Journal of Humanistic Mathematics

Volume 8 | Issue 1 January 2018

What Makes a Theory of Infinitesimals Useful? A View by Klein and Fraenkel

Vladimir Kanovei

Karin Katz

Mikhail Katz
Bar-Ilan University

Thomas Mormann

What Makes a Theory of Infinitesimals Useful? A View by Klein and Fraenkel

Vladimir Kanovei

V. Kanovei, IPPI, Moscow, and MIIT, Moscow, RUSSIA
kanovei@googlemail.com

Karin U. Katz

Department of Mathematics, Bar Ilan University, Ramat Gan 52900, ISRAEL
katzmik@math.biu.ac.il

Mikhail G. Katz

Department of Mathematics, Bar Ilan University, Ramat Gan 52900, ISRAEL
katzmik@macs.biu.ac.il

Thomas Mormann

Department of Logic and Philosophy of Science, University of the Basque Country UPV/EHU, 20080 Donostia San Sebastian, SPAIN
ylxmomot@sf.ehu.es

Abstract

Felix Klein and Abraham Fraenkel each formulated a criterion for a theory of infinitesimals to be successful, in terms of the feasibility of implementation of the Mean Value Theorem. We explore the evolution of the idea over the past century, and the role of Abraham Robinson's framework therein.

Keywords: *infinitesimal; Felix Klein; Abraham Fraenkel; hyperreal; Mean Value Theorem*

1. Introduction

Historians often take for granted a historical continuity between the calculus and analysis as practiced by the 17^{th}–19^{th} century authors, on the one hand, and the arithmetic foundation for classical analysis as developed starting with the work of Cantor, Dedekind, and Weierstrass around 1870, on the other.

We extend this continuity view by exploiting the Mean Value Theorem (MVT) as a case study to argue that Abraham Robinson's framework for analysis with infinitesimals constituted a continuous extension of the procedures of the historical infinitesimal calculus. Moreover, Robinson's framework provided specific answers to traditional preoccupations, as expressed by Klein and Fraenkel, as to the applicability of rigorous infinitesimals in calculus and analysis.

This paper is meant as a modest contribution to the prehistory of Robinson's framework for infinitesimal analysis. To comment briefly on a broader picture, in a separate article by Bair *et al.* [1] we address the concerns of those scholars who feel that insofar as Robinson's framework relies on the resources of a logical framework that bears little resemblance to the frameworks that gave rise to the early theories of infinitesimals, Robinson's framework has little bearing on the latter.[1] For an analysis of Klein's role in modern mathematics see Bair *et al*—. [2]. For an overview of recent developments in the history of infinitesimal analysis see Bascelli *et al.* [3].

2. Felix Klein

In 1908, Felix Klein formulated a criterion of what it would take for a theory of infinitesimals to be successful. Namely, one must be able to prove an MVT for arbitrary intervals (including infinitesimal ones). Klein writes: "there was lacking a method for estimating . . . the increment of the function in the finite

[1] Such a view suffers from at least two misconceptions. First, a hyperreal extension results from an ultrapower construction exploiting nothing more than the resources of a serious undergraduate algebra course, namely the existence of a maximal ideal (see Section 5). Furthermore, the issue of the *ontological* justification of infinitesimals in a set-theoretic framework has to be distinguished carefully from the issue of the *procedures* of the early calculus which arguably find better proxies in modern infinitesimal theories than in a Weierstrassian framework; see further in Błaszczyk *et al.* [5].

interval. This was supplied by the *mean value theorem*; and it was Cauchy's great service to have recognized its fundamental importance and to have made it the starting point accordingly of differential calculus" [17, page 213]. A few pages later, Klein continues:

> The question naturally arises whether ... it would be possible to *modify* the traditional foundations of infinitesimal calculus, so as to include actually *infinitely small* quantities in a way that would satisfy modern demands as to rigor; in other words, to construct a non-Archimedean system. The first and chief problem of this analysis would be to prove the mean-value theorem
>
> $$f(x+h) - f(x) = h \cdot f'(x + \vartheta h)$$
>
> [where $0 \leq \vartheta \leq 1$] from the assumed axioms. I will not say that progress in this direction is impossible, but it is true that none of the investigators have achieved anything positive. [17, page 219] (emphasis added)

See also Kanovei *et al*. [14, Section 6.1]. Klein's sentiment that the axioms of the traditional foundations need to be modified in order to accommodate a true infinitesimal calculus were right on target. Thus, Dedekind completeness needs to be relaxed; see Section 5.2.

The MVT was still considered a research topic in Felix Klein's lifetime. Thus, in 1884 a controversy opposed Giuseppe Peano and Louis-Philippe Gilbert concerning the validity of a proof of MVT given by Camille Jordan; see Luciano [19], Mawhin [20], Besenyei [4], Smoryński [25] for details.

3. Abraham Fraenkel

Robinson noted in his book that in 1928, Abraham Fraenkel formulated a criterion similar to Klein's, in terms of the MVT. Robinson first mentions the philosopher Paul Natorp of the Marburg school: "during the period under consideration attempts were still being made to define or justify the use of infinitesimals in Analysis (e.g. Geissler [1904], Natorp [1923])" [23, page 278]. Robinson goes on to reproduce a lengthy comment from Abraham Fraenkel's 1928 book [7, pages 116–117] in German. We provide a translation of Fraenkel's comment:

> ... With respect to this test the infinitesimal is a complete failure. The various kinds of infinitesimals that have been taken into account so far and sometimes have been meticulously argued for, have contributed nothing to cope with even the simplest and most basic problems of the calculus. For instance, for [1] a proof of the mean value theorem or for [2] the definition of the definite integral. ... There is no reason to expect that this will change in the future." (Fraenkel as quoted in Robinson [23, page 279]; translation ours; numerals [1] and [2] added)

Thus Fraenkel formulates a pair of requirements: [1] the MVT and [2] definition of the definite integral. Fraenkel then offers the following glimmer of hope:

> Certainly, it would be thinkable (although for good reasons rather improbable and, at the present state of science, situated at an *unreachable distance* [in the future]) that a second Cantor would give an impeccable arithmetical foundation of new infinitely small number that would turn out to be mathematically useful, offering perhaps an easy access to infinitesimal calculus. (ibid., emphasis added)

Note that Fraenkel places such progress at unreachable distance in the future.

This is perhaps understandable if one realizes that Cantor–Dedekind–Weierstrass foundations, formalized in the Zermelo–Fraenkel (the same Fraenkel) set-theoretic foundations, were still thought at the time to be a primary point of reference for mathematics (see Section 1). Fraenkel concludes:

> But as long this is not the case, it is not allowed to draw a parallel between the certainly interesting numbers of Veronese and other infinitely small numbers on the one hand, and Cantor's numbers, on the other. Rather, one has to maintain the position that one cannot speak of the mathematical and therefore logical existence of the infinitely small in the same or similar manner as one can speak of the infinitely large.[2] (ibid.)

[2]The infinities Fraenkel has in mind here are Cantorian infinities.

An even more pessimistic version of Fraenkel's comment appeared a quarter-century later in his 1953 book *Abstract Set Theory*, with MVT replaced by Rolle's theorem [8, page 165].

4. Modern infinitesimals

Fraenkel's 1953 assessment of "unreachable distance" notwithstanding, only two years later Jerzy Łoś in [18] (combined with the earlier work by Edwin Hewitt [12] in 1948) established the basic framework satisfying the Klein–Fraenkel requirements, as Abraham Robinson realized in 1961; see [22]. The third, 1966 edition of Fraenkel's *Abstract Set Theory* makes note of these developments:

> Recently an *unexpected* use of infinitely small magnitudes, in particular a method of basing analysis (calculus) on infinitesimals, has become possible and important by means of a non-archimedean, non-standard, proper extension of the field of the real numbers. For this surprising development the reader is referred to the literature. [9, page 125] (emphasis added)

Fraenkel's use of the adjective *unexpected* is worth commenting on at least briefly. Surely part of the surprise is a foundational challenge posed by modern infinitesimal theories. Such theories called into question the assumption that the Cantor–Dedekind–Weierstrass foundations are an inevitable *primary* point of reference, and opened the field to other possibilities, such as the IST enrichment of ZFC developed by Edward Nelson; for further discussion see Katz–Kutateladze [15] and Fletcher *et al.* [6].

This comment of Fraenkel's is followed by a footnote citing Robinson, Laugwitz, and Luxemburg. Fraenkel's appreciation of Robinson's theory is on record:

> my former student Abraham Robinson had succeeded in saving the honour of infinitesimals - although in quite a different way than Cohen and his school had imagined. [10] (cf. [11, page 85])

Here Fraenkel is referring to Hermann Cohen (1842–1918), whose fascination with infinitesimals elicited fierce criticism by both Georg Cantor and Bertrand Russell. For an analysis of Russell's critique see Katz–Sherry [16, Section 11.1]. For more details on Cohen, Natorp, and Marburg neo-Kantianism, see Mormann–Katz [21].

5. A criterion

Both Klein and Fraenkel formulated a criterion for the usefulness of a theory of infinitesimals in terms of being able to prove a mean value theorem. Such a Klein–Fraenkel criterion is satisfied by the framework developed by Hewitt, Łoś, Robinson, and others. Indeed, the MVT

$$(\forall x \in \mathbb{R})(\forall h \in \mathbb{R})(\exists \vartheta \in \mathbb{R})\big(f(x+h) - f(x) = h \cdot g(x + \vartheta h)\big)$$

where $g(x) = f'(x)$ and $\vartheta \in [0,1]$, holds also for the natural extension ${}^{*}\!f$ of every real smooth function f on an arbitrary hyperreal interval, by the *Transfer Principle*; see Section 5.1. Thus we obtain the formula

$$(\forall x \in {}^{*}\mathbb{R})(\forall h \in {}^{*}\mathbb{R})(\exists \vartheta \in {}^{*}\mathbb{R})\big({}^{*}\!f(x+h) - {}^{*}\!f(x) = h \cdot {}^{*}\!g(x + \vartheta h)\big),$$

valid in particular for infinitesimal h.

5.1. Transfer

The *Transfer Principle* is a type of theorem that, depending on the context, asserts that rules, laws or procedures valid for a certain number system, still apply (i.e., are "transferred") to an extended number system. In this sense it is a formalisation of the Leibnizian *Law of Continuity*; such a connection is explored in Katz–Sherry [16].

Thus, the familiar extension $\mathbb{Q} \hookrightarrow \mathbb{R}$ preserves the property of being an ordered field. To give a negative example, the extension $\mathbb{R} \hookrightarrow \mathbb{R} \cup \{\pm\infty\}$ of the real numbers to the so-called *extended reals* does not preserve the field properties. The hyperreal extension $\mathbb{R} \hookrightarrow {}^{*}\mathbb{R}$ (see Section 5.2) preserves *all* first-order properties. The result in essence goes back to Łoś [18]. For example, the identity $\sin^2 x + \cos^2 x = 1$ remains valid for all hyperreal x, including infinitesimal and infinite inputs $x \in {}^{*}\mathbb{R}$. Another example of a transferable property is the property that for all positive x, y, if $x < y$ then $\frac{1}{y} < \frac{1}{x}$. The Transfer Principle applies to formulas like that characterizing the continuity of a function $f\colon \mathbb{R} \to \mathbb{R}$ at a point $c \in \mathbb{R}$:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\big[|x - c| < \delta \;\Rightarrow\; |f(x) - f(c)| < \varepsilon\big];$$

namely, formulas that quantify over *elements* of the field.

An element $u \in {}^*\mathbb{R}$ is called *finite* if $-r < u < r$ for a suitable $r \in \mathbb{R}$. Let ${}^\flat\mathbb{R} \subseteq {}^*\mathbb{R}$ be the subring consisting of finite elements of ${}^*\mathbb{R}$. There exists a function $\mathbf{st}\colon {}^\flat\mathbb{R} \to \mathbb{R}$ called *the standard part* (sometimes referred to as the *shadow*) that rounds off each finite hyperreal u to its nearest real number $u_0 \in \mathbb{R}$, so that $u_0 = \mathbf{st}(u)$ and $u \approx u_0$, where $a \approx b$ is the relation of infinite proximity (i.e., $a - b$ is infinitesimal).

5.2. *Extension*

The hyperreal extension $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ is the only modern theory of infinitesimals that satisfies the Klein–Fraenkel criterion. Here ${}^*\mathbb{R}$ can be obtained as the quotient of the ring of sequences $\mathbb{R}^{\mathbb{N}}$ by a suitable maximal ideal. The fact that it satisfies the criterion is due to the transfer principle. In this sense, the transfer principle can be said to be a "powerful new principle of reasoning". Note that ${}^*\mathbb{R}$ is not Dedekind-complete.

One could object that the classical form of the MVT is not a key result in modern analysis. Thus, in Lars Hörmander's theory of partial differential operators [13, p. 12–13], a key role is played by various multivariate generalisations of the following Taylor (integral) remainder formula:

$$f(b) = f(a) + (b-a)f'(a) + \int_a^b (b-x)f''(x)dx. \tag{1}$$

Denoting by $\mathcal{D}$ the differentiation operator and by $\mathcal{I} = \mathcal{I}(f, a, b)$ the definite integration operator, we can state (1) in the following more detailed form for a function f:

$$\begin{aligned} &(\forall a \in \mathbb{R})(\forall b \in \mathbb{R}) \\ &\quad f(b) = f(a) + (b-a)(\mathcal{D}f)(a) + \mathcal{I}\big((b-x)(\mathcal{D}^2 f), a, b\big) \end{aligned} \tag{2}$$

Applying the transfer principle to the elementary formula (2), we obtain

$$\begin{aligned} &(\forall a \in {}^*\mathbb{R})(\forall b \in {}^*\mathbb{R}) \\ &\quad {}^*\!f(b) = {}^*\!f(a) + (b-a)({}^*\mathcal{D}\,{}^*\!f)(a) + {}^*\mathcal{I}\big((b-x)({}^*\mathcal{D}^2\,{}^*\!f), a, b\big) \end{aligned} \tag{3}$$

for the natural hyperreal extension ${}^*\!f$ of f. The formula (3) is valid on every hyperreal interval of ${}^*\mathbb{R}$. Multivariate generalisations of (1) can be handled similarly.

5.3. Mean Value Theorem

We have focused on the MVT (and its generalisations) because, historically speaking, it was emphasized by Klein and Fraenkel. The transfer principle applies far more broadly, as can be readily guessed from the above. The mean value theorem is immediate from Rolle's theorem, which in turn follows from the extreme value theorem. For the sake of completeness we include a proof of the extreme value theorem exploiting infinitesimals; see Robinson [23, page 70, Theorem 3.4.13].

Theorem 1. *A continuous function f on $[0,1] \subseteq \mathbb{R}$ has a maximum.*

Proof. The idea is to exploit a partition into infinitesimal subintervals, pick a partition point x_{i_0} where the value of the function is maximal, and take the *shadow* (see below) of x_{i_0} to obtain the maximum.

In more detail, choose infinite hypernatural number $H \in {}^*\mathbb{N} \setminus \mathbb{N}$. The real interval $[0,1]$ has a natural hyperreal extension ${}^*[0,1] = \{x \in {}^*\mathbb{R} \colon 0 \leq x \leq 1\}$. Consider its partition into H subintervals of equal infinitesimal length $\frac{1}{H}$, with partition points $x_i = \frac{i}{H}, \quad i = 0, \ldots, H$. The function f has a natural extension ${}^*\!f$ defined on the hyperreals between 0 and 1. Among finitely many points, one can always pick a maximal value: $(\forall n \in \mathbb{N})\ (\exists i_0 \leq n)\ (\forall i \leq n)\, (f(x_{i_0}) \geq f(x_i))$. By transfer we obtain

$$(\forall n \in {}^*\mathbb{N})\ (\exists i_0 \leq n)\ (\forall i \leq n)\ \ ({}^*\!f(x_{i_0}) \geq {}^*\!f(x_i))\,, \tag{4}$$

where ${}^*\mathbb{N}$ is the collection of hypernatural numbers. Applying (4) to $n = H \in {}^*\mathbb{N} \setminus \mathbb{N}$, we see that there is a hypernatural i_0 such that $0 \leq i_0 \leq H$ and

$$(\forall i \in {}^*\mathbb{N})\big[i \leq H \Longrightarrow {}^*\!f(x_{i_0}) \geq {}^*\!f(x_i)\big]. \tag{5}$$

Consider the real point $c = \mathbf{st}(x_{i_0})$ where $\mathbf{st}$ is the *standard part function*; see Section 5.1. Then $c \in [0,1]$. By continuity of f at $c \in \mathbb{R}$, we have ${}^*\!f(x_{i_0}) \approx {}^*\!f(c) = f(c)$, and therefore $\mathbf{st}\,({}^*\!f(x_{i_0})) = {}^*\!f\,(\mathbf{st}(x_{i_0})) = f(c)$. An arbitrary real point x lies in an appropriate sub-interval of the partition, namely $x \in [x_i, x_{i+1}]$, so that $\mathbf{st}(x_i) = x$, or $x_i \approx x$. Applying the function $\mathbf{st}$ to the inequality in formula (5), we obtain $\mathbf{st}({}^*\!f(x_{i_0})) \geq \mathbf{st}({}^*\!f(x_i))$. Hence $f(c) \geq f(x)$, for all real x, proving c to be a maximum of f (and by transfer, of ${}^*\!f$ as well). $\square$

The partition into infinitesimal subintervals (used in the proof of the extreme value theorem) similarly enables one to define the definite integral as the shadow of an infinite Riemann sum, fulfilling Fraenkel's *second* requirement, as well; see Section 3.

The difficulty of the Klein–Fraenkel challenge was that it required a change in foundational thinking, as we illustrated.

6. Acknowledgments

V. Kanovei was partially supported by the RFBR grant no. 17-01-00705. M. Katz was partially supported by the Israel Science Foundation grant no. 1517/12.

References

[1] Jacques Bair, Piotr Błaszczyk, Robert Ely, Valérie Henry, Vladimir Kanovei, Karin U. Katz, Mikhail G. Katz, Taras Kudryk, Semen S. Kutateladze, Thomas McGaffey, Thomas Mormann, David M. Schaps, and David Sherry, "Cauchy, infinitesimals and ghosts of departed *quantifiers*," *Matematichnī Studīī*, Volume **47** (2017), Number 2, pages 115–144. Available at http://dx.doi.org/10.15330/ms.47.2.115-144 and https://arxiv.org/abs/1712.00226, last accessed on January 23, 2018.

[2] Jacques Bair, Piotr Błaszczyk, Peter Heinig, Mikhail G. Katz, Jan Peter Schäfermeyer, and David Sherry, "Klein *vs* Mehrtens: restoring the reputation of a great modern," *Matematichnī Studīī* (to appear).

[3] Tiziana Bascelli, Emanuele Bottazzi, Frederik Herzberg, Vladimir Kanovei, Karin Katz, Mikhail Katz, Tahl Nowik, David Sherry, and Steve Shnider, "Fermat, Leibniz, Euler, and the gang: The true history of the concepts of limit and shadow." *Notices of the American Mathematical Society*, Volume **61** (2014), Number 8, pages 848–864. Available at http://www.ams.org/notices/201408/rnoti-p848.pdf and http://arxiv.org/abs/1407.0233, last accessed on January 23, 2018.

[4] Ádám Besenyei, "A brief history of the mean value theorem", Talk slides, September 12, 2012. Available at http://abesenyei.web.elte.hu/publications/meanvalue.pdf, last accessed on January 23, 2018.

[5] Piotr Błaszczyk, Vladimir Kanovei, Karin Katz, Mikhail Katz, Semen Kutateladze, and David Sherry, "Toward a history of mathematics focused on procedures," *Foundations of Science*, Volume **22** (2017), Number 4, pages 763–783. Available at http://dx.doi.org/10.1007/s10699-016-9498-3 and https://arxiv.org/abs/1609.04531, last accessed on January 23, 2018.

[6] Peter Fletcher, Karel Hrbacek, Vladimir Kanovei, Mikhail Katz, Claude Lobry, and Sam Sanders, "Approaches to analysis with infinitesimals following Robinson, Nelson, and others," *Real Analysis Exchange*, Volume **42** (2017), Number 2, pages 193–252. Available at https://arxiv.org/abs/1703.00425 and http://msupress.org/journals/issue/?id=50-21D-61F, last accessed on January 23, 2018.

[7] Abraham Fraenkel, *Einleitung in die Mengenlehre*, Dover Publications, New York NY, 1946 [originally published by Springer, Berlin, 1928].

[8] Abraham Fraenkel, *Abstract Set Theory*, Studies in logic and the foundations of mathematics, North-Holland Publishing, Amsterdam, 1953.

[9] Abraham Fraenkel, *Abstract Set Theory*, Third revised edition, North-Holland Publishing, Amsterdam, 1966.

[10] Abraham Fraenkel, *Lebenskreise. Aus den Erinnerungen eines jüdischen Mathematikers*, Deutsche Verlags-Anstalt, Stuttgart, 1967.

[11] Abraham Fraenkel, *Recollections of a Jewish mathematician in Germany*. With a foreword by Menachem Magidor. With a foreword to the 1967 German edition by Y. Bar-Hillel. Edited and with a chapter "Afterword: 1933–1965" by Jiska Cohen-Mansfield. Translated from the German by Allison Brown. Birkhäuser/Springer, [Cham], 2016.

[12] Edwin Hewitt, "Rings of real-valued continuous functions. I," *Transactions of the American Mathematical Society*, Volume **64** (1948), pages 45–99.

[13] Lars Hörmander, *Linear Partial Differential Operators*, Springer Verlag, Berlin–New York, 1976.

[14] Vladimir Kanovei, Mikhail Katz, and Thomas Mormann, "Tools, objects, and chimeras: Connes on the role of hyperreals in mathematics," *Foundations of Science*, Volume **18** (2013), pages 259–296. Available at http://dx.doi.org/10.1007/s10699-012-9316-5 and http://arxiv.org/abs/1211.0244, last accessed on January 23, 2018.

[15] Mikhail Katz and Semen Kutateladze, "Edward Nelson (1932-2014)," *The Review of Symbolic Logic*, Volume **8** (2015), pages 607–610. Available at http://dx.doi.org/10.1017/S1755020315000015 and https://arxiv.org/abs/1506.01570, last accessed on January 23, 2018.

[16] Mikhail Katz and David Sherry, "Leibniz's infinitesimals: Their fictionality, their modern implementations, and their foes from Berkeley to Russell and beyond," *Erkenntnis*, Volume **78** (2013), pages 571–625. Available at http://dx.doi.org/10.1007/s10670-012-9370-y and https://arxiv.org/abs/1205.0174, last accessed on January 23, 2018.

[17] Felix Klein, *Elementary Mathematics from an Advanced Standpoint. Vol. I. Arithmetic, Algebra, Analysis*, Translation by E. R. Hedrick and C. A. Noble [Macmillan, New York, 1932] from the third German edition [Springer, Berlin, 1924]. Originally published as *Elementarmathematik vom höheren Standpunkte aus* (Leipzig, 1908).

[18] Jerzy Łoś, "Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres," in *Mathematical interpretation of formal systems*, 98–113, North-Holland Publishing, Amsterdam, 1955.

[19] Erika Luciano, "The treatise of Genocchi and Peano (1884) in the light of unpublished documents," (Italian) *Bollettino di Storia delle Scienze Matematiche*, Volume **27** (2007), pages 219–264.

[20] Jean Mawhin, "Some contributions of Peano to analysis in the light of the work of Belgian mathematicians," in *Giuseppe Peano between mathematics and logic*, pages 13–28, Springer Italia, Milan, 2011.

[21] Thomas Mormann and Mikhail Katz, "Infinitesimals as an issue of neo-Kantian philosophy of science," *HOPOS: The Journal of the International Society for the History of Philosophy of Science*, Volume **3** (2013), pages 236–280. Available at http://www.jstor.org/stable/10.1086/671348 and http://arxiv.org/abs/1304.1027, last accessed on January 23, 2018.

[22] Abraham Robinson, "Non-standard analysis," *Nederl. Akad. Wetensch. Proc. Ser. A* **64** = *Indag. Math.* **23** (1961), 432–440 [reprinted in *Selected Papers*; see Robinson [24], pages 3–11]

[23] Abraham Robinson, *Non-standard analysis*, North-Holland Publishing, Amsterdam, 1966.

[24] Abraham Robinson, *Selected papers of Abraham Robinson. Vol. II. Nonstandard analysis and philosophy.* Edited and with introductions by W. A. J. Luxemburg and S. Körner. Yale University Press, New Haven, Conn., 1979.

[25] Smoryński, C. *MVT: a most valuable theorem.* Springer, Cham, 2017.