A Multi-Succedent Sequent Calculus for Logical Expressivists

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Abstract: Expressivism in logic is the view that logical vocabulary plays a primarily expressive role: that is, that logical vocabulary makes perspicuous in the object language structural features of inference and incompatibility (Brandom, 1994, 2008). I present a precise, technical criterion of expressivity for a logic (§2). I next present a logic that meets that criterion (§3). I further explore some interesting features of that logic: first, a representation theorem for capturing other logics (§3.1), and next some novel logical vocabulary for *expressing* structural features of inference (§4).

Keywords: Inferentialism, Logical expressivism, Non-monotonic logic

1 Introduction: some philosophical background

In this paper I present a non-monotonic, multi-succedent sequent calculus that vindicates the ambitions of logical expressivists and semantic inferentialists. Expressivism in logic is the view that logical vocabulary plays a primarily expressive role: that is, that logical vocabulary makes perspicuous *in the object language* structural features of inference and incompatibility (Brandom, 1994, 2008). Brandom cashes this out with the slogan that logical vocabulary allows one to *say* (in the object language) what one was previously only able to *do* (in a pre-logical discursive practice). The result is that logical vocabulary should be understood as "LX", i.e. *(algorithmically) elaborated from* and *explicative of* a pre-logical consequence relation. (Algorithmic) elaboration is a pragmatic constraint: the ability to competently navigate such a pre-logical consequence relation already endows one with the abilities needed to navigate a consequence relation with logical vocabulary. That such vocabulary is explicative means that it must successfully encode structural features of that consequence relation.

[1]This work is the fruit of a joint-research project. I am indebted to the helpful feedback given by a research group run by Bob Brandom. The central philosophical and technical results I am reporting, however, are my own. A system with similar ambitions (in fact an earlier result in our project) can be seen in (Hlobil, 2016).

Logical expressivism is motivated in turn by two other significant philosophical theses: *semantic* inferentialism—-the view that the meaning of *non-logical* vocabulary is determined (at least, if not essentially) by its role in inference—and *logical* inferentialism–the view that the meaning of *logical* vocabulary is also so determined, paradigimatically by introduction and elimination rules (Brandom, 1994, 2008; Peregrin, 2014). It is the former of these two views that distinguishes expressivism from logical inferentialism. Since our ordinary discursive practices potentially include material and non-monotonic inferences, the logical expressivist wishes to understand logical vocabulary as expressive of *those* inferential practices.

Combining these three lines of thought produces some demands on a logical system. A commitment to semantic inferentialism means that (i) our logical systems should include within them material and pre-logical fragments, out of which logical vocabulary is to be *elaborated.* (ii) Such elaboration should in turn *naturally and conservatively* extend such a pre-logical consequence relation. The extension should be *conservative* in the sense that no new material implications are introduced as a result; further, the structural features that our logical vocabulary expresses should likewise be preserved in the logically extended consequence relation. The demand that the extension be *natural* means that specifying the inferential role of logical vocabulary should suffice *on its own* to extend and preserve the structural features in question (e.g. that no further structural rules are required). This demand (if met) justifies the claim that the abilities needed to use non-logical vocabulary *already* endow one with the abilities needed to use logical vocabulary. Finally, (iii) such systems should be capable of expressing *in the object language* those features of consequence that were discarded as a result of (i); they should, therefore, express when an implication is e.g. monotonic or classically valid.

The system I construct meets these demands. In order to show this, I begin by making precise what logical expressivism demands. That is, I argue for a precise criterion against which a logic may be tested (§2). I also argue for two additional criteria intended to make precise the idea that a logic preserves and expresses *structural* features of implication. Following this I construct a system that meets all of these constraints (§3). I start with a material, non-monotonic, multi-succedent base consequence relation over an atomic language: $\mid\!\sim_0 \subseteq \mathcal{P}(\mathcal{L}_0) \times \mathcal{P}(\mathcal{L}_0)$. The language is extended in the standard fashion to include $\{\&, \vee, \neg, \rightarrow\}$. The consequence relation is conservatively extended via familiar sequent rules $\mid\!\sim \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$. The extension is also natural in the sense sketched above. Local regions of monotonicity and classical validity are each preserved by the logical rules of our sequent calculus without the aid of any structural rules. This means, in the case of classicality, that the stipulation that our base consequence relation contain all atomic classical sequents (i.e. sequents of the form $\Gamma, p \mid\!\sim_0 p, \Theta$ for $p \in \mathcal{L}_0$, $\Gamma, \Theta \subseteq \mathcal{L}_0$) guarantees that our logically extended consequence relation be supra-classical via the logical rules alone. That monotonicity is *naturally* preserved means that if a base sequent is monotonic with respect to atoms $\forall \Delta_0, \Lambda_0 \subseteq \mathcal{L}_0(\Delta_0, \Gamma_0 \mid\!\sim_0 \Theta_0, \Lambda_0)$, then that sequent will also be monotonic with respect to logically complex sentences: $\forall \Delta, \Lambda \subseteq \mathcal{L}(\Delta, \Gamma_0 \mid\!\sim \Theta_0, \Lambda)$. And this result holds for *all* implications in the logically extended consequence relation:

$$\forall \Delta_0, \Lambda_0 \subseteq \mathcal{L}_0(\Delta_0, \Gamma \mid\!\sim \Theta, \Lambda_0) \Rightarrow \forall \Delta, \Lambda \subseteq \mathcal{L}(\Delta, \Gamma \mid\!\sim \Theta, \Lambda).$$

In addition, I introduce modal operators to encode these very same structural features in the object language (satisfying (iii) above). That is, I show how we can introduce a '$\boxed{M}$' operator which marks all and only those sequents which obey monotonicity:

$$\forall \Delta, \Lambda \subseteq \mathcal{L}(\Delta, \Gamma \mid\!\sim A, \Theta, \Lambda) \Leftrightarrow \Gamma \mid\!\sim \boxed{M} A, \Theta.$$

I also introduce an operator '$\boxed{K}$' which marks all and only classically valid implications:

$$\Gamma \vdash_{LK} A, \Theta \Leftrightarrow \Gamma \mid\!\sim \boxed{K} A, \Theta.$$

These operators are introduced via simple and straightforward sequent rules and they mark precisely the structural features they purport to mark in virtue of those sequent rules alone. I also explain how the techniques used to develop '$\boxed{M}$' and '$\boxed{K}$' may be generalized to other structural features (§4).It is in virtue of this that I claim we can see the logical vocabulary of my system as truly *expressive of* an underlying material consequence relation.

Finally, along the way I present a representation theorem which allows specification of exactly which implications must be included in the base if we want our extended consequence relation to *include* a potentially arbitrary, logically complex consequence relation (§3.1). If that consequence relation meets several modest syntactic constraints then we may specify *exactly* which base consequence relation will generate *exactly* that logically complex consequence relation. Thus, because the sequent rules I employ *naturally* extend an underlying material and pre-logical consequence relation, we should understand that logical vocabulary as truly algorithmically elaborated from that pre-logical consequence relation. And because my representation theorem allows us to see *exactly* which base implications in a pre-logical consequence relation are responsible for a given implication, we should understand those logically complex sequents as truly *expressive of* that underlying base consequence relation. My system therefore vindicates some core ambitions of logical expressivists.

2 Precisification of "expressivity"

I now seek to make the notion of "expression" more precise. Brandom understands expressivism in terms of what he calls an "LX relation", where a vocabulary B is "LX" of a vocabulary A if it is elaborated from and explicative of A. The first criterion (elaboration) has it that if one is able to successfully deploy vocabulary A then one already has the skills necessary to use B. That is, that B may be (algorithmetically) elaborated from A. The second criterion (explication) has it that B says something about (makes perspicuous in the object language) what one was doing by using A (minimally that B may encode the implications and incompatibilities of A). Logical vocabulary is said to be "universally LX" meaning that logical vocabulary stands in this relation to *all vocabularies*.

Let us make this relation more precise. First let $\mathcal{L}_0$ be an arbitrary vocabulary devoid of logical symbols (i.e. a set of atomic sentence letters). Let $\mid\!\sim_0$ be a consequence relation over $\mathcal{L}_0$ (i.e. $\mid\!\sim_0 \subseteq \mathcal{P}(\mathcal{L}_0)^2$). Note that while I call $\mid\!\sim_0$ and $\mid\!\sim$ (below) *consequence* relations I do not yet impose *any* restrictions on them. They should be treated, therefore, simply as two place relations between sets of sentences. As I discussed in the introduction, there are philosophically rich reasons for wanting a consequence relation that is e.g. non-monotonic or perhaps non-classical. In addition part of the motivation of expressivism is that where such features hold of consequence it is an *expression* of an underlying (material) relation of consequence.[2]

Next let $\mathbb{L}$ be our logic. Our logic consists of a finite set of logical symbols (e.g. $\{\&, \vee, \neg, \rightarrow\}$) and rules for expanding $\mathcal{L}_0$ to $\mathcal{L}$ (our language enriched with those logical symbols) and for expanding $\mid\!\sim_0$ to $\mid\!\sim$. Intuitively, we should think of $\mathbb{L}$ as a function from $\mid\!\sim_0$ to $\mid\!\sim$. That is, $\mathbb{L} : \mid\!\sim_0 \mapsto \mid\!\sim$. Then whether $\mathbb{L}$ is "LX" concerns the relationship between $\mid\!\sim_0$ and $\mid\!\sim$ (i.e. the behavior of $\mathcal{L}$ in relation to the behavior of $\mathcal{L}_0$).

[2] I am being brief here on the *justification* for treating $\mid\!\sim_0$ as I do. I primarily wish to stress here—in order to avoid confusion—that $\mid\!\sim_0$ need not have *any constraints*.

That the logical vocabulary be elaborated fixes a tight relationship from $\mid\!\sim_0$ to $\mid\!\sim$. That is, to get from $\mid\!\sim_0$ to $\mid\!\sim$ should require no more than a specification of the logical vocabulary. That is, given $\mid\!\sim_0$, $\mid\!\sim$ should be uniquely determined: $\mid\!\sim_0 \Rightarrow \mid\!\sim$. In prose, the behavior of $\mathcal{L}$ should be determined by the behavior of $\mathcal{L}_0$ simply by specifying the logical symbols.

That the logical vocabulary be explicative fixes a tight relationship from $\mid\!\sim$ to $\mid\!\sim_0$. Since this requires that the logical vocabulary enable us to *say something* about the underlying pre-logical consequence relation, we should require that it actually do what it purports to do: $\mid\!\sim \Rightarrow \mid\!\sim_0$. In prose, the behavior of $\mathcal{L}$ should *genuinely* say or *express* something about the behavior of $\mathcal{L}_0$. The behavior of $\mathcal{L}$ should therefore fix the behavior of $\mathcal{L}_0$. If $\mathcal{L}$ behaved differently then it would *express* something different about the behavior of $\mathcal{L}_0$. If such expression is to be genuine then the behavior of $\mathcal{L}_0$ (i.e. $\mid\!\sim_0$) would need to be different.

Together these two criteria have it that $\mid\!\sim_0 \Leftrightarrow \mid\!\sim$. The behavior of $\mathcal{L}$ is elaborated out of, but also explicative of the behavior of $\mathcal{L}_0$.

While this criterion has some naive plausibility, it must still be made more precise. In particular, if our logical vocabulary is to be *conservative*, then $\mid\!\sim_0 \subseteq \mid\!\sim$ and so the criterion will hold trivially. We may circumvent this problem by quantifying over possible $\mid\!\sim_0$. This might have already been anticipated since I mentioned that logical vocabulary is to have this relationship *universally* i.e. with respect to arbitrary vocabularies (and thus arbitrary $\mid\!\sim_0$). This gives rise to the following definition:

Definition 1 *Fix a logic $\mathbb{L}$, i.e. a function from $\mid\!\sim_0$ to $\mid\!\sim$. We say that $\mathbb{L}$ is* **expressive** *or that $\mid\!\sim$* **expresses** *a base consequence relation $\mid\!\sim_0$ iff:*

$$\begin{gathered}(\forall \Gamma, \Theta \subseteq \mathcal{L})(\exists \Gamma_1, \Theta_1, \ldots, \Gamma_n, \Theta_n \subseteq \mathcal{L}_0) \\ (\forall \mid\!\sim_0 \subseteq \mathcal{P}(\mathcal{L}_0)^2)(\forall \mid\!\sim \subseteq \mathcal{P}(\mathcal{L})^2(\mathbb{L} : \mid\!\sim_0 \mapsto \mid\!\sim)) \\ ((\Gamma \mid\!\sim \Theta) \Leftrightarrow (\Gamma_1 \mid\!\sim_0 \Theta_1 \bigwedge \Gamma_2 \mid\!\sim_0 \Theta_2 \bigwedge \cdots \bigwedge \Gamma_n \mid\!\sim_0 \Theta_n)).\end{gathered}$$

We also say $\Gamma \mid\!\sim \Theta$ **expresses** *$\Gamma_i \mid\!\sim_0 \Theta_i$ $(1 \leq i \leq n)$ (its expressientia).*

This definition says anytime $\Gamma \mid\!\sim \Theta$ this is in virtue of some set of implications present in the language prior to $\mathbb{L}$. So the logical vocabulary is said to be elaborated if $\Gamma \mid\!\sim \Theta$ occurs whenever those pre-logical implications obtain, and the logical vocabulary is said to be explicative if $\Gamma \mid\!\sim \Theta$ occurs only if those pre-logical implications obtain.

The above should be taken as a precise specification of a minimal constraint on logics to count as "expressive". But one of the central features of expression is the idea that logical vocabulary should be able to make perspicuous in the object language *structural features* of inference. By structural features I have in mind such things as monotonicity, classicality, reflexivity, contraction, etc., where each is understood to be capable of holding both globally (e.g. that $\mid\!\sim$ is monotonic) as well as locally (e.g. that $\Gamma \mid\!\sim \Theta$ is monotonic, though $\Delta \mid\!\sim \Lambda$ may not be). Expressivism says that it is distinctive of logical vocabulary to be able to express such features. This requires that (i) $\mathbb{L}$ be capable of *preserving* structural features and that (ii) $\mathbb{L}$ be capable of *expressing* those very structural features it preserves.

Definition 2 *Let $\mathfrak{Sf}$ be a structural feature. Let $\mathfrak{Sf}(\Gamma \mid\!\sim \Theta)$ be shorthand for $\Gamma \mid\!\sim \Theta$ obeys (is an instance of) $\mathfrak{Sf}$. Next, let $\Gamma \mid\!\sim \Theta$ be arbitrary with $\Gamma_i \mid\!\sim_0 \Theta_i$ $(1 \leq i \leq n)$ its expressientia (in accordance with Definition 1). We say that a logic $\mathbb{L}$* **preserves** *a structural feature $\mathfrak{Sf}$ iff:*

$$\mathfrak{Sf}(\Gamma \mid\!\sim \Theta) \Leftrightarrow \left(\mathfrak{Sf}(\Gamma_1 \mid\!\sim_0 \Theta_1) \bigwedge \cdots \bigwedge \mathfrak{Sf}(\Gamma_n \mid\!\sim_0 \Theta_n)\right).$$

A structural feature is *preserved* when an implication obeys that structural feature *iff* all of the implications it expresses also obey that structural feature. Thus, whether an implication obeys a structural feature should be seen as expressing something about the pre-logical implications that that implication expresses: it inherits those features from them and has those features in virtue of those implications alone. Next, I must explain what it means for a particular piece of logical vocabulary to express such structural features.

Definition 3 *Let $\mathfrak{Sf}$ be a structural feature. Suppose some logical operation '*' may be used to mark a sequent in some way (with the constraint that $\Gamma^* \mid\!\sim \Theta^*$ only if $\Gamma \mid\!\sim \Theta$). Then we say that '*' (or $\mathbb{L}$) expresses $\mathfrak{Sf}$ iff there exists a '*' in $\mathbb{L}$ such that:*

$$\Gamma^* \mid\!\sim \Theta^* \Leftrightarrow \mathfrak{Sf}(\Gamma \mid\!\sim \Theta).$$

Sf-Expression combines three ideas. (i) That a logic be capable of expressing an underling base consequence relation, (ii) that it be capable of preserving structural features of that base consequence relation, and finally (iii) that it be able to mark in the object language those very same features that it preserves.

3 A non-monotonic multi-succedent sequent calculus

I now construct a logic $\mathbb{L}$ which I will use to exhibit some of the above ideas. Let us fix $\mid\!\sim_0$. Let our logic include the symbols $\{\&, \vee, \neg, \rightarrow\}$ and let it expand $\mathcal{L}_0$ to $\mathcal{L}$ in the standard fashion. Then our logic $\mathbb{L}$ is given by the following sequent calculus, where proof trees are introduced by axioms:[3]

Axiom 1: If $\Gamma \mid\!\sim_0 \Theta$, then $\Gamma \mid\!\sim \Theta$ may form the base of a proof tree.

$$\frac{\Gamma \mid\!\sim \Theta, A \qquad B, \Gamma \mid\!\sim \Theta}{A \rightarrow B, \Gamma \mid\!\sim \Theta}\ \text{L}\!\rightarrow \qquad \frac{A, \Gamma \mid\!\sim \Theta, B}{\Gamma \mid\!\sim \Theta, A \rightarrow B}\ \text{R}\!\rightarrow$$

$$\frac{\Gamma, A, B \mid\!\sim \Theta}{\Gamma, A\&B \mid\!\sim \Theta}\ \text{L\&} \qquad \frac{\Gamma \mid\!\sim A, \Theta \qquad \Gamma \mid\!\sim B, \Theta}{\Gamma \mid\!\sim A\&B, \Theta}\ \text{R\&}$$

$$\frac{A, \Gamma \mid\!\sim \Theta \qquad B, \Gamma \mid\!\sim \Theta}{A \vee B, \Gamma \mid\!\sim \Theta}\ \text{L}\vee \qquad \frac{\Gamma \mid\!\sim A, B, \Theta}{\Gamma \mid\!\sim A \vee B, \Theta}\ \text{R}\vee$$

$$\frac{\Gamma \mid\!\sim A, \Theta}{\neg A, \Gamma \mid\!\sim \Theta}\ \text{L}\neg \qquad \frac{A, \Gamma \mid\!\sim \Theta}{\Gamma \mid\!\sim \neg A, \Theta}\ \text{R}\neg$$

Note that $\mid\!\sim_0$ and $\mid\!\sim$ here relate multisets. I treat things in this manner in order to avoid assuming *any* structural features absent permutation. I call $\mathbb{L}$ here NM-MS since its consequence relation is given by a Non-Monotonic Multi-Succedent sequent calculus.

Next I rehearse some important results for NM-MS.[4]

Theorem 1 *If* $\Gamma \mid\!\sim \Theta$ *may be arbitrarily weakened with atoms, then it may be arbitrarily weakened with logically complex sentences:*

$$\forall \Delta_0, \Lambda_0 \subseteq \mathcal{L}_0(\Delta_0, \Gamma \mid\!\sim \Theta, \Lambda_0) \Leftrightarrow \forall \Delta, \Lambda \subseteq \mathcal{L}(\Delta, \Gamma \mid\!\sim \Theta, \Lambda).$$

Proof. ($\Leftarrow$) is immediate. ($\Rightarrow$) is proven by induction on the complexity of $\Delta \cup \Lambda$ where complexity is understood in terms of the complexity of the most complex sentences in $\Delta \cup \Lambda$. □

[3]The rules are the same as Ketonen uses. The rules with two top sequents are additive; the rules with a single top-sequent are multiplicative. These are sometimes called "mixed" or "assorted" rules/connectives (see e.g. Dicher, 2016). It is similar to the system called G3cp discussed in (Negri, Von Plato, & Ranta, 2008, ch. 3) with a more standard treatment of negation and material axioms. As is well known, these rules are equivalent to the multiplicative and additive rules of linear logic given monotonicity and contraction (Girard, 2011).

[4]Note that many of these results have full proofs worked out in (Girard, 2011; Negri et al., 2008). Since my system is slightly different than the systems featured there, a more thorough treatment would require some minor modification.

A similar result is in the offing, namely that NM-MS preserves contraction.

Theorem 2 *If* $\Gamma \mathrel{|\!\sim} \Theta$ *allows contraction of atomic sentences, then it allows contraction of logically complex sentences.*

Proof. One direction is trivial, the other direction is provided by induction on the complexity of the contracted sentence. □

Since it is well known that the rules featured above are equivalent to both the additive and multiplicative rules of linear logic given contraction and monotonicity, we can actually locate the condition needed for our logic to be supra-classical.

Definition 4 *We say that* $\mathrel{|\!\sim}_0$ *obeys Containment (CO) if*

$$\forall \Delta, \Lambda \subseteq \mathcal{L}_0(\Delta, p \mathrel{|\!\sim}_0 p, \Lambda)$$

(i.e. if we have $\forall q \in \mathcal{L}_0(q \mathrel{|\!\sim}_0 q)$ *and all such sequents may be arbitrarily weakened; the fragment carved out by this stipulation will also obviously obey contraction). In short: let us define* $\mathrel{|\!\sim}_0^{CO}$ *such that* $\mathrel{|\!\sim}_0^{CO}$ *obeys reflexivity* $\forall q \in \mathcal{L}_0(q \mathrel{|\!\sim}_0 q)$*, weakening and contraction. And further stipulate that no proper subset of* $\mathrel{|\!\sim}_0^{CO}$ *obeys all of these conditions. A base consequence relation* $\mathrel{|\!\sim}_0$ *is said to obey CO iff it includes* $\mathrel{|\!\sim}_0^{CO}$*, i.e.* $\mathrel{|\!\sim}_0^{CO} \subseteq \mathrel{|\!\sim}_0$.

Theorem 3 *If* $\mathrel{|\!\sim}_0$ *obeys CO, then* $\mathrel{|\!\sim}$ *is supra-classical.*

Proof. The result is well known, but can be easily established by showing an equivalence with Gentzen's LK in the fragment of $\mathrel{|\!\sim}$ generated by $\mathrel{|\!\sim}_0^{CO}$. □

Finally, the next theorem is of particular import to the sections following this one.

Theorem 4 *All rules of NM-MS are reversible. That is, if* $\Gamma \mathrel{|\!\sim} \Theta$ *would be the result of the application of a rule to* $\Gamma^* \mathrel{|\!\sim} \Theta^*$ *(and possibly* $\Gamma^{**} \mathrel{|\!\sim} \Theta^{**}$*) then*

$$\Gamma \mathrel{|\!\sim} \Theta \Leftrightarrow \Gamma^* \mathrel{|\!\sim} \Theta^* (\textit{and} \quad \Gamma^{**} \mathrel{|\!\sim} \Theta^{**}).$$

Proof. Proof is straightforward by induction on proof height. □

From this my first gloss on logical expression follows immediately. In the next section I prove that the more precise sense (in Definition 1) also holds.

Corollary 1 *NM-MS is conservative. That is*

$$\Gamma \mathrel{|\!\sim}_0 \Theta \Leftrightarrow \Gamma \mathrel{|\!\sim} \Theta.$$

3.1 Representation Theorem

Next I show how consequence relations may be represented in NM-MS. First two central results concerning conjunctive and disjunctive normal forms.[5]

Proposition 1 *Let $CNF(A)$ be the conjunctive normal form representation of A. It follows that*

$$\Gamma \mid\!\sim \Theta, A \Leftrightarrow \Gamma \mid\!\sim \Theta, CNF(A).$$

Proof. Proof proceeds constructively. From theorem 4, we may deconstruct A until we have a number of sequents of the form: $\Gamma \mid\!\sim \Theta, \Lambda_1$; $\Gamma \mid\!\sim \Theta, \Lambda_2; \ldots \Gamma \mid\!\sim \Theta, \Lambda_n$ where $\Lambda_i (1 \leq i \leq n)$ contains only literals. We next construct $CNF(A)$ via repeated application of R$\vee$ and R&:

$$\Gamma \mid\!\sim \Theta, (\bigvee \Lambda_1)\&(\bigvee \Lambda_2)\& \cdots \&(\bigvee \Lambda_n),$$

i.e. $\Gamma \mid\!\sim \Theta, CNF(A)$. □

Proposition 2 *Let $DNF(A)$ be the disjunctive normal form representation of A. It follows that*

$$A, \Gamma \mid\!\sim \Theta \Leftrightarrow DNF(A), \Gamma \mid\!\sim \Theta.$$

Proof. Proof is identical to the previous proposition except the sets are on the left and we construct $DNF(A)$ via L& and L$\vee$. □

Theorem 5 (Representation Theorem 1) *Let CR be a consequence relation, i.e. $CR \subseteq \mathcal{P}(\mathcal{L})^2$. Then we may specify what must be included in $\mid\!\sim_0$ such that $CR \subseteq \mid\!\sim$.*

Proof. Proof proceeds constructively. For each $\Gamma \mid\!\sim \Theta$ in CR let us find an equivalent $CNF(A) \mid\!\sim CNF(B)$. This has the form:

$$(\&\Gamma_1) \vee \cdots \vee (\&\Gamma_a) \mid\!\sim (\bigvee \Theta_1)\& \cdots \&(\bigvee \Theta_b).$$

This holds just in case (for $1 \leq i \leq a$ and $1 \leq j \leq b$) $\Gamma_i \mid\!\sim_0 \Theta_j$. Thus we stipulate of the base that $\Gamma_i \mid\!\sim_0 \Theta_j$ for $1 \leq i \leq a$ and $1 \leq j \leq b$. If we do this for each implication in CR then we are guaranteed that $CR \subseteq \mid\!\sim$. □

[5]Note that the results in Propositions 1 and 2 follow closely the distribution properties Girard demonstrates for different connectives in linear logic (Girard, 1987, 2011).

Theorem 6 (Representation Theorem 2) *Let CR be a consequence relation. If CR is closed under some modest syntactic constraints,[6] then we may specify* $\mid\!\sim_0$ *such that* $CR = \mid\!\sim$.

Proof. Proof is identical to the first Representation Theorem except that the syntactic constraints on CR have it that $\mid\!\sim = CR$. □

These results give us a way of saying exactly how to reconstruct arbitrary consequence relations using my machinery and given some modest constraints how to reconstruct them *exactly*. It is this ability to reconstruct consequence relations *exactly* that will prove most important. For what it shows is that we are able to find exactly which pre-logical implications an arbitrary implication involving logical vocabulary *expresses*. That is, what I have shown is a method for finding exactly which implications in $\mid\!\sim_0$ are expressed by each implication in $\mid\!\sim$. We are thus in a position to prove the following straight away.

Theorem 7 (Expressivity) *o NM-MS is expressive. That is, we have*

$$\Gamma \mid\!\sim \Theta \Leftrightarrow (\Gamma_1 \mid\!\sim_0 \Theta_1 \bigwedge \cdots \bigwedge \Gamma_n \mid\!\sim_0 \Theta_n).$$

for some $\Gamma_1, \Theta_1, \ldots, \Gamma_n, \Theta_n$ *and arbitrary* $\mid\!\sim_0$.

Proof. Suppose $\Gamma \mid\!\sim \Theta$ and let it be equivalent to $DNF(A) \mid\!\sim CNF(B)$ for some A and B. This has the form:

$$(\&\Gamma_1) \vee \cdots \vee (\&\Gamma_a) \mid\!\sim (\bigvee\Theta_1)\& \cdots \&(\bigvee\Theta_b).$$

This holds just in case (for $1 \leq i \leq a$ and $1 \leq j \leq b$) $\Gamma_i \mid\!\sim_0 \Theta_j$. Next, let us enumerate $\langle i, j\rangle$ as $1, \ldots, n$. Then we have that:

$$\Gamma \mid\!\sim \Theta \Leftrightarrow (\Gamma_1 \mid\!\sim_0 \Theta_1 \bigwedge \cdots \bigwedge \Gamma_n \mid\!\sim_0 \Theta_n).$$ □

[6] In a more formal account I treat representation as of *theories*. Here I characterize it in terms of consequence relations, where we are able to precisely represent a consequence relation just in case it is closed under the rules of NM-MS. In the case where we wish to treat theories instead, then a theory T must meet the following constraints: &-composition and -decomposition ($A, B \in T$ iff $A\&B \in T$), Distribution (of $\vee$ over &) $(A \vee (B\&C)) \in T$ iff $(A \vee B)\&(A \vee C) \in T$, Conditional Equivalence ($A \rightarrow B = \sigma$ is a sub-formula of $\tau \in T$ iff $\neg A \vee B = \sigma'$ is a subformula of $\tau \in T$), both De-Morgan's Equivalences (likewise defined over sub-formulae) and involution (also defined over subformulae).

4 Expressing Other Features

I have so far shown how NM-MS is expressive in the sense made precise in Definition 1. Now I show how NM-MS may express particular structural features. First I introduce a schema for introducing a piece of logical vocabulary '$\boxed{\mathfrak{S}}$'.

First, let us enrich our sequent calculus by introducing a second turnstile $\mid\!\sim^{\mathfrak{S}}$. Now let $\mid\!\sim_0^{\mathfrak{S}}$ pick out some subset of $\mid\!\sim_0$. Later I will discuss principles for determining which subset, but for now I leave the details vague. We may introduce the following rules to our sequent calculus:[7]

Axiom 2: If $\Gamma \mid\!\sim_0^{\mathfrak{S}} \Theta$ then $\Gamma \mid\!\sim^{\mathfrak{S}} \Theta$.

$$\frac{A, \Gamma \mid\!\sim^{\mathfrak{S}} \Theta}{\boxed{\mathfrak{S}}A, \Gamma \mid\!\sim^{[\mathfrak{S}]} \Theta}\text{ L}\boxed{\mathfrak{S}} \qquad \frac{\Gamma \mid\!\sim^{\mathfrak{S}} \Theta, A}{\Gamma \mid\!\sim^{[\mathfrak{S}]} \Theta, \boxed{\mathfrak{S}}A}\text{ R}\boxed{\mathfrak{S}}$$

Lemma 1 *L$\boxed{\mathfrak{S}}$ and R$\boxed{\mathfrak{S}}$ are reversible rules.*

We thus have the following result.

Theorem 8 *Let $\mathfrak{Sf}$ be a structural rule. Suppose that $\mathfrak{Sf}$ is* preserved *(in the sense of Definition 2) and suppose further that $\mid\!\sim^{\mathfrak{S}}$ marks that structural feature exactly. We thus have: $\mathfrak{Sf}(\Gamma \mid\!\sim \Theta)$ iff $\Gamma \mid\!\sim^{\mathfrak{S}} \Theta$. It follows that $\boxed{\mathfrak{S}}$* expresses *(in the sense of Definition 3) $\mathfrak{Sf}$. Thus:*

$$\boxed{\mathfrak{S}}A, \Gamma \mid\!\sim \Theta \Leftrightarrow \mathfrak{Sf}(A, \Gamma \mid\!\sim \Theta)$$
$$\Gamma \mid\!\sim \Theta, \boxed{\mathfrak{S}}A \Leftrightarrow \mathfrak{Sf}(A, \Gamma \mid\!\sim \Theta, A)$$

Proof. I prove only the latter biconditional since the proof of the former is identical. By supposition $\mathfrak{Sf}(\Gamma \mid\!\sim \Theta, A)$ iff $\Gamma \mid\!\sim^{\mathfrak{S}} \Theta, A$. Because our R$\boxed{\mathfrak{S}}$ rule is reversible, we have that $\Gamma \mid\!\sim^{\mathfrak{S}} \Theta, A$ iff $\Gamma \mid\!\sim \Theta, \boxed{\mathfrak{S}}A$. Thus

$$\Gamma \mid\!\sim \Theta, \boxed{\mathfrak{S}}A \Leftrightarrow \mathfrak{Sf}(\Gamma \mid\!\sim \Theta, A). \qquad \square$$

The result of the above proof is a general method for introducing logical vocabulary that is *expressive* of structural features. If the rules for the logical vocabulary's introduction are reversible and the structural feature in question is *preserved* by $\mathbb{L}$, then the logical vocabulary will *express* that structural feature. I next rehearse two specific cases of this: an operator that marks monotonicity and an operator that marks classical validity.

[7] Note that the rest of our sequent calculus is altered such that our other rules preserve $\mid\!\sim^{\mathfrak{S}}$. E.g. R& requires that both top sequents have either $\mid\!\sim$ or $\mid\!\sim^{\mathfrak{S}}$ (I do not allow mixed cases).

4.1 Capturing Monotonicity '$\boxed{M}$' and Classicality '$\boxed{K}$'

The rules for monotonicity have the following form:

Axiom 2: If $\forall \Delta, \Lambda \subseteq \mathcal{L}_0(\Delta, \Gamma \mid\!\sim_0 \Theta, \Lambda)$ then $\Gamma \mid\!\sim^M \Theta$.

$$\frac{A, \Gamma \mid\!\sim^M \Theta}{\boxed{M}A, \Gamma \mid\!\sim^{[M]} \Theta}\ \text{L}\boxed{M} \qquad\qquad \frac{\Gamma \mid\!\sim^M \Theta, A}{\Gamma \mid\!\sim^{[M]} \Theta, \boxed{M}A}\ \text{R}\boxed{M}$$

I have already show in Theorem 1 that weakening is preserved by the rules of NM-MS. It therefore follows that:

Corollary 2 $\boxed{M}$ *expresses weakening/monotonicity. That is,*

$$\boxed{M}A, \Gamma \mid\!\sim \Theta \Leftrightarrow \forall \Delta, \Lambda(\Delta, A, \Gamma \mid\!\sim \Theta, \Lambda)$$
$$\Gamma \mid\!\sim \Theta, \boxed{M}A \Leftrightarrow \forall \Delta, \Lambda(\Delta, \Gamma \mid\!\sim \Theta, A, \Lambda)$$

This means that we may expand NM-MS (our $\mathbb{L}$) in order to mark *in the object language* which implications are persistent under arbitrary weakenings. Next, I demonstrate the same for "classicality", i.e. develop an operator that marks implications that are valid classically.

Axiom 2: If $\Gamma, p \mid\!\sim_0 p, \Theta$ then $\Gamma, p \mid\!\sim^K p, \Theta$ (where Γ, Θ may be possibly empty).

$$\frac{A, \Gamma \mid\!\sim^K \Theta}{\boxed{K}A, \Gamma \mid\!\sim^{[K]} \Theta}\ \text{L}\boxed{K} \qquad\qquad \frac{\Gamma \mid\!\sim^K \Theta, A}{\Gamma \mid\!\sim^{[K]} \Theta, \boxed{K}A}\ \text{R}\boxed{K}$$

Again, I have already shown in Theorem 3 that classicality is a feature NM-MS preserves. Thus any sequent which is derived from atomic sequents which are part of the CO (cf. Definition 4) fragment of $\mid\!\sim_0$ (regardless of whether $\mid\!\sim_0$ actually obeys CO) will be classically valid.

Corollary 3 *Let $\vdash_{LK}$ be the consequence relation instantiated by Gentzen's LK minus the rules for quantifiers (and with $\wedge$ substituted with $\&$, etc.). Then $\boxed{K}$ expresses classical validity, that is: o*

$$\boxed{K}A, \Gamma \mid\!\sim \Theta \Leftrightarrow A, \Gamma \vdash_{LK} \Theta$$
$$\Gamma \mid\!\sim \Theta, \boxed{K}A \Leftrightarrow \Gamma \vdash_{LK} \Theta, A$$

There are of course many further possibilities for such '$\boxed{\mathfrak{S}}$' operators. We may also introduce vocabulary for expressing inference that obey contraction, transitivity + weakening, more restricted weakening principles, and perhaps more.[8]

5 Some Defective Cases

So far I have introduced a more precise criterion for understanding logical expressivism and in particular for understanding how *structural* features of inference might be expressed. I then introduced a system that was not only *expressive* in this sense, but also successfully preserved and expressed several important structural features. In order to appreciate exactly what I am up to, however, it will be useful to look at some cases where each of these criteria fail.

Example 1 The multiplicative rules of linear logic are *not expressive*. I show that this is the case for the multiplicative conjunction $\otimes$:

$$\frac{\Gamma, A, B \vdash \Theta}{\Gamma, A \otimes B \vdash \Theta}\,\mathrm{L}\otimes \qquad\qquad \frac{\Gamma \vdash \Theta, A \qquad \Delta \vdash \Lambda, B}{\Gamma, \Delta \vdash \Theta, \Lambda, A \otimes B}\,\mathrm{R}\otimes$$

It is sufficient to show a case where the logic does not express particular implications in $\vdash_0$. Notice that there are potentially two ways to derive $p \otimes q \vdash p \otimes q$ where $p, q \in \mathcal{L}_0$:

$$\frac{\dfrac{p \vdash p \qquad q \vdash q}{p, q \vdash p \otimes q}\,\mathrm{R}\otimes}{p \otimes q \vdash p \otimes q}\,\mathrm{L}\otimes \qquad\qquad \frac{\dfrac{p, q \vdash q}{p \otimes q \vdash q}\,\mathrm{L}\otimes \qquad \vdash p}{p \otimes q \vdash p \otimes q}\,\mathrm{R}\otimes$$

Since the atomic sequents used to start each proof tree are different (in fact they are entirely different), it's possible that $\vdash_0$ includes one and $\vdash_0'$ includes the other and thus the presence of $p \otimes q \vdash p \otimes q$ does not guarantee the presence of either. In this sense, logics which include '$\otimes$' are not expressive in the relevant sense.

[8] Makinson for example considers a consequence relation which is supra-classical, monotonic, and obeys transitivity (Makinson, 2003, 2005). We could introduce an operator to express exactly this consequence relation along with some other consequence relations discussed therein.

It is also possible to find counter-examples to Sf-Preservation and Sf-Expression. Even using the rules of NM-MS such counter-examples will arise:

Example 2 Suppose we want to introduce an operator '$\boxed{R}$' to mark instances of reflexivity, i.e. $\phi \mid\!\sim \phi$. Then the rules for introducing such an operator should probably have the form:

Axiom 2: If $p \mid\!\sim_0 p$ then $p \mid\!\sim^R p$.

$$\frac{A, \Gamma \mid\!\sim^R \Theta}{\boxed{R}A, \Gamma \mid\!\sim^{[R]} \Theta}\ \text{L}\boxed{R} \qquad \frac{\Gamma \mid\!\sim^R \Theta, A}{\Gamma \mid\!\sim^{[R]} \Theta, \boxed{R}A}\ \text{R}\boxed{R}$$

Unfortunately, it is easy to show that NM-MS fails to preserve reflexivity and thus fails to express it. For example $A\&B \mid\!\sim A\&B$ is clearly an instance of reflexivity and thus we should want $A\&B \mid\!\sim \boxed{R}(A\&B)$. But clearly $A\&B \mid\!\sim A\&B$ must be derived from $A, B \mid\!\sim A$ and $A, B \mid\!\sim B$, neither of which are instances of reflexivity.[9]

There will therefore be logics which in general fail to be expressive and even among those that are expressive there will be structural features that fail to be preserved and thus expressed. Deciding how expressive one wants one's logic to be and which structural features ought to be preserved are therefore *not* independent questions.

6 Conclusion

In this paper I introduced a precisification of the notion of "logical expression". I also argued for two additional criteria for understanding when a structural feature is preserved and expressed. With these criteria in hand I introduced a system NM-MS. NM-MS is a sequent calculus without any structural features or restrictions placed on it. I showed that NM-MS is *expressive* in the precise, technical sense I argued for and I also exhibited some

[9]Though they are both found in the region of the consequence relation which allows reflexivity *together with weakening*, hence why we are able to have an operator to mark classicality.

It is also worth remarking that the above might also fail for independent reasons. For example, if we are able to derive $A\&B \mid\!\sim A\&B$, then we could also derive $A\&B \mid\!\sim B\&A$, but is the latter here an instance of the structural feature of reflexivity? It is not obvious that we should think so. In general, even when a sequent calculus preserves reflexivity, it needn't generate *only* reflexive sequents from the reflexive fragment of its axioms.

other interesting features it possesses. I next introduced some machinery for marking and thus expressing structural features in NM-MS where they occur. I showed that monotonicity and classicality are two features that may each be preserved and expressed in this way. I closed by exhibiting some cases where a logic *fails* to be expressive or *fails* to preserve and/or express a structural feature. The goal of the paper was to make the thesis of logical expressivism more precise and to introduce a logic which is actually expressive in the relevant sense. My hope is that providing such an account might help illuminate exchanges within the philosophy of logic: to proponents of expressivism, a clearer doctrine and a logic to call their own, and to those opposed, a clearer target.

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