The hidden use of new axioms

Deborah Kant

November 25, 2023

This paper analyses the hidden use of new axioms in set-theoretic practice with a focus on large cardinal axioms and presents a general overview of set-theoretic practices using large cardinal axioms. The hidden use of a new axiom provides extrinsic reasons in support of this axiom via the idea of verifiable consequences, which is especially relevant for set-theoretic practitioners with an absolutist view. Besides that, the hidden use has pragmatic significance for further important sub-groups of the set-theoretic community—set-theoretic practitioners with a pluralist view and set-theoretic practitioners who aim for ZFC-proofs. By describing this, the paper gives a more complete picture of new axioms in set-theoretic practice. These observations, for instance, show that set-theoretic practitioners interested in ZFC-proofs use tools that go beyond ZFC. The analysis is based on empirical data that was collected in an extensive interview study with set-theoretic practitioners.

Introduction

Philosophers of set theory are very interested in new axioms of set theory, typically related to the question of axiom adoption. In this article, I analyse part of set-theoretic practice by presenting the hidden use of new axioms, a specific way new axioms are used by set-theoretic practitioners. This relates to the question of axiom adoption, and I argue that the hidden use of a new axiom provides extrinsic reasons in support of this axiom via the idea of verifiable consequences, introduced by Gödel [1947]. This perspective is especially relevant for set-theoretic practitioners with an absolutist view who search for new justifiable axioms that can extend the standard theory ZFC. However, a focus on axiom adoption has two shortcomings. First, there is strong evidence that the view that extrinsic justification is valid reasoning for the truth of new axioms is not the default among set-theoretic practitioners. Rather, a part of the community is conclusively accepting ZFC and sees no need for further axiom adoption; they endorse a pluralist view. Second, a substantial proportion of set-theoretic practitioners are aiming for ZFC-proofs and are reluctant to explicitly using new axioms in proofs. One might conjecture that new axioms simply do not appear in the practice of these set theorists.

A philosophical focus on axiom adoption disregards these two areas of the reality of practising set theorists.

In a recent study of the set-theoretic community, I analysed the practice of set-theoretic practitioners from various research areas and backgrounds. One advantage of my methodology is that it is able to retrieve information on set-theoretic practices that is otherwise unavailable (i.e. not contained in published research), allowing this to be analysed for philosophical purposes. Some of these results provide the empirical basis of this article. The specific practice presented here is called *hidden use*, because the new axioms are eliminated from the proof. Besides the epistemic significance for set-theoretic practitioners with an absolutist view, such hidden use has pragmatic significance for set-theoretic practitioners with a pluralist view—*ZFC is enough and no new axioms should be adopted*—and for set-theoretic practitioners whose fundamental interest is in ZFC-proofs. The analysis, therefore, also sheds light on the parts of set-theoretic practice that are usually disregarded in philosophical discourse about the roles of axioms. This includes, for example, refuting the conjecture that new axioms do not appear in the practice of set-theoretic practitioners interested in ZFC-proofs. Moreover, from a social-epistemological perspective, the pragmatic significance is also epistemic, because it contributes to the extension of set-theoretic knowledge.

The hidden use of new axioms is a two-step procedure resulting in a ZFC-proof of some statement S. In the first step, some new axiom believed to be consistent with ZFC is used as a source of 'extra power' to prove S that is believed to be decidable in ZFC alone. At this point, set theorists learned that, if S is indeed decidable in ZFC, then S rather than its negation, $\neg S$, is provable. The second step involves an attempt to eliminate the new axiom, and if this is successful, the set-theoretic practitioner ends up with a ZFC-proof of S.

While every consistent new axiom can be used in this way, in this article, I focus on the hidden use of *large cardinal axioms*, because their use is widespread. Therefore, a large amount of data is available, and I believe initially focusing on a restricted class of axioms aids clarity. An overly general approach can also lead to some fuzziness about the detail. Large cardinal axioms are part of set-theoretic practice but they are not considered to be part of the standard axioms. I provide a summary of the data related to large cardinal axioms, which enables a proper embedding of the hidden use in the realm of different set-theoretic practices using large cardinal axioms.

The analysis is based on information gathered in an explorative interview study between 2017-2019 on set-theoretic independence with 28 set-theoretic practitioners. The interviewees, who work in various research areas and have fundamentally different views, were asked about a number of research-related topics. To guarantee anonymity, the source interviews for interview quotations are not indicated. This measure is necessary, because readers of this article probably know the interviewees and might identify them even with a small amount of information. More details on the method and important parts of the results are included in my dissertation.[1]

[1] *Independence and Naturalness in Set-theoretic Practice*, defended at the University of Konstanz in February 2023, to be published in Studies in Theoretical Philosophy, Vittorio Klostermann (2024).

The paper is structured as follows: Section 1 gives a brief review of relevant literature on the roles of axioms in mathematical practice. Section 2 summarises relevant data from the study to provide an overview of set-theoretic practices involving large cardinal axioms. The discussion in Section 3 focuses solely on the hidden use, especially of large cardinal axioms. It includes a conceptualisation of hidden use, provides two examples from published research literature, and describes the significance of the hidden use for set-theoretic practitioners. Section 4 concludes the paper and raises some open questions.

1 Studying the roles of axioms in mathematical practice

In the literature in the philosophy of mathematics, the roles of axioms are seldom considered from a purely practical point of view. Most questions relate to the issue of their justification in one way or another. While an orthodox view sees axioms as self-evident truths, this has been rejected by many philosophers. Here, Maddy highlights this development:

> [A]ssumptions once thought to be self-evident have turned out to be debatable, like the law of the excluded middle, or outright false, like the idea that every property determines a set. Conversely, the axiomatization of set theory has led to the consideration of axiom candidates that no one finds obvious, not even their staunchest supporters. [Maddy, 1988a, p. 481]

Self-evidence is a kind of intrinsic justification of axioms, and despite some problems, intrinsic justification is not off the table; today, it is usually related to an informal conception of mathematical objects (such as the iterative conception of set [Boolos, 1971]).

Easwaran proposes four necessary conditions different from self-evidence for the adoption of an axiom: An axiom should (1) be widely acceptable, (2) be useful in proving interesting consequences, (3) avoid philosophical problems, and (4) be independent from the other axioms [Easwaran, 2008, p. 387]. His main claim is that the adoption of axioms is a social practice based on the resolution of some philosophical problems while bracketing other philosophical disagreements. His points (1), (2), and (4), seem uncontroversial to me. Point (3) is more interesting. According to Easwaran, mathematical practitioners with different philosophical views and possibly for different reasons may accept the same axioms, thereby bracketing their disagreement on philosophical issues. Easwaran emphasises, moreover, that his analysis fits mathematical practice:

> [I]t seems that axioms are not chosen because they are inherently certain and let us make an uncertain result certain—they can certainly play this role, but that is not how or why they are chosen. Rather, they are uncontroversial and we use them to make a controversial result uncontroversial. I would like to suggest that this is the real role of axioms in mathematics—to stop arguing about our disagreements, and just work together on proving theorems. [Easwaran, 2008, p. 385]

Clearly, Easwaran objects to the view that axioms should be seen conclusively as self-evident statements. I agree with him on this point, as well as on the requirement that a philosophical analysis of the roles of mathematical axioms should be consistent with mathematical practice.

In the philosophy of set theory, the most prominent defender of a shift towards mathematical practice is Maddy, who reconstructs from mathematical practice the methodological principles, including the adoption of axioms, which govern mathematics (see for example [Maddy, 1997] and [Maddy, 2011]). My view regarding the focus on mathematical practice is completely in line with those of Easwaran and Maddy, but I go further still. While they are both mainly interested in the question of axiom adoption, my interest in the roles of axioms in mathematical practice is much wider and includes uses such as in the discovery process of mathematical proofs. This aligns with a social-epistemological perspective investigating the mechanisms within a scientific community that produce scientific knowledge.

That axioms play important roles in mathematical practice beside their adoption is supported by Schlimm [2013]. In his systematisation of these roles, he distinguishes between three dimensions of axiom systems—Presentation, Role, and Function—and argues that while the presentation (language and consequence relation) is fixed, role and function are usually dependent on the user and not inherent to the axiom system:

> [T]he power of axiom systems stems from the possibility of changing our perspective and using them in different ways. ... Putting forward an axiomatization does not commit mathematicians to one particular perspective. [Schlimm, 2013, p. 81]

I strongly agree with Schlimm's viewpoint, and aim to show how even the significance of very specific practices like the hidden use of new axioms varies according to the user.

2 Data on large cardinal axioms in set-theoretic practice

This section gives an overview of the relevant data included in the study regarding the roles of large cardinal axioms in set-theoretic practice. Some of the following data are probably known to people with set-theoretic expertise and are available in the research literature. Therefore, this section is brief and I then focus on data that are probably less well known. Before presenting some results of the study, I describe the participants who were interviewed. The reader might want to skip this part initially and only come back when questions about methodological details arise.

2.1 Sample set

This subsection gives some quantitative evaluations of the sample of 28 professional set theorists to show that the sample possibly represents more than 8% of the community, and that the sample is diverse in terms of research area, age, gender, location of home institution, and view on the possibility of extending ZFC.

First, regarding the size of the current set-theoretic community, there are no conclusive data on the total number of professional set theorists. But a preliminary hint at the community's size is given by the 'list of homepages of set theorists' managed by the set theorist Jean A. Larson, which lists 323 set theorists. I spoke to 8.6% of them (by adding one person to the list), and I invited 45 (13.6%) of them to participate in the study (by adding two people to the list).[2] If one extrapolates that per 43 set theorists, two are not listed, one obtains a total number of 338 set theorists, and 28 out of 338 is still 8.3%. As said, this is a preliminary evaluation owing to the lack of conclusive data.

Table 1: Distribution of research areas

Combinatorics	13
Descriptive set theory	11
Ergodic theory	4
Inner model theory	8
Forcing axioms	8
Large cardinals and forcing	8
Forcing	8
Set-theoretic and general topology	5
Cardinal characteristics	4
Determinacy and large cardinals	3
Recursion theory	3
Class forcing	2
Set theory of the reals (forcing)	2
Small research areas (very specific)	4

Second, regarding the specific research areas of the interviewees, all main research areas in set theory are represented (see Table 1). Each interviewee indicated between one and five research areas, including some additional smaller ones.

Regarding their age, the interviewees are all professional set theorists with a (past) permanent position as a professor of mathematics with a research focus on set theory. One shortcoming of the study in this respect is that the views of the younger generation are not represented. However, the study still represents different generations: Six of the interviewees obtained their PhD before 1980, four between 1980 and 1989, nine from 1990 to 1999, and nine after 1999. The years of obtaining the PhD were taken from the *Mathematics Genealogy Project*.[3]

With regard to gender, four of the interviewees (14%) are women according to my evaluation (the interviewees were not explicitly asked about their gender). The sample does not seem to be biased in this respect, because the majority of set theorists are male.

Regarding the location of their home institution, fifteen of the interviewees are affiliated to a European university, eleven to a university in the USA, and two to a university outside Europe and the USA.[4] There are groups outside Europe and the USA that were

[2]See the web page https://people.clas.ufl.edu/jal/set-theory-homepages/#I, accessed Aug 12, 2020.

[3]See https://mathgenealogy.org, accessed Nov 11, 2020.

[4]Specific locations are not indicated because of anonymity.

not included in the interviews, but there is no obvious corresponding bias, because most set-theoretic research seems to be concentrated in these Western regions.

Regarding their views on independence, eleven of the interviewees have an absolutist view and another eleven a pluralist view. Hence, the sample is not biased in this respect. The remaining six interviews do not contain a coding in one of these two categories.

These data show that the sample is broad, especially according to the range of different research areas, but also that it is limited in certain respects, for example, it captures mainly the European and US-American research context. As such it is not biased in some obviously misleading way. All in all, these data provide evidence that a broad cross-section of views are represented in the study.

2.2 Results

The interview data suggest that a majority, but not every set theorist, uses large cardinal axioms in their research. 23 of the 28 interview partners indicate that they use large cardinal axioms. This is more than those who report using forcing axioms and determinacy principles, which supports the assumption that large cardinal axioms are the most widely used new axioms in set theory.

The additional results summarised in this subsection identify different uses of as well as research questions about large cardinal axioms, and suggest consistency beliefs, occasional reluctance to use them, and partial acceptance.

Uses of large cardinal axioms. Large cardinal axioms are used by set-theoretic practitioners as tools to achieve some mathematical goal. Besides the insight into set-theoretic practices, the usefulness of new axioms is philosophically relevant, and even cited as evidence in favour of some new axioms (see, for instance, [Viale, 2019]). The following uses were each mentioned by one or more interviewees in the study. (The order of the list is arbitrary.) Large cardinal axioms are used:

∗ for certain *forcing arguments*, because they enable forcing constructions that are otherwise impossible. The research area called 'large cardinals and forcing' is dedicated to this research. An important research question in this context is how one preserves the large cardinal property.

∗ to *learn about independent statements*. Via their consequences, large cardinal axioms organise independent statements. This use is shared with other new axioms.

∗ *outside set theory*, which does not seem to be widespread, but even the rare cases are relevant. An example is the use of Vopěnka's principle in algebraic topology.

∗ for *consistency proofs* of other new axioms and principles: A theory is usually considered consistent if it is proven consistent relative to large cardinal axioms.

∗ as a *consistency measure* of new axioms and principles. As one interviewee expressed this:

> *The really surprising thing is that almost every concept that we come up with, whose consistency strength is beyond that of ZFC, aligns exactly with some large cardinal concept in its consistency strength. And this is just phenomenally bizarre. Somehow, this large cardinal concept ends up being a measuring stick for consistency strength.*[5]

* for "*methodological guidance*". A few interviewees noted that they do not use large cardinal axioms directly in their work but consider them to be useful as guiding principles. This guidance, one interviewee stated, consists in the assumption that if a statement in their research area is true assuming a large cardinal, it is probably true without. This use is discussed extensively as a case study below.

Research questions about large cardinal axioms. Large cardinal axioms are not only considered to be tools to solve problems but are also investigated as objects of study in themselves. The interviewees mentioned that set theorists investigate large cardinal axioms by asking about:

* the *structure* of large cardinals, which are, for instance, important in their use for forcing constructions.
* the *consequences* of large cardinal axioms, which is, among other things, used to evaluate the large cardinal axioms. Interviewees explained that the fact that the existence of Woodin cardinals implies generic absoluteness is itself an interesting fact about Woodin cardinals; and that the theorem, that an inner model with a supercompact cardinal would also contain all larger large cardinals, is a fascinating fact about supercompact cardinals.[6]
* the existence of a *canonical inner model* with some large cardinal in it. The existence of such an inner model is cited as the most important evidence in favour of the consistency of large cardinal axioms. The research area of 'inner model theory' is devoted to the construction of these models.

[5] Interview quotations are written in small italics.

[6] The most prominent results in this context are probably the theorems establishing a deep link between large cardinal axioms and determinacy principles. This was a major surprise at the time (in the 1980s). For instance, Larson describes this development as follows:

> In a dramatic development, the hypotheses for these results [determinacy hypotheses] would be significantly reduced through work of Woodin, Martin and John Steel. The initial impetus for this development was a seminal result of Matthew Foreman, Menachem Magidor and Saharon Shelah which showed, assuming the existence of a supercompact cardinal, that there exists a generic elementary embedding with well-founded range and critical point ω_1. [Larson, 2020, p. 6]

These results were quoted extensively by the interview partners as a major breakthrough. Set theorists were amazed that large cardinal assumptions have such neat consequences, to the extent that some of them became convinced of their truth. The quoted paper by Larson [2020] gives a detailed overview of the specific results.

* the *order* of large cardinals. The linearity of the large cardinals is also quoted as evidence in favour of their consistency.[7]
* *equivalent formulations* of large cardinal axioms, which are important for their use in proofs, because some equivalent formulations are better suited to certain applications. Moreover, it is interesting for the evaluation of large cardinal axioms in terms of plausibility or even acceptability. This research question is also posed in relation to other new axioms.

Consistency beliefs and reluctance to use large cardinal axioms. Using large cardinal axioms in the ways described presupposes some belief in their consistency. The data indeed suggest that set theorists generally believe in the consistency of large cardinal axioms, but that they, nevertheless, sometimes prefer to avoid their use.

To give more detail, no interviewee revealed any doubt about the consistency of large cardinal axioms. Some of them explained, moreover, why they think proof of inconsistency is very improbable. For example, if these axioms were inconsistent, one would obtain weird results because everything would be provable from them, one interviewee argued. Another pointed to the amount of time that has passed without any inconsistency being found. One may interpret this argument as suggesting an inductive justification for their consistency beliefs. This person concluded: "*I don't think that one can really say that large cardinal assumptions are on some sort of shaking ground or something.*" One peculiar observation is that although the existence of inner models is referred to as evidence in favour of the consistency of large cardinal axioms, consistency beliefs do not seem to be weaker for supercompact cardinals (for which no inner model has been constructed to date) than for smaller ones.

In contrast to these consistency beliefs in the set-theoretic community, the data also show that set theorists sometimes prefer to avoid the use of large cardinal axioms. One interviewee, for example, noted: "*Of course, if you prove something using some of these [large] cardinals, the question is always: Is this necessary?*"

But the data also show that a subtle evaluation takes place. For instance, one can observe that the usefulness of large cardinals works against the reluctance to use them, and that there may be more reluctance about using the large cardinal axioms than about using some of their consequences. This is expressed in the following quotation:

> *Every time I use the existence of a measurable, the existence of a supercompact, it's a big jump in faith, so to say. For me, it is like an extra effort. But it's an extra effort to the moment in which I realise that this assumption is giving me this combinatorics for free. And I feel happy to work with the given combinatorics, irrespective of where it comes from.*

Partial acceptance of large cardinal axioms. The use of large cardinal axioms in set-theoretic practice requires a belief in their consistency, but it does not require a belief in their truth or, formulated less strongly, any genuine acceptance of large cardinal axioms.

[7] Please note that there are exceptions to the linearity phenomenon.

Still, the data suggest that a substantial proportion of the community genuinely accept large cardinal axioms; some of the interviewees explicitly expressed their belief in them. The following quotation illustrates this observation:

> *Among those that are, let's say, Platonist, or have a point of view which is not too dissimilar to Platonism, I think, there [are] not [many] questions about the truth of large cardinal axiom[s].*

These results have provided the reader with some background on large cardinal axioms in set-theoretic practice. The next section is dedicated to an analysis of the use of large cardinal axioms for 'methodological guidance'. Although 'methodological guidance' is a suitable name for this use, I call it *hidden use*, because this name highlights the specific characteristic of interest: New axioms are used to prove a theorem, but they are eliminated afterwards, and therefore do not appear in the final ZFC-proof.

3 Discussion: The hidden use of large cardinal axioms

The use of large cardinal axioms for methodological guidance is only suitable in certain specific areas of set theory that are mostly interested in pure ZFC results (or results in even weaker systems). In these areas, the use of large cardinal axioms should be hidden: They should not appear in the final proof.

These areas are typically descriptive set theory, forcing on the reals, set-theoretic and general topology, or cardinal characteristics. In descriptive set theory, for example, interviewees gave very clear statements like: *"For descriptive set theory, as you know, we typically prove results that hold in ZF even."* or:

> *Most of the time when I'm working on something, I have built-in faith that it's not independent from ZF+DC. But that's just where I tend to work. Every so often, something comes up that's further out and then I don't know so much. And then, occasionally, something comes up where one can just see pretty quickly that there's some independence going on. But I would say 95% of the time I go on thinking that there is no chance that there is any independence phenomenon there, and I can't recall being wrong.*

However, one major conclusion of this article is that, nevertheless, new axioms that go beyond ZFC are used by set-theoretic practitioners in these areas. One of my interview questions asked about axioms used besides the ZFC axioms, here posed to a descriptive set theorist:

> *I: Which axioms do you use in your work apart from ZFC axioms? ...*
> *IP: So, I suppose, the Axiom of Determinacy plays a role, also weaker versions like Projective Determinacy. And occasionally Martin's Axiom comes in, ... either because it allows you to generalise that result that you have without ... .*[8] *Or, it*

[8] Taken out of the quote: *"perhaps [about] countable trees, or this could be about sets that are described using countable trees. And then, you might be able to generalise them to trees of higher cardinality, or sets described by trees of higher cardinality."*

> *can come in on a rare occasion, I think, in my research more, as opposed to my results. For what can happen sometimes is that in sort of desperation to try to get traction on a problem, ... I might say to myself that, well, suppose now, we had Martin's Axiom, and I could actually meet these κ many dense sets—and I'm well aware that I've now moved past /,*[9] *I want to prove something that should be just a ZFC theorem—but, in desperation, I might be looking for the extra power. And then, if it is a good idea, you might end up realising, well, you never really needed to meet more than countably many sets. So, it actually was a ZFC theorem.*

In their remark, this person elaborated on the use of Martin's Axiom, but they added that *"the extra power is more typically something like determinacy."*

So, although I focus on the hidden use of large cardinal axioms, forcing axioms and determinacy principles can also be, and indeed are, used in the same way.

Another interviewee working in descriptive set theory who is also an expert on forcing reflected on their use of large cardinal axioms:

> *Certainly, large cardinal axioms are always there. They may not be present in the work that I produce, but they at least serve as a methodological guide. I think it's extremely useful to have them; at least to have them as a methodological guide. Of course, if you actually use them in your work, then non-set theorists will take a dim view of it. So, it's better not to do it. But they're helpful on a methodological level.*

A third interviewee, working in set-theoretic and general topology, expressed a similar idea.

What is it that set-theoretic practitioners learn by using the *extra power* of a new axiom that can then be eliminated? If they assume that the statement in question is very probably decidable within ZFC, why don't they try to find a ZFC-proof directly? Let me present an illustrative quote about this stage of the proof-finding procedure:

> *If I start new at a problem, I really have to do this, we call it the Magidor strategy because he advocates it: On the even calendar day, you try 'yes', and on the odd day, you try 'no'. You should not waste time on one direction because the opposite might be true. And as long it's far from intuition, one really has to proceed both [ways]. Sometimes, I also have to give up after certain months or so. Then, I say 'I can't afford any more to work in vain on such and such problem.' I don't give up forever but sometimes for some years or so.*

At this stage of the proof-finding procedure, set-theoretic practitioners are tackling the question of whether some statement is probably true or not, and if they use an additional axiom to prove the statement or its negation, they have learned quite a lot. They can now abandon the *Magidor strategy*, because they know in which direction they should proceed. If they can prove some statement S using a new axiom, and they believe that S is not independent of ZFC, they can now try to find a proof of S without wasting any more thought on the possibility that the negation might be true.

---

[9] The sign '/' marks that the sentence was not finished.

3.1 Conceptualisation

The idea of hidden use is that assuming that some statement S is not independent of ZFC ($\mathrm{ZFC} \vdash S \vee \mathrm{ZFC} \vdash \neg S$), one first uses the *extra power* of an additional new axiom to prove either S or its negation, say S: $\mathrm{ZFC} + \mathrm{NA} \vdash S$ or, equivalently: $\mathrm{ZFC} \vdash \mathrm{NA} \rightarrow S$.[10] Subsequently, because it is assumed that S is not independent, set theorists try to eliminate the use of the new axiom in the proof, and if they succeed, they provide a ZFC-proof: $\mathrm{ZFC} \vdash S$.

If, for some statement S, set-theoretic practitioners assume that S is decidable in ZFC, then the following scenario is possible and conceptualises the hidden use of new axioms:

Assumption:	$\mathrm{ZFC} \vdash S \vee \mathrm{ZFC} \vdash \neg S$	*Question*: Is S true or false?
Step 1:	$\mathrm{ZFC} + \mathrm{NA} \vdash S$	*Conjecture*: S is true.
Step 2:	$\mathrm{ZFC} \vdash S$	*Proof of the conjecture and confirmation of the assumption.*

Note that hidden use does not reveal any logical connection between the new axioms and the statement S. For, if S was proven in ZFC, then it trivially holds that $\mathrm{ZFC}+\mathrm{NA} \vdash S$. The characteristic of hidden use is rather that there is a time when this proof is non-trivial because the new axioms are directly used.

One further remark on the assumption in the framework: In some cases, as described in the quotes above, bringing their extensive experience to bear, set-theoretic practitioners are sure that the statement S is not independent. Subsequently, they indeed succeed in finding a ZFC-proof of either S or its negation. In other cases, the assumption is rather a second question—Is S independent or not? This question can be answered in the negative by showing either that the use of the new axiom was actually necessary or that the negation of S is consistent. When discussing the examples in the next subsection, I come back to this possibility.

3.2 Exemplary proofs

The idea of the hidden use of a new axiom is that it is used in the discovery process of a theorem but not in the final proof and that it guides the search for a proof but can be eliminated in the end. Described like this, of course, it is very hard to find instances of hidden use, unless one asks practitioners for examples. Because the final proof does not mention the new axiom, as a reader of a final ZFC-proof, one does not learn about the new axiom's use in earlier stages. Because of this methodological difficulty, I draw on instances in which the first proof was published too. The two examples are Borel Determinacy and Cichoń's maximum, which were both first proved using large cardinal

[10] Wherever I refer to proofs, please note that I mean what De Toffoli introduced as *simil-proofs*, i.e. "arguments that look like proofs to the relevant experts" [De Toffoli, 2020, p. 824].

axioms.[11] They are from different times and research areas, and are about different kinds of statements. The ZFC-proof of Borel Determinacy was published in 1975. It is a plain statement about a property of sets of reals: *All Borel sets are determined.* The ZFC-proof of Cichoń's maximum was published in 2022. It is a consistency statement about separating the cardinal characteristics of the continuum.

3.2.1 Borel Determinacy

In this section, I summarise the historical development of Borel Determinacy. In 1953, Gale and Stewart studied two-player games on sets of reals and asked: "[H]ow pathological must the set [of reals] be for the game to be indeterminate?" [Gale and Stewart, 1953, p. 246].[12] They showed that sets on the first level of the Borel hierarchy, open sets, are determined. Some years later, in 1970, Martin proved that analytic sets are determined using a large cardinal axiom in his proof:

> We assume the existence of a measurable cardinal and prove that every analytic set is determinate. Our proof is fairly simple and makes a very direct use of the large cardinal assumption (we present it in terms of a Ramsey cardinal) and the fact that open games are determined. [Martin, 1970, p. 287]

Since every Borel set is analytic, Borel Determinacy holds too on the assumption of a measurable cardinal.

At the same time, Friedman [1971] showed that Borel Determinacy is independent of a fragment of ZFC that he calls Z, which excludes in particular the replacement schema.[13] A few years later, Martin showed that Borel Determinacy is provable in ZFC, so the measurable cardinal is not necessary. With reference to Friedman's work, he noted:

> Borel Determinacy is probably then the first theorem whose statement does not blatantly involve the axiom of replacement but whose proof is known to require the axiom of replacement. [Martin, 1975, p. 364]

In terms of our conceptualisation, in 1970, there was a proof of Borel Determinacy assuming a large cardinal axiom, but only a few years later, in 1975, there was a pure ZFC-proof of Borel Determinacy. In the meantime, set theorists were elaborating on details of the determinacy of sets on the Borel hierarchy and developing proving techniques.

[11] I owe the first example to Philipp Schlicht.

[12] I assume that the readers of this volume do have some set-theoretic expertise. But for comprehensiveness, a few words on determinacy. Determinacy is a property of sets of reals and of the related games. Let us take a set of reals and imagine two players who play a game in which they alternately choose natural numbers. This game results in an infinite sequence of natural numbers, which in turn refers to a real number. Now, player I wins the game if this real number is an element of the given set, otherwise player II wins. If one of the two players has a winning strategy, then we call the game as well as the set of reals *determined*.

[13] Martin's and Friedman's articles were written around the same time. Each refers to the other's work as unpublished.

The question remained: Is the large cardinal assumption necessary to prove Borel Determinacy? Friedman showed that Z is not sufficient to prove it, so it had to be something stronger. And finally, Martin found a ZFC-proof. A proof making direct use of a large cardinal axiom could be replaced by a proof in ZFC.

It is plausible to assume that, in this case, the question of whether or not Borel Determinacy is independent remained open throughout the process. Martin told me in a private communication: "When I learned about Friedman's theorem, I began a long attempt to find out whether or not Borel Determinacy is provable in ZFC. If my memory is correct, I mainly tried to prove that the answer is yes. I spent little time trying to show that Borel Determinacy could not be proved in ZFC." So, he conjectured that Borel Determinacy might be provable but was probably still aware that it might also be independent. If we consider the development regarding the stronger statement of Analytic Determinacy, then we move on to the alternative case, in which the use of some large cardinal strength was indeed found to be necessary. In 1978, Harrington [1978] proved that Analytic Determinacy implies $0^\sharp$. Hence, in parallel to solving the question of whether Borel Determinacy is true or not, the question of whether or not Borel Determinacy, resp. Analytic Determinacy, is indepedent from ZFC was solved.

3.2.2 Cichoń's maximum

In the case of proving Cichoń's maximum, a proof making direct use of large cardinal axioms could also be replaced by a proof in ZFC:

> It is consistent that all entries of Cichoń's diagram are pairwise different (apart from $\mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M})$, which are provably equal to other entries). However, the consistency proofs so far required large cardinal assumptions. In this work, we show the consistency without such assumptions. [Goldstern et al., 2022, p. 3951]

In an earlier paper from 2019, Goldstern, Kellner, and Shelah [Goldstern et al., 2019] assumed the existence of four compact cardinals to prove the consistency of the following statement: $\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}$.[14] They used the technique of forcing, and the compact cardinals provide Boolean ultrapower embeddings, which enable the construction of the respective forcing notions. The question of Cichoń's maximum is whether the cardinal characteristics of Cichoń's diagram can be separated simultaneously (respecting the two ZFC-provable equalities in the diagram), and the above-mentioned authors proved that they can.

In their first proof, the large cardinals are directly used for the forcing constructions. But the authors asked: "Can we prove the result without using large cardinals?" [Goldstern et al., 2019, p. 139], and (revealing the temporal incoherence between knowledge

[14] Again, for comprehensiveness, a few words on the content of this statement. The cardinal numbers in this inequality denote *cardinal characteristics*. In ZFC, it is provable that all of them are greater than $\aleph_0$ and less than or equal to $2^{\aleph_0}$. In research on cardinal characteristics, many forcing notions have been found that can show that certain pairs of cardinal characteristics can be separated. This question was subsequently generalised by trying to separate simultaneously more than two cardinal characteristics.

and published knowledge) the authors referred to a draft paper in which they had proved that this is indeed possible. In this paper, published in 2022, Goldstern, Kellner, Mejía, and Shelah "introduce a new method to control cardinal characteristics ... . This method can replace the Boolean ultrapower embeddings in previous constructions, so in particular [they] can get Cichoń's maximum without assuming large cardinals" [Goldstern et al., 2022, p. 3953].

The example of Cichoń's maximum is another prototype for the hidden use of large cardinal axioms. It is interesting to note, moreover, that the authors directly referred to this use when they conjectured that the large cardinal assumptions should be eliminable in their proof:

> It seems unlikely that any large cardinals are actually required; but a proof without them would probably be considerably more complicated. It is not unheard of that ZFC results first have (simpler) proofs using large cardinal assumptions. [Goldstern et al., 2019, p. 116]

The authors considered it a common pattern that a ZFC result is sometimes first proved assuming large cardinal axioms.

3.3 Significance of hidden use for set-theoretic practitioners

The set-theoretic community is not a homogeneous group of scholars with similar research interests and framework beliefs on the nature of mathematics. Therefore, I present the various perspectives of different sub-groups of the set-theoretic community on the relevance of the hidden use of new axioms. The three sub-groups to be considered are set-theoretic practitioners with an absolutist view, set-theoretic practitioners with a pluralist view, and set-theoretic practitioners whose fundamental interest is in ZFC-proofs. We have already heard several voices from this latter sub-group above. The sections that follow start by characterising each of the first two sub-groups, and then present the significance of the hidden use for them.

3.3.1 Extrinsic justification of new axioms

Hidden use has an epistemic significance in the context of axiom justification. I discuss this claim and argue that a successful hidden use is a case of the kind of verification that is suggested by Gödel. From the perspective of set-theoretic practitioners with an absolutist view, this is important. They believe that ZFC should be extended by new axioms, and these axioms should typically be justified by convincing, extrinsic reasons.

Absolutist views of set-theoretic practitioners. The convictions of set-theoretic practitioners with an absolutist view are varied. Some believe in the existence of a set-theoretic universe, some believe that set theory being a mathematical foundation implies that every sentence must be either true or false, and others simply believe that the independence phenomenon is surmountable by adding new axioms to ZFC. Examples of these views are presented below.

An absolutist view is often related to a realist position, for instance when set-theoretic practitioners talk about a *"real set-theoretic universe"* or describe set-theoretic research as follows:

> *We are actually describing a reality, which is the mathematical world or set-theoretic world, which is real, as real as the physical world. We are acquiring real knowledge about something, which we don't know quite yet what it is but / .*

With regard to independence proofs (comprising two consistency proofs),[15] set-theoretic practitioners with an absolutist view often consider searching for new axioms to be more valuable than proving consistency results: *"The most interesting consistency results have been proved and in my perspective it's better to try to find new arguments to choose among the many possibilities which is the right one"*. They are *"more interested in searching for new axioms and the like than in finding out what is consistent with ZFC"*. This research goal is consistent with the view that ZFC axioms are not sufficient: *"We know [ZFC] doesn't tell you enough; there's not enough to settle important problems like CH and so on"*.[16]

Gödel's idea of verification and extrinsic justification. Gödel proposed that new axioms may be justified not only intrinsically but also extrinsically via their consequences:

> [D]isregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its 'success,' that is, its fruitfulness in consequences and in particular in 'verifiable' consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs. The axioms for the system of real numbers, rejected by the intuitionists, have in this sense been verified to some extent owing to the fact that analytical number theory frequently allows us to prove number theoretical theorems which can subsequently be verified by elementary methods. A much higher degree of verification than that, however, is conceivable. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems ... that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory. [Gödel, 1947, p. 521]

There are two points worth noting in this quotation. First, for Gödel, proofs of "verifiable consequences" that use new axioms are much simpler, but these consequences can be demonstrated without the new axiom. Second, his examples are number theoretical consequences of the theory of real numbers, which were later proven by elementary means.

[15] To prove that a statement A is independent of ZFC, one must prove that ZFC is consistent with A and also with $\neg A$.

[16] CH denotes the famous continuum hypothesis.

The first point directly applies to the hidden use of large cardinal axioms. A first and simpler proof that uses some large cardinal axiom was later replaced by a somewhat more complicated proof without assuming large cardinal axioms. Recall the quotation in the example of Cichoń's maximum: "It is not unheard of that ZFC results first have (simpler) proofs using large cardinal assumptions" [Goldstern et al., 2019, p. 116]. Gödel's example is also analogous to our examples: Consequences of a stronger theory were later verified by the weaker theory.

Philosophers of set theory interested in the extrinsic justification of axioms have tried to explicate the notion of the fruitfulness or success of an axiom. In this context, verifiable consequences support the notion that an axiom is fruitful. Therefore, people tried to explicate *verification*. In an analysis of Gödel's works on independence, van Atten and Kennedy explicated verifiable consequences as arithmetically verifiable consequences [van Atten and Kennedy, 2009, p. 341]. In the above quotation, the example refers to such consequences. Hidden use, however, would not correspond to this conception, because ZFC-provable statements are not necessarily arithmetically verifiable. The examples given above do not count as such arithmetical consequences. Borel Determinacy is about sets of reals, and Cichoń's maximum is about cardinal characteristics, which are both set-theoretic subject matters. But it does not seem necessary to restrict the notion of verifiable consequences conclusively to arithmetical consequences and non-set-theoretic subject matters. Gödel does not reject notions of verifiability that go beyond arithemetical consequences, and van Atten and Kennedy also refer to Borel Determinacy as an example:

> Borel Determinacy is a 'verifiable consequence,' in Gödel's sense of the phrase here, i.e., it was proved without using measurables, and the measurables in turn were verified by their having led to the 'correct' result." [van Atten and Kennedy, 2009, p. 343]

Hence, in their explication, van Atten and Kennedy do not actually restrict verifiable consequences to arithmetical consequences. In conclusion, I consider it a reasonable proposal to explicate 'verifiable consequence' as 'ZFC-provable consequence'. One final observation is that it is only the consequences of an axiom that can be verified and never the new axiom itself (if it really goes beyond ZFC).

The upshot of this subsection is a proposal for the clarification of an important aspect of extrinsic justification. I argue that the hidden use of large cardinal axioms identifies verifiable consequences of large cardinal axioms and that 'verifiable consequences' of an axiom are the ZFC-provable consequences. In other words, ZFC is considered to be a mathematical theory that is suitable for the verification of more uncertain components of a possible theory extension. Every such verifiable consequence of a large cardinal axiom is considered to be an extrinsic reason in favour of this axiom.

Hidden use compared to Martin's mathematical evidence. Similar observations concerning justification and verification are provided by Martin [1998].[17] In his chapter,

[17] Thanks to the anonymous referee who pointed me towards this reference.

Martin presents two mathematical examples which, according to him, "count[ ] as evidence for mathematical truth" [Martin, 1998, p. 215] of determinacy principles such as Projective Determinacy. Similar to the hidden use of new axioms, the determinacy principle is used in addition to ZFC to prove some general statement. In Martin's examples, specific instances of this more general statement are verified, and so, Martin argues, the examples provide a case of prediction and confirmation. The determinacy principle predicts a general statement and all specific instances of it that were 'tested' are confirmed.[18] Martin's first example is the Cone Lemma, which states that if AD(PD/BD) holds, then every (projective/Borel) set of Turing degrees either contains a cone or its complement does. Martin comments:

> When I discovered the Cone Lemma, I became very excited. I was certain that I was about to achieve some notoriety within set theory by deducing a contradiction from AD. In fact I was pretty sure of refuting Borel Determinacy. I had spent the preceding five years as a recursion theorist, and I knew many sets of degrees. I started checking them out, confident that one of them would ... give me my contradiction. But this did not happen. For each set I considered, it was not hard to prove, from the standard ZFC axioms, that it or its complement contained a cone. ...
> I take it to be intuitively clear that we have here an example of prediction and confirmation. [Martin, 1998, p. 224]

The Cone Lemma was proved in 1968, and as the quote makes clear, there is a close relation to the developments concerning Borel Determinacy. Martin's attempts to refute Borel Determinacy, as described here, happened before he had proved that the existence of measurable cardinals implies Analytic Determinacy. But after his proof and Friedman's result [Friedman, 1971], he tried to prove Borel Determinacy. This reconstruction of events only strengthens the epistemic significance of the use of the measurable in proving Borel Determinacy in 1970, which seemed to have strongly supported the conjecture that Borel Determinacy might actually be provable in ZFC.

It is worth making a final note on Martin's examples and their relation to the hidden use of axioms. The examples are similar but not quite the same. The difference is that the use of the new axiom is not eliminated in Martin's examples as it is in hidden use. In the Cone Lemma, the conclusion is just as strong as the assumption. If one assumes AD, then the conclusion holds for every set of Turing degrees; if one assumes Projective or Borel Determinacy, then the conclusion only holds for Projective or Borel sets of Turing degrees. So, no eliminable extra power was used to prove the conclusion for Borel sets.

The characteristic of Martin's examples is rather that a general implication schema was proved, and that the conclusion could be proven for many specific instances without the assumption of some strong determinacy principle. This is what he desribes in the quote above. Hence, he argues that this verification of the specific instances provides evidence

[18] In the first version of this chapter, I said 'prediction-use' instead of 'hidden use'. I changed my mind, because although one characteristic of hidden use is indeed that the truth of S is predicted, the more important characteristic is that the new axiom can be eliminated.

of the general stronger determinacy principle that predicted all specific instances. (Of course, since AD contradicts AC, Martin does not consider his examples to be evidence in favour of AD, but, with hindsight, in favour of PD/$\mathrm{AD}^{L(\mathbb{R})}$.)

Consistency or truth? This paragraph addresses the reviewers' requests for more clarity regarding the consistency or truth of axioms. I think that this can be clarified by again considering different sub-groups of the set-theoretic community. One of my main observations of set-theoretic practice is that the community is heterogeneous when it comes to the framework beliefs of its members. Regarding more foundational questions on set-theoretic axioms, it is simply disingenuous to make statements of the sort: 'Set-theoretic practitioners think this or that'. Such a statement presupposes a homogeneity that is not found in reality.

In this respect, an important distinction is the one between consistency beliefs about axioms on the one hand and actual beliefs in axioms or beliefs in the truth of axioms on the other. When I talk of *believing axioms* (in the same sense that Maddy [1988a] uses the notion), this is actual belief in axioms and can be identified with believing in the truth of axioms. Scholars who prefer to avoid completely the notion of truth in the context of mathematical statements can use the notion of 'belief in axioms' instead.

If we now consider set-theoretic practitioners with an absolutist view, then we observe that they believe in the ZFC axioms, and most of them believe in large cardinal axioms and in Projective Determinacy. This is not only about believing in the consistency of these axioms but about believing them to be true. After all, the philosophical question of a justification of new axioms does not ask whether they are consistent—consistent new axioms would not solve the continuum hypothesis or any other question that is not answered in ZFC—it asks about the truth of these axioms. For members of this sub-group of the set-theoretic community, hidden use might have provided evidence in favour of the consistency of new axioms years ago when questions of consistency were more controversial. But, today, the question about new axioms for set-theoretic practitioners with an absolutist view is not about their consistency but about their truth, and, thus, hidden use is, as described above, a good way to provide support for the truth of axioms.

3.3.2 Pragmatic significance

In this section, I describe the pragmatic significance of the hidden use of new axioms. Because the significance of hidden use in providing extrinsic evidence is not relevant for set-theoretic practitioners without an absolutist view, pragmatic significance here refers specifically to their view on hidden use. But, of course, pragmatic significance is relevant for set-theoretic practitioners with an absolutist view, too.

Hidden use is pragmatically relevant for set-theoretic practitioners with a pluralist view, but also, as described above, for set-theoretic practitioners who are interested in ZFC-proofs. These two sub-groups are not identical. Set-theoretic practitioners aiming for ZFC-proofs can have absolutist or pluralist views, or indeed neither. Let me present some observations from set-theoretic practitioners with a pluralist view.

Pluralist views of set-theoretic practitioners. For set-theoretic practitioners with a pluralist view, the idea of a set-theoretic universe does not make sense. They say, for instance: *"I don't really know what is meant by it"*, and they think that *"we don't have a universe of ZFC anyway"*.

Typically, set theorists with a pluralist view are not annoyed by the independence phenomenon: *"The thing is, I guess, the independence phenomenon doesn't bother me"*; for them, independence is a *"fact of life"*. This acceptance of the current situation of set-theoretic independence as somehow settled is also reflected in their views on ZFC as the right and conclusive theory for sets—*"in some sense, you could say I believe in ZFC"*—that, according to them, does not need to be extended by further axioms: *"I don't really feel the need to sort of choose between axiom systems either.*[19] *They're all out there and you can study all of them, and I think that's all worthy of study."*

Set-theoretic practitioners with a pluralist view are usually convinced that no new axioms beyond ZFC will be accepted by the mathematical community: *"If I had a guess, I don't really expect any axioms beyond ZFC to be accepted and have this status in the mathematical community that the axioms of ZFC do"*.

Furthering set-theoretic progress. The pragmatic significance of the hidden use of new axioms is simply that it leads to new theorems that extend set-theoretic knowledge. The axioms are useful in this sense because they provide support during the proof-finding process. Other things are needed to end up with a finished proof. For example, the direct use of the new axioms in the first proof might inform the final ZFC-proof, but does not necessarily do so. In some cases, an essentially new proof idea is needed.[20] Of course, in the first case, the new axiom is even more useful.

Regarding consistency and truth, the pragmatic significance of the hidden use of new axioms does not relate to any question about the truth of new axioms, but it crucially involves consistency beliefs. Regarding the conceptualisation given in 3.1, if a new axiom is not taken to be consistent and $\mathrm{ZFC} + \mathrm{NA} \vdash S$ is proven in **Step 1**, then one cannot conjecture that S is true, because NA could be inconsistent and S could be false. In this case, the new axiom would not be of any help; only new axioms believed to be consistent are actually useful. The pragmatic significance shows how new axioms are useful even for set theorists who do not want to go beyond ZFC, but the new axioms are only as useful as far as they are believed to be consistent. In the case of large cardinal axioms, in 2, I presented some data supporting the view that set-theoretic practitioners do not doubt the consistency of large cardinal axioms, which in turn supports their pragmatic usefulness.

Regarding the epistemic value of hidden use, I want to add that pragmatic significance is also an epistemic significance from the social-epistemological viewpoint. Although for set-theoretic practitioners who do not believe in extrinsic justification, the hidden use of new axioms does not provide reasons justifying new axioms as true, it nevertheless

[19] Because set theorists work with various axiom systems that extend ZFC, the notion of *"axiom system"* in this quotation can be understood as referring to extensions of ZFC and not to axiom systems in general.

[20] Thanks to Benedikt Löwe for raising this point in discussion.

leads to the extension of set-theoretic knowledge by answering valuable research questions among set-theoretic practitioners, because new true, valuable theorems are epistemically significant.

4 Conclusion

By analysing interview data on set-theoretic practices, this article provides insight into a novel role of axioms in mathematical practice. The case study of the hidden use of large cardinal axioms presents publicly unavailable information on mathematical practices. Moreover, this case study is relevant for the ever-growing research on mathematical practices as well as for the literature on the justification of mathematical axioms.

The set-theoretic community is rather heterogeneous when it comes to more foundational beliefs. Some set theorists have an absolutist view and accept the extrinsic justification of new axioms, while other set theorists have a pluralist view and do not accept extrinsic justification. A third relevant sub-group are set-theoretic practitioners who are aiming for ZFC-proofs. I have showed in this article that the hidden use of large cardinal axioms has significance for all of them. It can provide extrinsic reasons in favour of new axioms and it can be used in the discovery process of a ZFC-proof. Easwaran argued that mathematicians adopt axioms to bracket their philosophical disagreements, and I add the related observation that mathematicians agree on the usefulness of axioms while bracketing philosophical disagreements. The hidden use of new axioms is signficant either way. My analysis, moreover, illustrates Schlimm's thesis that using "an axiomatization does not commit mathematicians to one particular perspective".

Although the different perspectives on the significance of hidden use have been presented separately, the set-theoretic community should not be seen as being divided into disjoint sub-groups that each have their own individual perspective on the hidden use of new axioms. While the exploitation of the hidden use of new axioms in finding reasons in favour of these axioms is only open to people who endorse extrinsic justification, pragmatic significance is relevant to everyone, regardless of their beliefs on the question of axiom justification. One might plausibly assume that all set-theoretic practitioners by their nature are interested in furthering set-theoretic progress, and that pragmatic significance is valued and accepted by all of them.

I conclude with a few open questions and initial commentary on them as potential avenues to fruifully continue the work of this paper:

More mathematical examples. For a deeper insight into the hidden use of new axioms, informal examples of it would be necessary. This might be reports from practitioners in set theory. It would, moreover, be informative to know how many set theorists use which new axioms in this way. When does it work, when does it not work—how widely applicable is this use?

Is the hidden use of large cardinal axioms particularly well suited for consistency statements? Of the two examples cited of the hidden use of large cardinal axioms, one is a consistency statement and the other is not. Hence, hidden use can be successful in both cases. However, there might be differences according to the kind of statement. The two interviewees in the study who mentioned the hidden use of large cardinal axioms are both experts in forcing. Moreover, we have seen that large cardinals are especially useful in enabling certain forcing constructions. These two considerations might suggest that the hidden use of large cardinal axioms is probably more successful in cases where the statement of interest, S, is a consistency statement.

The hidden use of other new axioms. Both determinacy principles and forcing axioms might play a similar role to some extent. Since many determinacy principles are implied by large cardinal axioms, their role would possibly be different within the details of the proof, but not in the logical strength of the assumptions. Regarding the axiom of determinacy and Martin's axiom, we have seen quotations confirming that set-theoretic practitioners working in descriptive set theory use forcing axioms and determinacy principles in the same way. It would be interesting to elaborate on the hidden use of these new axioms and see whether there are differences to the hidden use of large cardinal axioms. My conjecture is that the hidden use of other new axioms is as effective as that of large cardinal axioms, because they are all well-researched new axioms whose application is no longer difficult. I further conjecture that there are differences with regard to the research areas in which certain new axioms are applicable. One descriptive set theorist mentioned that they typically use AD, for instance. For forcing-related questions, large cardinal axioms are possibly used more. And forcing axioms seem to be often applied to topological questions.

Acknowledgements.

I am very grateful to the book's editors, Carolin Antos, Neil Barton, and Giorgio Venturi, for inviting me to contribute a chapter. I recall the seminar series organised as part of the preparation of this volume as a fruitful endeavour. I am very grateful for the detailed and thoughtful reports that I received from two anonymous referees. They helped to improve the paper substantially. I am also very grateful for the discussions with the audiences when I presented this work at the Logic Colloquium 2023 in Milano, and at the Mathematical Logic Seminar in Hamburg.

References

George Boolos. The iterative conception of set. *Journal of Philosophy*, 68(8):215–231, 1971.

Silvia De Toffoli. Groundwork for a fallibilist account of mathematics. *The Philosophical Quarterly*, 71(4), 12 2020.

Kenny Easwaran. The role of axioms in mathematics. *Erkenntnis*, 68(3):381–391, 2008. doi: 10.1007/s10670-008-9106-1.

Harvey M. Friedman. Higher set theory and mathematical practice. *Annals of Mathematical Logic*, 2(3):325–357, 1971. doi: 10.1016/0003-4843(71)90018-0.

D. Gale and F. M. Stewart. Infinite games with perfect information. *Annals of Math. Studies*, 28:245–266, 1953.

Kurt Gödel. What is Cantor's continuum problem? *The American Mathematical Monthly*, 54(9):515–525, 1947.

Martin Goldstern, Jakob Kellner, and Saharon Shelah. Cichoń's maximum. *Annals of Mathematics*, 190(1):113 – 143, 2019. doi: 10.4007/annals.2019.190.1.2.

Martin Goldstern, Jakob Kellner, Diego A. Mejía, and Saharon Shelah. Cichoń's maximum without large cardinals. *J. Eur. Math. Soc.*, 24(11):3951–3967, 2022.

Leo Harrington. Analytic determinacy and $0^\sharp$. *The Journal of Symbolic Logic*, 43(4): 685–693, 1978. URL http://www.jstor.org/stable/2273508.

Paul B. Larson. A brief history of determinacy. In Alexander S. Kechris, Benedikt Löwe, and John R. Steel, editors, *Large Cardinals, Determinacy and Other Topics: The Cabal Seminar, Volume IV*, volume 4 of *Lecture Notes in Logic*, page 3–60. Cambridge University Press, 2020. doi: 10.1017/9781316863534.002.

Penelope Maddy. Believing the axioms. I. *Journal of Symbolic Logic*, 53(2):481–511, 1988a.

Penelope Maddy. Believing the axioms. II. *The Journal of Symbolic Logic*, 53(03):736–764, 1988b.

Penelope Maddy. *Naturalism in Mathematics*. Oxford University Press, Oxford, 1997.

Penelope Maddy. *Defending the Axioms: On the Philosophical Foundations of Set Theory*. Oxford University Press, Oxford, 2011.

Donald A. Martin. Measurable cardinals and analytic games. *Fundamenta Mathematicae*, 66:287–291, 1970.

Donald A. Martin. Borel determinacy. *Annals of Mathematics*, 102(2):363–371, 1975.

Donald A. Martin. *Mathematical evidence*, pages 215–231. Oxford University Press, Usa, New York, 1998.

Dirk Schlimm. Axioms in mathematical practice. *Philosophia Mathematica*, 21(1):37–92, 2013. doi: 10.1093/philmat/nks036.

Mark van Atten and Juliette Kennedy. "Gödel's Modernism: On Set-Theoretic Incompleteness," Revisited. In Sten Lindström, Erik Palmgren, Krister Segerberg, and Viggo Stoltenberg-Hansen, editors, *Logicism, Intuitionism, and Formalism: What has Become of Them?*, pages 303–355. Springer Netherlands, Dordrecht, 2009. doi: 10.1007/978-1-4020-8926-8_15.

Matteo Viale. Useful axioms, 2019. arXiv, 1610.02832, math.LO.