, A}.

Now we find min max{A, B}

$= \min\{\max\{\begin{pmatrix} \text{good} & \text{best} \\ \text{bad} & \text{fair} \end{pmatrix}, \begin{pmatrix} \text{bad} & \text{best} \\ \text{very good} & \text{very bad} \end{pmatrix}\}\}$

$= \begin{pmatrix} \min\{\max\{\text{good, bad}\}, & \min\{\max\{\text{good, best}\}, \\ \max\{\text{best, very good}\}\} & \max\{\text{best, very bad}\}\} \\ \min\{\max\{\text{bad, bad}\} & \min\{\max\{\text{bad, best}\}, \\ \{\text{fair, very good}\}\} & \max\{\text{fair, very bad}\}\} \end{pmatrix}$

$= \begin{bmatrix} \min\{\text{good, best}\} & \min\{\text{best, best}\} \\ \min\{\text{bad, very good}\} & \min\{\text{best, fair}\} \end{bmatrix}$

$= \begin{pmatrix} \text{good} & \text{best} \\ \text{bad} & \text{fair} \end{pmatrix}$ …I

Consider min{max{B, A}}

$$= \min\{\max\{\begin{pmatrix} \text{bad} & \text{best} \\ \text{very good} & \text{very bad} \end{pmatrix}, \begin{pmatrix} \text{good} & \text{best} \\ \text{bad} & \text{fair} \end{pmatrix}\}\}$$

$$= \begin{bmatrix} \min\{\max\{\text{bad}, \text{good}\}, \max[\text{best}, \text{bad}\}\} & \min\{\max\{\text{bad}, \text{best}\}, \max\{\text{best}, \text{fair}\}] \\ \min\{\max\{\text{very good}, \text{good}\}, \max\{\text{very bad}, \text{bad}\} & \min\{\max\{\text{very good}, \text{best}\}, \max\{\text{very bad}, \text{fair}\} \end{bmatrix}$$

$$= \begin{bmatrix} \min\{\text{good}, \text{best}\} & \min\{\text{best}, \text{best}\} \\ \min\{\text{very good}, \text{bad}\} & \min\{\text{best}, \text{fair}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{good} & \text{best} \\ \text{bad} & \text{fair} \end{bmatrix} \text{..II}$$

We see this pair is commutative.

We will provide some X and Y $\in$ M with

$\min\{\max\{X, Y\}\} \neq \min\{\max \{Y,X\}\}$.

Consider $X = \begin{pmatrix} \text{bad} & \text{good} \\ \text{fair} & \text{very bad} \end{pmatrix}$ and $Y = \begin{pmatrix} \text{fair} & \text{bad} \\ \text{best} & \text{good} \end{pmatrix} \in M$

$$\min\{\max\{X, Y\}\} = \min\{\max\{\begin{pmatrix} \text{bad} & \text{good} \\ \text{fair} & \text{very bad} \end{pmatrix}, \begin{pmatrix} \text{fair} & \text{bad} \\ \text{best} & \text{good} \end{pmatrix}\}\}$$

$$= \begin{pmatrix} \min\{\max\{\text{bad}, \text{fair}\}, & \min\{\max\{\text{bad}, \text{bad}\}, \\ \max\{\text{good}, \text{best}\}\} & \max\{\text{good}, \text{good}\} \\ \min\{\max\{\text{fair}, \text{fair}\}, & \min\{\max\{\text{best}, \text{good}\}, \\ \max\{\text{best}, \text{very bad}\}\} & \max\{\text{good}, \text{very bad}\}\} \end{pmatrix}$$

$$= \begin{pmatrix} \min\{\text{fair}, \text{best}\} & \min\{\text{bad}, \text{good}\} \\ \min\{\text{fair}, \text{best}] & \min\{\text{best}, \text{good}\} \end{pmatrix} = \begin{pmatrix} \text{fair} & \text{bad} \\ \text{fair} & \text{good} \end{pmatrix} \quad \ldots\text{I}$$

$\min\{\max\{Y, X\}\}$

$$= \min\{\max\{\begin{pmatrix} \text{fair} & \text{bad} \\ \text{best} & \text{good} \end{pmatrix}, \begin{pmatrix} \text{bad} & \text{good} \\ \text{fair} & \text{very bad} \end{pmatrix}$$

$$= \begin{pmatrix} \min\{\max\{\text{fair}, \text{bad}\}, \max\{\text{bad}, \text{fair}\}\} & \min\{\max\{\text{fair}, \text{good}\}, \max\{\text{bad}, \text{very bad}\}\} \\ \min\{\max\{\text{best}, \text{bad}\}, \max\{\text{good}, \text{fair}\}\} & \min\{\max\{\text{best}, \text{good}\}, \max\{\text{good}, \text{very bad}\}\} \end{pmatrix}$$

$$= \begin{pmatrix} \min\{\text{fair}, \text{fair}\} & \min\{\text{good}, \text{bad}\} \\ \min\{\text{best}, \text{good}\} & \min\{\text{best}, \text{good}\} \end{pmatrix}$$

$$= \begin{pmatrix} \text{fair} & \text{bad} \\ \text{good} & \text{good} \end{pmatrix} \quad ..\text{II}$$

I and II are not equal.

Thus $\min\{\max\{X, Y\}\} \neq \min\{\max\{Y, X\}\}$.

Hence {M, min, min, max} is a non commutative linguistic topological space of finite order of 2 × 2 linguistic matrices.

Thus we have provided non commutative linguistic topological spaces of both finite and infinite order.

Now we spell out the main condition under which linguistic topological spaces commutative or non commutative of finite or infinite order.

The linguistic set S associated with the linguistic variable V should be totally ordered set.

That it is mandatory for us to define linguistic topological spaces (S, ≤) must be a totally ordered set.

Now we proceed onto define the notion of linguistic topological spaces.

Definition 2.5. *Let V be a ling variable such that the linguistic set S associated with V is a totally ordered set. {S, min, max} is a linguistic topological space if ;*

i) *For all $s_1, s_2 \in S$; $\min\{s_1, s_2\}$ and $\max\{s_1, s_2\} \in S$.*

ii) *S has an element $s_\ell \in S$ called the least element of S it is such that $\max\{s, s_\ell\} = s$ for all $s \in S$ so that {S, max} is a monoid.*

iii) *S contains the greatest element $s_g \in S$ such that {S, min} is a monoid that is $\min\{s, s_g\} = s$ for all $s \in S$ are satisfied.*

We say {S, min, max} is a finite linguistic topological space if order of S is finite. {S, min, max} is a linguistic topological space of infinite order if order of S is infinite.

Clearly as $\min\{a, b\} = \min\{b, a\}$ and $\max\{a, b\} = \max\{b, a\}$ for all $a, b \in S$ we have {S, min, max} is a commutative linguistic topological space.

Now we define non commutative linguistic topological spaces.

Definition 2.6. *Let V be a linguistic variable and S be a linguistic set associated with the linguistic variable V. To define a linguistic topological space basically we need S to be a totally ordered set.*

Now define M = {collection of all n × n linguistic matrices with entries from S; 2 ≤ n < ∞}, {M, min, min max} {M, min, max min}, {M, max, min max} and {M, max, maxmin} are non commutative linguistic topological spaces as

i) {M, max} is a commutative linguistic semigroup.

ii) {M, minmax} and ({M, maxmin}) are both non commutative ling semigroups.

So {M, max, min max} and all other four ling topological spaces are non commutative.

If M is of finite order that is, S is of finite order so is M, then M will contribute to finite order non commutative linguistic topological spaces.

If M is of infinite order that is, S is of infinite order then M will contribute a non finite order non commutative ling topological spaces.

We have provided examples of them so that it is easy for the reader to understand them.

Next we proceed on to describe and develop the notion of commutative and non commutative linguistic special subset topological spaces of both finite and infinite order.

Example 2.24. Let V be a linguistic variable describing the height of people. The linguistic set S associated with this ling variable V is a linguistic continuum given by

S = [shortest, tallest].

Let P(S) be the linguistic power set of S including S and ϕ. Suppose

A = {tall, short, very short, just tall, medium, very tall} and

B = {ϕ, tall, tallest, shortest, just medium, just short} ∈ P(S).

We find min{A, B} and max{A, B}.

min{A, B} = min{{tall, short, very short, just tall, medium, very tall}, {ϕ, tall, tallest, shortest, just medium, just short}}

= {min {tall, ϕ}, min{tall, tall}, min{tall, tallest}, min{tall, shortest}, …, min{very tall, ϕ}, min{very tall, tall}, …, min{very tall, just short}}

= {ϕ, shortest, just medium, just short, short, very short, just tall, medium, very tall} ∈ P(S).

Now max{A, B} = max{{tall, short, very short, just tall, medium, very tall}, {ϕ, tall, tallest, shortest, just medium, just short}} = {max{tall, ϕ}, max{tall, tall}, max{tall, tallest}, …, max{very tall, ϕ}, max{very tall, tall}, max{very tall, tallest}, …, max{very tall, just short}}

= {tall, tallest, short, just medium, just short, very short, just tall, medium, very tall} ∈ P(S).

Thus {P(S), min, max} is a special subset topological linguistic space of infinite order.

Clearly {P(S), min, max} is a commutative special subset topological linguistic space of infinite order.

It is mandatory the ling set S associated with the ling variable for us to define a special subset linguistic topological space should be a totally ordered set, that is (S, ≤) is a totally order set.

Now we proceed on to describe a special subset linguistic topological space of finite order by an example.

Example 2.25. Let V be the linguistic variable associated with the height of some 12 children in various age groups.

Let S = {tall, very tall, short, just short, medium, just tall, very short}

be the linguistic set. These values are not arbitrary the height is assessed relative to age.

Clearly the linguistic set S is a totally ordered set with the total order given by

very short < short < just short < medium < just tall ≤ tall ≤ very tall.

Let P(S) be the power set of S which includes S and ϕ, ϕ the empty linguistic set.

Let A = {just short, short, tall, very tall, medium} and

B = {very short, very tall, just tall, just short, medium} ∈ P(S).

We find min{A, B} = min{{tall, just short, short, very tall, medium}, {very short, very tall, just tall, just short, medium}}

= {min {tall, very short}, min{tall, very tall}, min{tall, just tall}, min{tall, just short}, min{tall, medium}, …, min{medium, very short}, min{medium, very tall}, min{medium, just tall}, …, min{medium, medium}}

= {very short, just short, tall, just tall, medium, very tall} ∈ P(S).

Now we find max{A, B} = max{{just short, short, tall, very tall, medium}, {very short, very tall, just tall, just short, medium}}

= {just short, very tall, just tall, medium, short, very tall, medium} ∈ P(S).

This T = {P(S), min, max} is a special subset linguistic topological space of finite order which is commutative. Infact order of T = |P(S)| = 2^7.

Now here also to define special subset linguistic topological spaces of finite or infinite order it is mandatory we need to have the underlying linguistic set to be a totally ordered set.

Next we proceed onto describe special subset linguistic topological space which is non commutative by examples.

Example 2.26. Let us consider the ling variable V temperature of water from ice state to vapours state.

Let S = {lowest, highest} be the ling set associated with V. Clearly S is a totally ordered.

Now let P(S) be the linguistic power set of S including S and the empty linguistic set ϕ.

Let M = {collection of all 3 × 3 linguistic matrices with entries from S}

$$= \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} / \text{a, b, c, d, e, f, g, h, I} \times \text{S} \right\}.$$

Let us define two operations min and min max on M.

Consider $A = \begin{bmatrix} \text{low} & \text{high} & \text{low} \\ \text{very low} & \text{just high} & \text{high} \\ \text{low} & \text{medium} & \text{very high} \end{bmatrix}$ and

$$B = \begin{bmatrix} \text{high} & \text{low} & \text{low} \\ \text{very high} & \text{high} & \text{high} \\ \text{very low} & \text{high} & \text{low} \end{bmatrix} \in M$$

min{A, B}

$$= \min\left\{ \begin{bmatrix} \text{low} & \text{high} & \text{low} \\ \text{very low} & \text{just high} & \text{high} \\ \text{low} & \text{medium} & \text{very high} \end{bmatrix}, \begin{bmatrix} \text{high} & \text{low} & \text{low} \\ \text{very high} & \text{high} & \text{high} \\ \text{very low} & \text{high} & \text{low} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \min\{\text{low, high}\} & \min\{\text{high, low}\} & \min\{\text{low, low}\} \\ \min\{\text{very low, very high}\} & \min\{\text{just high, high}\} & \min\{\text{high, high}\} \\ \min\{\text{low, very low}\} & \min\{\text{medium, high}\} & \min\{\text{very high, low}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{low} & \text{low} & \text{low} \\ \text{very low} & \text{just high} & \text{high} \\ \text{very low} & \text{medium} & \text{low} \end{bmatrix}.$$

It is easily verified min{A, B} = min{B, A};

thus {M, min} is a commutative linguistic semigroup of infinite order.

Now for the same A and B we find

min{max{A, B}}

$$= \min\{\max\{\begin{bmatrix} \text{low} & \text{high} & \text{low} \\ \text{very low} & \text{just high} & \text{high} \\ \text{low} & \text{medium} & \text{very high} \end{bmatrix},$$

$$\begin{bmatrix} \text{high} & \text{low} & \text{low} \\ \text{very high} & \text{high} & \text{high} \\ \text{very low} & \text{high} & \text{low} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \min\{\max\{\text{low}, \text{high}\}, \max\{\text{high}, \text{very high}\}, \\ \max\{\text{low}, \text{very low}\}\} \\ \min\{\max\{\text{very low}, \text{high}\}, \max\{\text{just high}, \\ \text{very high}\}, \{\text{high}, \text{very low}\}\} \\ \min\{\max\{\text{low high}\}, \max\{\text{medium}, \text{very high}\}, \\ \max\{\text{very high}, \text{very low}\} \end{bmatrix}$$

$$\begin{matrix} \min\{\max\{\text{low},\text{low}\},\max\{\text{high},\text{high}\}, \\ \max\{\text{low},\text{high}\}\} \\ \min\{\max\{\text{very low},\text{low}\},\max\{\text{just high}, \\ \text{high}\},\max\{\text{high},\text{high}\}\} \\ \min\{\max\{\text{low},\text{low}\},\max\{\text{medium},\text{high}\}, \\ \max\{\text{low},\text{low}\}\} \end{matrix}$$

$$\left.\begin{matrix} \min\{\max\{\text{low},\text{low}\},\max\{\text{high},\text{high}\}, \\ \max\{\text{low},\text{low}\}\} \\ \min\{\max\{\text{very low},\text{low}\},\max\{\text{just high}, \\ \text{high}\},\max\{\text{high},\text{low}\}\} \\ \min\{\max\{\text{low},\text{low}\},\max\{\text{medium},\text{high}\}, \\ \max\{\text{very high},\ \text{low}\}\} \end{matrix}\right]$$

$$= \begin{bmatrix} \min\{\text{high},\text{very high}, & \min\{\text{low},\text{high}, & \min\{\text{low},\text{high}, \\ \text{low}\} & \text{high}\} & \text{low}\} \\ \min\{\text{high},\text{just high}, & \min\{\text{low},\text{high}, & \min\{\text{low},\text{high}, \\ \text{very high}\} & \text{high}\} & \text{high}\} \\ \min\{\text{high},\text{very high}, & \min\{\text{low},\text{high}, & \min\{\text{low},\text{high}, \\ \text{very high}\} & \text{very high}\} & \text{very high}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{low} & \text{low} & \text{low} \\ \text{just high} & \text{low} & \text{low} \\ \text{high} & \text{low} & \text{low} \end{bmatrix} \qquad \ldots \text{I}$$

$$\min\{\max\{\text{B, A}\}\} = \min\{\max \begin{bmatrix} \text{high} & \text{low} & \text{low} \\ \text{very high} & \text{high} & \text{high} \\ \text{very low} & \text{high} & \text{low} \end{bmatrix},$$

$$\begin{bmatrix} \text{low} & \text{high} & \text{low} \\ \text{very low} & \text{just high} & \text{high} \\ \text{low} & \text{medium} & \text{very high} \end{bmatrix} \}$$

$$= \begin{bmatrix} \min\{\max\{\text{high}, \text{low}\}, \max\{\text{low}, \text{very low}\}, \max\{\text{low}, \text{low}\}\} & \min\{\max, \{\text{high}, \text{high}\}, \max\{\text{low}, \text{just high}\}, \max\{\text{low}, \text{medium}\}\} & \min\{\max\{\text{high}, \text{low}\}, \max\{\text{low}, \text{high}\}, \max\{\text{low}, \text{very high}\}\} \\ \min\{\max\{\text{very high}, \text{low}\}, \max\{\text{high}, \text{very low}\}, \max\{\text{high}, \text{low}\} & \min\{\max\{\text{very high}, \text{high}\}, \max\{\text{high}, \text{just high}\}, \max\{\text{high}, \text{medium}\}\} & \min\{\max\{\text{very high}, \text{low}\}, \max\{\text{high}, \text{high}\}, \max\{\text{high}, \text{very high}\} \\ \min\{\max\{\text{very low}, \text{low}\}, \max\{\text{high}, \text{very low}\}, \max\{\text{low}, \text{low}\}\} & \min\{\max\{\text{very low}, \text{high}\}, \max\{\text{high}, \text{just high}\}, \max\{\text{low}, \text{medium}\}\} & \min\{\max\{\text{very low}, \text{low}\}, \max\{\text{high}, \text{high}\}, \max\{\text{low}, \text{very high}\}\} \end{bmatrix}$$

$$= \begin{bmatrix} \min\{\text{high}, \text{low}, \\ \text{low}\} & \min\{\text{high}, \text{just high}, \\ \text{medium}\} & \min\{\text{high}, \text{high}, \\ \text{very high}\} \\ \min\{\text{very high}, \\ \text{high}, \text{high}\} & \min\{\text{very high}, \\ \text{high}, \text{high}\} & \min\{\text{very high}, \\ \text{high}, \text{high}\} \\ \min\{\text{low}, \text{high}, \text{low}\} & \min\{\text{high}, \text{high}, \\ \text{medium}\} & \min\{\text{low}, \text{high}, \\ \text{very high}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{low} & \text{medium} & \text{high} \\ \text{high} & \text{high} & \text{high} \\ \text{high} & \text{medium} & \text{low} \end{bmatrix} \qquad \ldots \text{II}$$

Clearly I and II are distinct hence

$\min\{\max\{A, B\}\} \neq \min\{\max\{B, A\}\}$.

Clearly {M, min max} is a non commutative semigroup of infinite order. Hence {M, min, min max} is a non commutative linguistic topological space of infinite order.

On similar lines we can prove {M, max, max min} is also a non commutative linguistic topological space of infinite order.

Now we will give one example of non commutative linguistic topological space of finite order.

Example 2.27. Let V be the linguistic variable associated with the performance aspects of factor workers.

Let S = {good, bad, fair, very bad, just bad, very fair, just fair, very good, best, just good, very very good}

be the linguistic set associated with the linguistic variable V.

S is a totally ordered set for;

very bad < bad < just bad < just fair < fair < very fair < just good < good < very good < very very good < best.

Let P = {collection of all 2 × 2 linguistic matrices with entries from S}

$$= \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} / a, b, c, d \in S\}.$$

We show {P, max} is a commutative semigroup of finite order.

For if $A = \begin{pmatrix} \text{good} & \text{fair} \\ \text{bad} & \text{best} \end{pmatrix}$ and

$B = \begin{pmatrix} \text{bad} & \text{best} \\ \text{good} & \text{fair} \end{pmatrix} \in P$, then max{A, B}

$$= \max \left\{\begin{pmatrix} \text{good} & \text{fair} \\ \text{bad} & \text{best} \end{pmatrix}, \begin{pmatrix} \text{bad} & \text{best} \\ \text{good} & \text{fair} \end{pmatrix}\right\}$$

$$= \begin{pmatrix} \max\{\text{good}, \text{bad}\} & \max\{\text{fair}, \text{best}\} \\ \max\{\text{bad}, \text{good}\} & \max\{\text{best}, \text{fair}\} \end{pmatrix}$$

$$= \begin{pmatrix} \text{good} & \text{best} \\ \text{good} & \text{best} \end{pmatrix} \in P.$$

Thus {P, max} is a commutative semigroup.

$$\text{maxmin}\{A, B\} = \max\{\min\{\begin{pmatrix} \text{good} & \text{fair} \\ \text{bad} & \text{best} \end{pmatrix}, \begin{pmatrix} \text{bad} & \text{best} \\ \text{good} & \text{fair} \end{pmatrix}\}\}$$

$$= \begin{pmatrix} \max\{\min\{\text{good}, \text{bad}\}, \min\{\text{fair}, \text{good}\}\} \\ \max\{\min\{\text{bad}, \text{bad}\}, \min\{\text{best}, \text{good}\}\} \end{pmatrix.$$

$$\begin{matrix} \max\{\min\{\text{good}, \text{best}\} \min\{\text{fair}, \text{fair}\}\} \\ \max\{\min\{\text{bad}, \text{best}\}, \min\{\text{best}, \text{fair}\}\} \end{pmatrix}$$

$$= \begin{pmatrix} \max\{\text{bad}, \text{fair}\} & \max\{\text{good}, \text{fair}\} \\ \max\{\text{bad}, \text{good}\} & \max\{\text{bad}, \text{fair}\} \end{pmatrix} = \begin{pmatrix} \text{fair} & \text{good} \\ \text{good} & \text{fair} \end{pmatrix} \ldots \text{I}$$

$\max\{\min\{B, A\}\}$

$$= \max\{\min \begin{bmatrix} \text{bad} & \text{best} \\ \text{good} & \text{fair} \end{bmatrix}, \begin{bmatrix} \text{good} & \text{fair} \\ \text{bad} & \text{best} \end{bmatrix}\}\}$$

$$\begin{pmatrix} \max\{\min\{\text{bad}, \text{good}\}, \min\{\text{best}, \text{bad}\}\} \\ \max\{\min\{\text{good}, \text{good}\}, \min\{\text{fair}, \text{bad}\}\} \end{matrix}$$

$$\begin{matrix} \max\{\min\{\text{bad}, \text{fair}\}, \min\{\text{best}, \text{best}\}\} \\ \max\{\min\{\text{good}, \text{fair}\}, \min\{\text{fair}, \text{best}\}\} \end{pmatrix}$$

$$= \begin{pmatrix} \max\{\text{bad}, \text{bad}\} & \max\{\text{bad}, \text{best}\} \\ \max\{\text{good}, \text{bad}\} & \max\{\text{good}, \text{fair}\} \end{pmatrix} = \begin{pmatrix} \text{bad} & \text{best} \\ \text{good} & \text{good} \end{pmatrix} \text{..II}$$

I and II are distinct hence $\max\{\min\{A,B\}\} \neq \max\{\min\{B,A\}\}$.

Thus {M, maxmin} is a non commutative semigroup of finite order.

So we claim {M, max, maxmin} is a non commutative linguistic topological space of finite order.

Now we proceed onto develop and describe the notion of linguistic topological subspace by some examples.

Example 2.28. Let V be the linguistic variable associated with age of people. S = [youngest, oldest] be the linguistic set associated with V. S is a linguistic continuum hence S is a totally ordered set.

We know {S, max min}is a linguistic topological space of infinite order.

Take P = [youngest, middle age] $\subseteq$ [youngest, oldest] = S.

Now {P, maxmin} is also a linguistic topological space of infinite order.

We call {P, max, min} as a linguistic topological subspace of {S, max, min} of infinite order.

Consider T = {young, just young, very young, old, very old, middle age, oldest} $\in$ S

is a finite subset of S but clearly is not a linguistic continuum but however T is a totally ordered set given by

very young < just young < young < middle age < old < very old < oldest.

Now {T, max min} is a linguistic topological space of finite order which is commutative.

Infact T $\subseteq$ S and {T, max min} is defined as the linguistic topological subspace of {S, max min} but of finite order.

This example infact let us know in case of linguistic topological spaces of infinite spaces we can have nontrivial linguistic topological subspaces of finite order as well as linguistic topological subspaces of infinite order.

All these linguistic topological spaces as well as linguistic topological subspaces are commutative.

We now provide one example of linguistic topological subspace of a non commutative linguistic topological space.

Example 2.29. Let V be the linguistic variable associated with the weight of people. The associated linguistic set of V is given by S = [lowest, highest] which is a linguistic continuum.

Now let M = {all 5 × 5 linguistic matrices with entries from S}

{M, min, minmax} is a noncommutative linguistic topological space of infinite order.

Consider N = {collection of all 5 × 5 linguistic matrices with entries from P = [low, high] $\subseteq$ S = [lowest, highest]. Clearly P is a linguistic continuum of infinite order.

Infact {N, min, minmax} is a linguistic topological subspace of N of infinite order which is noncommutative.

If take the set W = {low, lowest, high, very low, just low, just high, medium, just medium, highest, very high} $\subseteq$ M, and let B = {collection of all 5 × 5 linguistic matrices with entries from W} ⊆ M,

{B, min min max} is a non commutative linguistic topological subspace of M and is of finite order.

Now having seen both commutative and non commutative linguistic topological spaces using square linguistic matrices we now proceed onto define appropriate operations define again a new class of m × n linguistic topological spaces of matrices of any order.

We proceed onto give some examples of them in the following.

Example 2.30. Let V be a linguistic variable associated with age. Let S = [youngest, oldest] be the linguistic set associated with the linguistic variable V. Clearly S is a linguistic continuum. Infact S is a totally ordered set.

$$\text{Let } M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle/ a_i \in S, 1 \leq i \leq 7 \right\}$$

= {collection of all 1 × 7 column linguistic matrices with entries from S}.

Now define on M the two operation min and max as follows.

Let $A = \begin{bmatrix} \text{old} \\ \text{young} \\ \text{just old} \\ \text{middle age} \\ \text{just young} \\ \text{old} \\ \text{young} \end{bmatrix}$ and $B = \begin{bmatrix} \text{young} \\ \text{old} \\ \text{medium} \\ \text{just old} \\ \text{just young} \\ \text{old} \\ \text{young} \end{bmatrix} \in M$.

We define max {A, B} = $\max\left\{ \begin{bmatrix} \text{old} \\ \text{young} \\ \text{just old} \\ \text{middle age} \\ \text{just young} \\ \text{old} \\ \text{young} \end{bmatrix}, \begin{bmatrix} \text{young} \\ \text{old} \\ \text{medium} \\ \text{just old} \\ \text{just young} \\ \text{old} \\ \text{young} \end{bmatrix} \right\}$

$$= \begin{bmatrix} \max\{\text{old, young}\} \\ \max\{\text{young, old}\} \\ \max\{\text{just old, medium}\} \\ \max\{\text{just young, just young}\} \\ \max\{\text{old, old}\} \\ \max\{\text{young, young}\} \end{bmatrix} = \begin{bmatrix} \text{old} \\ \text{old} \\ \text{just old} \\ \text{just young} \\ \text{old} \\ \text{young} \end{bmatrix} \in M.$$

Thus we see {M, max} is a commutative linguistic semigroup.

We see youngest is the least element in S.

$$\text{If } I = \begin{bmatrix} \text{youngest} \\ \text{youngest} \\ \text{youngest} \\ \text{youngest} \\ \text{youngest} \\ \text{youngest} \\ \text{youngest} \end{bmatrix} \in M\text{; it is such that}$$

max{A, I} = max{I, A} = A for all A ∈ M.

We call, the linguistic matrix I as the identity of M under the max operation.

Thus we conclude {M, max} is a commutative linguistic monoid of infinite order.

Take same A and B ∈ M, we now find

$$\min\{A, B\} = \min\{\begin{bmatrix} \text{old} \\ \text{young} \\ \text{just old} \\ \text{middle age} \\ \text{just young} \\ \text{old} \\ \text{young} \end{bmatrix}, \begin{bmatrix} \text{young} \\ \text{old} \\ \text{medium} \\ \text{just old} \\ \text{just young} \\ \text{old} \\ \text{young} \end{bmatrix}\}$$

$$= \begin{bmatrix} \min\{\text{old, young}\} \\ \min\{\text{young, old}\} \\ \min\{\text{just old, medium}\} \\ \min\{\text{middle age, just old}\} \\ \min\{\text{just young, just young}\} \\ \min\{\text{old, old}\} \\ \min\{\text{young, young}\} \end{bmatrix} = \begin{bmatrix} \text{young} \\ \text{young} \\ \text{middle age} \\ \text{middle age} \\ \text{just young} \\ \text{old} \\ \text{young} \end{bmatrix}.$$

We see {M, min} is a linguistic commutative semigroup.

$$\text{The matrix J} = \begin{bmatrix} \text{oldest} \\ \text{oldest} \\ \text{oldest} \\ \text{oldest} \\ \text{oldest} \\ \text{oldest} \\ \text{oldest} \end{bmatrix} \in M$$

where the linguistic term oldest is the greatest value and we see for all A ∈ M, is such that min{A, J} = A.

So we can claim {M, min} is a commutative linguistic monoid of finite order.

Thus {M, min, max} is a commutative linguistic topological space of infinite order.

This will also be known as commutative linguistic topological space of column linguistic matrices of infinite order.

Now we provide an example of a linguistic topological space of column linguistic matrices of finite order.

Example 2.31. Let V be the linguistic variable associated with weight of 20 children.

Let S = {low, very low, high, very high, medium (normal), just low, just high, very medium}

be the linguistic set associated with the linguistic variable V weight of children. When we give the linguistic values we keep in mind the approximate age of those children for which we have found the weight.

Clearly S is a totally ordered set for

very low < low < just low < very medium < just medium < just high < high < very high.

Now let N = {collection of all 6 × 1 column linguistic matrices with entries from S} = $\left\{\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} / a, b, c, d, e, f \in S\right\}$.

Let A = $\begin{bmatrix} \text{low} \\ \text{high} \\ \text{just low} \\ \text{low} \\ \text{high} \\ \text{very high} \end{bmatrix}$ and B = $\begin{bmatrix} \text{high} \\ \text{very high} \\ \text{low} \\ \text{low} \\ \text{medium} \\ \text{very low} \end{bmatrix}$

be two ling column matrices in N.

We calculate min {A, B}

$$= \min\{\begin{bmatrix} \text{low} \\ \text{high} \\ \text{just low} \\ \text{low} \\ \text{high} \\ \text{very high} \end{bmatrix}, \begin{bmatrix} \text{high} \\ \text{very high} \\ \text{low} \\ \text{low} \\ \text{medium} \\ \text{very low} \end{bmatrix}\}$$

$$= \begin{bmatrix} \min\{\text{low}, \text{high}\} \\ \min\{\text{high}, \text{very high}\} \\ \min\{\text{just low}, \text{low}\} \\ \min\{\text{low}, \text{low}\} \\ \min\{\text{high}, \text{medium}\} \\ \min\{\text{very high}, \text{very low}\} \end{bmatrix} = \begin{bmatrix} \text{low} \\ \text{high} \\ \text{low} \\ \text{low} \\ \text{medium} \\ \text{very low} \end{bmatrix}.$$

Clearly min{A, B} $\in$ N.

We see S is a totally ordered set so has the greatest and the least element very low is the least element and very high is the greatest element.

If I = $\begin{bmatrix} \text{very high} \\ \text{very high} \\ \text{very high} \\ \text{very high} \\ \text{very high} \\ \text{very high} \end{bmatrix}$ $\in$N is a linguistic matrix such that min{A, I} = A for all A $\in$ N.

Thus we conclude {N, min} is a finite linguistic monoid which is commutative {N, min max} is a finite commutative linguistic topological space of linguistic column matrices.

It is important to note that for one to define a commutative linguistic topological space of linguistic column matrices finite or infinite order it is mandatory that the underlying linguistic set S on which the linguistic matrices are built must be a totally ordered set.

With this in view we have the following result.

Theorem 2.1. *Let V be a linguistic variable, S the linguistic set associated with V (S can be a infinite continuum or if S is a finite set it should be totally ordered).*

M = {collection of all $n \times 1$ linguistic column matrices with entries from S}. Then {M, min max} is a commutative linguistic topological space of $n \times 1$ matrices (finite if $|S| < \propto$ and infinite if S is a linguistic continuum).

Proof is left as an exercise to the reader.

Next we proceed onto give examples of linguistic topological spaces of linguistic row matrices.

Example 2.32. Let V be the linguistic variable associated with the height of people. Let S = [shortest, tallest] be the linguistic continuum (set) associated with V.

Let P = $\{(a_1, a_2, \ldots, a_9) / a_i \in S; 1 \leq i \leq 9\}$

= {collection of all 1×9 linguistic row matrices with entries from S}.

Define on P the operation min as follows; let A = (tall, just fall, short, very short, tall, very tall, short, tall, short) and

B = (short, medium height, very short, tall, just short, tall, short, just tall, short) ∈ P.

We now find min{A, B} = min{(tall, just tall, short, very short, tall, very tall, short, tall, short), (short, medium height, very short, tall, just short, tall, short, just tall, short)}

= (min(tall, short), min(just tall, medium height), min(short, very short), min(very short, tall), min(tall, just short), min(very tall, tall) min(short, short), min(tall, just tall), min(short, short)}

= (short, medium height, very short, very short, just short, tall, short, just tall, short) ∈ P.

Clearly it can be easily verified min{A, B} = min{B, A}; that is min operation is commutative.

We see as S is a totally ordered set; S has the least element and the greatest element.

The greatest element of S is tallest and the least element of S as shortest.

J = (tallest, tallest, tallest, tallest, tallest, tallest, tallest, tallest, tallest) ∈ P is such that min{J, A} = A for all A ∈ P.

Thus {P, min} is a linguistic commutative monoid of infinite order.

Now for the same A, B ∈ P we define the max operation; max{A, B} = max{(tall, jus tall, short, very short, tall, very tall, short, tall, short), (short, medium height, very short, tall, just short, tall, short, just tall, short)}

= (max tall, short) max (just tall, medium height), max(short, very short), max(very short, tall), max(tall, just short), max(very tall, tall), max(short, short), max(tall, just tall), max(short, short)}

= (tall, just tall, short, tall, tall, very tall, short, tall, short) ∈ P. Thus max(A, B) ∈ P.

Further max(A, B) = max (B, A).

Finally as S is a linguistic continuum so it has the least element, viz shortest.

I = (shortest, shortest, shortest, shortest, shortest, shortest, shortest, shortest, shortest) is in P and we see max{A, I} = A for all A ∈ P.

Thus {P, max} is a commutative linguistic monoid of row linguistic matrices infinite order.

In view of all these we see {P, max min} is an infinite linguistic topological space of linguistic row matrices which is commutative and is of infinite order.

Now we provide one example of a finite linguistic topological row matrices which is commutative.

Example 2.33. Let V be a linguistic variable associated with the temperature of weather for 16 consecutive hours a day. The linguistic set

S = {hot, just hot, just cold, cold, very cold, very hot, hottest, coldest}

is a finite totally ordered set given by

coldest < very cold < cold < just cold > just hot ≤ hot ≤ very hot ≤ hottest.

Now let T = {collection of all 1 × 5 row matrices with entries from S}.

T is of finite order S has hottest to be the largest linguistic term of S and coldest to be the least linguistic term of S.

That is T = {(a_1, a_2, a_3, a_4, a_5) / a_i ∈ S; $1 \leq i \leq 5$}.

Let A, B ∈ J where A = (hot, cold, just hot, hottest, hot) and

B = (cold, hottest, hot, cold, just hot) ∈ T.

min{A, B} = min{(hot, cold, just hot, hottest, hot), (cold, hottest, hot, cold, just hot)}

= (min{hot, cold}, min{cold, hottest}, min{just hot, hot}, min{hottest, cold}, min{hot, just hot}) = (cold, cold, just hot, cold, just hot) ∈ T.

Thus {T, min} is a commutative linguistic semigroup of finite order.

Now consider X = (coldest, coldest, coldest, coldest, coldest) ∈ T;

clearly min{A, X} = X for all A as coldest is the least linguistic element of S.

Further if I = (hottest, hottest, hottest, hottest, hottest) ∈ T where hottest is the greatest linguistic element of T.

Now min{A, I} = A for all A ∈ T. So {T, min} infact a commutative linguistic monoid of finite order.

Now for the same A, B ∈ T

max{A, B} = max{(hot, cold, just hot, hottest, hot), (cold, hottest, hot, cold, just hot)}

= (max{hot, cold}, max{cold, hottest}, max{just hot, hot}, max{hottest, cold}, max{hot, just hot})

= (hot, hottest, hot, hottest, hot) ∈ T.

Further for X ∈ T we have max{A, X} = A for all A ∈ T.

Thus {T, max} is a finite commutative linguistic monoid of row matrices.

Now having seen examples of both finite and infinite order commutative ling row matrices we now proceed onto give a result on commutative linguistic row matrices in the following:

Now {T, max} and {T, min} are commutative linguistic monoids.

In view of this we have {T, min, max} is a commutative linguistic topological space of $1 \times n$ row matrices.

Theorem 2.2. *Let V be a linguistic variable such that its associated linguistic set S is such that S is totally order set with ℓ as its least linguistic element and g as its greatest linguistic* *element. Suppose R = {collection of all 1 × n linguistic row matrices with entries from S} = {(a_1, …, a_n) / $a_i \in S$} then*

i) *{R, min} is a linguistic commutative monoid of finite order if |S| , ∞ and infinite order if S is a linguistic continuum.*

ii) *{R, max} is a linguistic commutative monoid of finite or infinite order as in (i)*

iii) *{R, max, min} is a linguistic commutative topological space of order as in (i)*

Proof is left as an exercise to the reader.

Next we proceed onto give examples of linguistic topological space of m × n (m ≠ n) linguistic matrices of both finite and infinite order which are commutative.

Example 2.34. Let V be the ling variable corresponding to height of people. S = [shortest, tallest] be the ling set associated with V.

S is a linguistic continuum.

So S is a totally ordered set with shortest as the least element and tallest as the greatest element.

$$\text{Let } M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{bmatrix} \middle/ a_i \in S;\ 1 \leq i \leq 18 \right\}$$

be the collection of all 6 × 3 linguistic matrices with entries from S.

Now we define min and max operation on M.

$$\text{Let } A = \begin{bmatrix} \text{short} & \text{tall} & \text{tall} \\ \text{tall} & \text{just short} & \text{very tall} \\ \text{just tall} & \text{very tall} & \text{tall} \\ \text{short} & \text{just tall} & \text{short} \\ \text{tall} & \text{tallest} & \text{shortest} \\ \text{medium} & \text{tall} & \text{short} \end{bmatrix}_{6\times 3} \text{ and}$$

$$B = \begin{bmatrix} \text{very tall} & \text{short} & \text{medium} \\ \text{medium} & \text{tall} & \text{tall} \\ \text{short} & \text{short} & \text{very tall} \\ \text{very short} & \text{tall} & \text{just short} \\ \text{very tall} & \text{tall} & \text{tall} \\ \text{short} & \text{just short} & \text{very short} \end{bmatrix}_{6\times 3} \in M.$$

$$\text{We find } \min\{A, B\} = \min\{\begin{bmatrix} \text{short} & \text{tall} & \text{tall} \\ \text{tall} & \text{just short} & \text{very tall} \\ \text{just tall} & \text{very tall} & \text{tall} \\ \text{short} & \text{just tall} & \text{short} \\ \text{tall} & \text{tallest} & \text{shortest} \\ \text{medium} & \text{tall} & \text{short} \end{bmatrix}_{6\times 3},$$

$$\begin{bmatrix} \text{very tall} & \text{short} & \text{medium} \\ \text{medium} & \text{tall} & \text{tall} \\ \text{short} & \text{short} & \text{very tall} \\ \text{very short} & \text{tall} & \text{just short} \\ \text{very tall} & \text{tall} & \text{tall} \\ \text{short} & \text{just short} & \text{very short} \end{bmatrix}_{6\times 3} \}$$

$$= \begin{bmatrix} \min\{\text{short, very tall}\} & \min\{\text{tall, short}\} & \min\{\text{tall, medium}\} \\ \min\{\text{tall, medium}\} & \min\{\text{just short, tall}\} & \min\{\text{very tall, tall}\} \\ \min\{\text{just tall, short}\} & \min\{\text{very tall, short}\} & \min\{\text{tall, very tall}\} \\ \min\{\text{short, very short}\} & \min\{\text{just tall, tall}\} & \min\{\text{short, just short}\} \\ \min\{\text{tall, very tall}\} & \min\{\text{tallest, tall}\} & \min\{\text{shortest, tall}\} \\ \min\{\text{medium, short}\} & \min\{\text{tall, just short}\} & \min\{\text{short, very short}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{short} & \text{short} & \text{medium} \\ \text{medium} & \text{just short} & \text{tall} \\ \text{short} & \text{short} & \text{tall} \\ \text{very short} & \text{just tall} & \text{short} \\ \text{tall} & \text{tall} & \text{shortest} \\ \text{short} & \text{just short} & \text{very short} \end{bmatrix} \in \text{M}.$$

On similar times

$$\max\{A,B\} = \begin{bmatrix} \text{very tall} & \text{tall} & \text{tall} \\ \text{tall} & \text{tall} & \text{very tall} \\ \text{just tall} & \text{very tall} & \text{very tall} \\ \text{short} & \text{tall} & \text{just short} \\ \text{very tall} & \text{tallest} & \text{tall} \\ \text{medium} & \text{tall} & \text{short} \end{bmatrix} \in M.$$

Since S is a total lest ordered set S has the least linguistic element viz shortest and a greatest linguistic element tallest.

$$I = \begin{bmatrix} \text{shortest} & \text{shortest} & \text{shortest} \\ \text{shortest} & \text{shortest} & \text{shortest} \\ \text{shortest} & \text{shortest} & \text{shortest} \\ \text{shortest} & \text{shortest} & \text{shortest} \\ \text{shortest} & \text{shortest} & \text{shortest} \\ \text{shortest} & \text{shortest} & \text{shortest} \end{bmatrix}_{6\times 3} \in M$$

is such that min{A, I} = I for all A ∈ M and max{A, I} = A for all A in M.

Thus {M, max} is a commutative linguistic monoid of infinite order.

$$\text{Now } J = \begin{bmatrix} \text{tallest} & \text{tallest} & \text{tallest} \\ \text{tallest} & \text{tallest} & \text{tallest} \\ \text{tallest} & \text{tallest} & \text{tallest} \\ \text{tallest} & \text{tallest} & \text{tallest} \\ \text{tallest} & \text{tallest} & \text{tallest} \\ \text{tallest} & \text{tallest} & \text{tallest} \end{bmatrix}_{6\times 3} \in M \text{ is such that}$$

max{A, J} = J for all A ∈ M and min{A, J} = A for all A ∈ M.

Hence {M, min} is a linguistic commutative monoid of infinite order. Further {M, min, max} is the linguistic commutative topological space of infinite order.

We can by take the linguistic set S to be of finite order and S is such that S is totally ordered built any $m \times n$ matrix $m \neq n$ then that collection say W will be a linguistic topological space under min and max and W will be commutative and be of finite order.

Thus we can have the following result.

Theorem 2.3. *Let V be a linguistic variable such that the linguistic set S associated with V is totally ordered set of finite or of infinite order. M = {collection of all m × n linguistic matrices with entries from S(m ≠ n)}.*

Then the following results are true

- *i)* *{M, min} is a linguistic commutative monoid of finite or infinite order.*
- *ii)* *{M, max} is a linguistic commutative monoid of finite or infinite order.*
- *iii)* *{M, min, max} is a linguistic commutative topological space of finite or infinite order.*

Proof is left as an exercise to the reader.

Now we see only in case of square linguistic matrices one can have both non commutative and commutative linguistic topological spaces of square matrices.

Already we have defined developed and described linguistic topological spaces of square matrices which are non commutative. Here we provide examples of commutative linguistic topological spaces of square matrices.

Example 2.35. Let V be the linguistic variable associated with the performance aspects of students in general in the classroom atmosphere. The linguistic set associated with V is given by

$$S = [\text{worst, best}].$$

Clearly S is a linguistic continuum with worst as its least linguistic element and best as the greatest linguistic element.

$$\text{Let } M = \{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} / a, b, c, d, e, f, g, h, i \in S\}$$

be the collection of all 3 × 3 linguistic square matrices with entries from S. Define on M the max operation.

$$\text{For} \quad A = \begin{bmatrix} \text{worst} & \text{good} & \text{fair} \\ \text{bad} & \text{very bad} & \text{good} \\ \text{bad} & \text{fair} & \text{fair} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \text{good} & \text{fair} & \text{bad} \\ \text{worst} & \text{best} & \text{good} \\ \text{very good} & \text{fair} & \text{bad} \end{bmatrix} \in M.$$

We have min{A, B} = $\max\{\begin{bmatrix} \text{worst} & \text{good} & \text{fair} \\ \text{bad} & \text{very bad} & \text{good} \\ \text{bad} & \text{fair} & \text{fair} \end{bmatrix},$

$$\begin{bmatrix} \text{good} & \text{fair} & \text{bad} \\ \text{worst} & \text{best} & \text{good} \\ \text{very good} & \text{fair} & \text{bad} \end{bmatrix}\}$$

$$= \begin{bmatrix} \max\{\text{worst}, \text{good}\} & \max\{\text{good}, \text{fair}\} & \max\{\text{fair}, \text{bad}\} \\ \max\{\text{bad}, \text{worst}\} & \max\{\text{very bad}, \text{best}\} & \max\{\text{good}, \text{good}\} \\ \max\{\text{bad}, \text{very good}\} & \max\{\text{fair}, \text{fair}\} & \max\{\text{fair}, \text{bad}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{good} & \text{good} & \text{fair} \\ \text{bad} & \text{best} & \text{good} \\ \text{very good} & \text{fair} & \text{fair} \end{bmatrix} \in \text{M}.$$

Thus {M, max} is a commutative linguistic semigroup of linguistic square matrices infinite order.

Now for the same A, B ∈ M we find min{A, B} =

$$\min\{\begin{bmatrix} \text{worst} & \text{good} & \text{fair} \\ \text{bad} & \text{very bad} & \text{good} \\ \text{bad} & \text{fair} & \text{fair} \end{bmatrix}, \begin{bmatrix} \text{good} & \text{fair} & \text{bad} \\ \text{worst} & \text{best} & \text{good} \\ \text{very good} & \text{fair} & \text{bad} \end{bmatrix}\}$$

$$= \begin{bmatrix} \min\{\text{worst}, \text{good}\} & \min\{\text{good}, \text{fair}\} & \min\{\text{fair}, \text{bad}\} \\ \min\{\text{bad}, \text{worst}\} & \min\{\text{very bad}, \text{best}\} & \min\{\text{good}, \text{good}\} \\ \min\{\text{bad}, \text{very good}\} & \min\{\text{fair}, \text{fair}\} & \min\{\text{fair}, \text{bad}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{worst} & \text{fair} & \text{bad} \\ \text{worst} & \text{very bad} & \text{good} \\ \text{bad} & \text{fair} & \text{bad} \end{bmatrix} \in \text{M}.$$

Thus {M, min} is a linguistic commutative semigroup of square linguistic matrices.

$$\text{We see I} = \begin{bmatrix} \text{best} & \text{best} & \text{best} \\ \text{best} & \text{best} & \text{best} \\ \text{best} & \text{best} & \text{best} \end{bmatrix}$$

in M is such that all entries of I are the greatest ling term of S.

Hence min{A, I} = A for all A $\in$ M making the ling commutative semigroup {M, min} into a ling commutative monoid of square matrices as I = (best) servers as the linguistic identity matrix for the min operat9ion.

Similarly J = (worst) is such that max{A, J} = A for all A $\in$ M making {M, max} into a linguistic commutative monoid with J as its linguistic matrix identity.

$$\text{Here J} = \begin{pmatrix} \text{worst} & \text{worst} & \text{worst} \\ \text{worst} & \text{worst} & \text{worst} \\ \text{worst} & \text{worst} & \text{worst} \end{pmatrix} = (\text{worst}).$$

Thus {M, max, min} is a linguistic commutative topological space of square matrices which is commutative and is of infinite order.

By considering the linguistic set S to be finite order and if S is a totally ordered set then certainly collection of square matrices with entries from S will be a commutative finite linguistic topological space of square matrices. With this mind we can build infinite collection of finite and infinite order linguistic topological spaces of square matrices.

Next we proceed onto describe the notion of linguistic topological subspaces of both finite and infinite order by some examples.

Example 2.36. Let V be the linguistic variable associated with growth of the height of the paddy plants. Let S be the linguistic set associated with V; that is

S = [shortest (most stunted), tallest).

Clearly S is a linguistic continuum with shortest or most stunted as the least linguistic element and tallest as greatest linguistic element.

Suppose {S, min} is the linguistic commutative monoid under the operation min. Then tallest is the linguistic identity for the linguistic operation min.

Consider P = [shortest, medium height] $\subseteq$ S to be linguistic subinterval of S.

Clearly {P, min} is again a linguistic monoid but now with medium height as the linguistic identity for min {p, medium height} = p as all elements p in P \ {medium height} is clearly less than medium height.

Now we claim {P, min} is a linguistic submonoid of S. The vital difference is only the linguistic identity of S and P are different.

Now consider the max operation on S, {S, max} is a linguistic monoid with the linguistic operation max. Further shortest is the linguistic identity of {S, max}.

For max{x, shortest} = x for all $x \in S$ as shortest $\leq$ x.

Consider M = [just short, tall] $\subseteq$ S.

Clearly {M, max} is a linguistic monoid which is commutative with linguistic identity just short and infact

{M, max} is a linguistic submonoid of {S, max}, the only information to note is that the linguistic identities of M and S are different.

Further max{just short, x} = x for all $x \in M$.

We can have several linguistic submonoids of the monoids {S, max} and {S, min}. The only important factor to observe is these submonoids in most cases have linguistic identities which are different from that of the monoids {S, max} and {S, min} of infinite order.

Infact the linguistic monoids {S, min} and {S, max} both have linguistic submonoids of finite order.

For consider T = {short, very short, just short, medium height, tall, just tall, very tall, very very tall, tallest} ∈ S.

We see both {T, max} and {T, min} are linguistic submonoids of finite order.

Further the linguistic identity of {T, max} is very short and that {T, min} is very very tall. Infact we can have an infinite collection of linguistic submonoid of finite order.

Now having seen examples of commutative linguistic submonoids of commutative linguistic monoid.

We give examples of commutative linguistic topological spaces of linguistic matrices using mainly the concept of linguistic monoids.

Example 2.37. Let V be the linguistic variable weight of dogs. The linguistic set associated with this V is given by

S = [lightest, heaviest].

$$\text{Let } M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix}_{6\times 2} \Big/ a_i \in S; 1 \leq i \leq 12 \right\}$$

= {collection of all 6 × 2 linguistic matrices with entries from S}.

Now it is easily verified min{A, B} and max{A, B} ∈ M and $I = \begin{bmatrix} \text{lightest} & \text{lightest} \\ \text{lightest} & \text{lightest} \\ \vdots & \vdots \\ \text{lightest} & \text{lightest} \end{bmatrix}_{6\times 2}$ = (lightest) $\in$ M

is such that max{A, I} = A and min{A, I} = I.

Now $J = \begin{bmatrix} \text{heaviest} & \text{heaviest} \\ \text{heaviest} & \text{heaviest} \\ \vdots & \vdots \\ \text{heaviest} & \text{heaviest} \end{bmatrix}_{6\times 2}$

= (heaviest) $\in$ M is such that max{A, J}

= J and min{A, J} = A for all A $\in$ M.

Thus J = $(\text{heaviest})_{6 \times 2}$ is the linguistic identity matrix for the min operation on M and I = {lightest) is the linguistic identity matrix for max operation on M.

Thus {M, max} and {M, min} are linguistic monoids of matrices.

Take P = [light, medium weight] $\subseteq$ S = [lightest, heaviest]; clearly {P, max} and {P, min} are linguistic sub monoids of {S, max} and {S, min} respectively.

Further for {P, max}, I_p = (light)

Clearly R = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} \right.$ / $a_i \in P$; $1 \leq i \leq 12\} \subseteq M$

is such that {R, min} and {R, max} are linguistic submonoids of {M, min} and {M, max} respectively but the ling matrix identities are

$$I_p = \begin{bmatrix} \text{light} & \text{light} \\ \text{light} & \text{light} \\ \vdots & \vdots \\ \text{light} & \text{light} \end{bmatrix}_{6\times 2} = (\text{light}) \in R$$

is such that max{A, I_p} = A for all A ∈ R.

$$\text{Now } J_P = \begin{bmatrix} \text{medium weight} & \text{medium weight} \\ \text{medium weight} & \text{medium weight} \\ \vdots & \vdots \\ \text{medium weight} & \text{medium weight} \end{bmatrix}_{6\times 2}$$

= (medium weight) ∈ R is such that J_P is the linguistic identity matrix of {R, min} for min {A, J_P} = A for all A ∈ R.

Thus {R, min} is a linguistic submonoid of linguistic matrices.

Now {M, max, min} is the linguistic topological space of linguistic matrices and {R, max, min} is the linguistic topological subspaces of linguistic matrices.

We can by this way construct infinite number of linguistic topological subspaces of linguistic matrices.

Now we can also build linguistic topological subspaces of finite order.

We will illustrate this situation by an example.

Take W = {light, lighter, just light, very light, medium weight, very heavy, heavy, just heavy} ∈ [lightest, heaviest].

We see W is a ordered set for

lighter < very light < light < just light < medium weight < just heavy < heavy < very heavy < heaviest

We see lighter is the least elements in W and heaviest is the greatest element of W.

$$V = \{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ \vdots & \vdots \\ a_{11} & a_{12} \end{bmatrix} / a_i \in W; 1 \leq i \leq 12\}$$

be the collection of all 6 × 2 linguistic matrices with entries from W.

Clearly order of V is finite and

$$A = \begin{bmatrix} \text{lighter} & \text{lighter} \\ \text{lighter} & \text{lighter} \\ \vdots & \vdots \\ \text{lighter} & \text{lighter} \end{bmatrix} \in V \text{ is such that A is least matrix of V.}$$

Thus for the linguistic submonoid {V, max}; A is the linguistic identity matrix such that

max{A, D} = A for all D ∈ V.

$$B = \begin{bmatrix} \text{heaviest} & \text{heaviest} \\ \text{heaviest} & \text{heaviest} \\ \vdots & \vdots \\ \text{heaviest} & \text{heaviest} \end{bmatrix} \in V;$$

B is the linguistic matrix identity of V for the linguistic submonoid {V, min}; for min{D, B} = D for all D ∈ V.

We see {V, max, min} is a linguistic topological subspace of M.

Clearly V is a find ordered commutative linguistic topological subspace of M.

Thus we can easily prove {M, min, max} the commutative infinite topological space of 6 × 2 linguistic matrices infinite number of commutative linguistic topological subspaces of linguistic matrices of finite order.

Likewise {M, max, min} has infinite number of topological subspaces of linguistic matrices of infinite order.

We just give one or two examples of non commutative linguistic topological subspect of matrices of both infinite or finite order.

Example 2.38. Let V be the linguistic variable age of people. Let S = [youngest, oldest] be the linguistic set which is clearly a linguistic continuum. So S is a totally ordered set.

i) We know {S, min, max} commutative is a linguistic topological space of infinite order.

ii) If P(S) is the power set of S then {P(S), ∪, ∩} is again a commutative linguistic topological space of infinite order.

iii) {P(S), min, max} is a special type of linguistic topological space of infinite order.

Recall if A = {young, old, very old, just young, very young, middle age} and

B = {youngest, just old, just middle age, just young, very old} are two distinct linguistic sets of P(S); then

min{A, B} = min{{young, old, very old, just young, very young, middle age}, {youngest, just old, just middle age, just young, very old}}

= {youngest, just young, just old, just middle age, very old, just young, very young, middle age}. …I

max{A, B} = max{{young, old, very old, just young, very young, middle age}, {youngest, just old, just middle age, just young, very old}}

= {young, just old, just middle age, very old, old, just young, very young, middle age}. ...II

I and II are distinct.

Thus {P(S), max, min} we have defined as the special type of subset linguistic topological space of infinite order which is commutative.

Now using S we can define a class of linguistic row matrices as R = {collection of all $1 \times n$ ling row matrices with entries from S} = $\{(a_1, a_2, \ldots, a_n) / a_i \in S; 1 \leq i \leq n\}$.

We see {R, min, max} is again a linguistic topological space of linguistic row matrices.

Clearly {R, min, max} is commutative and is of infinite order.

Let C= {collection of all $m \times 1$ linguistic column matrices with entries from S}

$$= \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} / a_i \in S; 1 \leq i \leq m \right\}.$$

Clearly {C, min, max} is a linguistic topological space of column linguistic matrices of infinite order.

Likewise if M = {collection of all $p \times q$ linguistic matrices $p \neq q$ with entries from

$$S = \left\{ \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1q} \\ a_{21} & a_{22} & \ldots & a_{2q} \\ \vdots & \vdots & \ldots & \vdots \\ a_{p1} & a_{p2} & \ldots & a_{pq} \end{bmatrix} \right.$$

where $a_{ij} \in S$; $1 \leq i \leq P$ and $1 \leq j \leq q\}$.

It can be verified {M, min, max} is a commutative linguistic topological space of infinite order p × q or rectangular linguistic matrices.

Finally we consider N = {collection of all n × n linguistic square matrices with entries from S}

$$= \{\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{m} \end{bmatrix} \text{ where } a_{ij} \in S; 1 \leq i, j \leq n\}.$$

We see {N, max, min} is a linguistic topological space (of square linguistic matrices) which is commutative and is of infinite order.

Now we can define 4 different non commutative linguistic topological spaces using only linguistic square matrices.

We list out the 4 different non commutative linguistic topological spaces of squares matrices which are of infinite order.

{N, max, min max}, {N, max, max min}, {N, min, min max} and {N, min, max min}.

For the sake of better understanding we will illustrate all the four situations by some example for n = 3.

Let N = {collection of all 3 × 3 linguistic matrices with entries from S}

$$= \{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} / a_{ij} \in S;\ 1 \leq i, j \leq 3\}.$$

We define min max operation.

It is know {N, max} is a linguistic commutative monoid of infinite order.

Now we illustrate how the operation min max is carried out on N.

$$\text{Let } A = \begin{pmatrix} \text{young} & \text{old} & \text{young} \\ \text{just young} & \text{young} & \text{very old} \\ \text{very old} & \text{very young} & \text{old} \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} \text{old} & \text{young} & \text{old} \\ \text{very old} & \text{very young} & \text{young} \\ \text{just young} & \text{oldest} & \text{youngest} \end{pmatrix} \in N.$$

We find min max{A, B}

$$= \min\{\max\{\begin{pmatrix} \text{young} & \text{old} & \text{young} \\ \text{just young} & \text{young} & \text{very old} \\ \text{very old} & \text{very young} & \text{old} \end{pmatrix},$$

$$\begin{pmatrix} \text{old} & \text{young} & \text{old} \\ \text{very old} & \text{very young} & \text{young} \\ \text{just young} & \text{oldest} & \text{youngest} \end{pmatrix}\}\}$$

$$= \left(\begin{array}{c} \min\{\max\{\text{young}, \text{old}\}, \max\{\text{old}, \text{very old}\}, \\ \max\{\text{young}, \text{just young}\}\} \\ \min\{\max\{\text{just young}, \text{old}\}, \max\{\text{young}, \text{very old}\}, \\ \max\{\text{very old}, \text{just young}\}\} \\ \min\{\max\{\text{very old}, \text{old}\}, \max\{\text{very young}, \text{very old}\}, \\ \max\{\text{old}, \text{just young}\}\} \end{array} \right.$$

$$\begin{array}{c} \min\{\max\{\text{young}, \text{young}\}, \max\{\text{old}, \text{very young}\}, \\ \max\{\text{young}, \text{oldest}\}\} \\ \min\{\max\{\text{just young}, \text{young}\}, \max\{\text{young}, \text{very young}\}, \\ \max\{\text{very old}, \text{oldest}\}\} \\ \min\{\max\{\text{very old}, \text{old}\}, \max\{\text{very young}, \text{very old}\}, \\ \max\{\text{old}, \text{oldest}\}\} \end{array}$$

$$\left. \begin{array}{c} \min\{\max\{\text{young}, \text{old}\}, \max\{\text{old}, \text{young}\}, \\ \max\{\text{young}, \text{youngest}\} \\ \min\{\max\{\text{just young}, \text{old}\}, \max\{\text{young}, \text{young}\} \\ \max\{\text{very old}, \text{youngest}\}\} \\ \min\{\max\{\text{very old}, \text{old}\}, \max\{\text{very young}, \text{young}\}, \\ \max\{\text{old}, \text{youngest}\}\} \end{array} \right)$$

$$= \left[\begin{array}{l} \min\{\text{old}, \text{very old}, \text{just young}\} \\ \min\{\text{old}, \text{very old}, \text{very old}\} \\ \min\{\text{very old}, \text{very old}, \text{old}\} \end{array} \right.$$

$$\begin{array}{l} \min\{\text{young}, \text{old}, \text{oldest}\} \\ \min\{\text{just young}, \text{young}, \text{oldest}\} \\ \min\{\text{very old}, \text{very old}, \text{oldest}\} \end{array}$$

$$\left.\begin{array}{l} \min\{\text{old, old young}\} \\ \min\{\text{old, young, very old}\} \\ \min\{\text{very old, young, old}\} \end{array}\right]$$

$$= \begin{pmatrix} \text{just young} & \text{young} & \text{young} \\ \text{old} & \text{just young} & \text{young} \\ \text{old} & \text{very old} & \text{young} \end{pmatrix} \qquad \ldots \text{I}$$

This is the way min max operation is performed.

We now find out the value of minmax{B,A}

$$= \min\{\max\{ \begin{pmatrix} \text{old} & \text{young} & \text{old} \\ \text{very old} & \text{very young} & \text{young} \\ \text{just young} & \text{oldest} & \text{youngest} \end{pmatrix},$$

$$\begin{pmatrix} \text{young} & \text{old} & \text{young} \\ \text{just young} & \text{young} & \text{very old} \\ \text{very old} & \text{very young} & \text{old} \end{pmatrix}\}\}$$

$$= \left(\begin{array}{c} \min\{\max\{\text{old, young}\}, \max\{\text{young, just young}\}, \\ \max\{\text{old, very old}\}\} \\ \min\{\max\{\text{very old, young}\}, \max\{\text{very young,} \\ \text{just young}\}, \min\{\text{young, very old}\}\} \\ \min\{\text{just young, young}\}, \max\{\text{oldest, just young}\} \\ \max\{\text{youngest, veryold}\}\} \end{array}\right.$$

$$\begin{array}{c} \min\{\max\{\text{old}, \text{old}\}, \max\{\text{young}, \text{young}\}, \\ \max\{\text{old}, \text{very young}\}\} \\ \min\{\max\{\text{very old}, \text{old}\}, \max\{\text{very young}, \\ \text{young}\}, \max\{\text{young}, \text{very young}\}\} \\ \min\{\max\{\text{just young}, \text{old}\}, \max\{\text{oldest}, \text{young}\} \\ \max\{\text{youngest}, \text{very young}\}\} \end{array}$$

$$\left. \begin{array}{c} \min\{\max\{\text{old}, \text{young}\}, \max\{\text{young}, \text{very old}\}, \\ \max\{\text{old}, \text{old}\}\} \\ \min\{\max\{\text{very old}, \text{young}\}, \max\{\text{very young}, \\ \text{very old}\}, \max\{\text{young}, \text{old}\}\} \\ \min\{\max\{\text{just young}, \text{young}\}, \max\{\text{oldest}, \text{very old}\}, \\ \max\{\text{youngest}, \text{old}\}\} \end{array} \right)$$

$$\begin{pmatrix} \min\{\text{old}, \text{young}, & \min\{\text{old}, \text{young}, & \min\{\text{old}, \\ \text{very old}\} & \text{old}\} & \text{very old}\} \\ \min\{\text{very old}, \text{just} & \min\{\text{very old}, & \min\{\text{very old}, \\ \text{young}, \text{very old}\} & \text{young}, \text{young}\} & \text{very old}, \text{old}\} \\ \min\{\text{young}, \text{oldest}, & \min\{\text{old}, \text{oldest}, & \min\{\text{young}, \\ \text{very old}\} & \text{very young}\} & \text{oldest}, \text{old}\} \end{pmatrix}$$

$$= \begin{pmatrix} \text{young} & \text{young} & \text{old} \\ \text{just young} & \text{young} & \text{old} \\ \text{young} & \text{very young} & \text{young} \end{pmatrix} \qquad \ldots \text{II}$$

We see I and II are distinct, that is

minmax{A, B} ≠ minmax{B, A}.

Hence {N, minmax} is a noncommutative linguistic semigroup of 3 × 3 matrices.

Thus {N, max, min max} is a non commutative linguistic topological space of 3 × 3 linguistic matrices of infinite order.

In view of this we can say {N, min, min max} also is a non commutative linguistic topological space of 3 × 3 linguistic matrices of infinite order different from {N, max, min max}.

Now we know {N, max} is a commutative monoid of linguistic 3 × 3 matrices of infinite order.

Consider max min operation on N for the same A and B we now find out max{min {A, B}}

$$= \max\ \{\min\{ \begin{pmatrix} \text{young} & \text{old} & \text{young} \\ \text{just young} & \text{young} & \text{very old} \\ \text{very old} & \text{very young} & \text{old} \end{pmatrix},$$

$$\begin{pmatrix} \text{old} & \text{young} & \text{old} \\ \text{very old} & \text{very young} & \text{young} \\ \text{just young} & \text{oldest} & \text{youngest} \end{pmatrix}\}\}$$

$$= \left(\begin{array}{c} \max\{\min\{\text{young}, \text{old}\}, \min\{\text{old}, \text{very old}\} \\ \min\{\text{young}, \text{just young}\} \\ \max\{\min\{\text{just young}, \text{old}\}, \min\{\text{young}, \text{very old}\} \\ \min\{\text{very old}, \text{just young}\}\} \\ \max\{\min\{\text{very old}, \text{old}\}, \min\{\text{very young}, \text{very old}\} \\ \min\{\text{old}, \text{just young}\}\} \end{array} \right.$$

$$\begin{array}{c}
\max\{\min\{\text{just young}, \text{old}\}, \min\{\text{young}, \text{very old}\} \\
\min\{\text{very old}, \text{just young}\} \\
\max\{\min\{\text{just young}, \text{young}\}, \min\{\text{young}, \text{very} \\
\text{young}\}, \min\{\text{very old}, \text{oldest}\} \\
\max\{\min\{\text{just young}, \text{old}\}, \min\{\text{young}, \text{young}\}, \\
\min\{\text{very old}, \text{youngest}\}\}
\end{array}$$

$$\left.\begin{array}{c}
\max\{\min\{\text{very old}, \text{old}\}, \min\{\text{very young}, \\
\text{very old}\}, \min\{\text{old}, \text{just young}\} \\
\max\{\min\{\text{very old}, \text{young}\}, \min\{\text{very young}, \\
\text{very young}\}, \min\{\text{old}, \text{oldest}\} \\
\max\{\min\{\text{very old}, \text{old}\}, \min\{\text{very young}, \\
\text{young}, \min\{\text{old}, \text{youngest}\}\}
\end{array}\right)$$

$$= \left(\begin{array}{cc}
\max\{\text{young}, \text{old}, & \max\{\text{just young}, \text{young}, \\
\text{just young}\} & \text{just young}\} \\
\max\{\text{just young}, \text{young}, & \max\{\text{just young}, \text{very} \\
\text{just young}\} & \text{young}, \text{very old}\} \\
\max\{\text{old}, \text{very young}, & \max\{\text{just young}, \text{young} \\
\text{just young}\} & \text{youngest}\}
\end{array}\right.$$

$$\left.\begin{array}{c}
\max\{\text{old}, \text{very young}, \text{just young}\} \\
\max\{\text{young}, \text{very young}, \text{old}\} \\
\max\{\text{old}, \text{very young}, \text{youngest}\}
\end{array}\right)$$

$$= \begin{pmatrix}
\text{old} & \text{young} & \text{old} \\
\text{young} & \text{very old} & \text{old} \\
\text{old} & \text{just young} & \text{old}
\end{pmatrix} \qquad \ldots\text{III}$$

Now we find $\max\{\min\{B,A\}\}$

$$= \max\{\min\{\begin{pmatrix} \text{old} & \text{young} & \text{old} \\ \text{very old} & \text{very young} & \text{young} \\ \text{just young} & \text{oldest} & \text{youngest} \end{pmatrix},$$

$$\begin{pmatrix} \text{young} & \text{old} & \text{young} \\ \text{just young} & \text{young} & \text{very old} \\ \text{very old} & \text{very young} & \text{old} \end{pmatrix}\}\}$$

$$= \begin{pmatrix} \max\{\min\{\text{old, young}\}, \min\{\text{young, just young}\}, \min\{\text{old, very old}\} \\ \max\{\min\{\text{very old, young}\} \min\{\text{very young, just young}\}, \min\{\text{young, very old}\} \\ \max\{\min\{\text{just young, young}\} \min\{\text{oldest, just young}\}, \min\{\text{youngest, very old}\}\} \end{pmatrix}$$

$$\begin{matrix} \max\{\min\{\text{old,old}\}, \min\{\text{young, young}\}, \min\{\text{old, very young}\} \\ \max\{\min\{\text{very old, old}\}, \min\{\text{very young, young}\}, \min\{\text{young, very young}\} \\ \max\{\min\{\text{just young, old}\}, \min\{\text{oldest, young}\}, \min\{\text{youngest, very young}\}\} \end{matrix}$$

$$\begin{pmatrix} \max\{\min\{\text{old, young}\}, \min\{\text{young, very old}\} \min\{\text{old, old}\}\} \\ \max\{\min\{\text{very old, young}\}, \min\{\text{very young, very old}\}, \max\{\text{young, old}\}\} \\ \max\{\max\{\text{just young, young}\}, \min\{\text{oldest, very old}\}, \min\{\text{youngest, old}\}\} \end{pmatrix}$$

$$= \begin{pmatrix} \max\{\text{young, just young, old}\} & \max\{\text{old, young, very young}\} & \max\{\text{young, young, old}\} \\ \max\{\text{young, very young, young}\} & \max\{\text{old, very young, very young}\} & \max\{\text{young, very young, young}\} \\ \max\{\text{just young, just young, youngest}\} & \max\{\text{just young, young, youngest}\} & \max\{\text{just young, very old, youngest} \end{pmatrix}$$

$$= \begin{pmatrix} \text{old} & \text{old} & \text{old} \\ \text{young} & \text{old} & \text{young} \\ \text{just young} & \text{young} & \text{very old} \end{pmatrix} \qquad \ldots IV$$

Clearly equations (III) and (IV) are distinct.

Hence max{min{A, B}} ≠ max{min{B, A}}.

Thus {N, max min} is a linguistic non commutative semigroup of infinite order.

Hence {N, min, max min} is a non commutative linguistic topological space of (3 × 3 linguistic square matrices) infinite order.

This linguistic noncommutative topological space is different from the other two topological spaces

{N, min, min max} and {N, max, min max}.

Finally we see {N, max, max min} is a different non commutative linguistic topological space of 3 × 3 square linguistic matrices of infinite order.

This linguistic topological space {N, max, min max} is different from {N, max, max min}, {N, min, minmax} and {N, min, max min}, the three different non commutative linguistic topological spaces.

Now we proceed onto give a few linguistic topological subspaces of these four linguistic topological spaces of both finite and infinite order.

Consider the linguistic subinterval

P = [young, old] ⊆ [youngest, oldest] = S.

Clearly as P is a linguistic subinterval of the linguistic interval S. P is also a totally ordered linguistic set.

Let T = {collection of all 3 × 3 linguistic matrices with entries from P ⊆ S}

$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} / a_{ij} \in P; 1 \leq i, j \leq 3 \right\}.$$

{P, max, min max}, {P, max, max min}, {P, min, min max} and {P, min, max min} are 4 distinct linguistic topological subspaces of the linguistic non commutative topological spaces, {N, max, min max}, {N, max, max min}, {N, min, min max} and {N, min, max min} respectively.

All these 4 linguistic topological subspaces are also non commutative and infact all of them are of infinite order.

Finally we provide example linguistic noncommutative topological subspaces of {N, max, min max}, {N, max, maxmin}, {N, min, minmax} and {N, min, maxmin} of finite order.

Consider W = {old, young, just old, very young, very old, very very old, youngest, just young, middle age} ∈ S; clearly W is a finite linguistic subset of S.

Further W is also a totally ordered set; the ordering of W is as follows:

youngest ≤ very young ≤ just young ≤ young ≤ middle age ≤ just old ≤ old ≤ very old ≤ very very old.

Now let V = {collection of 3 × 3 linguistic matrices with entries from W}

$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Big/ a_{ij} \in W,\ 1 \leq i, j \leq 3 \right\}.$$

{V, max, min max}, {V, max, max min}, {V, min, minmax} and {V, min, maxmin} are finite order non commutative ling topological subspaces of {N, max, min max}, {N, max, max min}, {N, min, min max} and {N, min, maxmin} respectively.

Now having seen both finite and infinite order linguistic topological subspaces which are non commutative we proceed onto describe the fact all totally ordered linguistic sets are linguistic semirings under max and min operations as the operations are distributive.

Infact it is not possible to develop the notion of linguistic group; however one should find by some means to develop a group structure. This will be taken up as a very big task in due course of time.

However we know {P(S), ∪, ∩} is a linguistic topological space akin to the classical topological space.

Now using P(S) we can construct subset linguistic topological spaces of special type using linguistic subset matrices.

We first recall the properties of usual subset matrices in reals and describe them by examples.

Example 2.39. Let R be the real lien P(R) be the power set of R. Now{P(R), ∩, ∪} is the classical topological space.

Consider A = {collection of all 1 × 4 matrices with entries from P(R)}} = $\{(a_1, a_2, a_3, a_4) / a_i \in P(R); 1 \leq i \leq 4\}$.

Now if

x = ({3, 4, 7}, {8, 9, 11, 4}, {2, 1}, {7, 8, 9, 11, 12}) and

y = ({3, 7, 20, 11}, {8, 9, 1, 2, 3}, {2, 5, 7, 1}, {8, 9, 11, 12}} ∈ A.

x ∩ y = ({3, 4, 7}, {8, 9, 11, 4}, {2, 1}, {7, 8, 9, 11, 12}) ∩ ({3, 7, 11, 20}, {8, 9, 1, 2, 3}, {2, 5, 7, 1}, {8, 9, 11, 12})

= ({3, 4, 7} ∩ {3, 7, 11, 20}, {8, 9, 4, 11} ∩ {8, 9, 1, 2, 3}, {2, 1} ∩ {2, 5, 7, 1}, {7, 8, 9, 11, 2} ∩ {8, 9, 11, 12})

= ({3, 7}, {8, 9}, {2, 1}, {8, 9, 11, 12}) ∈ A.

Thus {A, ∩} is a commutative monoid as for any x ∈ A

x ∩ (R) = x for all x ∈ A, where (R) = (R, R, R, R).

Now we show ∪ defined on A is again a commutative monoid. For the same x and y we find

x ∪ y = ({3, 4, 7}, {8, 9, 11, 4}, {2, 1}, {7, 8, 9, 11, 12}) ∪ ({11, 3, 7, 20}, {8, 9, 12, 3}, {2, 5, 7, 1}, {8, 9, 11, 12})

= ({3, 4, 7} ∪ {3, 7, 11, 20}, {8, 9, 11, 4} ∪ {8, 9, 12, 3}, {2, 1} ∪ {2, 5, 7, 1}, {7, 8, 9, 11, 12} ∪ {8, 9, 11, 12})

= ({3, 4, 7, 11, 20}, {8, 9, 11, 4, 12, 3}, {2, 1, 5, 7}, {7, 8, 9, 11, 12}) ∈ A.

Thus it is easily verified {A, ∪} is a subset monoid of row matrices; infact (ϕ, ϕ, ϕ, ϕ) = (ϕ) acts as ling row matrix identity.

For x ∪ (ϕ) = x for all x∈ A.

Thus {A, ∪, ∩} is a subset topological space of matrices or subset matrix topological space.

Now we give using same set R, that is reals define P(R) the power set of R.

Let B = {collection of all 5 × 1 subset column matrices with entries from P(R)}

$$= \{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} / a_i \in P(R); 1 \leq i \leq 5\}.$$

Now we show {P(R), ∪, ∩} is a subset column matrix topological space.

$$\text{Let } x = \begin{bmatrix} \{7,-1,5,8,3,4,1\} \\ \{3,4,7,8\} \\ \{\phi\} \\ \{4,-8,-9,-18,9\} \\ \{1,3,2,1-4,-9,9,27\} \end{bmatrix} \text{ and } y = \begin{bmatrix} \{7,1,9\} \\ \{9,18,27\} \\ \{40,-9\} \\ \{\phi\} \\ \{1,2,3,4,9\} \end{bmatrix} \in B.$$

$$x \cap y = \begin{bmatrix} \{7,-1, 5, 8, 4, 3, 1\} \\ \{3,4,7,8\} \\ \{\phi\} \\ \{4,-8,-9,-18,9\} \\ \{4,3,2,1,-4,-9,9,27\} \end{bmatrix} \cap \begin{bmatrix} \{7,1,9\} \\ \{9,18,27\} \\ \{40,-9\} \\ \{\phi\} \\ \{1,2,3,4,9\} \end{bmatrix}$$

$$= \begin{bmatrix} \{7,1\} \\ \{\phi\} \\ \{\phi\} \\ \{\phi\} \\ \{1,2,3,4,9\} \end{bmatrix} \in B.$$

Thus $\{B, \cap\}$ is a commutative subset monoid of column subset matrices.

$$\begin{bmatrix} R \\ R \\ R \\ R \\ R \end{bmatrix} = (R) \text{ serves as the subset matrix identity of B as}$$

$(R) \cap x = x$ for all $x \in B$.

Now we can define the operation $\cup$ on B as follows.

We consider the same $x, y \in B$;

$$x \cup y = \begin{bmatrix} \{7,-1,5,8,3,4,1\} \\ \{3,4,7,8\} \\ \{\phi\} \\ \{4,-8,-9,-18,9\} \\ \{4,3,2,1,-4,9,-9,27\} \end{bmatrix} \cup \begin{bmatrix} \{7,1,9\} \\ \{9,18,27\} \\ \{40,-9\} \\ \{\phi\} \\ \{1,2,3,4,9\} \end{bmatrix}$$

$$= \begin{bmatrix} \{7,-1,5,8,3,4,1\} \cup \{7,1,9\} \\ \{3,4,7,8\} \cup \{9,18,27\} \\ \{\phi\} \cup \{40,-9\} \\ \{4,-8,-9,-18,9\} \cup \{\{\phi\} \\ \{4,3,2,1,-4,9,-9,27\} \cup \{1,2,3,4,9\} \end{bmatrix}$$

$$= \begin{bmatrix} \{7,1,-1,5,8,3,4,9\} \\ \{3,4,7,8,9,18,27\} \\ \{40,-9\} \\ \{4,-8,-9,-18,9\} \\ \{4,3,2,1-4,9,-9,27\} \end{bmatrix} \in B.$$

Thus {B, ∪} is a monoid of column subset matrices.

We see {B, ∪, ∩} is the topological space of subset matrices of infinite order.

Now we can have topological subspace of subset matrices both of finite and infinite order.

We know Q ⊆ R where Q is the set of rationals.

Further P(Q) ⊆ P(R) their power sets also satisfy the containment relation.

If D = {collection of all 5 × 1 matrices with entries from

$$P(Q) = \{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} / a_i \in P(Q); 1 \le i \le 5\} \{D, \cap\} \subseteq \{B, \cap\}, \{D, \cup\}$$

⊆ {B, ∪} and {D, ∪, ∩} ⊆ {B, ∪, ∩}.

Thus PD, ∪, ∩} is the subset topological subspace of 5 × 1 column matrices of {B, ∪, ∩}, {D, ∪, ∩} is also a subset topological subspace of matrices of infinite order.

Let T = {1, 10, –1, 5, 7, 8, 9, 16, 20, 22, 19, 18, 27, –9} $\subseteq$ R be a finite subset of R.

Clearly P(T) $\subseteq$ P(R) and P(T) is of finite order.

If G = {collection of all 5 × 1 matrices with entries from P(T)} $\subseteq$ D $\subseteq$ B.

{G, $\cup$, $\cap$} is a subset topological subspace of subset 5 × 1 matrices of finite order.

Thus we have given examples of finite and infinite order subset topological subspaces of matrices.

Example 2.40. Let Z_{20} be the modulo integers $P(Z_{20})$ be the power set of Z_{20}.

$$M = \{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} / a_i \in P(Z_{20})\ 1 \leq i \leq 9\}$$

= {collection of all 3 × 3 subset matrices}.

M is of finite order.

Let $$A = \begin{pmatrix} \{0,2,3,9\} & \{8,9,11\} & \{4,3,8\} \\ \{9,11,16,17\} & \{3,6,12\} & \{9,0,1\} \\ \{1,4,5\} & \{8,6,0\} & \{7,1\} \end{pmatrix}$$

and $$B = \begin{pmatrix} \{7,2,4\} & \{10,5,7\} & \{6,4,2,0\} \\ \{1,3,0\} & \{0,1,9\} & \{7,9\} \\ \{6,8,4\} & \{1,2\} & \{6,5,9,8\} \end{pmatrix} \in M.$$

We see A ∩ B = $\begin{pmatrix} \{0,2,3,9\} & \{8,9,11\} & \{4,3,8\} \\ \{9,11,16,17\} & \{3,6,12\} & \{9,0,1\} \\ \{1,4,5\} & \{8,6,0\} & \{7,1\} \end{pmatrix}$

$$\cap \begin{pmatrix} \{7,2,4\} & \{10,5,7\} & \{6,4,2,0\} \\ \{1,3,0\} & \{0,1,9\} & \{7,9\} \\ \{6,8,4\} & \{1,2\} & \{6,5,9,8\} \end{pmatrix}$$

$$\begin{pmatrix} \{0,2,3,9\} \cap \{7,2,4\} & \{8,9,11\} \cap \{10,5,7\} & \{4,3,8\} \cap \{6,4,2,0\} \\ \{9,11,16,17\} \cap \{1,3,0\} & \{3,6,12\} \cap \{0,1,9\} & \{9,0,1\} \cap \{7,9\} \\ \{1,4,5\} \cap \{6,8,4\} & \{0,6,8\} \cap \{1,2\} & \{7,1\} \cap \{6,5,9,8\} \end{pmatrix}$$

$$= \begin{pmatrix} \{2\} & \{\phi\} & \{4\} \\ \{\phi\} & \{\phi\} & \{9\} \\ \{4\} & \{\phi\} & \{\phi\} \end{pmatrix} \quad \ldots \text{I}$$

$$A \cup B = \begin{pmatrix} \{0,2,3,9\} \cup \{7,2,4\} & \{8,11,9\} \cup \{10,5,7\} & \{4,3,8\} \cup \{6,4,2,0\} \\ \{9,11,16,17\} \cup \{1,3,0\} & (3,6,12\} \cup \{0,1,9\} & \{9,0,1\} \cup \{7,9\} \\ \{1,4,5\} \cup \{6,8,4\} & \{8,6,0\} \cup \{1,2\} & \{7,1\} \cup \{6,5,9,8\} \end{pmatrix}$$

$$= \begin{pmatrix} \{0,2,3,4,7,9\} & \{8,9,11,5,7,10\} & (4,3,8,6,2,0\} \\ \{9,11,16,17,1,3,0\} & \{3,6,12,0,1,9\} & \{9,0,1,7\} \\ \{1,4,5,6,8\} & \{8,6,0,1,2\} & \{7,1,6,5,9,8\} \end{pmatrix} \quad \ldots \text{II}$$

I and II are distinct.

Thus $\{P(Z_{20}), \cup, \cap\}$ is a subset topological space of square matrices of finite order.

Now we can also define noncommutative operations which will be dealt later.

We provide yet another example.

Example 2.41. Consider S = Z_{18} the modulo integer P(S) = $P(Z_{18})$ be the power set of Z_{18}.

Let B = {collection of all 5 × 3 matrices with entries from

$$P(Z_{18})\} = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle/ a_i \in P(Z_{18});\ 1 \leq i \leq 15 \right\}$$

Now we can define $\cup$ and $\cap$ on $P(Z_{18})$ and see

$\{P(Z_{18}, \cup, \cap\}$ is again a topological subset matrix space of finite order.

This is also commutative.

Hence all subset topological subspaces will also be only of finite order and will be commutative.

Now in this case of subset matrices we can define both + and × as well as min and max.

We will describe them first by examples.

Example 2.42. Let S = Z the set of integer. P(S) be the power set of S. Consider D = {collection of all 1 × 6 subset matrices with entries from P(S)} = $\{(a_1, a_2, \ldots, a_6) / a_i \in P(S); 1 \leq i \leq 6\}$.

We know how to define ∪ and ∩ on D.

Infact we have proved {D, ∪, ∩} is a subset topological space of subset row matrices.

We define product on D; consider

a = ({3, 4, 9}, {0, 2, 9, 18}, {9}, {2, 0, –4, –9}, {7, 8, 9, 0, 6, –9}) and

b = ({1, 2, 3, 7}, {8}, {15, 24, 3} and {4, 1}, {0} ∈ D we now find

a+ b = ({3, 4, 9}, {0, 2, 9, 18}, {9}, {2, 0, –4, –9}, {7, 8, 9, 0, 6, –9}) + ({1, 2, 3, 7}, {8}, {15, 24, 3}, {4, 1}, {9})

= ({3, 4, 9} + {1, 2, 3, 7}, {0, 2, 9, 18} + {8}, {9} + {15, 24, 3}, {2, 0, –4, –9} + {4, 1}, {7, 8, 9, 0, 6, –9} + {0})

= ({4, 5, 6, 10, 7, 11, 12, 16}, {8, 10, 17, 26}, {12, 24, 33}, {6, 4, 0, –5, 3, 1, –3, –8}, {7, 8, 9, 0, 6, –9}) …I

This is the way '+' operation on D is performed and it is easily verified a + b = b + a for all a, b ∈ D and a + b ∈ D,

{D, +} is infact a subset semigroup.

We can claim {D, +} to be subset monoid by taking (0) = ({0}, {0}, {0}, {0}, {0}) ∈ D as the additive subset identity as a + (0) = (0) + a = a for all a ∈ D.

Now define × on D as follows take the same a and b in D.

a × b = ({3, 4, 9}, {0, 2, 9, 18}, {9}, {2, 0, –4, –9}, {7, 8, 9, 0, 6, –9}) × ({1, 2, 3, 7}, {8}, {15, 24, 3}, {4, 1}, {9})

= ({3, 4, 9} × {1, 2, 3, 7}, {0, 2, 9, 18} × {8}, {9} × {15, 24, 3}, {2, 0, –4, –9} × {4, 1}, {7, 8, 9, 0, 6, –9} × {9})

= ({3, 6, 9, 21, 4, 8, 12, 28, 18, 27, 63}, {0, 16, 72, 144}, {27, 135, 216}, {8, 0, –16, –36, 2, –4, –9}, {0} ∈ d.

We see {D, ×} is also a monoid of subset row matrices with I = ({1}, {1}, {1}, {1}, {1}) ∈ D.

a × I = a for all a ∈ D.

Thus {D, ×} is a commutative monoid of subset row matrices.

Clearly {D, ×, +} is a commutative topological space of subset matrices.

Nof if

M = {collection of all 3 × 5 subset matrices with entries from P(Z)}

$$= \{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} / a_i \in P(Z); 1 \leq i \leq 15\}.$$

We show how product and sum operation are performed. Infact we assume $\cup$ and $\cap$ are know operations M.

We consider

$$A = \begin{bmatrix} \{3,7,9\} & \{4,3\} & \{1\} & \{0,1\} & \{6,2,3,7,9\} \\ \{8\} & \{5,6,7,-1\} & \{3\} & \{7,9,12\} & \{3,4,0\} \\ \{9,2\} & \{0,1,2,3\} & \{4,2,0\} & \{8,0,1,4\} & \{0\} \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \{2,7,6\} & \{5\} & \{2,0,1\} & \{3,0,2\} & \{3,4,5,7,0\} \\ \{7\} & \{5,6,7,9,0\} & \{3,-1,0,2\} & \{9\} & \{8,9,0\} \\ \{0,1,2,3,4\} & \{8,0\} & \{1,0,5,9\} & \{1,0,3,4\} & \{0\} \end{bmatrix} \in M$$

$$A \cap B = \begin{bmatrix} \{\phi\} & \{\phi\} & \{1\} & \{0\} & \{3,7\} \\ \{\phi\} & \{5,6,7\} & \{3\} & \{9\} & \{0\} \\ \{2\} & \{0\} & \{0\} & \{0,1,4\} & \{0\} \end{bmatrix} \quad \text{...I}$$

$$A \cup B = \begin{bmatrix} \{3,7,9,2,6\} & \{4,3,5\} & \{2,0,1\} \\ \{8,7\} & \{5,6,7,-1,9,0\} & \{3,-1,0,2\} \\ \{0,1,2,3,4,9\} & \{0,1,2,3,8\} & \{1,5,0,2,4\} \end{bmatrix}$$

$$\left.\begin{array}{cc}
\{1,2,3,0\} & \{6,2,3,7,9,4,5\} \\
\{7,9,12\} & \{8,9,0,7,12\} \\
\{8,0,1,4,3\} & \{8,0,1,4\}
\end{array}\right] \quad \ldots \text{II}$$

$$A + B = \left[\begin{array}{ccc}
\begin{array}{c}\{5,10,9,14,13,\\ 11,16,15\}\end{array} & \{9,8\} & \{3,1,2\} \\
\{15\} & \begin{array}{c}\{10,11,12,14,5,13,15\\ 6,16,7,4,8,-1\}\end{array} & \{6,2,3,5\} \\
\begin{array}{c}\{9,10,11,12,13,\\ 2,3,5,6\}\end{array} & \{0,1,2,3,9,10,11\} & \begin{array}{c}\{0,5,1,9,4,13\\ 3,7,6\}\end{array}
\end{array}\right.$$

$$\left.\begin{array}{cc}
\{3,0,2,1,3,4\} & \begin{array}{c}\{6,2,3,7,9,5,11,10,\\ 12,13,8,14,16\}\end{array} \\
\{16,18,21\} & \{0,3,4,8,11,12,9,13\} \\
\begin{array}{c}\{8,0,1,4,9,2,5,\\ 11,3,7,12\}\end{array} & \{0\}
\end{array}\right] \quad \ldots \text{III}$$

$$A \times B = \left[\begin{array}{ccc}
\begin{array}{c}\{6,21,18,14,49,\\ 42,36\}\end{array} & \{15,20\} & \{2,0,1\} \\
\{56\} & \begin{array}{c}\{-5,-6,-7,-9,0,25,\\ 30,35,36,42,54,\\ 49,63\}\end{array} & \{9,-3,0,6\} \\
\begin{array}{c}\{0,2,4,6,8,9,\\ 18,27,36\}\end{array} & \{0,8,16,24\} & \begin{array}{c}\{4,0,20,36,\\ 2,10,18\}\end{array}
\end{array}\right.$$

$$\left.\begin{array}{cc} \{0,3,2\} & \begin{array}{c}\{18,6,9,21,27,24,8,\\ 12,28,36,30,10,15,\\ 49,63,21,35,45,42,14\}\end{array} \\ \{81,63,108\} & \{0,24,32,27,36\} \\ \begin{array}{c}\{8,0,1,4,24,3,12\\ 16,32\}\end{array} & \{0\} \end{array}\right] \quad \ldots IV$$

All the four operations are distinct.

In view of this we can have $4C_2 = \frac{4.3}{1.2} = 6$ distinct subset topological spaces of subset matrices given by;

$\{M , \cup, \cap\}$, $\{M, +, \times\}$, $\{M, \cup, \cap\}$, $\{M, \cap, +\}$, $\{M, \cup, \times\}$ and $\{M, \cap, \times\}$.

Now proceed to define min and max operation on subset 2×3 matrices with entries from P(Z).

Let N = {collection of all 2×3 subset matrices with entries from P(Z)}

$$= \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} / a_i \in P(Z);\ 1 \leq i \leq 6\right\}.$$

Consider $\quad A = \begin{bmatrix} \{0,1,2,5,7\} & \{0\} & \{5,7,9\} \\ \{3,-1,0,1\} & \{1,2,4,6,8\} & \{7,9\} \end{bmatrix}$

and $\quad B = \begin{bmatrix} \{1,4,7,8,9\} & \{9,7,8,-4\} & \{-7,2\} \\ \{9,7,1,8,6\} & \{7,2,-1,-3\} & \{8,-1,0\} \end{bmatrix}$

be two 2 × 3 subset matrices

$$\min\{A, B\} = \begin{bmatrix} \{0,1,2,4,5,7\} & \{0,-4\} & \{-7,2,5,7\} \\ \{1,3,-1,0,1\} & \{-1,-3,1,2,4,6,7\} & \{-1,0,7,8\} \end{bmatrix}$$

$$\in P(Z) \quad \ldots V$$

$$\max\{A, B\} = \begin{bmatrix} \{1,4,7,8,9,5\} & \{9,7,8,-4\} & \{5,7,9\} \\ \{1,6,7,8,9,3\} & \{7,2,4,6,8,1\} & \{7,9,8\} \end{bmatrix} \quad \ldots VI$$

It is pertinent to see {N, min} and {N, max} are subset matrix subset monoids of infinite order.

There are 6 distinct operation on N or M or for that matter for any collection of subset matrices.

We can have $6C_2 = \frac{6.5}{1.2} = 15$ distinct subset topological space of subset matrices.

If M is the collection of subset matrices with entries from the power sets P(Z) or P(Q) or P(R) or P(C) or $P(Z_n)$ then

{M, ∪, ∩}, {M, max, min}, {M, ×, +}, {M, ∪, max}, {M, ∪, min}, {M, ∪, ×}, {M, ∪, +}, {M, ∩, max}, {M, ∩, +}, {M, ∪, min}, {M, ∩, ×}, {M, +, min}, {M, +, max}, {M, ×, min} and {M, ×, max} are the 15 distinct subset topological space of subset matrices.

Now all these 15 subset topological space of subset matrices are commutative. We now give examples of non commutative operations on square subset matrices whose entries are from P(Z) or P(Q) or P(R).

$$\text{Let N} = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \Big/ a_i \in P(Z)\ 1 \leq i \leq 9 \right\}$$

= {collection of all 3 × 3 subset matrices with entries from P(Z) or P(Q) or P(R)}.

$$\text{Let A} = \begin{pmatrix} \{3,9,2\} & \{0\} & \{1,2,3\} \\ \{4,6\} & \{1,0\} & \{0\} \\ \{2\} & \{7,0,6\} & \{4,6,2,0\} \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} \{1,0,5\} & \{1,2,3\} & \{6,0,2\} \\ \{5,1,0\} & \{9,0\} & \{1,2,3,0\} \\ \{5,2\} & \{0,1,2\} & \{5,6\} \end{pmatrix} \in P(Z) \text{ or } P(Q) \text{ or } P(R).$$

We now find out usual matrix product (that is classical product).

$$A \times B = \begin{pmatrix} \{3,9,2\} & \{0\} & \{1,2,3\} \\ \{4,6\} & \{1,0\} & \{0\} \\ \{2\} & \{7,0,6\} & \{4,6,2,0\} \end{pmatrix} \times$$

$$\begin{pmatrix} \{1,0,5\} & \{1,2,3\} & \{6,0,2\} \\ \{5,1,0\} & \{9,0\} & (1,2,3,0\} \\ \{5,2\} & \{0,1,2\} & \{5,6\} \end{pmatrix}$$

$$= \begin{pmatrix} \{3,9,2\} \times \{1,0,5\} + \{0\} \times \{5,1,0\} & \{3,9,2\} \times \{1,2,3\} + \{0\} \\ +\{1,2,3\} \times \{5,2\} & \times\{9,0\} + \{1,2,3\} \times \{0,1,2\} \\ \{4,6\} \times \{1,0,5\} + \{1,0\} \times \{5,1,0\} & \{4,6\} \times \{1,2,3\} + \{1,0\} \\ +\{0\} \times \{5,2\} & \times\{9,0\} + \{0\} \times \{0,1,2\} \\ \{2\} \times \{1,0,5\{+\{1,0,6\} \times \{5,1,0\} & \{2\} \times \{1,2,3\} + \{7,0,6\} \\ +\{4,6,2,0\} \times \{5,2\} & \times\{9,0\} + \{4,6,2,0\} \times \{0,1,2\} \end{pmatrix}$$

$$\begin{matrix} \{3,9,2\} \times \{6,0,2\} + \{0\} \\ \times\{1,2,3,0\} + \{1,2,3\} \times \{5,6\} \\ \{4,6\} \times \{6,0,2\} + \{1,0\} \times \\ \{1,2,3,0\} + \{0\} \times \{5,6\} \\ \{2\} \times \{6,0,2\} + \{7,0,6\} \times \\ \{1,2,3,0\} + \{4,6,2,0\} \times \{5,6\} \end{matrix} \Bigg]$$

$$= \begin{bmatrix} \{3,9,2,0,10,15,4,5\} + \{0\} + \{5,10,15,2,4,6\} \\ \{4,6,0,20,30\} + \{5,1,0\} + \{0\} \\ \{2,0,10\} + \{7,0,6,35,30\} + \{20,30,10,0,8,12,4\} \end{matrix}$$

$$\begin{matrix} \{3,9,2,1,18,4,6,27\} + \{0\} + \{1,2,3,0,4,6\} \\ \{4,6,8,12,18\} + \{9,0\} + \{0\} \\ \{2,4,6\} + \{0,63,54\} + \{0,4,6,2,8,12\} \end{matrix}$$

$$\begin{matrix} \{18,54,12,0,6,4\} + \{0\} + \{5,10,15,6,12,18\} \\ \{24,36,0,8,12\} + \{1,2,3,0\} + \{0\} \\ \{12,0,4\} + \{7,0,6,14,12,21,18\} + \\ \{20,30,10,0,36,24,12\} \end{matrix} \Bigg]$$

$$= \left[\begin{array}{c} \{8,14,7,5,15,20,50,13,19,12,10,25,55,9,5,11 \\ 4,2,47,17,7,13,6,14,49,16,21,51\} \\ \{9,11,0,25,35,5,6,1,21,31\} \\ \{29,22,28,57,50,37,30,36,65,60,21,39,32,38, \\ 67,47,40,46,4,24,35,0,75,70,20,30,45,25,27, \\ 26,55,37,36,19,12,18,7,6,20,10,50,17,16,45, \\ 9,2,8,53,14,43,42,12,52,4,13,6,14,43,51,51,13,41\} \end{array}\right.$$

$$\begin{array}{c} \{3,9,2,6,18,4,0,27,4,10,7,19,5,28,15, \\ 11,8,20,29,12,21,30,13,22,31,24,33\} \\ \{4,6,8,12,18,13,15,17,21,27\} \\ \{2,4,6,65,67,69,56,58,60,8,10,71,12,63,62 \\ 64,71,73,75,66\} \end{array}$$

$$\left.\begin{array}{c} \{23,59,17,5,11,9,28,64,22,10,16,14,33,69, \\ 27,15,21,19,40,24,60,18,6,12,36,72\} \\ \{25,37,1,9,13,26,38,2,10,14,27,39,3,11,15,0\} \\ \{0,7,6,14,12,21,18,11,4,10,16,25,22,19,12,26, \\ 24,33,30,27,34,32,41,38,31,36,45,42,37,44, \\ 51,48,40,46,61,35,52,49,42,37,44,51,48,40, \\ 46,61,35,52,49,42,56,54,63,60,17,28,21,20, \\ 29,36,34,43,50,57,47,58,55,62,69,66,35,23\} \end{array}\right]$$

It is left as an exercise for the reader to find the classical matrix product of B × A and show B × A ≠ A × B.

However for the sake of readers satisfaction we would take a subset matrix of order 2 × 2 and show A × B ≠ B × A.

Consider $A = \begin{bmatrix} \{3,0,6,1\} & \{4,7,9,1\} \\ \{1,2,0\} & \{1,5,0,10\} \end{bmatrix}$ and

$$B = \begin{bmatrix} \{2,1,0,10\} & \{4,7,9,1\} \\ \{1,2,0\} & \{1,5,0,10\} \end{bmatrix}$$ two 2 × 2 subset matrix from

P = {collection of all 2 × 2 subset matrices with entries from P(Z) or P(Q) or P(R)

$$= \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \;/\; a_i \in P(Z) \text{ or } P(Q) \text{ or } P(R) : 1 \leq i \leq 4 \right\}.$$

We find the classical product A × B and B × A and find if A × B = B × A or not.

$$\text{Let } A = \begin{bmatrix} \{0,1\} & \{2,5\} \\ \{6,10\} & \{1\} \end{bmatrix} \; B = \begin{bmatrix} \{5,4\} & \{3,10\} \\ \{7\} & \{1,20\} \end{bmatrix} \in P.$$

$$\text{We find } A \times B = \begin{bmatrix} \{0,1\} & \{2,5\} \\ \{6,10\} & \{1\} \end{bmatrix} \times \begin{bmatrix} \{5,4\} & \{3,10\} \\ \{7\} & \{1,20\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,1\} \times \{5,4\} + \{2,5\} \times \{7\} & \{0,1\} \times \{3,10\} + \{2,5\}\{1,20\} \\ \{6,10\} \times \{5,4\} + \{1\} \times \{7\} & \{6,10\} \times \{3,10\} + \{1\} \times \{1,20\} \end{bmatrix}$$

$$= \begin{bmatrix} \{5,4,0\} + \{14,35\} & \{0,3,10\} + \{2,5,40,100\} \\ \{30,50,24,40\} + \{7\} & \{18,30,100,60\} + \{1,20\} \end{bmatrix}$$

$$= \begin{bmatrix} \{19,18,14,35,39,40\} & \begin{matrix}\{2,5,40,100,8,43,103, \\ 12,15,50,110\}\end{matrix} \\ \{37,57,31,47\} & \begin{matrix}\{19,31,101,61,38,50, \\ 120,80\}\end{matrix} \end{bmatrix} \quad \ldots \text{I}$$

$$B \times A = \begin{bmatrix} \{5,4\} & \{3,1,0\} \\ \{7\} & \{1,20\} \end{bmatrix} \times \begin{bmatrix} \{0,1\} & \{2,5\} \\ \{6,10\} & \{1\} \end{bmatrix}$$

$$\begin{bmatrix} \{5,4\}\times\{0,1\}+\{3,10\}\times\{6,10\} & \{5,4\}\times\{2,5\}+\{3,19\}\times\{1\} \\ \{7\}\times\{0,1\}+\{1,20\}\times\{6,10\} & \{7\}\times\{2,5\}+\{1,20\}\times\{1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,5,4\}+\{18,60,30,100\} & \{10,8,25,20\}+\{3,10\} \\ \{0,7\}+\{6,10,120,200\} & \{14,35\}+\{1,20\} \end{bmatrix}$$

$$= \begin{bmatrix} \{18,60,30,100,23,65,35 & \{13,11,28,23,20,18,25 \\ 105,22,64,34,104\} & 30\} \\ \{6,10,120,200,13,17, & \{15,36,34,55\} \\ 127,207\} & \end{bmatrix} \quad \ldots \text{II}$$

Clearly B × A ≠ A × B evident from the equation I and II. Thus {P, × (classical)} is a non commutative subset monoid of infinite as

$I_2 = \begin{bmatrix} \{1\} & \{0\} \\ \{0\} & \{1\} \end{bmatrix} \in$ P serves as the subset identity matrix.

Suppose $I_n = \begin{bmatrix} \{1\} & \{0\} & \ldots & \{0\} \\ \{0\} & \{1\} & \ldots & \{0\} \\ \vdots & \vdots & \ldots & \vdots \\ \{0\} & \{0\} & \ldots & \{1\} \end{bmatrix}$ is the subset identity under classical matrix product of n × n square subset matrices.

We see {P, min × (classical matrix product} is a non commutative subset topological space of subset square matrices.

Next we proceed onto describe minmax and maxmin product in subset square matrices.

Let P = {2 × 2 subset matrices with entries from P(Z)} = {collection of 2 × 2 subset matrices with entries from P(Z)}.

Let A = $\begin{bmatrix} \{3,7,8\} & \{0,4\} \\ \{1,2\} & \{5,6\} \end{bmatrix}$ and B = $\begin{bmatrix} \{2,1\} & \{1,3,0\} \\ \{0,5\} & \{7,2\} \end{bmatrix}$ ∈ P.

min{max{A, B}}

$$= \min\{\max\{ \begin{bmatrix} \{3,7,8\} & \{0,4\} \\ \{1,2\} & \{5,6\} \end{bmatrix}, \begin{bmatrix} \{2,1\} & \{1,3,0\} \\ \{0,5\} & \{7,2\} \end{bmatrix} \}\}$$

$$= \begin{bmatrix} \min\{\max\{\{3,7,8\},\{2,1\}\},\max\{\{0,4\}\{0,5\}\} \\ \min\{\max\{\{1,2\}\{2,1\}\},\max\{\{6,5\},\{0,5\}\}\} \end{bmatrix}$$

$$\begin{matrix} \min\{\max\{\{3,7,8\},\{1,3,0\}\},\max\{\{0,4\}\{7,2\}\} \\ \min\{\max\{\{1,2\}\{1,3,0\}\},\max\{\{6,5\},\{7,2\}\} \end{matrix} \Bigg]$$

$$= \begin{bmatrix} \min\{\{3,7,8\},\{0,4,5\}\} & \min\{\{3,7,8\},\{7,2,4\}\} \\ \min\{\{1,2\},\{6,5\}\} & \min\{\{1,2,3\},\{6,5,7\}\} \end{bmatrix}$$

$$= \begin{bmatrix} \{0,3,4,5\} & \{3,2,4,7\} \\ \{1,2\} & \{1,2,3\} \end{bmatrix}. \qquad \ldots I$$

min{max{B,A}}

$$= \min\{\max\{ \begin{bmatrix} \{2,1\} & \{1,3,0\} \\ \{0,5\} & \{7,2\} \end{bmatrix}, \begin{bmatrix} \{3,7,8\} & \{0,4\} \\ \{1,2\} & \{5,6\} \end{bmatrix} \}\}$$

$$\begin{bmatrix} \min\{\max\{\{1,2\},\{3,7,8\}\},\max\{\{1,3,0\}\{1,2\}\} & \min\{\max\{2,1\},\{0,4\}\},\max\{\{1,3,0\}\{5,6\}\} \\ \min\{\max\{0,5\},\{3,7,8\}\},\max\{\{7,2\},\{1,2\}\} & \min\{\max\{0,5\}\{0,4\}\},\max\{\{7,2\},\{5,6\}\} \end{bmatrix}$$

$$= \begin{bmatrix} \min\{\{3,7,8\},\{1,3,2\}\},\min\{\{2,1,4\}\{5,6\}\} \\ \min\{\{3,7,8,5\},\{7,2\}\},\min\{\{7,5,6\},\{0,4,5\}\} \end{bmatrix}$$

$$= \begin{bmatrix} \{1,3,2\} & \{2,1,4\} \\ \{2,3,5,7\} & \{0,4,5\} \end{bmatrix} \qquad \ldots \text{II}$$

Clearly I and II are distinct, hence

$\min\{\max\{A, B\}\} \neq \min\{\max\{B, A\}$.

So min max operation on subset square matrices is non commutative.

In view of this we claim {P, min, min max}, {P, max, min max}, {P, ∪, min max}, {P, ∩, min max}, {P, +, min max} and {P, ×, min max} are 6 distinct subset non commutative topological spaces.

If we take on the other hand

{P, ×-classical product of subset matrices, min max}

is defined as the doubly non commutative subset topological space as both the subset semigroups {P, min max} and

{P, ×-classical subset matrix product} are non commutative.

Now consider the operation max min on subset square matrices A and B just given earlier

max{min{A, B}}

$$= \max\{\min\{\begin{bmatrix} \{3,7,8\} & \{0,4\} \\ \{1,2\} & \{5,6\} \end{bmatrix}, \begin{bmatrix} \{2,1\} & \{1,3,0\} \\ \{0,5\} & \{7,2\} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \max\{\min\{\{3,7,8\},\{2,1\}\},\min\{\{0,4\},\{0,3\}\}\} & \max\{\min\{\{3,7,8\},\{1,3,0\}\},\min\{\{0,4\},\{7,2\}\}\} \\ \max\{\min\{\{1,2\},\{2,1\}\},\min\{\{5,6\},\{0,5\}\}\} & \max\{\min\{1,2\},\{1,3,0\}\},\min\{\{5,6\},\{7,2\}\}\} \end{bmatrix}$$

$$= \begin{bmatrix} \min\{\{3,7,8\},\{7,2,4\}\} \\ \min\{\{1,2,3\},\{6,5,7\}\} \end{bmatrix} = \begin{bmatrix} \{1,2,3\} & \{1,0,3,4,2\} \\ \{1,2,5\} & \{2,5,6\} \end{bmatrix} \quad \ldots \text{III}$$

max{min{B,A}}

$$= \max\{\min\{\begin{bmatrix} \{2,1\} & \{1,3,0\} \\ \{0,5\} & \{7,2\} \end{bmatrix}, \begin{bmatrix} \{3,7,8\} & \{0,4\} \\ \{1,2\} & \{5,6\} \end{bmatrix}\}\}$$

$$= \begin{bmatrix} \max\{\min\{\{2,1\},\{7,3,8\}\},\min\{\{1,3,0\},\{1,2\}\}\} & \max\{\min\{\{2,1\},\{0,4\}\},\min\{\{1,3,0\},\{5,6\}\}\} \\ \max\{\min\{\{0,5\},\{3,7,8\}\},\min\{\{7,2\},\{1,2\}\}\}\} & \max\{\min\{\{0,5\},\{0,4\}\},\min\{\{7,2\},\{5,6\}\}\} \end{bmatrix}$$

$$= \begin{bmatrix} \max\{\{1,2\},\{0,1,2\}\} & \max\{\{0,1,2\},\{0,1,3\}\} \\ \max\{0,3,5\},\{1,2\}\} & \max\{\{0,4\},\{2,5,6\}\} \end{bmatrix}$$

$$= \begin{bmatrix} \{1,2\} & \{0,1,2,3\} \\ \{3,5,1,2\} & \{2,5,6,4\} \end{bmatrix} \quad \ldots IV$$

III and IV are distinct hence the operation maximum is non commutative.

In view of this we can say {P, max min} is a non commutative semigroup.

Thus this leads to the fact that {P, max, max min}, {P, min, maxmin}, {P, ∪, maxmin}, {P, ∩, max min}, {P, + max min} and {P, × maxmin} are all non commutative topological spaces of subset square matrices.

Finally we will using these concepts in the following chapter to building non commutative topological spaces of linguistic square matrices.

We suggest the following problems.

SUGGESTED PROBLEMS

1. Let V be the linguistic variable related with the height of people.

   a) Is the linguistic set associated with the ling variable V a linguistic continuum or a linguistic partially ordered set or a linguistic totally ordered set?

   b) How many linguistic topological can be built using this V?

2. Let V be the linguistic variable associated with the yield of paddy.

   a) Find the linguistic set S associated with it.

   b) Is S a linguistic continuum justify your claim?

   c) Will S be a totally ordered set?

   d) Can we say S is just a finite set? Substantiate your claim!

   e) Will S be atleast a partially order set? Justify.

   f) Will S be having several linguistic continuums? Justify your claim.

3. Distinguish between the linguistic topological spaces and the classical topological spaces.

4. Characterize those linguistic variables which give way to linguistic topological spaces.

5. Enumerate all those common features enjoyed by both classical topological spaces and linguistic topological spaces.

6. Give an example of non commutative linguistic topological spaces.

7. Let V be the linguistic variable associated with age of people. Let S = [youngest, oldest] be the linguistic set associated with V.

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} \middle/ a_i \in S \right.$$

= [youngest, oldest] $1 \leq i \leq 25\}$ be the 5×5 linguistic square matrices.

i) Prove {M, max} is a linguistic commutative monoid.

ii) What is the linguistic identity of M under max operation?

iii) Prove {M, min} is a linguistic commutative monoid of infinite order.

iv) What is the linguistic identity of M with respects to min operation?

v) Compare the linguistic identities in II and IV.

vi) Prove {M, max, min} is a linguistic topological space of finite order and it is commutative

vii) Is {M, min max} a commutative operation?

viii) Prove {M, max min} is a linguistic semigroup.

ix) Is {M, max min} a ling monoid? Justify your claim.

x) Can we say {M, min max min} and {M, max, max min} are linguistic topological spaces of infinite order which is noncommutative? Prove your claim.

xi) Prove {M, min max} is a linguistic semigroup and not a linguistic monoid.

xii) Prove {M, max, min max} and {M, min min max} are non commutative linguistic topological spaces of infinite order.

xiii) Will {M, max, min, min max} be a linguistic topological space? Justify your claim.

xiv) Can {M, max, min max} the linguistic topological space have finite order linguistic topological subspaces?

xv) Can {M, max min, min max} be defined as strongly non commutative linguistic topological space? Prove or justify your claim.

8. Can we have linguistic topological spaces which are not linguistic topological square ling matrices?

9. Let V be the linguistic variable associated with height people. S = [shortest, tallest] be the linguistic variable associated with V.

$$N = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{bmatrix} \middle/ a_i \in S; 1 \leq i \leq 18 \right\}$$

be the 3 × 6 ling matrices.

i) Prove {N, min} and {N, max} all linguistic monoids of infinite order.

ii) Prove {N, min, max} is a linguistic topological space of infinite order which is commutative.

iii) Can any linguistic submonoid of {N, min} be a linguistic submonoid of {N, max}? Justify your claim.

iv) Can {N, min, max} have linguistic topological subspaces of finite order? Substantiate your claim.

v) Give four linguistic topological subspaces of {N, min, max} which are of infinite order?

vi) Prove or disprove every subinterval

I $\subseteq$ [shortest, tallest] = S is a linguistic topological subspace {M min, max} of {N, min, max}, where

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \end{bmatrix} \Big/ a_i \in I; 1 \leq i \leq 18 \right\}.$$

vii) Will {M, max} and {M, min} be linguistic sub monoids of infinite order?

viii) Will every proper subset P of finite order in S be such that

$$\{J = \begin{bmatrix} a_1 & a_2 & \dots & a_6 \\ a_7 & a_8 & \dots & a_{12} \\ a_{13} & a_{14} & \dots & a_{18} \end{bmatrix} / a_i \in P;\ 1 \leq i \leq 18\} \subseteq$$

M, min, max} be a linguistic topological subspace of {N, min, max} of finite order. Prove your claim.

ix) Will {J, max} be a linguistic submonoids of finite order? What is the linguistic matrix identity of {S, max}?

x) Study question (ix) relative to {J, min}.

xi) Find any other special features enjoyed by {J, min, max} {J given in ix}.

10. Will the linguistic variable V; colour of the eyes of internationals pave way to a totally orderable linguistic set S.

Can we say {S, max} and {S, min} can be defined?

i) If P(S) is the powerset of S; will {P(S), min} be a linguistic monoid?

ii) Will {P(S), max} be the linguistic monoid?

iii) What are the linguistic identities in the linguistic monoids given in (i) and (ii)?

iv) Will {P(S), min max} be a linguistic topological space analogous to classical topological space.

11. Find all the possible linguistic topological spaces associated with the linguistic variable height of persons.

    a) How is this linguistic variable different from the linguistic variable growth of paddy plants and its yield?

    b) Can we build ling monoids using the variable mentioned in (a)? Justify your claim.

    c) Show by examples that all linguistic variables cannot be yield itself to build algebraic structures as its associated linguistic sets.

12. Let V be the linguistic variable weight of people. S = [lowest, highest] be the linguistic set associated with it.

    Suppose W = {collection of all 9 × 9 linguistic square matrices with entries from S}.

    i) Prove {W, ∪}, {W, ∩}, {W, max} and {W, min} are linguistic monoids. Are these distinct?

    ii) Prove {W, min max} and {W, max min} are linguistic semigroups of infinite order which are not commutative.

    iii) Find lying subsemigroups of finite and infinite order in case of linguistic semigroups mentioned in (ii)

    iv) Prove {W, ∪, ∩} and {W, ∪, min max}, {W, ∩, max min}, {W, ∪, max min} and {W, ∩, min max} are 5 distinct linguistic topological spaces of square linguistic matrices.

v) Prove only {W, ∪, ∩} is a commutative ling topological space of square linguistic matrices.

vi) What is doubly non commutative linguistic topological space?

vii) Can we say {W, min max, max min} is a doubly non commutative linguistic topological space of infinite order.

viii) Obtain any other special feature enjoyed by the doubly non commutative linguistic topological space in general.

ix) Can we say {W, min max, max min} be a linguistic semiring? Justify your claim.

x) Can we say {W, ∪, min max} is a linguistic non commutative semiring of infinite order on the linguistic square matrices.

xi) Is {W, ∪, ∩} a linguistic commutative semiring of linguistic square matrices?

xii) Can we see {W, ∩, ∪} = {W, min max}? Justify your claim!

xiii) Will {W, min max} be a linguistic monoid of linguistic square matrices?

Characterize linguistic topological subspaces of {N, max, min max} where N = {collection of all 7 × 7 linguistic matrices with entries from S} where S = [lowest, highest] the linguistic set associated with the linguistic variable V; the temperature (whether) report for 10 days.

a) Find atleast 5 linguistic topological subspaces of {N, max, min max} of finite order.

b) What can be least order of the linguistic topological subspaces of {N, max, min max}?

c) Can we say {N, max, min max} has infinite number of linguistic topological subspaces of matrices? Justify your claim.

14. Obtain any other property enjoyed by linguistic topological spaces which are non commutative square matrices.

15. Is it possible to construct non commutative linguistic topological spaces in any other way other than the ones mentioned in this chapter?

16. Construct a subset topological space using P(Q) (Q the field of rationals).

i) Prove or disprove {P(Q), ∪} and {P(Q), max} are different.

ii) Will {P(Q), ∪, max} be a special subset topological space?

iv) Can we say {P(Q), ∪, max}, {P, ∪, min} and {P(Q), max, min} are subset semirings? Justify your claim.

17. Let M = { $\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$ / $a_i \in$ P(Z), P(Z) the power set of integers $1 \le i \le 6$} be the subset matrix.

i) Prove {M, ∪, ∩} is a special topological space of subset matrices.

ii) Prove {M, ∪, max} and {M, ∩, max} are also subset topological spaces of subset matrices.

iii) Can we say {M, min, max} is a subset semiring of subset matrices?

iv) Will {M, min max} be a subset topological space? Justify your claim.

v) Can we define any noncommutative operation on M?

18. Let B = {collection of all 5 × 5 subset square matrices with entries from P(R); P(R) the power set of reals R}.

i) Prove {B, max min} and {B, min max} are two distinct non commutative subset semigroups of matrices?

ii) Can we say {B, min max} and {B, max min} are subset monoids? If so find the respective identity matrices?

iii) Prove {B, ∪, min max} is a non commutative special topological space of square subset matrices.

iv) How many special topological space of square subset matrices can be built using B?

v) Will {B, ∪, min max}, {B, ∩, min max}, {B, min minmax}, {B, max, min max} be non-commutative subset semirings of subset square matrices? If no why?

vi) Can we say {B, max min, min max} is a doubly non commutative subset special topological space of square matrices.

vii) Will {B, max min, × (usual product)} be a non-commutative doubly non commutative subset special topological space of subset matrices?

viii) Will {B, min max, ×} be a subset special semiring of subset square matrices which is non commutative?

Chapter Three

SPECIAL SUBSET LING TOPOLOGICAL SPACES USING LING SQUARE MATRICES

In this chapter, we for the first time, introduce the notion of ling-special subset topological spaces using ling matrices and ling square matrices. We call them special because we do not use the classical way of dealing with ling subsets.

We show first examples of the operation on ling-subsets and then extend it to ling-subset matrices.

Example 3.1. Let V be the ling variable height of people. S = [shortest, tallest] is the ling set associated with V. Clearly S is a ling continuum and S is a totally ordered set.

Let P(S) be the power set of S. Let

A = {short, very short, tall, just tall, very tall, medium height, very very short, just short} and

B = {short, just short, very very tall, tallest, just tall} $\in$ P(S) min {A, B} = min{(short, very short, tall, just tall, very tall, medium height, very very short, just short), (short, just short, very very tall, tallest, just tall)}

= (min{short, short}, min{very short, short}, min{tall, short}, min{just tall, short}, min{very tall, short}, min{medium height, short}, min{very very short, short}, min{just short, short}, min{short, just short}, … min{just short, just short}, …, min{short, just tall}, min{very short, just tall}, …, min{just short, just tall})

= (short, very short, just short, just tall, tall, very tall, medium height, very very short) ∈ P(S) …I

max{A, B} = max{(short, very short, tall, just tall, very tall, medium height, very very short, just short), (short, just short, very very tall, tallest, just tall)}

= {max {short, short}, max{short, just short}, max{short, very very tall}, max{short, tallest}, max{short, just tall}, max{very short, short}, …, max{very short, just tall}, …, max{just short, short}, …, max{just short, just tall}}

= {short, very very tall, tallest, just tall, just short, tall, very tall, medium height} ...II

max{A, B} ∈ P(S).

Clearly max{A, B} ≠ min{A, B} as the equations I and II are distinct.

Now we use this max and min operations are ling subset matrices. It is pertinent to keep on record {P(S), max} and

{P(S), min} are subset ling monoids of infinite order. Further {P(S), min, max} is a special subset topological space which is commutative and is of infinite order.

We first illustrate how we build ling subset matrices and define max or min operations on them.

Example 3.2. Let V be the ling variable associated with age of people. S = [youngest, oldest] be the ling set which is the ling continuum associated with V. P(S) be the ling power set of the ling continuum S.

Suppose M = {collection of all 1 × 4 ling subset row matrices}
= $\{(a_1, a_2, a_3, a_4) \mid a_i \in P(S), 1 \leq i \leq 4\}$.

We define on M min and max operations.

Let A = ({young, old, very old}, {just young, just old}, {very old, old, very young}, {very old, old}) and

B = ({old, just old, middle age}, {old, young}, {very old, young}, {young, old, middle age}) ∈ N

min {A, B} = min{({young, old, very old}, {just young, just old}, {very old, old, very young}, {very old, old}), ({old, just old, middle age}, {old, young}, {very old, young}, {young, old, middle age})}

= (min{{young, old, very old}, {old, just old, middle age}}, min{{just young, just old}, {old, young}}, min{{very old, old, very young}, {very old, young}}, min{{very old, old}, {young, old, middle age}})

= ({young, old, just old, middle age}, {(just young, just old, young}, {very old, old, very young, young}, {old, young, middle age}) ∈ M …I

Thus {M, min} is ling subset row matrix semigroup, in fact a ling subset row matrix monoid

I = ({S}, {S}, {S}, {S}) ∈ M is such that min{I, A} = A for all A ∈ M.

Now for the same A, B ∈ S we find

max {A, B} = max{({young, old, very old}, {just young, just old}, {very old, old, very young}, {very old, old}), ({old, just old, middle age}, {old, young}, {very old, young}, {young, old, middle age})}

= {(max{{young, old, very old}, {old, just old, middle agge}}, max{{just young, just old}, {old, young}}, max{{very old, old, very young}}, max{{very old, young}, {young, old, middle age}})

= ({old, just old, middle age, very old}, {old, young, just old}, {very old, old, young}, {very old, old, young, middle age}) …II

Clearly I and II are distinct max{A, B} ∈ M.

Further {M, max} is a long subset matrix monoid. The ling subset matrix identity being (ϕ) = ({ϕ}, {ϕ}, {ϕ}, {ϕ}) ∈ M is such that max{A, (ϕ)} = A for all A ∈ M.

Clearly {M, min, max} is a special subset topological space of subset row matrices.

Now for this special subset topological space we can have special subset topological subspaces of both finite and infinite order. They are infinitely many.

We will provide one or two examples of the same and proceed to develop different types of operations on them.

Consider the ling subcontinuum S_1 = [middle age, oldest] $\subseteq$ S, $P(S_1)$ the ling power set of S_1; $P(S_1) \subseteq P(S)$.

Let N ={collection of all 1 × 4 ling subset row matrix with entries from $P(S_1)$} = $\{(d_1, d_2, d_3, d_4) \mid d_i \in P(S_1); 1 \leq i \leq 4\} \subseteq M$

$$= \{(a_1, a_2, a_3, a_4) \mid a_i \in P(S); 1 \leq i \leq 4\}.$$

It can be verified {N, max} is a ling subset submonoid of ling subset row matrix of infinite order of {M; max}. Similarly {N, min} is again a ling subset submonoid is ling subset submonoid of ling subset row matrices of infinite order of {M, max}. Similarly {N, min} is again a ling subset submonoid of ling subset row matrices of infinite order of {M, min}.

Thus {N, min, max} is a ling special subset topological subspace of {M, min, max} of ling submatrices of infinite order.

Having seen ling sub structure of infinite order we proceed onto give examples of ling substructures of finite order.

Consider S_2 ={young, just young, old, very old, middle age, very very young, just old} $\subseteq$ S (S_2 is a finite subset of S since S is a totally order ling set so is S_2).

$P(S_2)$ be the ling powerset of S_2 $P(S_2) \subseteq P(S)$, is also a finite order ling subsets of P(S).

Let Q = {collection of all 1×4 ling subset row matrices with entries from

$$P(S_2)\} = \{(C_1, C_2, C_3, C_4) \mid C_i \in P(S_2); 1 \leq i \leq 4\} \subseteq M$$

is a finite collection of ling subset row matrices. {Q, max} is a ling subset matrix submonoid of {M, max} and {Q, min} is again a ling subset matrix submonoid of {M, min}. Infact {Q, min, max} is a ling special topological subset subspace of ling subset matrices of finite order.

Having seen special subset topological subspaces of both finite and infinite order, we proceed to work for noncommutative structure.

Example 3.3. Let V be the ling variable associated with the weight of people. S be the ling set associated with V.

S = [lightest, heaviest] be the ling set which is a ling continuum, so S is a totally ordered set.

Let T = {collection of all 6×1 ling subset column matrices with entries from P(S); where P(S) is the ling power set of S}

$$= \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} / a_i \in P(S); 1 \leq i \leq 6 \right\}$$

{T, min} and {T, max} are ling subset monoids of subset ling column matrices {T, min, max} is a ling subset special topological space of subset ling column matrices.

We leave the task of finding finite and infinite substructure in them.

Now we give yet another example of a 5 × 3 ling subset matrices.

Let L = {collection of all 5 × 3 ling subset matrices with entries from P(S)}

$$= \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle/ a_i \in P(S); 1 \leq i \leq 15 \right\}.$$

{L, min} and {L, max} are ling subsets monoids of 5 × 3 ling matrices.

We see {L, min, max} is a special ling subset topological space of ling subset matrices.

Let D = {collection of all 3 × 3 ling subset matrices with entries from

$$P(S)\} = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \middle/ a_i \in P(S); 1 \leq i \leq 9 \right\}.$$

It is easily verified {D, min} and {D, max} are commutative ling subset square matrix monoid of infinite order.

We see {D, min, max} is clearly a ling subset square matrix special topological space of infinite order.

Only using these square ling matrices we can build non commutative special subset ling matrices of monoid and consequently build special subset topological spaces of square ling subset matrices.

Now we find min{max{A, B}} where A and B ∈ E.

where E = {collection of all 2 × 2 subset ling matrices with entries from P(S)}

$$= \{\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} / a_i \in P(S); 1 \leq i \leq 4\}.$$

Consider $A = \begin{pmatrix} \{\text{light, very light,} & \{\text{light,} \\ \text{heavy}\} & \text{very heavy}\} \\ \{\text{very heavy,} & \{\text{heavy, very} \\ \text{heavy, light}\} & \text{light}\} \end{pmatrix}$ and

$B = \begin{pmatrix} \{\text{very light, very} & \{\text{heavy, light,} \\ \text{heavy, medium weight}\} & \text{very heavy}\} \\ \{\text{light,} & \{\text{light, heavy,} \\ \text{very light}\} & \text{very light}\} \end{pmatrix}$ in E.

We find min{max {A, B}}

$$= \min\{\max\{\begin{pmatrix} \{\text{light, very light heavy}\} & \{\text{light, very heavy}\} \\ \{\text{very heavy, heavy, light}\} & \{\text{heavy, very light}\} \end{pmatrix},$$

$$\begin{pmatrix} \{\text{very light, very heavy medium weight} & \{\text{heavy, light, very heavy}\} \\ \{\text{light, very light}\} & \{\text{light, heavy, very light}\} \end{pmatrix}\}\}$$

$$= \begin{pmatrix} \begin{matrix}\min\{\max\{\{\text{light, very light, heavy}\}, \\ \{\text{very light, very heavy, medium wt}\}\}, \\ \max\{\{\text{light, very heavy}\}, \{\text{very light, light}\}\}\end{matrix} \\ \begin{matrix}\min\{\max\{\{\text{very heavy, heavy, light}\}, \\ \{\text{very light, very heavy, medium weight}\}\}, \\ \max\{\{\text{very light, heavy}\}, \{\text{light, very light}\}\}\}\end{matrix} \end{pmatrix}$$

$$\begin{pmatrix} \begin{matrix}\min\{\max\{\{\text{light, very light, heavy}\}, \\ \{\text{heavy, light, very heavy}\}\}, \max\{\{\text{light, very heavy}\}, \{\text{light, very light, heavy}\}\}\end{matrix} \\ \begin{matrix}\min\{\max\{\{\text{very heavy, heavy light}\}, \\ \{\text{heavy, light, very heavy}\}\}, \max\{\{\text{heavy, very light}\}, \{\text{light, heavy, very light}\}\}\}\end{matrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{matrix}\min\{\{\text{light, very heavy, medium wt,} \\ \text{very light, heavy}\}, \{\text{light, very heavy}\}\}\end{matrix} & \begin{matrix}\min\{\{\text{heavy, light, very heavy}\}, \{\text{light,} \\ \text{very heavy, heavy}\}\}\end{matrix} \\ \begin{matrix}\min\{\{\text{very heavy, heavy, light}, \{\text{very light,} \\ \text{light, medium wt}\}, \text{heavy}\}\}\end{matrix} & \begin{matrix}\min\{\{\text{very heavy, heavy, light}\}, \{\text{heavy,} \\ \text{light, very light}\}\}\end{matrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{matrix}\{\text{light, very light,} \\ \text{heavy, very light, medium wt}\}\end{matrix} & \{\text{light, heavy, very heavy}\} \\ \begin{matrix}\{\text{light, very light, heavy} \\ \text{medium wt}\}\end{matrix} & \{\text{heavy, light, very light}\} \end{pmatrix} \quad \ldots I$$

$\min\{\max\{A, B\}\} \in E$.

$\min\{\max\{B, A\}\} =$

$$\min\{\max\{\begin{pmatrix} \begin{matrix}\{\text{very light, very heavy,} \\ \text{medium wt}\}\end{matrix} & \begin{matrix}\{\text{heavy, light, very} \\ \text{heavy}\}\end{matrix} \\ \{\text{light, very light}\} & \begin{matrix}\{\text{light, heavy, very} \\ \text{light}\}\end{matrix} \end{pmatrix}, \begin{pmatrix} \begin{matrix}\{\text{light, very light,} \\ \text{heavy}\}\end{matrix} & \begin{matrix}\{\text{light,} \\ \text{very heavy}\}\end{matrix} \\ \begin{matrix}\{\text{very heavy,} \\ \text{heavy, light}\}\end{matrix} & \begin{matrix}\{\text{heavy, very} \\ \text{light}\}\end{matrix} \end{pmatrix}\}\}$$

$$= \left(\begin{array}{c} \min\{\max\{\{\text{very light, very heavy,} \\ \text{medium wt}\},\{\text{light, very light, heavy}\}\} \\ \max\{\text{heavy, light, very heavy}\},\{\text{very heavy,} \\ \text{heavy, light}\}\}\} \\ \min\{\max\{\{\text{light, very light}\},\{\text{light, very light, heavy}\}\}, \\ \max\{\{\text{light, heavy very light}\},\{\text{heavy, very heavy,} \\ \text{light}\}\}\} \end{array} \right.$$

$$\left. \begin{array}{c} \min\{\max\{\{\text{very light, very heavy, medium wt}\}\}, \\ \{\text{light, very heavy}\}\},\max\{\text{heavy, light, very heavy}\}, \\ \{\text{heavy, very light}\}\}\} \\ \min\{\max\{\{\text{light, very light}\},\{\text{light, very heavy}\}\}, \\ \max\{\{\text{light, heavy, very light}\},\{\text{heavy, very light}\}\}\} \end{array} \right) =$$

$$\begin{pmatrix} \begin{array}{c}\min\{\{\text{light, very light, heavy} \\ \text{very heavy, medium wt}\}\end{array} & \begin{array}{c}\min\{\{\text{light, very heavy, medium} \\ \text{wt}\},\{\text{heavy, light, very heavy}\}\}\end{array} \\ \begin{array}{c}\min\{\{\text{light, very light,} \\ \text{heavy}\},\{\text{light, very light} \\ \text{heavy}\}\}\end{array} & \begin{array}{c}\min\{\{\text{light, very heavy}\}, \\ \{\text{light, very light, heavy}\}\}\end{array} \end{pmatrix}$$

$$= \begin{bmatrix} \begin{array}{c}\{\text{light, very light, heavy,} \\ \text{very heavy, medium wt}\}\end{array} & \begin{array}{c}\{\text{light, heavy, very heavy,} \\ \text{medium wt}\}\end{array} \\ \{\text{light, very light, heavy}\} & \{\text{light, heavy, very light}\} \end{bmatrix} \quad \ldots\text{II}$$

Clearly equations I and II are distinct.

Hence $\min\{\max\{A, B\}\} \neq \min\{\max\{B,A\}\}$.

Thus we see {E, min max} is a ling subset square matrix semigroup which is non commutative and is of infinite order 1.

Now for the same A and B in E we find

max{min {A, B}} and max{min{B, A}}.

max{min{A, B}}

$$= \max\{\min\{\begin{pmatrix} \{\text{light, very light, heavy}\} & \{\text{light, very heavy}\} \\ \{\text{very heavy, light heavy}\} & \{\text{heavy, very light}\} \end{pmatrix}$$

$$\begin{pmatrix} \{\text{very light, very heavy, medium wt}\} & \{\text{heavy, light, very heavy}\} \\ \{\text{light, very light}\} & \{\text{light, heavy, very light}\} \end{pmatrix}\}\}$$

$$= \left(\begin{array}{c} \max\{\min\{\{\text{very light, light, heavy}\}, \\ \{\text{very light, very heavy, medium wt}\}\}, \\ \min\{\{\text{light, very heavy}\}, \{\text{light, very light}\}\}\} \\ \max\{\min\{\{\text{very heavy, light, heavy}\}, \{\text{very} \\ \text{light, very heavy}\}\}, \text{medium wt}\}, \\ \min\{\{\text{heavy, very light}\}, \{\text{light, very light}\}\}\} \end{array} \right.$$

$$\left. \begin{array}{c} \max\{\min\{\{\text{light, very light, heavy}\}, \\ \{\text{very light, very heavy, medium wt}\}\}, \\ \min\{\{\text{light, very heavy}\}, \{\text{light, heavy} \\ \text{very light}\}\}\} \\ \max\{\min\{\{\text{very heavy, light, heavy}\}\{\text{heavy, light,} \\ \text{very heavy}\}\}, \min\{\text{heavy, very light}\}, \{\text{light, heavy,} \\ \text{very light}\}\}\} \end{array} \right)$$

$$= \begin{pmatrix} \max\{\{\text{light, very light, medium wt, heavy}\}, \{\text{light, very light}\}\} & \max\{\{\text{light, very light, heavy}\}, \{\text{light, very light, heavy}\}\} \\ \max\{\{\text{heavy, very heavy, very light, light, medium wt}\}, \{\text{light, very light}\}\} & \max\{\{\text{very heavy, heavy, light}\} \{\text{heavy, very light, light}\}\} \end{pmatrix}$$

$$= \begin{pmatrix} \{\text{heavy, light, very light, medium}\} & \{\text{light, very light, heavy}\} \\ \{\text{heavy, light, very light, very heavy, medium wt}\} & \{\text{very heavy, heavy light}\} \end{pmatrix} \quad \ldots\text{III}$$

Clearly max{min{A, B}} $\in$ E. We find

max{min{B,A}} =

$$\max\{\min\{\begin{pmatrix} \{\text{very light, very heavy, medium wt}\} & \{\text{heavy, light, very heavy}\} \\ \{\text{light, very light}\} & \{\text{light, heavy, very light}\} \end{pmatrix},$$

$$\begin{pmatrix} \{\text{light, very light, heavy}\} & \{\text{light, very heavy}\} \\ \{\text{very heavy, heavy light}\} & \{\text{heavy, very light}\} \end{pmatrix}\}\}.$$

$$= \begin{pmatrix} \max\{\min\{\{\text{very light, very heavy, medium wt}\}, \\ \{\text{light, very light, heavy}\}\}, \min\{\{\text{heavy, light,} \\ \text{very heavy}\}, \{\text{very heavy, heavy, light}\}\}\} \\ \max\{\min\{\{\text{light, very light}\}, \{\text{light, very light,} \\ \text{heavy}\}\}, \min\{\{\text{light, heavy, very livht}\}, \{\text{very} \\ \text{heavy, heavy, light}\}\} \end{pmatrix}$$

$$\begin{pmatrix} \max\{\min\{\{\text{very light, very heavy, medium wt}\{, \\ \{\text{light, very heavy}\}\}, \min\{\{\text{heavy, light,} \\ \text{very heavy}\}, \{\text{heavy, very light}\}\} \\ \{\max\{\min\{\{\text{light, very light}\}, \{\text{light, very heavy}\}, \\ \max\{\min\{\text{light, heavy, very light}\}, \\ \{\text{heavy, very light}\}\} \end{pmatrix} =$$

$$\begin{pmatrix} \max\{\text{light, very light, heavy, medium weight}\}, \\ \{\text{light, heavy, very heavy}\}\} \\ \max\{\{\text{light, very light}\}, \{\text{light, very light,} \\ \text{heavy}\}\} \end{pmatrix}$$

$$\begin{pmatrix} \max\{\{\text{very light, light, very heavy, medium wt}\}, \\ \{\text{heavy, very light, light}\}\} \\ \max\{\{\text{light, very light}\}, \{\text{very light,} \\ \text{light, heavy}\}\} \end{pmatrix}$$

$$= \begin{pmatrix} \{\text{light, heavy, medium wt,} & \{\text{heavy, very light, light,} \\ \text{very heavy}\} & \text{very heavy, medium wt}\} \\ \{\text{light, very light, heavy}\} & \{\text{light, very light, heavy}\} \end{pmatrix} \ldots \text{IV}$$

Clearly $\max\{\min\{B,A\} \in E$.

Further the equation III and IV are distinct. Thus max min{A,B} $\neq$ max{min{B,A}}.

Hence {E, max min} is a subset ling semigroup of subset ling square matrices which is non commutative.

We see in view of this {E, max, max min} and {E, min, max min} are non commutative special subset ling topological space of ling subset square matrices.

The formal abstract definition is given in the following.

Definition 3.1. *Let V be a ling variable S be the ling set associated with this ling variable V such that S is a totally ordered ling set.*

Let η_1 and η_2 be two distinct operations defined on S. We define {S, η_1, η_2} to be doubly non commutative ling topological space if and only if

i) {S, η} is a ling noncommutative semigroup.

ii) {S, η_2} is again a ling non commutative semigroup.

We can prove the existence of such doubly non commutative ling topological spaces.

The following theorem infact proves the existence of a large class of doubly non commutative ling topological spaces.

Theorem 3.2. *Let V be a ling variable such that the ling set S associated with V is a totally ordered set.*

P(S) be the ling power set of S.

Let M = {Collection of all n × n ling subset square matrices with entries from P(S)}

$$= \{ \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & \vdots & \dots & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{bmatrix} \mid m_{ij} \in P(S);\ 1 \leq i, j \leq n\}.$$

Then {M, min max, max min} is a doubly noncommutative special subset topological space of subset ling square matrices.

Proof. Clearly {M, min max} is a subset ling semigroup of ling subset square matrices as for any A, B ∈ M; it can be clearly proved min{max{A, B} ≠ min{max{B, A}}. Hence the claim. Similarly max{min{A, B} ≠ max{min{B,A}} their by making the ling subset semigroup of square matrices to be non commutative.

Thus {M, max min, min max} is a doubly non commutative special subset ling topological space of subset ling square matrices. Hence the claim.

Now we can have special ling topological subset subspaces of both finite and infinite order.

We can have only 4 non commutative special subset ling topological spaces of ling subset square matrices give by {M, min, max min}, {M, max, max min}, {M, min, min max} and {M, max, min max}. However in case of reals or rationals using square matrices we have several of them which is elaborately discussed in chapter II of this book.

We have introduced them and described and discussed them mainly keeping in mind this very notion is new and needs some sort of repetition will help the reader to work and understand these concepts easily.

We can have only one doubly non commutative ling special subset topological space of subset ling square matrices.

On the contrary in case of ling special subset topological spaces or special ling non commutative special subset topological spaces can be many for depending on the ling variable we get them.

For instance we can build ling doubly non commutative special subset topological spaces of square ling subset matrices for long variables such as height of people, age of people, weight of people, performance aspects of students or teachers or workers. Weather report (like temperature or rainfall 'or' used in the mutually exclusive sense), density of plantation and so on.

In all these cases basically we need ling set associated with the ling variable to be a totally ordered set.

If we do not have the ling variable contributing to a totally ordered set still we can have the classical type of topological spaces constructed using the ling power set P(S) (S the ling set associated with the ling variable).

Now {P(S), $\cup$ $\cap$} will be the classical ling topological space. We call them classical as S is any set and in particular here S is a ling set. So classicality plays a vital role in its construction.

We will for the sake of completeness give some examples.

Example 3.4. Let V be the ling variable associated with the quality of mango fruits from a particular mango grew. Mangoes vary in size, sweetness, colour and so on.

Let S = {big, small, very small, very ripe, just ripe, ripe, sweet, very sweet, bitter, insipid, very big, just big, green, yellow, orange and so on} be the ling set.

Clearly S is not a totally ordered ling set. Let P(S) be the ling powerset of S.

{P(S), $\cup$, $\cap$} is the classical ling topological space.

Example 3.5. Consider the ling variable colour of the eyes of itnernationals. S = {brown, dark brown, light brown, black, green, blue, amber} be the ling set. P(S) be the ling power set of S. {P(S), $\cup$, $\cap$} is the finite order ling classical topological space.

Example 3.6. Let V be the ling variable associated with the weather in a month. S = {rain, heavy rain, light rain, drizzle, hot, very hot, just hot, neither hot, not cold, just cold, very cold, very breezy, thunderstorm and so on}.

We cannot order them fully.

However {P(S), $\cup$, $\cap$} is a ling classical topological space.

Thus having seen classical as well as non classical ling topological spaces we proceed on to suggest a few problem for the reader. By solving these problems the reader may have a better understanding of these new concepts.

SUGGESTED PROBLEMS

1. Give an example of ling variable which can yield only a classical ling topological space.

2. Give examples of ling variables which can yield only a ling topological space which is classical and is commutative but different from problem (1).

3. Give examples of ling variables which give way to special subset noncommutative ling topological spaces of finite order.

4. Consider the ling variable V temperature of weather in 24 hours. S = [oldest, hottest] be the ling continuum associated with the ling variable V.

   i) Prove {S, min max} is akin to a classical ling topological space.

   ii) Prove {P(S), $\cup$, $\cap$} can be a classical ling topological space or otherwise depending on how $\cup$ and $\cap$ are defined on elements of P(S).

   iii) Let M = {collection of 5 $\times$ 5 ling subset square matrices with entries from

$$P(S)\} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{15} \\ a_{21} & a_{22} & \dots & a_{25} \\ \vdots & \vdots & \dots & \vdots \\ a_{51} & a_{52} & \dots & a_{55} \end{bmatrix} / a_{ij} \in P(S)$$

$1 \leq i, j \leq 5\}$.

a. Prove {M, min} is a subset ling square matrix monoid which is commutative. What is its ling matrix identity?

b. Is {M, max} a ling subset square matrix monoid? Does it have a ling matrix identity?

c. Prove {M, $\cap$} is a ling monoid of subset square matrices.

d. Establish {M, $\cap$} is different from {M, max} and {M, min}

e. Prove {M, $\cup$} is a ling monoid of subset square matrices.

   Find the ling identity of {M, $\cup$}.

f. Can we say $(\phi) = \begin{bmatrix} \phi & \phi & \dots & \phi \\ \phi & \phi & \dots & \phi \\ \vdots & \vdots & \dots & \vdots \\ \phi & \phi & \dots & \phi \end{bmatrix} \in M$

   is the ling identity with respect to $\cap$?

g. Prove {M, max}, {M, min} and {M, ∩} are 3 different ling subset square matrix monoids different from {M, ∪}; the ling subset square matrix monoid.

h. Will {M, max} and {M, ∪} have the same subset ling square matrix as its ling identity.

i. Does the ling square matrices {M, ∩} and {M, min} enjoy same type of ling square identity? (Justify or prove your claim).

iv) Prove {M, max, min} is a ling topological space of subset square ling matrix.

a) Is {M, max, min} a commutative ling special topological space? Justify your claim.

v) Prove {M, ∪, ∩} is a ling topological space of subset square matrices (under usual classical intersection of subsets).

vi) Are the two ling topological spaces {M, max, min} and {M, ∪, ∩} identical or distinct? Prove your claim.

vii) Prove {M, max min} is a ling subset monoid of ling subset square matrices.

a. Is {M, max min} commutative? Prove your claim.

viii) Prove {M, max, max min} is a ling subset special topological space of ling subset square matrices?

a. Is {M, max, max min} a commutative special topological subset space? Prove your claim.

ix) Prove {M, min, max min}, {M, ∪, max min} and {M, ∩, max min} are 3 distinct subset special ling topological space of square matrices different from {M, max, max min},

x) Is {M, min, min max} a special subset ling topological space of subset square ling matrices?

a. Prove {M, min, min max} is a special subset non commutative topological space.

xi) Prove {M, min, min max}, {M, max, minmax}, {M, ∪, min max} and {M, ∩, min max} are 4 distinct special ling topological subset spaces of ling square subset matrices.

xii) Describe a doubly non commutative special subset ling topological space of ling square subset matrices.

xiii) Can we say {M, min max, max min} is a ling special subset doubly non commutative topological space of ling square subset matrices? Justify / prove your claim.

xiv) Obtain any other interesting properties associated with special subset ling topological spaces in general and in particulars subset special ling topological spaces which are non commutative and doubly non commutative.

5. Let V be the ling variable associated with the performance aspects of factory workers. S = [worst, best] be the ling continuum associated with V.

   i. Prove S is a totally ordered ling set,

   ii. Prove{S, ∪, ∩} is a topological ling space like classical topological spaces.

   iii. Is {S, min, max} a ling topological space?

   iv. Are these ling topological spaces {S, ∪, ∩} and {S, min max} identical or distinct? Prove your claim.

   v. Find finite ordered ling subtopological subspaces?

   vi. Is the number of infinite ling subtopological spaces of {S, ∪, ∩} infinite or finite? Prove your claim.

   vii. Prove {P(S), ∩, ∪} is akin to the classical topological space?

   viii. Find the differences between {P(S), ∪, ∩} and {S, ∪, ∩}.

ix. Find finite order ling subspace of {P(S), ∪, ∩},

x. Are they finite in number or infinite in number? Justify your claim.

xi. Prove there are infinite number of infinite ordered ling topological subspaces of {P(S), ∪, ∩}.

xii. Prove {P(S), min, ∪} is a ling topological space of subsets (where min is defined in the special way described earlier).

xiii. Prove {P(S), max, ∪} is also a ling subset topological space.

xiv. Can we say {P(S), max, ∪} and {P(S), min, ∩} are distinct? Justify / prove your claim.

xv. Prove {P(S), min, ∪} is a subset ling topological space which is different from {P(S), max, ∪} and {P(S), min ∪}.

xvi. Is {P(S), min, ∩} a special subset topological ling space different from {P(S), min, ∪} {P(S), max, ∪} and {P(S), max, ∩}? Substantiate / prove your claim.

xvii. Obtain any other special feature enjoyed by these four types of ling subset special topological spaces.

xviii. Let B = $\{\begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \end{bmatrix}$ / $a_i \in S$;

$1 \leq i \leq 18\}$ be the collection of ling 2×9 matrices.

a. Prove {B, $\cup$} is a ling commutative monoid. Find its ling identity matrix.

b. Is {B, max} different from {B, $\cup$} substantiate / prove your claim.

c. Is {B, $\cap$} a ling commutative monoid of 2×9 ling matrices?

d. Is {B, min} different from {B, $\cap$}? Justify/prove your claim.

xix. Prove {B, $\cup$, min} is a ling commutative topological space of 2×9 ling matrices.

a. Find ling commutative topological subspaces of {B, $\cup$, min} of both finite and infinite order.

xx. Is {B, $\cap$, min} a ling topological space of ling matrices? Justify your claim.

xxi. Is {B, $\cap$, min} different from {B, $\cup$, min}? Substantiate your answer.

xxii. Compare {B, $\cup$, max} with {B, $\cap$, max}? Are they different? If so prove it!

xxiii. Prove all the four ling topological spaces are distinct.

xxiv. If D = $\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} / a_i \in P(S); 1 \leq i \leq 10\}$

be the ling subset 5 × 2 matrices is such that {D, ∪} is a ling subset 5 × 2 matrix monoid.

xxv. Prove {D, ∪}, {D, ∩}, {D, max} and {D, min} four distinct ling subset 5 × 2 matrix monoids of infinite order

xxvi. Can {D, min, max} and {D, max min} be made into ling subset monoids of 5 × 2 ling subset matrices commutative or non commutative prove your claim?

xxvii. Suppose N = {collection of all 3 × 3 ling matrices with entries from S}

$= \{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \text{ where } a_i \in S; 1 \leq i \leq 9\}.$

a. Prove {N, min}, {N, max}, {N, ∪} and {N, ∩} are four distinct ling square matrices of infinite order which are commutative!.

xxviii. Can we prove {N, min max} and {N, max min} are two distinct noncommutative ling matrix monoids? Substantiate / Justify your claim.

a. Find ling matrix submonoids of both finite and infinite order.

xxix. Prove {N, ∪, max min}, {N, ∩, max min}, {N, max, max min} and {N, min, max min} are four non commutative ling topological space of square matrix.

xxx. Find both finite and infinite ling topological sub spaces of square matrices.

xxxi. Let T = {collection of all 4 × 4 ling subset square matrices with entries from P(S)

$$= \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{pmatrix} \right. / a_i \in P(S);$$

$$1 \leq i \leq 16\}.$$

a. Prove {T, ∪} is a ling subset square matrices monoid of infinite order.

b. How is it different from {N, ∪}?

c. Prove {T, ∩}, {T, min} and {T, max} are different ling subset square matrices monoids of infinite order which are commutative.

d. Prove {T, max min} and {T, min max} are two distinct ling subset square matrix monoids which are non commutative.

e. If A =

$$\begin{bmatrix} \{\text{worst, good}\} & \{\text{good}\} & \{\text{best, good, very good}\} & \{S\} \\ \{\text{very bad, bad, fair}\} & \{\text{very bad}\} & \{\text{good}\} & \{\text{good, bad, just bad}\} \\ \{\text{very good, bad, just bad, good}\} & \{\text{good, bad, very bad}\} & \{\phi\} & \{\text{good, worst, bad}\} \\ \{\text{bad, good}\} & \{\text{fair, good, bad, just bad}\} & \{\text{bad}\} & \{\text{bad, good}\} \end{bmatrix}$$

$\in$ T. Find A^T.

f. Prove min {max{A, A^T}} $\neq$ min {max{A^T, A}}.

g. Prove max{min {A, A^T}} $\neq$ max{min{A^T, A}}

xxxii. Prove {T, ∪, max min}, {T, max, max min}, {T, ∩, max min} and {T, min, max min} are four distinct noncommutative special ling subset topological space of subset square matrices.

xxxiii. Prove {T, ∪, min max}, {T, ∩, min max}, {T, min, min max} and {T, max, min max} are four different ling special topological space subset square ling matrices.

xxxiv. Show the four special subset topological spaces defined in xxxii and xxxii are different. Justify your claim.

xxxv. Find special subset topological subspaces of both finite and infinite order given in xxxiii.

6. Let V be the ling variable associated with the height of 20 people. Let S be the ling set associated with V.

i. Is S a totally ordered set?

ii. Will S be finite or infinite?

iii. Show {S, min} and {S, max} are ling monoids of finite order.

iv. P(S) be the power set of S. Prove {P(S), min}, {P(S), max}, {P(S), ∩} and {P(S), ∪} are four distinct ling monoids of finite order and are distinct.

(If A = {tall, very tall, short} B = {short, just short, medium height tall} ∈ P(S).

A ∩ B = {tall, short}

A ∪ B = {tall, short, very tall, just short, medium ht}

min {A, B} = {short, just short, medium height, tall}

max {A, B} = {tall, just short, very tall, short, medium height})

v. Prove {P(S), min max} and {P(S), ∪, ∩} are ling topological spaces of finite order.

vi. Will {P(S), ∪, min}, {P(S), ∪, max}, {P(S), ∩, min} and {P(S), ∩, max} be ling topological subset spaces of finite order? Justify your claim.

vii. Let T = {collection of all 3 × 5 ling matrices with entries from S}; prove {T, ∪}, {T, ∩} and {T, ∪, ∩} are ling monoids of matrices and ling topological spaces respectively

viii. If M = {Collection of all 6 × 6 ling matrices with entries from S}; prove {M, ∪}, {M, ∩}, {M, max} and {M, min} are ling commutative subset monoids of finite order. Are they distinct? Justify your claim.

ix. Prove or disprove {M, ∪, max}, {M, ∩, max}, {M, ∪, min}, {M, ∩, min}, {M, ∩, ∪} and {M, min, max} are 6 distinct ling subset matrix topological spaces of finite order.

x. Prove {M, max min} and {M, min max} are distinct ling subset non commutative monoids of finite order.

xi. Are {M, ∪, max min}, {M, ∪, max min}, {M, max, max min} and {M, min, max min} distinct non commutative ling topological spaces? Prove your claim.

xii. Show {M, min max} and {M, max min} ling non commutative monoids of square matrices of finite order? What is order of {M, min max} and {M, max min}}.

xiii. Prove {M, min max, max min} is a doubly noncommutative ling topological space of ling square matrices of finite order.

xiv. Let N = { $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$ / $a_i \in P(S)$; $1 \leq i \leq 9$}

be the ling subset square matrices

a. Will {N, ∪} {N, ∩}, {N, max} and {N, min} be 4 distinct commutative subset ling monoids of finite order?

b. Are {N, max min} and {N, min max} ling subset commutative monoids of ling subset square matrices of finite order?

c. Will {N, max, max min}, {N, max, min max}, {N, min, max min}, {N, min, min max}, {N, ∩, max min}, {N, ∩, min max}, {N, ∪, max min} and {N, ∪, min max} be ling non commutative special subset topological space of ling subset square matrices? Justify your claim.

d. Prove {N, max min, min max} is a doubly noncommutative special subset ling topological space of subset square matrices.

FURTHER READING

1. Broumi S, Ye J, Smarandache F. An extended TOPSIS method for multiple attribute decision making based on interval neutrosophic uncertain linguistic variables. Infinite Study; 2015 Jul 1.

2. Dat LQ, Thong NT, Ali M, Smarandache F, Abdel-Basset M, Long HV. Linguistic approaches to interval complex neutrosophic sets in decision making. IEEE access. 2019 Mar 4;7:38902-17.

3. Fan C, Feng S, Hu K. Linguistic neutrosophic numbers einstein operator and its application in decision making. Mathematics. 2019 Apr 28;7(5):389.

4. Fan C, Ye J, Hu K, Fan E. Bonferroni mean operators of linguistic neutrosophic numbers and their multiple attribute group decision-making methods. Information. 2017 Sep 1;8(3):107.

5. Fang Z, Ye J. Multiple attribute group decision-making method based on linguistic neutrosophic numbers. Symmetry. 2017 Jul 7;9(7):111.

6. Hall, H.S, Stevens F. H., A School Geometry, Macmillan and Company, Limited, 1921

7. Hebisch, Udo and Hanns Joachim Weinert, Semirings and Semifields, Handbook of Algebra, Vol.1, Elsevier Scienice, Amsterdam, 1996.

8. Herstein., I.N., Topics in Algebra, Wiley Eastern Limited, (1975).

9. Kamacı H. Linguistic single-valued neutrosophic soft sets with applications in game theory. International Journal of Intelligent Systems. 2021 Aug;36(8):3917-60.

10. Lang, S., Algebra, Addison Wesley, (1967).

11. Li YY, Zhang H, Wang JQ. Linguistic neutrosophic sets and their application in multicriteria decision-making problems. International Journal for Uncertainty Quantification. 2017;7(2).

12. Lidl, Rudolf, and Günter Pilz. Applied abstract algebra. Springer Science & Business Media, (2012).

13. Louis Dala, Monic and monic free ideals in a polynomial semiring, PAMS, 56, pp. 45-50, 1976.

14. Meyer A, Zimmermann HJ. Applications of fuzzy technology in business intelligence. International Journal of

Computers Communications & Control. 2011 Sep 1;6(3):428-41.

15. Peng X, Dai J, Smarandache F. Research on the assessment of project-driven immersion teaching in extreme programming with neutrosophic linguistic information. International Journal of Machine Learning and Cybernetics. 2022 Oct 10:1-6.

16. Simmons George, F. "Topology and Modern Analysis.(1963)."

17. Smarandache F, Broumi S, Singh PK, Liu CF, Rao VV, Yang HL, Patrascu I, Elhassouny A. Introduction to neutrosophy and neutrosophic environment. InNeutrosophic Set in Medical Image Analysis 2019 Jan 1 (pp. 3-29). Academic Press.

18. Smarandache F. Linguistic paradoxes and tautologies. Collect. Pap. Vol. III. 2000;3:122.

19. Smarandache, Florentin, and W. B. Kandasamy. "Special type of subset topological spaces, Educational Publisher Inc., Ohio, (2013).

20. Smarandache, Florentin, Neutrosophic Logic—Generalization of the Intuitionistic Fuzzy Logic, Special Session on Intuitionistic Fuzzy Sets and Related Concepts, International EUSFLAT Conference, Zittau, Germany, 10-12 September 2003.

21. Tian ZP, Wang J, Zhang HY, Chen XH, Wang JQ. Simplified neutrosophic linguistic normalized weighted

Bonferroni mean operator and its application to multi-criteria decision-making problems. Filomat. 2016 Jan 1;30(12):3339-60.

22. Unnisa, Iqbal, WB Vasantha Kandasamy, and Florentin Smarandache. Supermodular lattices. Infinite Study, 2012.

23. Vasantha Kandamy, W.B., Semivector Spaces over semifields, Zeszyly Nauwork Polilechniki, Vol.17, pp. 43-51, 1993.

24. Vasantha Kandasamy WB, Smarandache F, Amal K. Fuzzy Linguistic Topological Spaces. arXiv e-prints. 2012 Mar:arXiv-1203.

25. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Linguistic functions, Editorial Global knowledge, (2022).

26. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Linguistic matrices, Editorial Global knowledge, (2022).

27. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Linguistic Graphs and their applications, Editorial Global knowledge, (2022).

28. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Linguistic Geometry, Editorial Global knowledge, (2022).

29. Vasantha Kandasamy, W.B., Ilanthenral, K., and Smarandache, F., Linguistic semivector spaces, Editorial Global knowledge, (2022).

30. Vasantha Kandasamy, W.B., Linear algebra and Smaradache linear algebra, Bookman Publishing, 2003.

31. Vasantha Kandasamy, W.B., On a new class of semivector spaces, Varahmihir, J. of Math and Sci., Vol.1, pp. 23-30, 2001.

32. Vasantha Kandasamy, W.B., On chain semirings, Journal of Bangladesh Academy of Sciences, Vol.16, pp. 257-58, 1992.

33. Vasantha Kandasamy, W.B., Semivector spaces over semifields, Mathematica, 188, pp. 43-50, 1993.

34. Vasantha Kandasamy, W.B., Smarandache Semirings and semifields, Smarandache Notions Journal, Vol.7, 1-2-3, pp. 88-91, 2001.

35. Vasantha Kandasamy, W.B., Smarandache semirings, Amer. Research Press, Rehboth, 2002.

36. Vasantha Kandasamy, W.B., Smarandache Semirings, semifields and semivector spaces, Amer. Research Press, Rehoboth, 2002.

37. Vasantha Kandasamy, W.B., Zero square group semirings Buletinal Institutului Politechnic Bucuresti, Vol.11, No.3-4, pp. 7-9, 1990.

38. Vasantha, W.B., Ilanthenral, K. and Florentin Smarandche. MOD Natural Neutrosophic Subset Topological Spaces and Kakutani's Theorem. EuropaNova ASBL, 2016.

39. Wasserman, S., & Faust, K. Social network analysis: Methods and applications. Cambridge University Press, (1994).

40. Ye J. Some aggregation operators of interval neutrosophic linguistic numbers for multiple attribute decision making. Journal of Intelligent & Fuzzy Systems. 2014 Jan 1;27(5):2231-41.

41. Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning—I. Information sciences. 1975 Jan 1;8(3):199-249.

42. Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning—II. Information sciences. 1975 Jan 1;8(4):301-57.

43. Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning-III. Information sciences. 1975 Jan 1;9(1):43-80.

44. Zimmermann HJ. Description and optimization of fuzzy systems. International journal of general System. 1975 Jan 1;2(1):209-15.

45. Zimmermann HJ. Fuzzy set theory. Wiley interdisciplinary reviews: computational statistics. 2010 May;2(3):317-32.

46. Zimmermann HJ. Fuzzy set theory—and its applications. Springer Science & Business Media; 2011 Jun 27.

47. Zoltan Esik and Kuich, Wenner, Inductive * semirings, BRICS Report Series RS-00-27, Univ. of Aashrhus, Denmark, 2000.

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In this book, authors, for the first time, introduce the new notion of special subset linguistic topological spaces using linguistic square matrices. This book is organized into three chapters. Chapter One supplies the reader with the concept of ling set, ling variable, ling continuum, etc. Specific basic linguistic algebraic structures, like linguistic semigroup linguistic monoid, are introduced. Also, algebraic structures to linguistic square matrices are defined and described with examples. For the first time, non-commutative linguistic topological spaces are introduced. The notion of a linguistic special subset of doubly non-commutative topological spaces of linguistic topological spaces is introduced in this book.

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