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The co-existence of this work

and its author

is due largely to the mediation of Renée,

to whom

the dissertation

is affectionately dedicated.

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## PREFACE

Frege's semantical theories, and especially his distinction between sense (Sinn) and denotation (Bedeutung), have come in for increasing attention as philosophers have become aware of the special problems posed by the so-called oblique or intensional contexts (even the term oblique is from Frege's "Ungerade"). Such frequently used phrases as "John believes that", "John knows that", "it is surprising that", "it is necessary that" all pose logical and interpretive difficulties which have only recently come in for intensive investigation. The development of quantified modal logic, which had its beginnings only in 1946, has made the problem of interpretation especially acute.

In his article "Über Sinn und Bedeutung" (1892), Frege outlined his theory of sense and denotation and provided a special treatment of oblique contexts using the notion of indirect denotation. This article, which is concerned with the analysis of ordinary German, is written in a highly condensed, informal way; and though the sense and denotation distinction is mentioned in a few other of his writings, the doctrine of indirect denotation and thus his treatment of oblique contexts seems to be confined to this one source. Frege's work itself, therefore, is more in the nature of a proposal than a fully developed doctrine.

Though widely known, Frege's theory has not, in general, been received with favor. No one denies its importance; it is just that few like it. Alonzo Church, who has been its most consistent (and perhaps sole ) prominent champion, has called repeatedly for a precise development of the ideas according to contemporary logical and semantical standards. And surely, if a fair evaluation of this important theory is to be reached, such a development is required. In 1951, Church attempted to axiomatize the theory in his article "A Formulation of the Logic of Sense and Denotation", but that formalization has serious deficiencies. The present work is another attempt to formalize Frege's theory; this time depending more heavily on semantical methods first developed by Alfred Tarski than on the axiomatic method.

I had the great good fortune to be one of a few UCLA students who attended graduate courses in semantics given first by Rudolf Carnap and later by Alonzo Church. Each man lectured on his as yet unpublished ideas on intensional logic. The theories seemed complementary; Church's formal language and Carnap's interpretation of intensions, each offered solutions to difficulties in the other's theory. In this dissertation, the foundations of intensional logic are developed in a way which, hopefully, partakes of the best features of both Carnap's and Church's theories.

In the formulation presented herein, intensional logic becomes a branch of the theory of models, founded by Alfred Tarski and cultivated intensely since 1950 by a number of logicians. Those familiar with model theory will recognize a number of our intensional entities under other names. An especially close relation holds with the model theoretic notion of a direct product of models.

The research leading to this dissertation was supported by the National Science Foundation under N.S.F. G-13226, N.S.F. G-19830, and N.S.F. GP-1603. The work has benefited by criticisms, suggestions, and encouragement from my teachers Rudolf Carnap, Donald Kalish, and Richard Montague, who in addition are responsible for my philosophical style. Literary style and philosophical content are my own responsibility.

ABSTRACT OF THE DISSERTATION

Foundations of Intensional Logic

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A number of languages based on Gottlob Frege's distinction of sense (Sinn) and denotation (Bedeutung) are constructed. The semantics of these languages is developed purely within set theory. Axioms are provided and completeness and decidability results are obtained for certain of the languages. The languages provide facilities for treating oblique contexts but are fully extensional in the sense that the replacement of any part of a well formed expression by another with the same denotation leaves the denotation of the whole unchanged, where the denotation of a name, predicate, or sentence may be taken to be the thing named, the class of things to which the predicate applies, and the truth value of the sentence respectively.

The syntax of the languages is based on that of the language constructed in Alonzo Church's "A formulation of the logic of sense and denotation" in <u>Structure, method</u>

and meaning: essays in honor of Henry M. Sheffer, edited by Henle, Kallen and Langer, New York, 1951. The semantical interpretation of intensions is based on that of Rudolf Carnap's Meaning and necessity, Chicago, 1947.

Preliminary sections provide a formulation of what are taken to be the fundamental ideas of Frege's semantical theory, especially with regard to the analysis of oblique contexts. In these terms, languages of direct discourse are distinguished from languages of indirect discourse, and some advantages of the former are suggested. The languages of sense and denotation are developed as languages of direct discourse, and their similarity with other languages of direct discourse, in particular formalized semantical metalanguages, is emphasized.

# CHAPTER 1

# INTRODUCTION

## 1. The General Program

Our primary goal is to construct a language based on Frege's distinction of sense and denotation, and to describe the semantics of such a language within set theory. The language is to provide facilities for treating so-called oblique contexts but is to be fully extensional (in a sense to be given in section 3 below). The syntax of the language is based on that of Church [6] with certain modifications to conform to the semantical point of view adopted. The semantical interpretation (especially of intensions) is based on that of Carnap [4], [5], [6] with certain modifications to conform to the syntax of the language.

Preliminary sections provide a formulation of what are taken to be the fundamental ideas of Frege's semantical theory, especially in regard to the analysis of oblique contexts. In these terms, languages of direct discourse are distinguished from languages of indirect discourse and some disadvantages of the latter are suggested. The language of sense and denotation is developed as a language of direct discourse and its similarity with other languages of direct discourse (especially formalized semantical metalanguages) is emphasized.

## 2. Semantical Systems

We understand a _semantical system_ to be composed of two elements: a language and a semantical relation for that language.

### 2.1. Languages

A _language_ consists of a class of expressions which we call well formed expressions (wfe's) with a structure which determines a part-whole relation. This structure is often conveniently given in terms of a _type function_, a class of _atomic_ wfe's, and a set of syntactical operations on wfe's which we call _formation rules_ (or formation operations). The type function assigns a grammatical category or _type_ (for example: term, formula, two-place predicate of individuals, etc.) to each wfe. Each formation rule is a 1-1 function with all wfe's of a given type as its domain (or, in the case of an n-place ($n > 1$) formation rule, each domain consits of all wfe's of a given type) and its range included in the set of wfe's of a single type. The formation rules have disjoint ranges and a wfe is atomic just in case it is not in the range of any formation rule.[1] Every wfe is to be obtainable as the last element of a _construction_, that is, a finite sequence of wfe's each element of which is either atomic or the result of applying a formation rule to previous elements.[2] In these terms we can define the part-whole relation among wfe's. If $\alpha$ and $\beta$ are wfe's, we say that <u>$\alpha$ is a part of $\beta$</u> just in case $\alpha$ is an element of every construction of which $\beta$ is an element.

We have so far avoided reference to primitive symbols and concatenation, in order to avoid confounding the part-whole relation with which we are concerned with an analogous relation within pure syntax. In general, that a given wfe $\beta$ may be obtained from a wfe $\alpha$ by concatenation is no guarantee that $\alpha$ is a part of $\beta$ in our (semantical) sense. Merely scrutinizing a wfe will not reveal its parts; in fact, given a class of wfe's many different structures are possible (as indicated in the following examples).

Consider the expression '$(P \to Q)$'. We may consider 'P' and 'Q' as the only parts (other than '$(P \to Q)$' itself), and the entire formula simply as the result of combining these parts in a certain manner. Here, the signs '(', ')', '$\to$' are all thought of as syncategorematic, or as Church puts it <u>improper symbols</u>.[3] They merely express a mode of connection between 'P' and 'Q'. 'P' and 'Q' are thought of as having a certain semantical value (perhaps a truth value, or a meaning), as is '$(P \to Q)$'. But no independent value is assigned to '(', ')', or '$\to$'. Another possible structure for '$(P \to Q)$' would make 'P', 'Q', and '$\to$' all parts. Under this interpretation, '$\to$' would be thought of as having an independent semantical value, and the entire formula as the result of combining three wfe's into a certain compound.

We understand the part-whole relation as definable only for a whole language, which we may think of as a quadruple <E,T,A,F> where E is the class of wfe's, T the type function, A the class of atomic formulas and F the set of formulation rules.[4]

We do not claim that our notion of a language has special advantages over any other, except for the present explicitly semantical purposes.[5] For these purposes we are interested in which are the semantically relevant expressions (namely the wfe's) and which pieces of compound expressions are such that their semantical values are thought to be relevant to the semantical value of the whole.[6]

In this sense, we would ordinarily consider 'John' a part of 'John is tall', and 'cat' not a part of 'cattle'. The question as to whether 'John' should be considered a part of ''John'' seems to be generally answered in the negative. The question as to whether 'John' should be considered a part of 'It is necessary that John=John' has generally been answered in the affirmative, but there are important difficulties which arise from this decision.

If we are to associate transformation rules with a language, we would formulate them in terms of the structure of the wfe's. Thus, the difficulties which are avoided by not treating certain pieces as parts (as in the last example of the preceding paragraph) arise again, in that this treatment precludes these pieces from the scope of the transformation rules.

Here again we emphasize the importance of the abstract structure of the wfe's. If one is to describe the actual designs of the wfe's, many additional considerations come into play (such as the convention placing the identity predicate between its terms). For purposes of convenience we will usually describe languages in the conventional way, providing simultaneous recursive definitions of the class of wfe's and the type function in terms of atomic formulas and formation rules. The latter are only implicitly indicated by the clauses of the recursive definitions; however, it will be easily seen how to express the languages in terms of explicit formation rules. In particular, the essential feature of the formation rules, that they provide a unique decomposition of each wfe into its atomic parts, will be obvious.

## 2.2. Semantical Relations

A semantical relation (or, more exactly, a semantical function) for a language L assigns some entity to each wfe of L. For example, the relations which associate with each sentence of English, its truth value, its cognitive content, its emotive content, etc. (we take the word "entity" in a very broad sense) are all taken to be semantical relations for a language whose wfe's are English sentences.

## 3. Fregean Semantical Systems

When we are given a complete semantical system $\langle L,R \rangle$ we can meaningfully ask, are the values of compound wfe's functions of the values of the parts? That is, if $\alpha$ is a part of $\beta$, $\gamma$ is of the same type as $\alpha$, $\delta$ is the result of substituting $\gamma$ for one or more occurrences of $\alpha$ in $\beta$, and $R(\alpha) = R(\gamma)$, must $R(\beta) = R(\delta)$?[7] If this condition holds for all wfe's $\alpha, \beta, \gamma, \delta$, we call $\langle L,R \rangle$ _Fregean_. Our nomenclature is motivated by what Carnap has called _Frege principles of interchangeability_,[8] namely that the denotation of a compound wfe is to be a function of the denotation of the parts, and the sense of a compound wfe is to be a function of the sense of the parts.[9,10] We refer to the generalized interchangeability requirement (with respect to an arbitrary semantical relation) as _Frege's principle_.

Most familiar symbolic languages are Fregean under a natural analysis of the part-whole relation and a natural semantical relation. In fact, we often take as a principle of logic, axioms which make explicit these assumptions.

Consider, for example, Leibniz' Law:

$$\ulcorner \text{If } \alpha = \beta, \text{ then } \phi_\alpha \text{ if and only if } \phi_\beta \urcorner.$$

If we treat $\alpha$, $\beta$, $\phi_\alpha$, $\phi_\beta$ as parts of this expression, and take a semantical relation R which assigns to terms, the designated entity and to formulas, truth values, then

Leibniz' Law tells us that when $R(\alpha) = R(\beta)$, $R(\phi_\alpha) = R(\phi_\beta)$.

Euclid's Law can be understood as making a similar assertion about compound terms, with terms as parts. Interchange of Equivalents, though sometimes formulated in such a way that $\ulcorner \phi$ if and only if $\psi \urcorner$ is required to be a theorem in order to allow the inference of $\ulcorner \chi_\phi$ if and only if $\chi_\psi \urcorner$, can also be formulated in a weaker fashion so as to allow the interchange on the basis of the truth of $\ulcorner \phi$ if and only if $\psi \urcorner$. When formulated in this form, it can be understood as asserting that the truth values of compound formulas with formulas as parts is to be a function of the truth values of the parts.

If we turn to the familiar developments of the semantics of, say, first order logic, we see an even more explicit reflection of Frege's principle in the recursive definitions of 'satisfaction' and 'value' or 'designation'.[11]

If a semantical system <L,R> is Fregean, the semantical relation R can be put in the form <T*,A*,F*>, where T* is a function which assigns to each type of L a non empty set (the universe from which wfe's of that type take their values), A* is a function which assigns to each atomic wfe $\alpha$ of L an element of the universe of the type of $\alpha$ (that is, if $\alpha$ has type $\tau$, $A^*(\alpha) \in T^*(\tau)$) and F* is a function which assigns to every formation rule of L a corresponding function on elements in the universes of the types. The connection between Fregean systems and the <T*,A*,F*>

better $T^* \cup A^* \cup F^*$ ?

Too bad not to take types single-mindedly as sets. Then $\tau$ for $W(\tau)$.

representation is perhaps more easily seen in the following (where we write '$W(\tau)$' for 'the set of all wfe's of the language L of type $\tau$').

T1. If $L = \langle E,T,A,F\rangle$ is a language, and R is a semantical relation for L, then $\langle L,R\rangle$ is Fregean if and only if there are T*,A*,F* fulfilling the following conditions:

(1) T* is a function whose domain is the set of types of L (that is, the range of T) and which assigns to each such type a non-empty set as universe;

(2) A* is a function whose domain is the set of atomic wfe's of L (that is, A) and which assigns to each atomic wfe $\alpha$, an element of the universe of the type of $\alpha$ (that is, an element of $T^*(T(\alpha))$ );

(3) F* is a function whose domain is the set of formation rules of L (that is, F), and which is such that if $\tau_0,\ldots,\tau_r$ are types of L and f is a formation rule which assigns to every r-tuple of wfe's $\langle\alpha_1,\ldots,\alpha_r\rangle \in (W(\tau_1)\times\cdots\times W(\tau_r))$ a compound wfe $\in W(\tau_0)$, then $F^*(f)$ is a function from $T^*(\tau_1)\times\cdots\times T^*(\tau_r)$ into $T^*(\tau_0)$;

(4) if $\alpha$ is an atomic wfe of L, then $R(\alpha) = A^*(\alpha)$;

(5) if f is a formation rule of L, and $\langle\alpha_1,\ldots,\alpha_r\rangle$ is in the domain of f, then $R(f(\alpha_1,\ldots,\alpha_r)) = \big(F^*(f)\big)(R(\alpha_1),\ldots,R(\alpha_r))$.

T2. If $L = \langle E,T,A,F\rangle$ is a language, and R, T*, A*, F* are

Repeats half of T1?

as in clauses (1)-(5) of T1, then $\langle L\ R\rangle$ is Fregean. In particular, if $\alpha$ is any wfe of L, $R(\alpha)$ is an element of the universe of the type of $\alpha$ (that is, $R(\alpha) \in T^*(T(\alpha))$ ).

T3. If $L = \langle E,T,A,F\rangle$ is a language, and $T^*$, $A^*$, $F^*$ are as in clauses (1)-(3) of T1, then there is a unique semantical relation R for L which satisfies (4) and (5).

## 4. Denotation

We understand <u>denotation</u> as a semantical relation defined originally only for names (and descriptions) and assigning to each such name the thing named (or described). Let us call this relation "restricted-denotation". Barring certain peculiar constructions (the so-called oblique contexts), the restricted-denotation of compound names is a function of the restricted-denotation of the parts.

As a simple example consider the language $L_1 = \langle E,T,A,F\rangle$ where T assigns to each wfe the type <u>name</u>, $A = \{$'Aristotle', 'Leibniz', 'Frege'$\}$, $F = \{m,f\}$ where $m(\alpha) = \ulcorner$the mother of $\alpha\urcorner$, $f(\alpha) = \ulcorner$the father of $\alpha\urcorner$, and E is the closure of A under the formation rules.
Consider the triple $\langle T^*,A^*,F^*\rangle$, where $T^*$ assigns to the type <u>name</u> the set of humans, $A^*$ ('Aristotle') = Aristotle, $A^*$ ('Leibniz') = Leibniz, $A^*$ ('Frege') = Frege, $F^*(m)$ = that function which assigns to each human his mother, $F^*(f)$ = that function which assigns to each human his father. Clearly, if $R(\alpha)$ is the restricted-denotation of $\alpha$;

R, T*, A*, F* satisfy the conditions of theorem 1. Hence, $\langle L_1, R \rangle$ is Fregean.

Suppose now that we wish to extend the semantical theory of the relation of restricted-denotation. This may be done in essentially two different ways. We may leave the class of wfe's to which we apply the relation untouched and extend the class of formation rules, thus enlarging the part-whole relation. If we proceed by this method, we continue to apply the semantical relation only to names, but we may "break-down" compound names in new ways. For example, in the name 'the person whom John believes to be the president', 'John' would ordinarily be treated as a part, but we may now also attempt to treat 'the president' as a part. A second alternative for extending the semantical theory of restricted-denotation, is to enlarge the class of wfe's to which the relation is applied by including expressions of different types.

The first alternative seems quickly to lead to failures of Frege's principle and will be discussed in section 6. Let us consider here the second method of extending the relation of restricted-denotation and let us call the extended relation (simply) "denotation". Let us attempt to extend the class of wfe's by including sentences. We must now search for an appropriate universe of entities to serve as denotations for sentences. What considerations should guide such a search? Suppose that we have such an extended language L. Then beyond the obvious requirement that the relations of restricted-denotation and denotation should agree for wfe's of the type name, we shall propose two criteria:

C1: The semantical system $\langle L, \text{denotation} \rangle$ should be Fregean,

C2: The number of distinct entities which serve as denotations for sentences, should be minimized.

Since in general there are a number of different ways of assigning values to sentences which are compatible with C1, we propose C2 in order to require that we choose one of the most simple methods. In general, C1 requires us to assign distinct denotations to certain pairs of sentences, and C2 requires us to assign the same denotation to a pair of sentences unless C1 requires that their denotations be distinct.

Given an appropriately complex structure on L (in terms of the part-whole relation), C1 immediately rules out certain possibilities since it requires that for every formation rule we be able to find a corresponding semantical operation. For example, if Id is a formation rule of L (where $\mathrm{Id}(\alpha,\beta) = \ulcorner(\alpha = \beta)\urcorner$), C1 requires that we be able to find a semantical operation Id* such that den $(\mathrm{Id}(\alpha\beta)) = \mathrm{Id}^*(\text{den }(\alpha), \text{den }(\beta)) = \mathrm{Id}^*(\text{rden }(\alpha), \text{rden }(\beta))$ (where we write 'den $(\alpha)$' for 'the denotation of $\alpha$' and 'rden $(\alpha)$' for 'the restricted-denotation of $\alpha$'). Thus we cannot take the meaning of a sentence as its denotation, since the meaning of an identity sentence is not determined solely by the individuals named; we need to know in addition <u>how</u> they are named. Compare, for example, the meanings of Id ('Hesperus', 'Phosphorus') and Id ('Hesperus', 'Hesperus'). The meanings clearly differ, although rden ('Hesperus') = rden ('Phosphorus'). We may treat this argument as showing that the relation between a sentence and its meaning is not the natural analogue of restricted denotation.[12]

The degree to which C1 rules out possibilities depends of course on the complexity of the structure on L. If this is given in such a way that no name contains a sentence as a part (although sentences may contain names and other sentences as parts) then C1 permits, and hence C2 requires, that all sentences take the same value. Suppose, however, that L contains compound names of the form $\ulcorner$the unique individual who is identical with $\alpha$ if $\phi$, and who is identical with $\beta$ if it is not the case that $\phi\urcorner$ with the names $\alpha$, $\beta$ and the sentence $\phi$ as parts. Suppose, in fact, that we have a formation rule g which yields the given compound name when applied to any wfe's $\alpha$, $\beta$, $\phi$ of the appropriate types.[13] Let g* be the corresponding semantical operation, that is, let den $(g(\alpha, \beta, \phi))$ = g* (den($\alpha$), den ($\beta$), den ($\phi$)). The denotations of $\alpha$, $\beta$, and $g(\alpha, \beta, \phi)$ are already determined by the fact that they are all names and our requirement that the denotation of a name be the same as its restricted-denotation. Hence we know that if $\phi$ is true, den $(g(\alpha, \beta, \phi)$ = rden $(g(\alpha, \beta, \phi)$ = rden $(\alpha)$, and if $\phi$ is false, den$(g(\alpha, \beta, \phi))$ = rden $(\beta)$. It follows that if L contains names $\alpha$, $\beta$ such that rden $(\alpha) \neq$ rden $(\beta)$, then C1 requires that if $\phi, \psi$ differ in truth value, den $(\phi) \neq$ den $(\psi)$. Since if den $(\phi)$ = den $(\psi)$ when, say $\phi$ is true and $\psi$ false, rden $(\alpha)$ = den $(g(\alpha, \beta, \phi)$ $= g^*($den $(\alpha)$, den $(\beta)$, den $(\phi)) = g^*$ (den$(\alpha)$, den$(\beta)$, den $(\psi))$ = den $(g(\alpha, \beta, \psi))$ = rden $(\beta)$.

Assuming that L contains compound names like $g(\alpha, \beta, \phi)$ but is so structured that allowing den $(\phi)$ to be the truth value of $\phi$ satisfies C1 (thus L contains no oblique contexts), then C2 <u>requires</u> us to assign the same value to sentences with the same truth value.[14]

Church has argued that the natural extension of the relation between a name and the thing named (which relation he calls 'denotation') to sentences would assign truth values to sentences.[15] He uses essentially two criteria, as do we. One corresponds to our C1, but where we would use C2 Church uses the seemingly gratuitous assumption that logically equivalent sentences have the same denotation. Actually Church's assumption, though not so general as our C2, is more closely directed to the commonly held opinion that the "natural" semantical relation is that which assigns propositions to sentences and individuals to names. It is to combat such beliefs, which have hindered the acceptance of semantical relations which are natural by our criteria but which countenance such seemingly "unnatural" entities as individual concepts, that the foregoing sketchy account of the genesis of general (that is, applying to more than one type of wfe) semantical relations was introduced.

We understand by 'denotation' an extension in accordance with C1 and C2 of the relation of restricted denotation. We take truth values to be the denotation of sentences, functions from individuals to truth values to be the denotation of (one-place) predicates, truth functions to be the denotation of sentential connectives, etc.[16] Note that the fact that we have extended the relation of restricted-denotation to apply to sentences and expressions of other grammatical categories is no better described as "treating sentences as names" than a similar extension of the relation between a sentence and its truth value to apply to names would be described as "treating names as sentences."[17] It is similarly misleading to refer to sentences as "naming" their truth value, just as it would be to refer to the restricted-denotation of a name as its "truth value". Of course, we do not wish to deny that the extension of a restricted semantical relation is based on a certain analogy between it and the extended relation, but the (cognitive) content of this analogy is given in our two criteria.[18] It should not be supposed that when we speak of the "denotation" of a sentence we are abandoning the traditional segregation of expressions into grammatical categories.[19]

## 5. Sense

Although the semantical relation we call "denotation" has probably received the fullest and most adequate treatment, other relations such as that between a sentence and its meaning, that between a sentence and its emotive content, and even that between an expression and itself (the so-called autonymous use) have also received a good deal of attention. Let us consider the relation between a sentence and its meaning. We shall call this relation "restricted-meaning" (on the analogy of our earlier "restricted-denotation"), and similarly speak of the restricted-meaning of a sentence. We use this language to call attention to the fact that originally we think of sentences as the only vehicles of meaning. However, if we wish to provide a fuller treatment of this relation we may attempt to extend it to other grammatical categories of expressions (just as we did the denotation relation). Here, as before, we are guided by C1 and C2 (in their general form).[18] In this way we come to speak of the meaning of predicates, names, connectives, etc., in such a way that the meaning of a compound expression will be a function of the meaning of its parts, or, in the earlier language, we develop a semantical relation M (meaning) such that $\langle L,M\rangle$ is Fregean.

That the meaning of a name can not be identified with the denotation of the name was clearly enunciated by Frege, who introduced his discussion of the two semantical relations: denotation and meaning, by asking how $\ulcorner(\alpha = \beta)\urcorner$, if true, can differ in meaning from $\ulcorner(\alpha = \alpha)\urcorner$. If we identify the meaning of a name with its denotation, then the truth of $\ulcorner(\alpha = \beta)\urcorner$, which presumably tells us that the denotation of $\alpha$ is the same as that of $\beta$, identifies the meaning of $\alpha$ with that of $\beta$ and hence, by Frege's principle, the meaning of $\ulcorner(\alpha = \beta)\urcorner$ with that of $\ulcorner(\alpha = \alpha)\urcorner$. We shall follow Frege in using "sense" (from Frege's "Sinn") for the meaning relation. Thus we shall speak of the sense of a sentence, the sense of a predicate, the sense of a name, etc. We follow Church [6] in also using the word "concept" to refer to those entities capable of being senses of expressions. Suppose an expression $\alpha$ has X as its denotation and S as its sense, then we say that <u>$\alpha$ expresses S</u>, <u>$\alpha$ denotes X</u>, and <u>S is a concept of X</u>.[20]

## 6. Extensionality

A notion which has been much discussed in the literature and which is related to our notion of a Fregean semantical system is that of extensionality. Our notion differs from those commonly found in that it is more general. We consider an arbitrary semantical relation R, whereas extensionality is usually only discussed with respect to the relation we call denotation. However, the more general notion is at the heart of Frege's program, as we see it, and plays an explicit role in his discussions of the extended notion of meaning or sense. In addition, some authors seem to consider certain kinds of <u>entities</u> as extensional and others as intensional; for example, sets are often called extensional and properties intensional. But this conception is based on the mistaken belief that only certain kinds of entities can function as denotations of wfe's and only certain other kinds can function as senses, so that by examining the kinds of entities taken as semantical values of the wfe's, we could determine whether the semantical relation is that of denotation. Against this conception we remark that although not every kind of entity can be the sense of a wfe, any entity can be denoted by some wfe. In particular, if $\alpha$ names a wfe, then the sense of that wfe is certainly denoted by the expression $\ulcorner$the sense of $\alpha\urcorner$, and may also be denoted by the expression "John's favorite concept" (for further discussion of this point see section 8).

We will call a semantical system <u>extensional</u> if the system is Fregean and the semantical relation is that of denotation.

Our argument that truth values be taken as the denotations of sentences depended on choosing a language with a structure sufficiently rich to contain compound names like $g(\alpha, \beta, \emptyset)$ but not so rich as to contain compound names like 'the individual whom Jane believes to be the 35th president' (with both 'Jane' and 'the 35th president' as parts). In fact, if our language structure is too rich, the question of extending restricted-denotation to a semantical relation which provides an extensional semantical system does not even arise, since even that part of the language which contains only names as wfe's may be so rich that the semantical relation restricted-denotation already gives a non-Fregean semantical system.

It seems clear that Jane may be sufficiently out of touch with current events for 'the individual whom Jane believes to be the 35th president' to name Richard Nixon, but not be so confused that 'the individual whom Jane believes to be John F. Kennedy' names Richard Nixon. Since 'the 35th president' and 'John F. Kennedy' name the same individual, we see that if a language contains such compound names, the semantical relation of restricted-denotation will give a non-Fregean semantical system, hence also a non-extensional semantical system.

In addition to complex wfe's formed using such psychological expressions as 'believes', 'doubts', 'asserts', etc. there are many other contexts that produce failures of extensionality. By a "context" we mean a formation rule (although the exact structure of the rule is often only implicit in our examples). Thus every "context" determines (in part) a part-whole relation. If f is a context for the language L and there is no semantical function f* such that den $(f(\alpha_1,\ldots,\alpha_r)) = f^*(\mathrm{den}(\alpha_1),\ldots,\mathrm{den}(\alpha_r))$ for all $\langle\alpha_1,\ldots,\alpha_r\rangle$ in the domain of f, then we say that f is an oblique context for L.[21] Here we again follow Frege, who called such contexts "ungerade."

It seems best to regard the notion of an oblique context in L, for arbitrary languages L, as a theoretical primitive of semantics. We then regard the failures of extensionality as described above as well as various other "tests" that have been proposed, such as the failure of the validity of existential generalization (first suggested in Quine [1], see also Church's review, Church [2]), as being indicative but not definitive of obliquity. Many contexts can be constructed which pass all the familiar "tests," yet still seem more naturally classified as oblique. For example, suppose that Mr. Jones has the singular good fortune to have all and only those beliefs which are true. Should such a contingency require us to classify the context $\ulcorner$Mr. Jones believes that $\phi\urcorner$ as non-oblique? Note that the context is now extensional. A definition of obliquity in terms of the <u>possibility</u> of a failure of extensionality seems hardly satisfactory in view of the obliquity of possibility. And even if this objection were not telling, still further cases are available which pass even such a test. For these reasons we prefer to regard the various tests only as providing sufficient conditions for the presence of an oblique context.

Once sensitized to the presence of oblique contexts, they seem ubiquitous in ordinary speech. Quine, in particular, has been remarkably successful in identifying and exposing some of the most subtle of such contexts.[22] He has also developed machinery for classifying and notationally exhibiting some of their peculiar features. But his interest in such matters seems primarily for purposes of quarantine, to avoid their working mischief in our more mundane preoccupations. With respect to a "logic" of such contexts, he has often expressed scepticism.[23]

Among other non-extensional contexts are $\ulcorner$It is necessary that $\phi\urcorner$, $\ulcorner$'$\alpha$'$\urcorner$, $\ulcorner$It is provable in L that $\phi\urcorner$.[24] In view of the importance of such contexts, the question immediately arises, can we provide an adequate treatment of the notions involved (e.g., modality) within an extensional semantical system? The affirmative position on this question is known as the <u>thesis of extensionality</u>, expressed by Carnap as follows: "for any nonextensional system there is an extensional system into which the former can be translated".[25] Without involving ourselves in the difficulties of providing an exact explication for 'translation', let us now attempt an analysis of oblique contexts along Frege's lines.

## 7. Obliquity and Ambiguity

Consider the following two names:

(1) the number of syllables in: venus

(2) the number of syllables in: the morning star

It appears that (2) can be obtained from (1) by replacing one name by another with the same denotation. Since (1) denotes two and (2) denotes four, the context ⌜the number of syllables in: $\alpha$⌝ seems to be oblique. But if it were claimed that any language incorporating (1) and (2) must be non-extensional one might reply that in (1) and (2) the expressions 'venus' and 'the morning star' were being used in an unusual way. Here, they are being used to denote themselves, not the planet. Thus in the present context they do not have the same denotation, and hence (1) and (2) can not provide a counter instance to Frege's principle. The reply may be put in another way. The expression 'venus' is ambiguous in English; it is usually taken as denoting a certain planet, but may in special circumstances be taken as denoting something else (for example, a picture, or the word itself). If the language we are analyzing contains such ambiguities, we must of course withdraw, or at least modify, any claim of extensionality. But *such* failures of extensionality do not in general pose deep theoretical problems for semantics. In a constructed language the apparent simplicity attained by allowing the same expression to function in different ways in different contexts, would probably be outweighed by the relative complexity of the semantical rules and the transformation rules. In particular, semantical systems which are Fregean have a certain simplicity of their own. Thus it seems profitable to first attempt to retain extensionality by revising the language so as to remove ambiguities. In this line we may insist on writing (1) and (2) as:[26]

(3) the number of syllables in 'venus'

(4) the number of syllables in 'the morning star'

It seems to have been Frege's belief, and we take it as a tenet of that semantical tradition stemming from his work, that all of the oblique contexts were susceptible of an analysis in terms of ambiguities, along the above lines.[27]

Although Frege particularly called attention to those cases of ambiguity where names denote either themselves or their (usual) senses rather than their usual denotations, he did not propose any language reform along the lines of (3) and (4). Possibly this was due to the fact that he never attempted a formal treatment of any language adequate to express the oblique contexts.

## 8. Direct and Indirect Discourse

If we adopt Frege's point of view, namely that failures of extensionality are due to ambiguity, we may distinguish two approaches to the formalization of a language adequate to express oblique contexts.[28] If we follow the method of indirect discourse, we will not insist on any language revision but will attempt to avoid paradox by carefully restricting the transformation rules; for example, existential generalization on two occurrences of the same name would not be allowed in certain contexts (those in which the name has two different denotations). If we follow the method of direct discourse, we first require that distinct uses of expressions be marked by some distinction in the expressions themselves. By introducing a multiplicity of expressions to avoid paradoxes, we can maintain a (relatively) standard form for the transformation rules.[29]

The two methods may be illustrated by reference to modal logic (a subject not discussed by Frege). Suppose we want to formalize a modal logic with identity and the names 'Hesperus' and 'Phosphorus'. Following the method of indirect discourse, we would treat such expressions as

(5) Hesperus = Phosphorus, and it is not necessary that Hesperus = Phosphorus

as wfe's. But we would restrict applications of Leibniz' Law so that

(6) It is not necessary that Hesperus = Hesperus

could not be obtained from (5) by the transformation rules. Similarly, we would be cautious about allowing the inference of

(7) There is an x, such that x = Phosphorus, and it is not necessary that x = Phosphorus

from (5), at least in the absence of some special semantical treatment which validates such inferences.

Following the method of direct discourse we first note that the failure of the inference of (6) from (5), indicates that in (5) the names 'Hesperus' and 'Phosphorus' are used ambiguously. Hence we will have to introduce a pair of names corresponding to each of the names used in the previous method. But first we must decide what the terms denote in the necessity context. Two possibilities immediately occur: they may denote themselves or they may denote their (usual) senses. Suppose we take the first possibility, we then note that the fact that necessity is not truth functional indicates that sentences do not have their usual denotations (namely, truth values). Hence we must replace the second occurrence of the identity predicate with a different sign, which when combined with two terms yields not a sentence (that is, an expression denoting a truth value) but rather an expression which unambiguously denotes whatever we understand the ambiguous use of the sentence to denote. Since we have chosen to understand the terms as denoting themselves, it is natural to understand the sentence in the same way. Hence in the present method we retain the expressions 'Hesperus' and 'Phosphorus' in their usual use, and add new expressions, say '$\text{Hesperus}_1$' and '$\text{Phosphorus}_1$', similarly we retain '=' but add an operation symbol '$=_1$' (where we understand '$=_1$' as denoting the syntactical operation Id of section 4). The counterpart of (5) is now expressed as:

(8) Hesperus = Phosphorus, and ($Hesperus_1 =_1 Phosphorus_1$) is not necessary.

There remains no temptation to infer any analogue to (6) by Leibniz' Law, (note that the expression 'Hesperus' is not considered a part of the expression '$Hesperus_1$') or any analogue to (7) by existential generalization. Thus we can use transformation rules (and also provide a semantical interpretation) of a relatively simple character.[30]

As matters stand, we have described the relative merits of the two approaches as involving a choice between simplicity of the structure of the wfe's on the one hand, and simplicity of the transformation and semantical rules on the other hand. There are, however, more profound differences. If we follow the method of indirect discourse and think of expressions in modal contexts as denoting themselves, we have the result that two expressions which in modal contexts denote the same expression, are themselves identical. We are thus deprived of the means of expressing arguments which turn on the use in modal contexts of non-synonymous expressions to denote the same expression. Consider, for example, the following informal argument given in direct discourse.[31]

Assume that John's favorite sentence is 'Hesperus = Hesperus'. Then presumably

(9) John's favorite sentence is necessary

is true. But

(10) (9) is necessary

is not only not necessary but false, since the truth of (9) depends on a contingent assumption.[32] However,

(11) 'Hesperus = Hesperus' is necessary

is true, and so is

(12) (11) is necessary.

Thus one might conclude that the principle that what is necessary is necessarily so, sometimes holds (as with (11)) and sometimes fails (as with (9)).

If we attempt to formulate the preceding argument in a modal logic of indirect discourse (that is, where 'is necessary' is preceded not by the name of a sentence but by a sentence itself) we will be able to express (11) as

(13) (Hesperus = Hesperus) is necessary

and (12) as

(14) ((Hesperus = Hesperus) is necessary) is necessary;

but no means is available to express (9).

The argument turns on the fact that John's favorite sentence = 'Hesperus = Hesperus', but 'John's favorite sentence' ≠ ''Hesperus = Hesperus''; in fact, 'John's favorite sentence' is not even synonymous with ''Hesperus = Hesperus''.

If we treat expressions in oblique contexts as denoting their (usual) senses (rather than themselves) and develop our language by the method of indirect discourse we have the analogous result that two expressions which in modal contexts denote the same sense are themselves synonymous (that is, they have the same sense). It may have been this fact that has led a number of authors (among them apparently Carnap and Quine)[33] to the mistaken belief that there is something in the nature of senses which makes them incapable of being denoted by non-synonymous expressions. On the contrary, if $\alpha$ and $\beta$ denote distinct entities of any kind whatever, we can always construct another name $g(\alpha, \beta, \emptyset)$ (using the formation rule g of section 4) such that $\ulcorner g(\alpha, \beta, \emptyset) = \alpha \urcorner$ is true but contingent. Thus we see that if we choose the method of direct discourse, we may introduce in addition to the one new expression which replaces the ambiguous use of an old expression, a number of new expressions (perhaps compound, as with $g(\alpha, \beta, \emptyset)$) all with the same denotation but with different senses.

Our comparisons of the two approaches to the formalization of a language adequate to treat oblique contexts, and indeed even the description of the two methods, has been based on a semantical treatment of such contexts along Frege's lines, namely: that the denotation of compound expressions is always a function of the denotation of the parts, but in some cases the denotation of an expression may vary with the context. However, some authors have probably chosen what we call the indirect discourse method primarily to avoid committing themselves to such a semantical analysis. If we replace occurrences of 'Hesperus' in modal contexts by occurrences of '$\text{Hesperus}_1$' where the latter is thought of as denoting the sense of the former, and especially if we then go on to take the natural step of introducing variables for which 'Hesperus$_1$' is a substutuend, we seem committed to the Fregean analysis along with its commitment to an entity called "the sense of the word 'Hesperus'". Unfortunately, many authors who have chosen the indirect discourse method have avoided committing themselves to any semantical analysis whatever, preferring to focus attention on transformation rules which inhibit the derivation of highly implausible conclusions from highly plausible premises, and treating formulas of dubious meaning by a combination of suggestion and revelation.[34]

Our aim is primarily semantical, and hence we adopt the method of direct discourse. In so doing we seek to emphasize the fundamental similarity of different treatments within this method (for example, that treatment wherein the new expressions are taken as denoting senses, and that treatment wherein the new expressions are taken as denoting other expressions) as opposed to the fundamental differences between direct and indirect discourse treatments. Although in certain cases one direct discourse treatment has certain advantages over another, these distinctions have perhaps been over-emphasized in the literature so as to neglect the connection of all such treatments with the Frege tradition.[35] We will attempt to show the similarity of different direct discourse treatments by developing a language of direct discourse which admits of both an interpretation whereby the expressions in oblique contexts denote (other) expressions (we call this the *syntactical* treatment) and an interpretation whereby the expressions in oblique contexts denote senses (we call this the *intensional* treatment).

# CHAPTER 2

## FIRST FORMULATIONS

### 9. Semantical Systems for Oblique Contexts

We conceive of languages adequate to treat oblique contexts as extensions formed in the following way of the more familiar languages. We begin with a base language, and add whatever apparatus is needed in order to express the oblique contexts of the wfe of the base language (we call such contexts <u>singly oblique</u>). We then close this language under the logical apparatus of the base language. If we refer to the base language as $L_o$ we may refer to the new language as $L_1$. It contains essentially the logical apparatus of $L_o$ with one layer of obliquity available. If we begin, for example, with the sentential calculus as $L_o$ and we are developing a treatment of modal logic, we would add all expressions of the form $\ulcorner N\,\emptyset\urcorner$ (where $\emptyset$ is either a formula of $L_o$ or the analogue to such a formula, depending on whether we choose the indirect discourse or the direct discourse methods). We then form the language $L_1$ by taking all the sentential combinations of the new wfe's with the formulas of $L_o$. The process of forming $L_1$ from $L_o$ can then be repeated to form $L_2$ from $L_1$. In $L_2$ we have available the apparatus to treat doubly oblique contexts (for example ⌜NNØ⌝). In this way we form a sequence of languages, each a sublanguage of its successor.

Among the earliest treatments of formalized languages for oblique contexts are the modal logics of C. I. Lewis [1], based on the sentential calculus. Among the most recent treatments is the system of Church [6], based on the simple theory of types. We shall choose a middle course, basing our systems essentially on the first order predicate calculus.[36] However, before constructing these languages, we shall introduce a hierarchy of languages $\mathcal{K}_o$, $\mathcal{K}_1$,... based on a logic which avoids the complexities involved with variables and variable binding operators. In the present chapter the languages $\mathcal{K}_o$ and $\mathcal{K}_1$ are introduced and the fundamental ideas behind their interpretations are discussed.

The languages $\mathcal{K}_o$, $\mathcal{K}_1$,... all share certain simple characteristics. The wfe's fall into two broad categories, those of simple type and those of complex type. The simple types are again divided into two hierarchies: $i$, $i_1$, $i_2$,... and $t$, $t_1$, $t_2$,... . Each simple type is associated with a certain universe of entities. For example, under the syntactical interpretation, wfe's of the simple type $i$ will denote individuals, those of the simple type $i_1$ will denote names of individuals, those of type $i_2$ will denote names of names of individuals, etc. Similarly, wfe's of the simple type t will denote truth values (thus, such wfe's are sentences), those of type $t_1$ will denote wfe's denoting truth values (that is, sentences), those of type $t_2$ will denote wfe's denoting wfe's denoting truth values, etc. Under the intensional interpretation the wfe's of type $i_1$ will denote concepts of individuals, those of type $i_2$ will denote concepts of concepts of individuals, etc. The wfe's of type $t_1$ will denote concepts of truth values (that is, propositions), those ot type $t_2$ will denote concepts of concepts of truth values, etc. The wfe's of complex type combine with wfe's of simple type to form compound wfe's of simple type. If a wfe $\eta$ combines with a wfe $\alpha$ of simple type i to form a compound wfe $\eta^\frown\alpha$ of simple type t, then $\eta$ will have the complex type <i, t> and will denote a function which assigns to every element of the universe of the type i an element of the universe of the type t.[37] Hence, in the present case, $\eta$ will denote a function from individuals to truth values and is therefore what is commonly called a one-place predicate. Aside from the introduction of variable-binding operators, all compound wfe's are formed by prefixing a wfe denoting a function to wfe's denoting its arguments. Thus the formation rules are all of essentially the same form (concatenation of a function expression with its argument expressions) and the corresponding semantical operations all amount to the application of a function to its arguments. Only atomic wfe's will have complex types; hence all compound wfe's will be of simple type. In addition, only wfe's of simple type may stand as argument expressions, so we have no second-order wfe's in any of our languages. Languages with a variety of simple types, such as ours, have been called "many sorted".

Our base language $\mathcal{K}_0$ will contain only the simple types i and t. When we move to the language $\mathcal{K}_1$ we will add wfe's of the simple types $i_1$, $t_1$ to replace the wfe's of types i and t in singly oblique contexts. Similarly, the step to $\mathcal{K}_2$ requires adding new wfe's of types $i_2$, $t_2$, etc.

## 10. The Language $\mathcal{K}_0$

The base language $\mathcal{K}_0$ is simply the first order predicate calculus with identity but without variables.

D1. The simple types of $\mathcal{K}_0$ are i and t (where i = 2 and t = 3).[38] The complex types of $\mathcal{K}_0$ consist of all finite sequences $\langle \tau_1, \ldots, \tau_r, \tau_0 \rangle$, where $\tau_0, \ldots, \tau_r$ are simple types of $\mathcal{K}_0$ and $r>0$.

D2. The atomic wfe's of $\mathcal{K}_0$ fall into the following categories:

(1) for each natural number p, a denumerable number of p-place operation symbols of which Opsymb (m,p) is the mth,[39]

(2) for each natural number p, a denumerable number of p-place predicates of which Pred (m,p) is the mth,

(3) the material conditional sign *Cond* and the negation sign *Neg*,

(4) the identity signs for individuals *Id(i)* and for the truth values *Id(t)*.[40]

D3. A 0-place operation symbol is called an *individual constant*, and a 0-place predicate is called a *sentential constant*.

D4. *The type of an atomic wfe of* $\underline{\mathcal{K}_o}$ is given by the following:

(1) for all m, Opsymb (m,o) has type i, Opsymb (m,1) has type <i,i>, Opsymb (m,2) has type <i,i,i>, etc.,

(2) for all m, Pred (m,o) has type t, Pred (m,1) has type <i,t>, Pred (m,2) has type <i,i,t>, etc.,

(3) Cond has type <t,t,t> and Neg has type <t,t>,

(4) Id(i) has type <i,i,t> and Id(t) has type <t,t,t>.

D5. *$\beta$ is a well formed expression (wfe) of* $\underline{\mathcal{K}_o}$ *of type* $\underline{\tau}$ if and only if:

(1) $\beta$ is an atomic wfe of $\mathcal{K}_o$ of type $\tau$, or

(2) there are wfe's $\eta$, $\alpha_1,\ldots,\alpha_r$ of $\mathcal{K}_o$ of types $\langle\tau_1,\ldots,\tau_r,\tau\rangle$, $\tau_1,\ldots,\tau_r$ respectively and $\beta$ is $\eta^\frown\alpha_1{}^\frown\cdots{}^\frown\alpha_r$.

For the sake of familiarity, we have excluded wfe's of such complex types as <t,t,i>, <i,t,i>, etc.,[41] and have also excluded sentential connectives other than the usual logical signs. In many ways, however, it might be more natural to admit descriptive constants of every type.

It is clear that $\mathcal{K}_0$ can be put in the form <E,T,A,F> discussed in Chapter 1 (especially section 2.1). Corresponding to every complex type $\langle \zeta_1, \ldots, \zeta_r, \zeta_0 \rangle$, we would have a formation rule which yields the compound $\eta^\frown\alpha_1{}^\frown\cdots{}^\frown\alpha_r$ given any $\eta, \alpha_1, \ldots, \alpha_r$ of the appropriate types. Since, as noted above, not all complex types of $\mathcal{K}_0$ are represented by wfe's, some of these formation rules (if thought of as functions) would just be the empty function. To fulfill the requirements laid down in section 2.1, we must establish that (1) each formation rule is one-one, (2) distinct formation rules have disjoint ranges, and (3) no atomic wfe is in the range of a formation rule. If '$\frown$', 'Pred', 'Opsymb', 'Cond', etc., are taken as primitive, we may simply lay down (1)-(3) as axioms. An alternative procedure is to identify wfe's with certain finite sequences of natural number and the concatenation of expressions with the concatenation of finite sequences, (1)-(3) are then derivable in terms of the following definitions.

**D6.** The atomic wfe's of $\mathcal{K}_0$ are defined as follows:

(1) Opsymb (m,p) = $\langle 2^{m+1}.3^{p+1}.5 \rangle$

(2) Pred (m,p) = $\langle 3^{m+1}.5^{p+1}.7 \rangle$

(3) Cond = $\langle 5 \rangle$

(4) Neg = $\langle 7 \rangle$

(5) Id(i) = $\langle 2^{2+1}.3 \rangle$

(6) $Id(t) = \langle 2^{3+1}\cdot 3\rangle$

D7. If $\alpha$, $\beta$ are finite sequences of lengths m and n respectively, then $\alpha\frown\beta$ is the finite sequence of length m+n, whose jth element is the jth element of $\alpha$ if $j<m$, and the (j-m)th element of $\beta$ if $j\geqslant m$.

## 11. The Semantics of $\mathcal{K}_o$

Instead of settling on some particular semantical relation for $\mathcal{K}_o$ we will describe the general characteristics of a large class of such relations, namely the denotation functions for $\mathcal{K}_o$. In essence, such a function must assign an individual to every wfe of type i, a truth value to every wfe of type t and a function of the appropriate kind to every wfe of complex type. In addition, it must assign certain particular functions to the logical atomic wfe's, that is, Cond, Neg, Id(i), and Id(t). Since we want our semantical systems to be extensional (that is, Fregean with respect to denotation), we will represent the semantical relations in a form closely related to the representation <T*,A*,F*> of section 3. It will be recalled that T* was to be a function assigning a universe to each type, A* a function assigning an element of the appropriate universe to each atomic wfe, and F* a function assigning a function of elements of the universes to each formation rule. We shall take as our fundamental semantical notions, <u>$\mathfrak{M}$ is a model for $\mathcal{K}_o$</u>, <u>the universe of the type $\tau$</u>

*in the model $\mathfrak{M}$ with respect to the language $\mathcal{K}_o$*, and *the value of the wfe $\alpha$ in the model $\mathfrak{M}$ with respect to the language $\mathcal{K}_o$*. A model for $\mathcal{K}_o$ will be an ordered couple $\langle DR \rangle$ where D is the universe of the type i (that is, the set of individuals) and R is a function which, like A*, assigns an appropriate entity to each atomic wfe.

D8. *$\mathfrak{M}$ is a model for $\mathcal{K}_o$* if and only if there are D,R such that:

(1) $\mathfrak{M} = \langle DR \rangle$

(2) D is a non-empty set

(3) R is a function which assigns an entity to each atomic wfe $\alpha$ of $\mathcal{K}_o$ in accordance with the following:

(a) if $\alpha$ is an individual constant, then $R(\alpha)$ is an element of D

(b) if $\alpha$ is a sentential constant, then $R(\alpha)$ is a truth value

(c) if $\alpha$ is a p-place operation symbol and $p>0$, then $R(\alpha)$ is a function which assigns an element of D to each p-tuple of elements of D

(d) if $\alpha$ is a p-place predicate and $p>0$, then $R(\alpha)$ is a function which assigns a truth value to each p-tuple of elements of D

(e) if $\alpha$ is the material conditional sign, then $R(\alpha)$ is that two-place truth function which assigns F to $\langle TF \rangle$ and T to all other pairs of truth values[42]

(f) if $\alpha$ is the negation sign, then $R(\alpha)$ is that one-place truth function which assigns F to T and T to F.

(g) if $\alpha$ is the identity sign for individuals, then $R(\alpha)$ is that function from pairs of elements of D which assigns T to a pair of identical elements and F to all other pairs

(h) if $\alpha$ is the identity sign for truth values, then $R(\alpha)$ is that function from pairs of truth values which assigns T to a pair of the same truth values and F to all other pairs.

D9. If $\mathfrak{M} = \langle DR \rangle$ is a model for $\mathcal{K}_o$, and $\tau$ is a type of $\mathcal{K}_o$, then <u>the universe of $\tau$ in $\mathfrak{M}$ with respect to $\mathcal{K}_o$</u> $(\mathcal{U}^{\mathcal{K}_o}_{\mathfrak{M}}(\tau))$ is given by the following:

(1) the universe of i is D

(2) the universe of t is $\{T,F\}$ (the set of truth values)

(3) if the universes of $\tau_o,\ldots,\tau_r$ are respectively $u_o,\ldots,u_r$ and $r>0$, then the universe of $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$ is the set of all functions from $(u_1 \times \cdots \times u_r)$ into $u_o$.

D10. If $\mathfrak{M} = \langle DR \rangle$ is a model for $\mathcal{K}_o$, and $\alpha$ is a wfe of $\mathcal{K}_o$, then <u>the value of $\alpha$ in $\mathfrak{M}$ with respect to $\mathcal{K}_o$</u> $(\mathrm{Val}^{\mathcal{K}_o}_{\mathfrak{M}}(\alpha))$ is given by the following:

(1) if $\alpha$ is an atomic wfe of $\mathcal{K}_o$, then the value of $\alpha$ is $R(\alpha)$

(2) if $\eta,\alpha_1,\ldots,\alpha_r$ are wfe's of $\mathcal{K}_o$ of types

$\langle\tau_1,\ldots,\tau_r,\tau_0\rangle, \tau_1,\ldots,\tau_r$ respectively with values $f, x_1,\ldots,x_r$ respectively, then the value of the compound wfe $\eta\frown\alpha_1\frown\cdots\frown\alpha_r$ is $f(x_1,\ldots,x_r)$.

T4. If $\alpha$ is a wfe of $\mathcal{K}_0$ with type $\tau$, and $\mathcal{M}$ is a model for $\mathcal{K}_0$, then the value of $\alpha$ in $\mathcal{M}$ is a member of the universe of $\tau$ in $\mathcal{M}$.

The notion of a denotation function, and hence of a Fregean semantical system, can now be reintroduced in terms of models for $\mathcal{K}_0$.

D11. If $\mathcal{M}$ is a model for $\mathcal{K}_0$, then <u>the denotation function for $\mathcal{K}_0$ corresponding to $\mathcal{M}$</u> is that function from the wfe's of $\mathcal{K}_0$ which assigns to each such $\alpha$, the value of $\alpha$ in $\mathcal{M}$.

T5. If $\mathcal{M}$ is a model for $\mathcal{K}_0$, and F is the denotation function for $\mathcal{K}_0$ corresponding to $\mathcal{M}$, then $\langle\mathcal{K}_0, F\rangle$ is a Fregean semantical system.

## 12. The Language $\mathcal{K}_1$

We now wish to construct an extension of the language $\mathcal{K}_0$ which will be adequate to treat oblique contexts of the wfe's of $\mathcal{K}_0$. Rather than actually introducing at this point some particular oblique contexts we will first aim at the attendant changes in or additions to $\mathcal{K}_0$ which such contexts would involve. Of course if the method of indirect discourse is followed, there are no such attendant changes.

However, we here wish to develop a language of direct discourse, and this involves (among other things) providing for each wfe of $\mathcal{K}_o$ an analogous wfe which will be used in our direct discourse treatment of oblique contexts to unambiguously denote what the original wfe would ambiguously denote in the indirect discourse treatment of oblique contexts. For example, if we think of a sentence $\phi$ as denoting a truth value in the context,

$\ulcorner$it is not the case that $\phi\urcorner$

but as denoting itself in the context

$\ulcorner$it is necessary that $\phi\urcorner$

we will retain $\phi$ in the former usage and provide a new wfe, $\bar{\phi}$, to replace $\phi$ in the latter usage. Thus we must provide for each wfe $\alpha$ of $\mathcal{K}_o$, a new wfe $\bar{\alpha}$; this new wfe is called <u>the analogue to $\alpha$</u>. Rather than simply introducing a new primitive as analogue for each wfe $\alpha$ of $\mathcal{K}_o$, we shall take advantage of the following simplifying assumption about the denotation of wfe's in an indirect discourse treatment of oblique contexts.

Assumption A: The denotation of a wfe $\alpha$ in an oblique context is the semantical value of $\alpha$ for some Fregean semantical relation R.

Thus, we may understand wfe's in oblique contexts to refer to themselves, or to their senses, etc. In any case we have the following result, where $\bar{\alpha}$ is to be the analogue to $\alpha$. The value of $\alpha$ by the Fregean semantical relation R will be the denotation of $\overline{\alpha}$ (or for short $R(\alpha) = \text{den } (\overline{\alpha})$). Thus, if f is a formation rule, den $\overline{(f(\beta,\gamma))} = R(f(\beta,\gamma))$. But since R is Fregean, there is some semantical operation $f^*$ corresponding to f such that $R\ (f(\beta,\gamma)) = f^*\ (R(\beta),R(\gamma))$. Therefore, since $R(\beta) = \text{den } (\overline{\beta})$ and $R(\gamma) = \text{den } (\overline{\gamma})$, we have den $\overline{(f(\beta,\gamma))} = f^*\ (\text{den } (\overline{\beta}), \text{den } (\overline{\gamma}))$ or, in words, the denotation of the analogue to a compound wfe is a function of the denotations of the analogues to the parts. Hence, instead of introducing a new primitive as analogue for each wfe $\alpha$ of $\mathcal{K}_0$, we shall introduce such new primitives only for the atomic wfe's of $\mathcal{K}_0$ and we will construct the analogue to a compound wfe such as $f(\beta,\gamma)$ by introducing a new formation rule, say $f_1$, such that $\overline{f(\alpha,\beta)} = f_1(\overline{\alpha},\overline{\beta})$. Such a procedure is justified by the result of assumption A.

In addition to providing the analogue expressions $\overline{\alpha}$ for every wfe $\alpha$ of $\mathcal{K}_0$, we also provide the means for expressing the relation between the denotation of $\alpha$ and the denotation of $\overline{\alpha}$. Thus, although we do not follow the indirect discourse approach of identifying the two expressions, neither do we completely ignore the relationship between their denotations. The relation will depend, of course, on what treatment of direct discourse we follow. That is, on what the Fregean semantical relation R is, such that we understand a wfe $\alpha$ in an oblique context as (ambiguously) denoting $R(\alpha)$. When we discuss semantical considerations with respect to $\mathcal{K}_1$ we will make explicit at least two possibilities.

12.1 The types of $\mathcal{K}_1$

Since $\mathcal{K}_o$ is to be a sublanguage of $\mathcal{K}_1$, we have in $\mathcal{K}_1$ the two simple types i, t of $\mathcal{K}_o$. In addition, we introduce two new simple types $i_1$, $t_1$ ($i_1 = 4$, $t_1 = 9$) for wfe's of $\mathcal{K}_1$ which are analogues to wfe's of $\mathcal{K}_o$ of the types i, t.[43] By the same reasoning it seems natural to introduce a new type $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle_1$ to correspond to each of the complex types $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$ of $\mathcal{K}_o$. We shall, however, take a different course. Although we consider expressions of complex type to be wfe's, our primary concern is with the wfe's of simple type. Another way of putting this is to remark that we understand our languages to have an essentially first order character. That is, they are so constructed that to add variables of simple types would be a natural step, but to add variables of complex types might require considerable revision. Our essential requirement, then, is that we have in $\mathcal{K}_1$ an analogue to every wfe $\alpha$ of $\mathcal{K}_o$ of <u>simple type</u>. Thus if $\eta$ is, say, a one-place operation symbol of $\mathcal{K}_o$, the essential role of $\overline{\eta}$ (the analogue to $\eta$) will be to combine with $\overline{\alpha}$ (the analogue to a wfe $\alpha$ of simple type) to form compound wfe's of the form $\overline{\eta}\frown\overline{\alpha}$ (that is, analogues to wfe's of simple-type of the form $\eta\frown\alpha$). To put it more generally, if $\eta$ is a wfe (of $\mathcal{K}_o$) of complex type $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$, it combines with wfe's of types $\tau_1,\ldots,\tau_r$ to form a compound wfe of type $\tau_o$. Hence in view of assumption A, $\bar{\eta}$ must have a type which combines with wfe's of types $\tilde{\tau}_1,\ldots,\tilde{\tau}_r$ (where $\tilde{\imath}$ is $i_1$, and $\tilde{t}$ is $t_1$) to form a compound wfe of type $\tilde{\tau}_o$. In view of these considerations we will assign the complex type $\langle\tilde{\tau}_1,\ldots\tilde{\tau}_r,\tilde{\tau}_o\rangle$ to the analogue to a wfe of type $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$. This procedure, which eliminates the need for an additional type $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle_1$, will effect a considerable simplification of the type system.

D12. <u>The simple types of $\mathcal{K}_1$</u> are $i$, $i_1$, $t$, and $t_1$. The <u>complex types of $\mathcal{K}_1$</u> consist of all finite sequences $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$, where $\tau_o,\ldots,\tau_r$ are simple types of $\mathcal{K}_1$ and $r>0$.

It will be convenient to be able to speak in general of the type of the analogue to a wfe of type $\tau$.

D13. If $\tau$ is a type of $\mathcal{K}_o$, then <u>the elevation of $\tau$</u> ($\tilde{\tau}$) is given by the following:

(1) the elevation of $i$ is $i_1$

(2) the elevation of $t$ is $t_1$

(3) if $\tau_o,\ldots,\tau_r$ are simple types of $\mathcal{K}_o$, $r>0$, and $\tilde{\tau}_o,\ldots,\tilde{\tau}_r$ are their respective elevations, then the elevation of $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$ is $\langle\tilde{\tau}_1,\ldots,\tilde{\tau}_r,\tilde{\tau}_o\rangle$.

Although the elevation of each type of $\mathcal{K}_o$ is a type of $\mathcal{K}_1$ not all of the new types of $\mathcal{K}_1$ are formed in this manner, for example $\langle i_1, i, t\rangle$ is not the elevation of any type.

12.2 The Well Formed Expressions of $\mathcal{K}_1$

As with $\mathcal{K}_0$ we will exclude all wfe's of certain complex types, and further we will exclude all primitive <u>descriptive</u> constants of higher types (the higher types are the types of $\mathcal{K}_1$ which are not also types of $\mathcal{K}_0$). In the present case, as distinct from $\mathcal{K}_0$, there are important theoretical considerations, rather than mere considerations of familiarity, behind this decision.

In addition to the analogues to the atomic wfe's of $\mathcal{K}_0$, we also introduce in $\mathcal{K}_1$ two new identity predicates (for wfe's of types $i_1$ and $t_1$), and two new predicates (the delta predicates) to stand for the relation which holds between the denotation of a wfe of $\mathcal{K}_0$ (of simple type) and the denotation of its analogue. (Further explanation of the delta predicates is forthcoming.)

D14. <u>The atomic wfe's of $\mathcal{K}_1$</u> fall into the following categories:

(1) all atomic wfe's of $\mathcal{K}_0$

(2) analogues to all atomic wfe's of $\mathcal{K}_0$, where
<u>Opsymb$_1$(m,p)</u> is the analogue to Opsymb (m,p),
<u>Pred$_1$(m,p)</u> is the analogue to Pred(m,p)
<u>Cond$_1$</u>, <u>Neg$_1$</u> are the analogues to Cond, Neg
<u>Id$_1$(i)</u>, <u>Id$_1$(t)</u> are the analogues to Id(i), Id(t)

(4) the identity signs for wfe's of types $i_1$ and $t_1$

$\underline{Id(i_1)}$, $\underline{Id(t_1)}$

(4) the delta predicates for wfe's of types $i_1$ and $t_1$

$\underline{Delta(i_1)}$, $\underline{Delta(t_1)}$.

D15. The type of an atomic wfe of $\mathcal{K}_1$ is given by the following:

(1) if $\alpha$ is an atomic wfe of $\mathcal{K}_0$, its type remains the same

(2) if $\alpha$ is an atomic wfe of $\mathcal{K}_0$, of type $\tau$, the type of the analogue to $\alpha$ is the elevation of $\tau$

(3) $Id(i_1)$ has type $<i_1,i_1,t>$ and $Id(t_1)$ has type $<t_1,t_1,t>$

(4) $Delta(i_1)$ has type $<i_1,i_,t>$ and $Delta(t_1)$ has type $<t_1,t,t>$

D16. $\beta$ is a well formed expression of $\mathcal{K}_1$ of type $\tau$ if and only if:

(1) $\beta$ is an atomic wfe of $\mathcal{K}_1$ of type $\tau$, or

(2) there are wfe's $\eta$, $\alpha_1,\ldots,\alpha_r$ of $\mathcal{K}_1$ of types $<\tau_1,\ldots,\tau_r,\tau>$, $\tau_1,\ldots,\tau_r$ respectively and $\beta$ is $\eta\frown\alpha_1\frown\ldots\frown\alpha_r$.

The bar notation ($\overline{\alpha}$) for the analogue to a wfe $\alpha$ of $\mathcal{K}_0$ can now be introduced in a precise way.

D17. The analogue to a wfe $\alpha$ of $\mathcal{K}_0$ ($\overline{\alpha}$) is given by the following:

(1) $\overline{\text{Opsymb}\ (m,p)} = \text{Opsymb}_1\ (m,p)$

(2) $\overline{\text{Pred}\ (m,p)} = \text{Pred}_1\ (m,p)$

(3) $\overline{\mathrm{Cond}} = \mathrm{Cond}_1$

(4) $\overline{\mathrm{Neg}} = \mathrm{Neg}_1$

(5) if $\tau = i$ or $\tau = t$, $\overline{\mathrm{Id}\ (\tau)} = \mathrm{Id}_1\ (\tau)$

(6) if $\eta$ is a wfe of $\mathcal{K}_0$ of complex type $<\tau_1,\ldots,\tau_r,\tau_0>$ and $\alpha_1,\ldots,\alpha_r$ are wfe's of $\mathcal{K}_0$ of simple types $\tau_1,\ldots,\tau_r$ respectively, then $\overline{\eta^\frown\alpha_1{}^\frown\ldots{}^\frown\alpha_r} = \overline{\eta}^\frown\overline{\alpha_1}{}^\frown\ldots{}^\frown\overline{\alpha_r}$.

T6. If $\alpha$ is a wfe of $\mathcal{K}_0$ of type $\tau$, then $\overline{\alpha}$ is a wfe of $\mathcal{K}_1$ of type $\tilde{\tau}$.

One must carefully distinguish such pairs of constants as $\mathrm{Id}(i_1)$ and $\mathrm{Id}_1(i)$. The former is the identity predicate for wfe's of type $i_1$; note that it has the type of a two-place predicate of such wfe's, $<i_1,i_1,t>$. In general, for each simple type $\tau$ we will introduce an identity sign $\mathrm{Id}(\tau)$ of type $<\tau,\tau,t>$ for wfe's of type $\tau$. The latter ( $= \overline{\mathrm{Id}(i)}$) is the analogue to the identity predicate for individuals; note that its type $<i_1,\ i_1,\ t_1>$ is not that of a predicate at all, since when combined with wfe's of type $i_1$ it does not form a sentence (that is, a wfe of type t), but rather the analogue to a sentence. In general, for each atomic wfe of $\mathcal{K}_0$, such as $\mathrm{Id}(i)$, we will introduce an analogue $\mathrm{Id}_1(i)$, an analogue to the analogue $\mathrm{Id}_2(i)$, an analogue to the analogue to the analogue $\mathrm{Id}_3(i)$, etc. In this way, each atomic wfe of $\mathcal{K}_0$ will ultimately generate an infinite hierarchy of analogues. But such wfe's as $\mathrm{Id}(i_1)$ do not appear in this hierarchy, instead, they stand at the base of a hierarchy of their own $\mathrm{Id}(i_1)$, $\mathrm{Id}_1(i_1)$, $\mathrm{Id}_2(i_1)$, etc.

Reference back to our conception of a language of direct discourse adequate to oblique contexts may help to clarify what we have done. We began with a base language $\mathcal{K}_0$ and added all the analogues to wfe's of the base language. In addition, we added some logical apparatus for the new wfe's (the new identity predicates and the delta expressions) and allowed any sentential combination of the enlarged class of wfe's of type t (for example, $\text{Neg}^\frown\text{Deta}(i_1)^\frown\overline{\text{Opsymb}(1,0)}^\frown\text{Opsymb}\ (2,0)$). We thus have the apparatus available to express singly oblique contexts of the wfe's of $\mathcal{K}_0$. In order to make the situation a little more concrete, let us now introduce such a context by adding clauses to the preceding definitions which introduce the modal operator of necessity.

(5) the necessity predicate, Nec, is an atomic wfe of $\mathcal{K}_1$ of type $\langle t_1, t\rangle$

## 13. The Semantics of $\mathcal{K}_1$

As indicated earlier, we will provide two interpretations of $\mathcal{K}_1$, a syntactical interpretation, according to which wfe's of the form $\overline{\alpha}$ will denote expressions of $\mathcal{K}_0$ and an intensional interpretation, according to which wfe's of the form $\overline{\alpha}$ will denote senses.

## 14. The Syntactical Interpretation of $\mathcal{K}_1$

### 14.1 Expressions and Syntactical Entities

Under the syntactical interpretation, the universe of the type $i_1$ will consist of all wfe's of $\mathcal{K}_o$ of type i, and the universe of the type $t_1$ will consist of all wfe's of $\mathcal{K}_o$ of type t. Thus, if $\alpha$ is a wfe of $\mathcal{K}_o$ of simple type i (t), we can require that $\overline{\alpha}$ (which will have simple type $i_1$ ($t_1$)) simply denote $\alpha$ itself. Accordingly, it seems natural to require also that $\overline{\eta}$ denote $\eta$, when $\eta$ is a wfe of $\mathcal{K}_o$ of complex type. We will not take this course. Instead, $\overline{\eta}$ will denote a certain function (called "the syntactical entity corresponding to $\eta$" or "Synt($\eta$)") which represents the complex wfe $\eta$.

We recall (section 12.1) that the essential role of the analogue $\overline{\eta}$ to a wfe $\eta$ of complex type is to combine with the analogues $\overline{\alpha_1}, \ldots, \overline{\alpha_r}$ to wfe's $\alpha_1, \ldots \alpha_r$ of simple type to form the analogue $\overline{\eta^\frown\alpha_1{}^\frown\ldots{}^\frown\alpha_r}$ to the compound wfe $\eta^\frown\alpha_1{}^\frown\ldots{}^\frown\alpha_r$ of simple type. The requirement on analogues to wfe's of simple type tells us that each $\overline{\alpha_j}$ denotes $\alpha_j$ and that $\overline{\eta^\frown\alpha_1{}^\frown\ldots{}^\frown\alpha_r}$ denotes $\eta^\frown\alpha_1{}^\frown\ldots{}^\frown\alpha_r$. Hence, since $\overline{\eta^\frown\alpha_1{}^\frown\ldots{}^\frown\alpha_r} = \overline{\eta}^\frown\overline{\alpha_1}{}^\frown\ldots{}^\frown\overline{\alpha_r}$, the denotation of $\overline{\eta}$ must combine with $\alpha_1, \ldots, \alpha_r$ to form $\eta^\frown\alpha_1{}^\frown\ldots{}^\frown\alpha_r$. This purpose is most simply served by requiring that $\overline{\eta}$ denote that formation rule (section 2.1) which forms from each r-tuple $\langle\alpha_1, \ldots, \alpha_r\rangle$ of wfe's of the appropriate types, the compound wfe $\eta^\frown\alpha_1{}^\frown\ldots{}^\frown\alpha_r$.[44] This formation rule is the syntactical entity corresponding to $\eta$. For purposes of uniformity, let us also refer to a wfe $\alpha$ of simple type as the syntactical entity corresponding to itself.

D18. If $\beta$ is a wfe of $\mathcal{K}_o$, then <u>the syntactical entity corresponding to $\beta$</u> (Synt($\beta$)) is given by the following:

(1) if $\beta$ is of simple type, then Synt($\beta$) is $\beta$ itself.

(2) if $\tau_o, \ldots \tau_r$ are simple types of $\mathcal{K}_o$, $r>o$, $\beta$ is of complex type $<\tau_1, \ldots, \tau_r, \tau_o>$, and $w_o, \ldots, w_r$ are the sets of all wfe's of $\mathcal{K}_o$ of types $\tau_o, \ldots, \tau_r$ respectively, then Synt($\beta$) is the unique function from $(w_1 x \ldots x w_r)$ into $w_o$ which assigns to each $<\alpha_1, \ldots \alpha_r>$ in its domain, the wfe $\beta^\frown\alpha_1^\frown\ldots^\frown\alpha_r$.

Our treatment of the denotation of $\overline{\eta}$ (namely, Synt($\eta$)) is in accord with our plan to have a wfe of complex type $<\tau_1, \ldots, \tau_r, \tau_o>$ denote a function which assigns to each r-tuple $<y_1, \ldots, y_r>$, where $y_j$ is an element of the universe of $\tau_j$, an element of the universe of $\tau_o$.

## 14.2 Universe and Value for $\mathcal{K}_1$

If we were to follow directly the pattern of definitions for the semantical notions in $\mathcal{K}_o$, we would be led to introduce the notion of a syntactical model for $\mathcal{K}_1$. Such a model might consist of a couple <DR>, where D again represented the universe of the type i and R assigned to each atomic wfe of $\mathcal{K}_1$ an appropriate denotation. But one of our motives in transferring attention from denotation functions for $\mathcal{K}_o$ to models for $\mathcal{K}_o$ was to simplify matters by considering just those essential places at which one denotation function could differ from another, namely in the assignment of denotations to descriptive atomic wfe's. (Actually we did not go as far in this direction as we might have, in that our models for $\mathcal{K}_o$ also assign denotations to the logical, that is, non-descriptive, atomic wfe's.) Similar considerations lead to a simplification of the semantical notions for $\mathcal{K}_1$.

Note that in the passage from $\mathcal{K}_o$ to $\mathcal{K}_1$, no <u>descriptive</u> atomic wfe's were added. Thus, the same notion of model will suffice for both languages, and the assignment of denotations to the new atomic wfe's can be accomplished through the new notion of value. Hence, for the syntactical interpretation of the language $\mathcal{K}_1$ we will introduce only the two semantical notions: the universe of a type, and the value of a wfe.

D19. If $\mathfrak{M} = \langle DR \rangle$ is a model for $\mathcal{K}_o$, and $\tau$ is a type of $\mathcal{K}_1$, then <u>the universe of $\tau$ in $\mathfrak{M}$ with respect to the Syntactical interpretation of $\mathcal{K}_1$</u> ($\mathcal{U}_{\mathfrak{M}}^{S,\mathcal{K}_1}(\tau)$) is given by the following:

(1) the universe of i is D

(2) the universe of t is $\{T,F\}$

(3) the universe of $i_1$ is the set of all wfe's of $\mathcal{K}_o$ of type i

(4) the universe of $t_1$ is the set of all wfe's of $\mathcal{K}_o$ of type t

(5) if the universes of $\tau_o,\ldots,\tau_r$ are respectively $u_o,\ldots,u_r$ and $r>o$, then the universe of $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$

is the set of all functions from $(u_1 x \ldots x u_r)$ into $u_o$.

Our definition of "the value of a wfe of $\mathcal{K}_1$" makes essential use of the earlier notion, the value of a wfe $\alpha$ in the model $\mathcal{M}$ with respect to the language $\mathcal{K}_o$ ($\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_o}(\alpha)$), which was introduced in D10.

D20. If $\mathcal{M} = \langle DR \rangle$ is a model for $\mathcal{K}_o$, and $\alpha$ is a wfe of $\mathcal{K}_1$, then <u>the value of $\alpha$ in $\mathcal{M}$ with respect to the Syntactical interpretation of $\mathcal{K}_1$</u> ($\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_1}(\alpha)$) is given by the following:

(1) if $\alpha$ is an atomic wfe of $\mathcal{K}_o$, then the value of $\alpha$ is $R(\alpha)$

(2) if $\alpha$ is an atomic wfe of $\mathcal{K}_o$, then the value of $\bar{\alpha}$ is $\mathrm{Synt}(\alpha)$

(3) if $\tau$ is $i_1$ or $t_1$, and $\alpha$ is $\mathrm{Id}(\tau)$, then the value of $\alpha$ is the unique f such that:

(a) f is a function from $(\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_1}(\tau) \times \mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_1}(\tau))$ into $\{T,F\}$

(b) if $\langle \beta,\gamma \rangle$ is in the domain of f, then $f(\beta,\gamma) = T$ if and only if $\beta = \gamma$.

(4) if $\tau$ is i or t, and $\alpha$ is $\mathrm{Delta}(\tilde{\tau})$, then the value of $\alpha$ is the unique f such that:

(a) f is a function from $(\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_1}(\tilde{\tau}) \times \mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_1}(\tau))$ into $\{T,F\}$

(b) if $\langle \beta, X \rangle$ is in the domain of f, then $f(\beta,X) = T$ if and only if $\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_o}(\beta) = X$.

(5) if $\alpha$ is Nec, then the value of $\alpha$ is the unique f such

that:

(a) f is a function from the set of all wfe's of $\mathcal{K}_o$ of type t into $\{T, F\}$

(b) if $\phi$ is in the domain of f, then $f(\phi) = T$ if and only if for all models $\mathcal{N}$ for $\mathcal{K}_o$, $Val_{\mathcal{N}}^{\mathcal{K}_o}(\phi) = T$.

(6) if $\eta, \beta_1, \ldots, \beta_r$ are wfe's of $\mathcal{K}_1$ of types $<\tau_1, \ldots, \tau_r, \tau_o>$, $\tau_1, \ldots, \tau_r$ respectively, with values f, $x_1, \ldots, x_r$ respectively, and $\alpha$ is $\eta^\frown\beta_1^\frown\ldots^\frown\beta_r$, then the value of $\alpha$ is $f(x_1, \ldots, x_r)$.

T7. If $\alpha$ is a wfe of $\mathcal{K}_1$ with type $\tau$, and $\mathcal{M}$ is a model for $\mathcal{K}_o$, then the value of $\alpha$ in $\mathcal{M}$ is a member of the universe of $\tau$ in $\mathcal{M}$ ($Val_{\mathcal{M}}^{S,\mathcal{K}_1}(\alpha) \in \mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_1}(\tau)$).

T8. If $\alpha$ is a wfe of $\mathcal{K}_o$, and $\mathcal{M}$ is a model for $\mathcal{K}_o$, then the value of $\alpha$ in $\mathcal{M}$ is the same for $\mathcal{K}_o$ and $\mathcal{K}_1$ ($Val_{\mathcal{M}}^{\mathcal{K}_o}(\alpha) = Val_{\mathcal{M}}^{S,\mathcal{K}_1}(\alpha)$).

T9. If $\mathcal{M}$ is a model for $\mathcal{K}_o$, and F is that function from the wfe's of $\mathcal{K}_1$ which assigns to each such $\alpha$, the value of $\alpha$ in $\mathcal{M}$($Val_{\mathcal{M}}^{S,\mathcal{K}_1}(\alpha)$); then $<\mathcal{K}_1, F>$ is a Fregean semantical system.

Our simple method of providing an analogue to every wfe of $\mathcal{K}_o$ was justified by Assumption A (section 12). The following theorem indicates that we have satisfied that assumption.

T10. If R is that function from the wfe's of $\mathcal{K}_o$ which assigns to each such $\alpha$, Synt($\alpha$); then $\langle \mathcal{K}_0, R \rangle$ is a Fregean semantical system.

14.3 Metalinguistic Features of $\mathcal{K}_1$

Under the syntactical interpretation, $\mathcal{K}_1$ may be thought of as a formalized metalanguage for $\mathcal{K}_0$ with the following features:

(1) The object language, $\mathcal{K}_0$, is a sublanguage of the metalanguage.

(2) The metalanguage contains type distinctions which make it impossible to express certain propositions (for example, that the expressions of the object language of different types are disjoint).

(3) The metalanguage contains two denotation predicates (one for names of the object language and one for sentences of the object language).

(4) Every name $\alpha$ and sentence $\phi$ of the object language has a <u>standard name</u> in the metalanguage (namely, $\overline{\alpha}$ and $\overline{\phi}$ respectively).

(5) p-place predicates and operation symbols (p>o) and sentential connectives are not treated as wfe's of the object language.

(6) The metalanguage contains a validity predicate, Nec, for sentences of the object language.

(7) The logical resources of the metalanguage are very weak.

$\mathcal{K}_1$ differs perhaps most strikingly from familiar formalized metalanguages with respect to points (2) and (5). In the more familiar metalanguages we have a single type for names of all wfe's of the object language and a single formation operation (concatenation). Thus we are able to form names of a wide class of expressions, in fact a wider class than the wfe's. Furthermore, we are able to express the fact that, for example, no sentential connective is an individual constant. In our form of metalanguage, the identity of a pair of expressions can be expressed in the metalanguage only when the identity of their denotations can be expressed in the object language. The second main departure from the familiar form of metalanguages is in the lack of names for expressions of $\mathcal{K}_0$ of complex type. In their place, corresponding to every such expression $\eta$ we have a name $\bar{\eta}$ of a formation rule which introduces $\eta$. Thus $\mathcal{K}_1$ treats $\eta$ as syncategorimatical; that is, $\eta$ is only indicative of a mode of combination of its argument expressions rather than being a full-fledged wfe with a denotation of its own. In this connection, note that $\mathcal{K}_1$ contains no denotation predicate for expressions of complex type. In its treatment of expressions of $\mathcal{K}_0$ of complex types, $\mathcal{K}_1$ provides a description of $\mathcal{K}_0$ which is alternative to that given in section 10.2 with respect to the part-whole relation. But both descriptions share the feature that only that part of the structure of the expressions of the object language which is relevant for semantical purposes is expressed.

It is this relative poverty of expressiveness which actually provides $\mathcal{K}_1$ its versatility. Recall that the denotation of $\overline{\alpha}$ was to be $R(\alpha)$ for some Fregean semantical relation R. We have kept the metalinguistic apparatus in $\mathcal{K}_1$ minimal so as to be able to accommodate, in a natural way, Fregean semantical relations other than that which assigns to an expression the expression itself. In particular, we will soon provide a quite different interpretation of $\mathcal{K}_1$, one in which wfe's of the form $\overline{\alpha}$ denote concepts. If we had constructed $\mathcal{K}_1$ in accordance with concatenation theory, we would then be faced with providing an interpretation for the "concatenation" of the sense of $\alpha$ and the sense of $\beta$, where $\alpha\frown\beta$ may not even be a well-formed expression.

In this connection, we recall our purpose in constructing $\mathcal{K}_1$: to provide a direct discourse language to treat singly oblique contexts of wfe's of $\mathcal{K}_0$. It is not intended to be adequate for other more far reaching purposes (although some languages adequate for other purposes may also prove adequate for ours). For our purpose the most important features of $\mathcal{K}_1$ are (1) and (4).

It is worth considering (4) at slightly greater length. In what sense is $\overline{\alpha}$ a standard name of $\alpha$? In many ways $\overline{\alpha}$ seems similar both to what Tarski has termed a structural descriptive name [45] and to a quotation name.

Like quotation names and (presumably) structural descriptive names, it differs from what we may call <u>contingent names</u> in that the fact that $\overline{\alpha}$ denotes $\alpha$ can be established on logical grounds alone. Expressions, like every other kind of entity, can be named by names which require empirical investigation to determine their denotation (for example, the name 'John's favorite sentence'). We understand the essence of the notion of a <u>standard name</u> to be the logical determinateness of its denotation. Thus, our claim in (4) is understood as justified by the following theorem:

T11. If $\alpha$ is a wfe of $\mathcal{K}_o$ of simple type, and $\mathcal{M}$ is any model for $\mathcal{K}_o$, then $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_1}(\overline{\alpha}) = \alpha$.

Theorem 11 can be slightly generalized as follows:

T12. If $\alpha$ is a wfe of $\mathcal{K}_o$, and $\mathcal{M}$ is any model for $\mathcal{K}_o$, then $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_1}(\overline{\alpha}) = \mathrm{Synt}(\alpha)$.

In connection with theorems 11 and 12 it is interesting to note that the universes of the higher types are also logically determinate.

T13. If $\tau$ is a type of $\mathcal{K}_o$, and $\mathcal{M}, \mathcal{N}$ are any models for $\mathcal{K}_o$, then $\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_1}(\widetilde{\tau}) = \mathcal{U}_{\mathcal{N}}^{S,\mathcal{K}_1}(\widetilde{\tau})$.

Although every kind of entity is capable of being named by a contingent name like 'John's favorite . . .', only those kinds of entities whose existence is not itself a contingent matter can be such that a given expression names them in every possible state of affairs. It is readily seen that the notion of a *contingent entity* is quite problematical, depending, as it does, on the notion of a possible state of affairs. However, it seems plausible to assume that the distinction between necessary and contingent entities approximately parallels that between abstract and concrete entities. Thus numbers, expressions (in the sense of type, not token, see Peirce [1]), concepts, and certain sets are capable of having standard names like '0', ''0'', 'the necessary proposition', '$\bigwedge$'; whereas for any name of a physical object, a set of physical objects, or a sense datum we seem to be able to imagine a possible state in which such a name either would be denotationless or would name something *other* than that which it in fact does name. From this point of view, we may regard theorem 13 as expressing the fact that the universes of the higher types consist of noncontingent entities.

### 15. The Intensional Interpretation of $\mathcal{K}_1$

In giving an intensional interpretation to $\mathcal{K}_1$, one of the first questions which arises is, "What is a concept?" If we do not wish to take "concept" (or "sense" or "intension" we use them interchangeably at this stage) as a primitive of the metalanguage we must offer an analysis in terms which are acceptable. There are a number of distinct notions which are suggested. These notions can be differentiated by various criteria.

## 15.1 Principles of Individuation

The criterion which has been most discussed is probably that of individuation. That is, under what circumstances shall we say that two expressions have the same sense. The proposal herein adopted is that of Carnap (for what he calls "intensions" in a technical sense) that <u>two expressions shall be said to have the same sense just in case the expressions are logically equivalent</u>. Thus, if $\alpha$ and $\beta$ are wfe's of $\mathcal{K}_0$ of type i, then the sense of $\alpha$ is identical with the sense of $\beta$ just in case the identity sentence, $\mathrm{Id}(i)^\frown\alpha^\frown\beta$, is logically true in $\mathcal{K}_0$. Similarly, if $\phi, \psi$ are wfe's of $\mathcal{K}_0$ of type t, the sense of $\phi$ is identical with the sense of $\psi$ just in case the biconditional of $\phi$ and $\psi$, $\mathrm{Id}(t)^\frown\phi^\frown\psi$, is logically true.

This proposal is referred to by Church in [3] and [6] as "Alternative Two." [46] Other principles of individuation have been considered (three are mentioned in Church [3] and [6]). In particular, there seem to be reasons for a tighter principle (that is, one which will individuate more senses) in certain contexts (for example, the oblique contexts formed by using predicates like 'belief'). There is, however, at least one excellent argument in favor of the present principle. It is clear. It seems very difficult to draw the line in a natural way at any other place. [47]

One may imagine two general methods for producing tighter principles. First, restrict the logical operations allowed in proving the equivalence. Looked at from a semantical point of view, this amounts to narrowing the category of logical signs. Principles produced by this method can probably be accommodated in the style of the present work.[48] A second method, initially quite appealing, is to identify only the senses of wfe's which are obviously logically equivalent. We might, for example, fix an upper bound to the length of the proof of the equivalence in some particularly natural system of logic. The primary difficulty with this method is that it does not produce an equivalence relation between wfe's having the same sense, since being obviously logically equivalent is not a transitive relation. The natural treatment of a notion explicated by this method is in terms of a degree of synonymy which in turn would call for a notion of degree of belief, here assigning different degrees to logically equivalent sentences.

At any rate, we should think of different principles of individuation as simply producing different notions of sense. Our immediate purpose is now to continue explicating one of these notions.

### 15.2 Empty Concepts

A second important criterion for distinguishing different notions of concept is concerned with the possibility of so-called empty concepts. Suppose we consider a name of type i (that is, the denotation of $\alpha$ is to be an individual). Now, according to Frege, every name must have a sense, so let $\bar{\alpha}$ denote the sense of $\alpha$. However, not every name has a denotation. For example, the name "Pegasus" is ordinarily thought of as being denotationless. If $\alpha$ is such a denotationless name, then $\bar{\alpha}$ will denote an empty concept. Note that $\bar{\alpha}$ is not denotationless; it denotes a certain concept, namely the sense of $\alpha$. But this concept does not apply to anything, that is, it is not a concept of anything.[49]

Thus it is seen that the problem of empty concepts is closely connected with that of denotationless names. If we admit denotationless names and at the same time insist that every name has a sense, we seem forced to admit empty concepts. Two alternative proposals for treating the problem are immediately apparent.

(1) Admit denotationless names and (thus) empty concepts.

(2) Admit no denotationless names and no empty concepts.

In general, alternative (2) is more convenient. It was in fact Frege who proposed that for the sake of simplicity in the syntax we should adopt rather artificial conventions which force a denotation on otherwise denotationless names. However, from the point of view of natural usage, alternative (1) seems superior.

It has been argued that to grasp a concept is to know to what it would apply given any state of affairs, and thus if one does not know to what the sense of "Pegasus" applies, one has not fully grasped the concept (or possibly has not grasped the full concept). Hence, the argument goes, a so-called empty concept is merely an incomplete concept (that is, only part of a concept). We shall adopt alternative (2) for purposes of logical simplicity. But it is worth mentioning that the above argument (except as a post facto analysis of the notion which has been formalized) does not seem completely satisfactory. One might equally well claim that to fully grasp a (full) concept is to know, given any state of affairs, either to what the concept applies, or that it applies to nothing.

As stated above, the present work is based on alternative (2). However, before going on we should mention that there are positions between (1) and (2). Let us call one of these (1.5).

(1.5) Admit no denotationless names but allow empty concepts (which of course would not be the sense of any names in the language) as values of variables of types $i_1$ and $t_2$.

This alternative is adopted in Church [6].[50] A modification of the systems of the present work to accommodate either alternative (1) or (1.5) does not appear, in principle, to impose serious difficulties.

15.3 The Carnap Interpretation

The preceding discussion of what it means to grasp a concept has already indicated a natural interpretation of concept. This interpretation, which was first suggested by Carnap,[51] but in a more exact way, is that a concept is a function from possible states of affairs to things in those states. Thus, for example, a concept of an individual (that is, an individual concept) is a function which assigns to each (possible) state of affairs some particular individual in that state. A concept of a truth value (that is, a proposition, the sense of a sentence) is a function which assigns to each state of affairs a truth value;[52] a concept of a set of individuals (that is, a property, the sense of a one-place predicate) is a function which assigns to each state of affairs a set of the individuals of that state.[54]

Let us review some of the previous discussion to see how well this interpretation fits with earlier decisions. According to our notion of a well formed expression (ignoring for the present, problems connected with the occurrence of free variables),

(1) Every wfe has a sense.

According to the principle of individuation,

(2) The senses of logically equivalent wfe's are identical.

According to the decision on empty concepts,

(3) Every wfe must have a denotation and no concept can be empty.

Now let us consider a particular name, let $\alpha$ = "the 35th president of the U.S.A." By (3), $\alpha$ has a denotation in every possible state of affairs. In the actual one $\alpha$

happens to denote John F. Kennedy, but in others it may denote, say, Richard Nixon, or some other possible individual who does not exist at all in the actual state of affairs. By (1), $\alpha$ has a sense, let us call it "Sense ($\alpha$)". By (3) again, Sense ($\alpha$) is a concept of something in each possible state of affairs. In the actual state it obviously is a concept of John F. Kennedy, that is, the actual denotation of $\alpha$. Thus, what is more natural than to let Sense ($\alpha$) be such that for any possible state of affairs it is a concept of the denotation of $\alpha$ for that state (which denotation must exist by (3)). It is now a simple step to let Sense ($\alpha$) <u>be</u> that function from possible states of affairs, which assigns to each state the denotation of $\alpha$ in that state. Suppose now that we consider another name, $\beta$, which differs from $\alpha$ but is logically equivalent. Repeating the previous argument, $\beta$ has a sense, Sense ($\beta$), a function from possible states of affairs which assigns to each state the denotation of $\beta$ in that state. From the hypothesis that $\alpha$ is logically equivalent to $\beta$ we conclude that for each possible state of affairs the denotation of $\alpha$ is the same as that of $\beta$. Hence the functions Sense ($\alpha$) and Sense ($\beta$) are the same, in accordance with (2).

15.4 States of Affairs

It is clear from the preceding discussion that under the Carnap interpretation the set of individual concepts, proposition, properties, etc., is determined by the set of "possible states of affairs". And further, that if we consider distinct alternative sets of possible states of affairs, they will generate distinct alternative sets of concepts.

We must now face the problem of a precise treatment of the notion <u>state of affairs</u> (or, more exactly, <u>possible</u> states of affairs). One method of doing this would be to treat the expression as a primitive and attempt to formulate axioms which would characterize the notion to a degree sufficient to establish certain general results about intensional logic. It is, however, one of our main aims to show that intensional logic can be developed in such a way that its semantics can be treated in a purely extensional language,[53] in fact in one of the familiar forms of set theory. Thus, we explicitly avoid the introduction of any "intensional" notions into the metalanguage. Instead we provide certain set theoretical entities to play the role of states of affairs.

Since the sense of a wfe $\alpha$ of $\mathcal{K}_0$ is to be that function which assigns to each state of affairs S the denotation of $\alpha$ in the state S, each such state must determine a unique denotation for each wfe $\alpha$. In addition, each state should determine a unique class of individuals (the individuals existing in that state of affairs ). In fact, each state of affairs determines a unique model for $\mathcal{K}_0$. Now suppose that two states of affairs $S_1$ and $S_2$ determine the same model of $\mathcal{K}_0$, then the sense of any wfe of $\mathcal{K}_0$ will assign the same entity to $S_1$ and $S_2$. Hence it seems natural to simply identify the states of affairs with the models which they determine.[55]

We now turn to a second question in connection with states of affairs, one which has been largely ignored in the preceding. What is a possible state of affairs? In particular, does every model for $\mathcal{K}_0$ constitute a possible state of affairs, independent of which model we take to constitute the actual state of affairs? If a negative answer is given, it would be natural to characterize an intensional model for $\mathcal{K}_1$ as a couple $\langle \mathfrak{M} M \rangle$ where $\mathfrak{M}$ is a model for $\mathcal{K}_0$ (the actual state) and M is a class of models for $\mathcal{K}_0$ (the "possible" states with respect to $\mathfrak{M}$). One such couple might be of the form $\langle \mathfrak{M} \{\mathfrak{M}\} \rangle$. With respect to this model for $\mathcal{K}_1$, all true sentences of $\mathcal{K}_1$ would be necessary. The question we are faced with is: should such possibilities be excluded on purely logical grounds? Church has indicated an inclination to allow such possibilities, and thus to answer our initial question negatively.[56] A number of other authors appear to be similarly inclined.[57] But since they are dealing with indirect discourse forms of modal logic one must be careful in interpreting their remarks. Carnap has in general given a modified positive answer to our question in that he usually has followed a procedure which amounts to specifying some set of models (possibly not *all* models in our sense) and then treating these as the possible states independently of what is taken as the actual state.[58]

We will follow Carnap's procedure. This decision was anticipated in the treatment of the necessity sign under the syntactical interpretation of $\mathcal{K}_1$. There, a sentence of $\mathcal{K}_0$ was said to be necessary just in case it was valid, that is, held in *every* model of $\mathcal{K}_0$. There seems no reason for providing a different treatment under the intensional interpretation. We assume that the class of all possible "individuals" forms a proper set in the set theory of our metalanguage. The class $M_0$ of *all* models of $\mathcal{K}_0$ then also forms a proper set as will various other classes to be introduced later.[59]

## 15.5 Concepts and Senses

Given the set $M_0$ of models for $\mathcal{K}_0$ we can now introduce the set of individual concepts determined by $M_0$ and the set of propositions determined by $M_0$.

D21. (1) *f is an individual concept (with respect to $M_0$)* if and only if f is a function on $M_0$ which assigns to each $\langle DR \rangle \in M_0$ an element of D. (2) *f is a proposition (with respect to $M_0$)* if and only if f is a function on $M_0$ which assigns to each $\mathfrak{M} \in M_0$ a truth value.

For wfe's $\mathcal{K}_o$ of simple type, we define "Sense" in accordance with our earlier suggestion that the sense of $\alpha$ is that concept which assigns to each model $\mathcal{M}$, the denotation of $\alpha$ with respect to $\mathcal{M}$.

D22. If $\alpha$ is a wfe of $\mathcal{K}_o$ of simple type, then <u>the sense of $\alpha$</u> (Sense($\alpha$)) is that function on $M_o$ which assigns to each $\mathcal{M} \in M_o$, $\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_o}(\alpha)$.

Following this line of development, let us now consider wfe's of $\mathcal{K}_o$ of complex type. If $\eta$ is such a wfe, we call the function which assigns to each $\mathcal{M} \in M_o$, the value of $\eta$ in $\mathcal{M}$, <u>the natural-sense of $\eta$</u>. Since wfe's of complex type denote functions, such a natural-sense would be a concept of a function, hence an assignment of a function in $\mathcal{M}$ to each $\mathcal{M} \in M_o$. But recall that in section 12.1 we decided to simplify the type system by rejecting a special type for the analogue to a wfe of complex type, and instead to have such analogues always denote functions on the universes of simple types. In conformity with this policy, $\bar{\eta}$ does not denote $\eta$ itself under the syntactical interpretation. Rather, it denotes a certain function which represents the expression $\eta$. Under the present interpretation, we will again treat $\bar{\eta}$ as denoting not a concept of a function, but rather a certain representative thereof.

Let us, for the moment, restrict our attention to concepts of functions from individuals to individuals.

Among such concepts are the senses of all one-place operation symbols of $\mathcal{K}_0$. Let us call f <u>a natural-concept of a function</u>, if f is itself a function whose domain is $M_0$ and which assigns to each $\langle DR \rangle \in M_0$ a function from D into D. Thus if $\eta$ is a one-place operation symbol of $\mathcal{K}_0$, the natural-sense of $\eta$ is a natural-concept of a function. Let us say that g is a <u>function on concepts</u>, if g is a function from the set of all individual concepts into the set of all individual concepts. If f is a natural concept of a function and g is a function on concepts, we say that <u>g represents f</u>, if for every model $\mathfrak{M} \in M_0$ and individual concept x, $(g(x))(\mathfrak{M}) = (f(\mathfrak{M}))(x(\mathfrak{M}))$. We shall also say of a function on concepts g, that it is <u>invariant</u>, if for all models $\mathfrak{M} \in M_0$ and individual concepts x, y, $(g(x))(\mathfrak{M}) = (g(y))(\mathfrak{M})$ whenever $x(\mathfrak{M}) = y(\mathfrak{M})$.

It is now an elementary matter to establish the following results:

(1) Each natural-concept of a function has a unique representative.

(2) Every representative of a natural-concept of a function is an invariant function on concepts.

(3) Every invariant function on concepts represents a unique natural-concept of a function.

(4) Not all functions on concepts are invariant.

Our method is to everywhere replace natural-concepts of functions by their representatives.

D23. If $\tau_0,\ldots,\tau_r$ are simple types of $\mathcal{K}_0$, $r>0$, $\eta$ is a wfe of $\mathcal{K}_0$ of complex type $\langle\tau_1,\ldots,\tau_r,\tau_0\rangle$, and each of $c_0,\ldots,c_r$ is either the set of all individual concepts or the set of all propositions depending on whether the corresponding element of $\tau_0,\ldots,\tau_r$ is i or t, then <u>the sense of $\eta$</u> (Sense $(\eta)$) is the unique g such that:

(1) g is a function from $(c_1 \times \cdots \times c_r)$ into $c_0$

(2) if $\mathcal{M} \in M_0$, and $\langle x_1,\ldots,x_r\rangle$ is the domain of g, then

$$(g(x_1,\ldots,x_r))(\mathcal{M}) = \mathrm{Val}^{\mathcal{K}_0}_{\mathcal{M}}(\eta)(x_1(\mathcal{M}),\ldots,x_r(\mathcal{M})).$$

This definition embodies our decision to replace natural-concepts of functions by their representatives, since it immediately implies that Sense$(\eta)$ represents the natural-sense of $\eta$.

The preceding definition considerably simplifies our theory. The idea of such a simplification was introduced in Church [6]. Unfortunately, the representation of natural-concepts of functions by invariant functions on concepts is there marred by two defects.

First, the notion of invariance used is too weak; it only requires of a function on concepts g, that for the <u>actual</u> state $\mathcal{M}$, for all individual concepts x, y, $(g(x))(\mathcal{M}) = (g(y))(\mathcal{M})$ whenever $x(\mathcal{M}) = y(\mathcal{M})$; rather than requiring it for <u>all</u> $\mathcal{M}$ in $M_0$. Thus the equivalence between natural-concepts of functions and invariant functions of concepts is lost. Church's axiom embodying the defective notion of invariance ($16^{\alpha,\beta}$), in combination with certain other natural principles which the system is required to fulfill, can be used to prove theorems within the system which assert that there is only one concept of each object (for example, only one true proposition and only one false one). We discussed earlier, the possibility of leaving such questions open by not assuming that the class of all possible states of affairs contains more than one member. Although our initial decision to make such an assumption may be somewhat questionable (Church, for example, explicitly opposes it) the contrary assumption, as embodied in the above mentioned results in Church's system, is surely untenable.

The second defect in Church's representation of natural-concepts of functions raises a second difficulty with the invariance property, but turns primarily on the fact that the underlying logic of Church's base language (recall section 9) is that of the simple theory of types. We must therefore represent natural-concepts of second order functions whose domains are sets of functions. What, for example, is to represent a concept of a second order function f which assigns an individual to every function from individuals to individuals? In accord with the general replacement principle, Church assigns an invariant function g whose domain is that of the representatives of natural-concepts of functions from individuals to individuals and whose range is that of concepts of individuals. But such a function will have in its domain <u>all</u> functions from concepts of individuals to concepts of individuals, since the primary virtue of the replacement principle is its identification of the type $\langle i,i\rangle_1$ (concepts of functions) with the type $\langle i_1,i_1\rangle$ (functions on concepts). It is easily seen (by (4), (2) p. 68) that not all functions from concepts of individuals to concepts of individuals represent natural-concepts of functions; thus our invariant function g contains in its domain entities which are not (and do not represent) concepts at all. Even if the first defect is repaired by revising the axioms to require the stronger notion of invariance (a revision which can easily be made), it is difficult to think of a natural requirement on the value of the function for those elements of the domain of g which are not concepts. One possibility is to require that an invariant function assign a designated element to all non-concepts in its domain, but this course would directly conflict with certain other basic principles of Church's system. If the problem is left unresolved (as it is in the article) and the invariance requirement is stated only with respect to <u>concepts</u> in the domain of g, we are again faced with the problem that a given concept of a (second-order) function will have more than one representative, which, as before, leads to conflicts with some of the leading ideas used in constructing the system. It would seem that if a system containing second order functions is to be developed, the best course may be to forego the immediate simplification which arises from the replacement principle in favor of a system with a more complicated type structure but also a more natural interpretation.

15.6 Universe and Value for $\mathcal{K}_1$

As in the case of the syntactical interpretation of $\mathcal{K}_1$, we need not introduce a new notion of model. Instead we simply introduce a new notion of value under the Intensional interpretation of $\mathcal{K}_1$ ($\mathrm{Val}^{I,\mathcal{K}_1}_{\mathfrak{M}}(\alpha)$) defined in terms of the notion of sense for wfe's of $\mathcal{K}_o$. Note that the latter notion was defined (as was $\mathrm{Val}^{S,\mathcal{K}_1}_{\mathfrak{M}}(\alpha)$) in terms of the notion of value for wfe's of $\mathcal{K}_o$ ($\mathrm{Val}^{\mathcal{K}_o}_{\mathfrak{M}}(\alpha)$). It is convenient to first introduce the universe of a type of $\mathcal{K}_1$ under the intensional interpretation.

D24. If $\mathfrak{M} = \langle DR \rangle$ is an element of $M_o$ and $\tau$ is a type of $\mathcal{K}_1$, then the universe of $\tau$ in $\mathfrak{M}$ with respect to the Intensional interpretation of the language $\mathcal{K}_1$ ($\mathcal{U}^{I,\mathcal{K}_1}_{\mathfrak{M}}(\tau)$) is given by the following:

(1) the universe of i is D

(2) the universe of t is T,F

(3) the universe of $i_1$ is the set of all individual concepts

(4) the universe of $t_1$ is the set of all propositions

(5) if the universes of $\tau_o, \ldots, \tau_r$ are respectively $u_o, \ldots, u_r$ and $r>0$, then the universe of $\langle \tau_1, \ldots, \tau_r, \tau_o \rangle$

is the set of all functions from $(u_1 x \cdots x u_r)$ into $u_o$.

D25. If $\mathcal{M} = \langle DR \rangle$ is an element of $M_o$, and $\alpha$ is a wfe of $\mathcal{K}_1$, then <u>the value of $\alpha$ in $\mathcal{M}$ with respect to the Intensional interpretation of $\mathcal{K}_1$</u> $(\mathrm{Val}^{I,\mathcal{K}_1}_{\mathcal{M}}(\alpha))$ is given by the following:

(1) if $\alpha$ is an atomic wfe of $\mathcal{K}_o$, then the value of $\alpha$ is $R(\alpha)$

(2) if $\alpha$ is an atomic wfe of $\mathcal{K}_o$, then the value of $\tilde{\alpha}$ is Sense $(\alpha)$

(3) if $\tau$ is $i_1$ or $t_1$, and $\alpha$ is $\mathrm{Id}(\tau)$, then the value of $\alpha$ is the unique f such that:

(a) f is a function from $(\mathcal{U}^{I,\mathcal{K}_1}_{\mathcal{M}}(\tau) \times \mathcal{U}^{I,\mathcal{K}_1}_{\mathcal{M}}(\tau))$ into $\{T,F\}$

(b) if $\langle g,h \rangle$ is in the domain of f, then $f(g,h) = T$ if and only if $g = h$

(4) if $\tau$ is i or t, and $\alpha$ is $\mathrm{Delta}(\tilde{\tau})$, then the value of $\alpha$ is the unique f such that:

(a) f is a function from $(\mathcal{U}^{I,\mathcal{K}_1}_{\mathcal{M}}(\tilde{\tau}) \times \mathcal{U}^{I,\mathcal{K}_1}_{\mathcal{M}}(\tau))$ into $\{T,F\}$

(b) if $\langle g,x \rangle$ is in the domain of f, then $f(g,x) = T$ if and only if $g(\mathcal{M}) = x$

(5) if $\alpha$ is Nec, then the value of $\alpha$ is the unique f such that:

(a) f is a function from the set of all propositions into $\{T,F\}$

(b) if g is in the domain of f, then $f(g) = T$ if and only if for all $\eta \in M$, $g(\eta) = T$

(6) if $\eta, \beta_1, \ldots, \beta_r$ are wfe's of $\mathcal{K}_1$ of types $\langle \tau_1, \ldots, \tau_r, \tau_o \rangle$, $\tau_1, \ldots, \tau_r$ respectively with values f, $x_1, \ldots, x_r$ respectively, and $\alpha$ is $\eta ^\frown \beta_1 ^\frown \cdots ^\frown \beta_r$, then the value of $\alpha$ is $f(x_1, \ldots, x_r)$.

The following are the counterparts to theorems 7-10 of section 14.2.

T14. If $\alpha$ is a wfe of $\mathcal{K}_1$ with type $\tau$, and $\mathcal{M} \in M_o$, then $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_1}(\alpha) \in \mathcal{U}_{\mathcal{M}}^{I,\mathcal{K}_1}(\tau)$

T15. If $\alpha$ is a wfe of $\mathcal{K}_o$, and $\mathcal{M} \in M_o$, then $\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_o}(\alpha) = \mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_1}(\alpha)$

T16. If $\mathcal{M} \in M_o$, and F is that function from the wfe's of $\mathcal{K}_1$ which assigns to each such $\alpha$, $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_1}(\alpha)$; then $\langle \mathcal{K}_1, F \rangle$ is a Fregean semantical system.

T17. If R is that function from the wfe's of $\mathcal{K}_o$ which assigns to each such $\alpha$, Sense $(\alpha)$; then $\langle \mathcal{K}_o, R \rangle$ is a Fregean semantical system.

## 15.7 Metalinguistic Features of $\mathcal{K}_1$

As with the syntactical interpretation of $\mathcal{K}_1$, under the present interpretation we may also think of $\mathcal{K}_1$ as a kind of formalized metalanguage for $\mathcal{K}_o$. But here the departure from conventional metalanguages is even greater than before. We now have no means for speaking directly of the wfe's of $\mathcal{K}_0$; we can speak directly only of the concepts they express. It is not uncommon for a metalanguage (at least informal metalanguages) to include the means to speak of the senses or meanings of the well formed expressions of the object language. This is done when we ask whether two expressions of the object language have the same meaning or whether everything expressible in one part of the object language is expressible in another part of the object language. But this is usually done indirectly by using names of the expressions and an operation expression such as 'the meaning of'. In the present treatment of $\mathcal{K}_0$ we speak directly of the concepts without mediation by way of the expressions.[60]

Some insight into the present interpretation can be gained by comparing, under the two interpretations, those features (enumerated in section 14.3) wherein $\mathcal{K}_1$ resembles a formalized metalanguage for $\mathcal{K}_0$. The inclusion of the object language in the metalanguage is of course unaffected by the interpretation, as are the features related to type distinctions mentioned in (2)(page 53).

In place of the reading $\ulcorner$the name $\alpha$ <u>denotes</u> the individual $\beta\urcorner$ for Delta$(i_1)^\frown\alpha^\frown\beta$, we now read this wfe $\ulcorner$the individual concept $\alpha$ is a <u>concept of</u> the individual $\beta\urcorner$.[61]

Similarly, we replace the reading ⌜the sentence $\emptyset$ denotes the truth value $\Gamma$⌝ of $\text{Delta}(t_1)^\frown\emptyset^\frown\Gamma$ by ⌜the proposition $\emptyset$ is a concept of the truth value $\Gamma$⌝. In the case of $\text{Delta}(t_1)$, the second argument expression, $\Gamma$, must be a sentence. Hence, we might instead have introduced a one-place predicate which when applied to a wfe $\emptyset$ of type $t_1$ forms a sentence $\text{Tr}(\emptyset)$ and which is such that $\text{Tr}(\emptyset)$ is true just in case, $\text{Delta}(t_1)^\frown\emptyset^\frown\Gamma$ is true whenever $\Gamma$ is true. The wfe $\text{Delta}(t_1)^\frown\emptyset^\frown\Gamma$ would then be equivalent to the material biconditional $(\text{Tr}(\emptyset) \equiv \Gamma)$. The syntactical interpretation would then provide the reading ⌜the sentence $\emptyset$ is true⌝ for $\text{Tr}(\emptyset)$ and the intensional interpretation would give the reading ⌜the proposition $\emptyset$ is a concept of Truth⌝.[62] Within the very limited resources of $\mathcal{K}_1$ we can only name propositions which are expressed by sentences of $\mathcal{K}_0$, and similarly we can only name individual concepts which are expressed by names of $\mathcal{K}_0$.

T18. If $\alpha$ is a wfe of $\mathcal{K}_1$ of type $i_1(t_1)$, then there is a wfe $\beta$ of $\mathcal{K}_0$ of type $i(t)$ such that $\alpha$ is $\beta$.

Thus, in the case of $\mathcal{K}_1$, we could use the reading ⌜the proposition $\emptyset$ <u>is expressed by a true sentence</u>⌝ for $\text{Tr}(\emptyset)$, and ⌜the individual concept $\alpha$ <u>is expressed by a name of</u> the individual $\beta$⌝, for $\text{Delta}(i_1)^\frown\alpha^\frown\beta$. However, when we turn to richer languages with variables these readings will no longer be available, since the variables will range over concepts which are not expressed by any wfc.

In general, the question of the truth of a given sentence or the denotation of a given name is an empirical one.[63] That is, although a given sentence may be true in the actual world, there are usually other possible states with respect to which it is false. And similarly, the question of the truth value of which a given proposition actually is a concept and the individual of which a given individual concept actually is a concept are also empirical. That is, although a given proposition may actually be a concept of Truth, there are usually other possible states with respect to which it is a concept of Falsehood. Thus, just as our fundamental notion of the relationship between an expression and its denotation is a relative one (the denotation of the expression $\alpha$ <u>in the model $\mathfrak{M}$</u>), so is our fundamental notion of the relation between a concept and that of which it is a concept. It will be convenient to introduce a brief way of expressing this relation. Let us call it the relation of <u>Determination</u>. As with denotation, we shall speak of <u>the Determination</u> of a concept, and say that each such concept <u>Determines</u> a unique entity (its Determination).[64] Note that under our treatment of intensional entities, each concept <u>is</u> a certain subrelation of the relative notion of Determination. That is, the Determination of the concept x <u>in the model $\mathfrak{M}$</u> is just x ($\mathfrak{M}$). The absolute notions of denotation and Determination (as when we speak simply of the (actual) denotation or Determination, without mentioning a state of affairs) can be introduced in terms of the relative notions with the help of the absolute notion of state of affairs (or model). If we can speak simply of the actual state $\mathfrak{M}^*$, then the absolute denotation of x (or Determination of x) is just the relative denotation (or Determination) with respect to $\mathfrak{M}^*$. It seems clear that Determination is the natural intensional counterpart to the syntactical relation denotation.

Returning once again to our reading of Delta$(\tau)^\frown\alpha^\frown\beta$, we now provide the readings $\ulcorner\alpha$ actually denotes $\beta\urcorner$ under the syntactical interpretation, and $\ulcorner\alpha$ actually Determines $\beta\urcorner$ under the intensional interpretation. It may therefore be seen that we could have expanded on the third feature of $\mathcal{K}_1$ as a formalized metalanguage for $\mathcal{K}_0$ (section 14.3) by remarking that the denotation (Determination) predicates have the sense of <u>absolute</u> denotation (Determination).[65]

We turn now to the fourth and perhaps most important metalinguistic feature of $\mathcal{K}_1$ (section 14.3). For every name or sentence $\alpha$ of $\mathcal{K}_0$, $\mathcal{K}_1$ contains an analogous wfe $\overline{\alpha}$. Under the syntactical interpretation, $\overline{\alpha}$ provides a standard name of $\alpha$ (theorems 11 and 12). Under the intensional interpretation, $\overline{\alpha}$ will provide a standard name of the sense of $\alpha$. Thereby, we also partially fulfill one of the basic requirements suggested by Church[66] for a language based on Frege's distinction between sense denotation; namely, that for each wfe $\alpha$ (without free variables) we provide a wfe $\beta$ which denotes the sense of $\alpha$. The following theorem is the counterpart to theorems 11 and 12.

T19. If $\alpha$ is a wfe of $\mathcal{K}_o$, and $\mathcal{M} \in M_o$, then $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_1}(\overline{\alpha}) =$ Sense $(\alpha)$.

As indicated earlier, standard names are possible only for non-contingent entities. The following counterpart to theorem 13 indicates that concepts are such entities.

T20. If $\tau$ is a type of $\mathcal{K}_o$, and $\mathcal{M}, \mathcal{N} \in M_o$, then $\mathcal{U}_{\mathcal{M}}^{I,\mathcal{K}_1}(\tilde{\tau}) = \mathcal{U}_{\mathcal{N}}^{I,\mathcal{K}_1}(\tilde{\tau})$.

## CHAPTER 3

## THE HIERARCHY OF LANGUAGES $\mathcal{K}_n$

We now begin our formal development of semantical systems for oblique contexts. The motivation for the particular line we take was given in the preceding chapters. Although formal definitions for some of the central notions have been given in Chapter 2, the following developments are self-contained. The task of the present chapter is the construction of a hierarchy of languages $\mathcal{K}_n$ with syntactical and intensional interpretations, thus generalizing to arbitrary $n$ the construction of Chapter 2 for $0$ and $1$.

In defining the semantical notions for the language $\mathcal{K}_{n+1}$ we will make essential use of those notions for the language $\mathcal{K}_n$. In the following chapter, we will consider a single language, $\mathcal{K}_\omega$, which includes all of the languages in the hierarchy. But the semantical notions for $\mathcal{K}_\omega$ are again defined with reference to the corresponding notions for the languages $\mathcal{K}_n$. Thus, the stepwise procedure of developing the hierarchy plays a vital role in our understanding of these notions. This procedure corresponds to the policy of language levels, that of always distinguishing object and metalanguages when semantical notions are considered.[67]

In contrast to the hierarchical dependence of the semantical notions, in the description of the languages themselves, that is, the notions of type, well-formed expression, atomic well-formed expression, etc., it seems more natural to first introduce the general notions for the language $\mathcal{K}_\omega$, and then subdivide them for each of the languages $\mathcal{K}_n$. Thus we will first define such notions as <u>$\tau$ is a simple type</u> and <u>the rank of the type $\tau$</u>, and then, making special use of the notion of rank, we will define <u>$\tau$ is a simple type of $\mathcal{K}_n$</u>.

## 16. The Language $\mathcal{K}_n$

### 16.1 The Types of $\mathcal{K}_n$

As in Chapter 2, the simple types fall into two hierarchies (i and t), and the complex types consist of finite sequences of simple types. The simple types are identified with numbers, although we continue to use the nomenclature "$i_j$" and "$t_j$".

D26. If j is a natural number, then

(1) $i_j = 2^{j+1}$

(2) $t_j = 3^{j+1}$

When the subscript to a type symbol is "o", we shall often omit it, writing "i" for "$i_o$" and "t" for "$t_o$".

D27.

(1) <u>$\tau$ is a simple type</u> if and only if $\tau = i_j$ or $\tau = t_j$ for some natural number j.

(2) <u>$\tau$ is a complex type</u> if and only if $\tau = \langle\tau_1,\ldots,\tau_r,\tau_o\rangle$,

where $\tau_o,\ldots,\tau_r$ are simple types and $r > o$.

(3) $\underline{\tau \text{ is a type}}$ if and only if $\tau$ is a simple type or $\tau$ is a complex type.

D28. If $\tau$ is a type, $\underline{\text{the rank of } \tau}$ is given by the following:

(1) the rank of $i_j = j$

(2) the rank of $t_j = j$

(3) if $\tau_o,\ldots,\tau_r$ are simple types with ranks $k_o,\ldots,k_r$ respectively, and $r > o$, then the rank of $<\tau_1,\ldots,\tau_r,\tau_o>$ = the maximum of: $k_o,\ldots,k_r$.

If a wfe $\alpha$ has type $\tau$, the type of the analogue to $\alpha$ will be the elevation of $\tau$; similarly, the type of the analogue to the analogue to $\alpha$ will be the elevation of the elevation of $\tau$, that is, the second elevation of $\tau$. In this way we are led to the general notion, $\underline{\text{the } k^{\underline{th}} \text{ elevation of the type } \tau}$.

D29. If $\tau$ is a type, and k is a natural number, then $\underline{\text{the } k^{\underline{th}} \text{ elevation of } \tau}$ is given by the following:

(1) the $k^{\underline{th}}$ elevation of $i_j = i_{j+k}$

(2) the $k^{\underline{th}}$ elevation of $t_j = t_{j+k}$

(3) if $\tau_o,\ldots,\tau_r$ are simple types, $r > o$, and $\tau_o^k,\ldots,\tau_r^k$ are the $k^{\underline{th}}$ elevations of $\tau_o,\ldots,\tau_r$ respectively, then the $k^{\underline{th}}$ elevation of $<\tau_1,\ldots,\tau_r,\tau_o>$ = $<\tau_1^k,\ldots,\tau_r^k,\tau_o^k>$.

The notions of rank and elevation are related in the following simple way.

T21. If $\tau$ is a type, and k is a natural number, then the rank of the k<u>th</u> elevation of $\tau$ is the rank of $\tau$, plus k.

Note that the 0<u>th</u> elevation of $\tau$ is just $\tau$ itself. It is often convenient to be able to speak in a simple manner of the first elevation of a type. For this purpose we use the superscripted wiggle ($\tilde{\tau}$), which is the counterpart to the superscripted bar ($\bar{\alpha}$) for the (first) analogue to a wfe. We also sometimes speak of $\tilde{\tau}$ simply as <u>the</u> elevation of $\tau$.

D30. If $\tau$ is a type, then $\tilde{\tau}$ is the 1<u>st</u> elevation of $\tau$.

We now introduce the restricted notion, <u>type of $\mathcal{K}_n$</u>, in terms of the general notion and the notion of rank. It will be seen that the rank of a type is the lowest level in the hierarchy at which the type appears. A similar remark applies to the notion of rank for wfe's, which will be introduced in the following section. We will not separately define <u>simple type of $\mathcal{K}_n$</u> and <u>complex type of $\mathcal{K}_n$</u>, but will understand these phrases to mean: a simple type which is a type of $\mathcal{K}_n$, a complex type which is a type of $\mathcal{K}_n$. The same method will be followed in the following section where we simply define <u>wfe of $\mathcal{K}_n$</u> without separate definitions of <u>atomic wfe of $\mathcal{K}_n$</u>, and <u>compound wfe of $\mathcal{K}_n$</u>.

D31. <u>$\tau$ is a type of $\mathcal{K}_n$</u> if and only if $\tau$ is a type and the rank of $\tau$ is less than or equal to n.

The following theorems follow immediately from the definitions and theorem 21.

T22. If $\tau$ is a type of $\mathcal{K}_n$, then $\tilde{\tilde{\tau}}$ is a type of $\mathcal{K}_{n+1}$.

T23. $\tau$ is a type of $\mathcal{K}_n$ if and only if $\tilde{\tilde{\tau}}$ is a type of $\mathcal{K}_{n+1}$.

T24. Not all types of $\mathcal{K}_{n+1}$ (for example, $\langle i_{n+1}, t\rangle$) which are not also types of $\mathcal{K}_n$ have the form $\tilde{\tilde{\tau}}$ for some type $\tau$ of $\mathcal{K}_n$.

T25. $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$ is a complex type of $\mathcal{K}_n$ if and only if $\tau_o,\ldots,\tau_r$ are simple types of $\mathcal{K}_n$ and $r>0$.

The following theorem provides a useful form of induction over the types of $\mathcal{K}_{n+1}$.

T26. $\tau$ is a type of $\mathcal{K}_{n+1}$ if and only if $\tau$ satisfies one of the following mutually exclusive conditions:

(1) $\tau$ is a simple type of $\mathcal{K}_o$

(2) $\tau = \tilde{\tilde{\tau'}}$, for some simple type $\tau'$ of $\mathcal{K}_n$

(3) $\tau = \langle\tau_1,\ldots,\tau_r,\tau_o\rangle$, for some simple types $\tau_o,\ldots,\tau_r$ of $\mathcal{K}_{n+1}$ and some $r>0$.

## 16.2 The well Formed Expressions of $\mathcal{K}_n$

The atomic wfe's are introduced with the following considerations in mind. We begin with the descriptive constants of $\mathcal{K}_0$, namely (1) the p-place operation symbols and (2) the p-place predicates. We then add the following logical signs: (3) the material conditional sign, (4) the negation sign, (5) an identity predicate for each simple type, (6) a delta predicate for each higher simple type (that is, each simple type other than $i_o$ and $t_o$), and (7) the necessity predicate. Finally, for each of the atomic wfe $\alpha$ so far introduced, we must add an analogue, an analogue to the analogue, an analogue to the analogue to the analogue, and so on; to put it more generally, we must add a j*th* analogue to $\alpha$ for every natural number j. In this way, each of the atomic wfe's introduced in steps (1)-(7) generates a denumerable hierarchy of atomic wfe's of which the original wfe may be considered the 0*th* member.

For definiteness, we may identify all expressions with finite sequences of numbers in the manner of section 10.2, being careful to distinguish an atomic wfe from its j*th* analogue.

D31. If m, p, j, $\tau$ are any natural numbers, then

(1) $\mathrm{Opsymb}_j(m,p) = \langle 2^{m+1} \cdot 3^{p+1} \cdot 5^{j+1} \rangle$

(2) $\mathrm{Pred}_j(m,p) = \langle 3^{m+1} \cdot 5^{p+1} \cdot 7^{j+1} \rangle$

(3) $\mathrm{Cond}_j = \langle 5^{j} \rangle$

(4) $\mathrm{Neg}_j = \langle 7^{j} \rangle$

(5) $\mathrm{Id}_j(\tau) = \langle 2^{\tau+1} \cdot 3^{j+1} \rangle$

(6) $\mathrm{Delta}_j(\tau) = \langle 2^{\tau+1} \cdot 5^{j+1} \rangle$

(7) $\mathrm{Nec}_j = \langle 11^{j+1} \rangle$

D32. If x and y are finite sequences, then $x^\frown y$ is their concatenation.

When the subscript to the symbol for a wfe is "o", we shall often omit it, writing "Pred(m,p)" for "$\text{Pred}_o$(m,p), and "Cond" for "$\text{Cond}_o$", etc. Opsymb(m,p) is the m<u>th</u> p-place operation symbol, Pred(m,p) is the m<u>th</u> p-place predicate, Cond is the material conditional sign, Neg is the negation sign, Id($\tau$) is the identity predicate for type $\tau$, Delta($\tilde{\tilde{\tau}}$) is the delta predicate for type $\tilde{\tilde{\tau}}$, and Nec is the necessity predicate.

The wfe $\text{Pred}_j$(m,p) is the j<u>th</u> analogue to Pred(m,p). Thus, the type of $\text{Pred}_j$(m,p) is the j<u>th</u> elevation of the type of Pred(m,p). Types are assigned to the other atomic wfe's in the same manner.

D33. <u>$\alpha$ is an atomic wfe</u> if and only if there are natural numbers m, p, j, and a simple type $\tau$ such that $\alpha$ is one of the following:

(1) $\text{Opsymb}_j$(m,p)

(2) $\text{Pred}_j$(m,p)

(3) $\text{Cond}_j$

(4) $\text{Neg}_j$

(5) $\text{Id}_j(\tau)$

(6) $\text{Delta}_j(\tilde{\tilde{\tau}})$

(7) $\text{Nec}_j$

D34. If $\alpha$ is an atomic wfe, then <u>the kth analogue to $\alpha$</u> is given by the following:

(1) the $k\underline{th}$ analogue to $Opsymb_j(m,p)$ is $Opsymb_{j+k}(m,p)$

(2) the $k\underline{th}$ analogue to $Pred_j(m,p)$ is $Pred_{j+k}(m,p)$

(3) the $k\underline{th}$ analogue to $Cond_j$ is $Cond_{j+k}$

(4) the $k\underline{th}$ analogue to $Neg_j$ is $Neg_{j+k}$

(5) the $k\underline{th}$ analogue to $Id_j(\tau)$ is $Id_{j+k}(\tau)$

(6) the $k\underline{th}$ analogue to $Delta_j(\tilde{\tau})$ is $Delta_{j+k}(\tilde{\tau})$

(7) the $k\underline{th}$ analogue to $Nec_j$ is $Nec_{j+k}$

D35. If $\alpha$ is an atomic wfe, <u>the type of $\alpha$</u> is given by the following:

(1) (a) the type of Opsymb(m,0) is i

(b) if $\tau_1,\ldots,\tau_p$ are each i, and p>0, then the type of Opsymb(m,p) is $<\tau_1,\ldots,\tau_p,i>$

(2) (a) the type of Pred(m,0) is t

(b) if $\tau_1,\ldots,\tau_p$ are each i, and p>0, then the type of Pred(m,p) is $<\tau_1,\ldots,\tau_p,t>$

(3) the type of Cond is $<t,t,t>$

(4) the type of Neg is $<t,t>$

(5) if $\tau$ is a simple type, the type of $Id(\tau)$ is $<\tau,\tau, t>$

(6) if $\tau$ is a simple type, the type of $Delta(\tilde{\tau})$ is $<\tilde{\tau},\tau, t>$

(7) the type of Nec is $<t_1,t>$

We introduce the notions of a wfe and the type of a wfe by a simultaneous recursion.

D36. <u>$\alpha$ is a wfe of type $\tau$</u> if and only if either

(1) $\alpha$ is an atomic wfe of type $\tau$, or

(2) there are $\eta$, $\beta_1,\ldots,\beta_r$, $\tau_1,\ldots,\tau_r$ such that $\beta_1,\ldots,\beta_r$ are wfe's of the simple types $\tau_1,\ldots,\tau_r$ respectively, $\eta$ is a wfe of type $<\tau_1,\ldots,\tau_r,\tau>$ and $\alpha$ is $\eta^\frown\beta_1^\frown\cdots^\frown\beta_r$.

D37. <u>$\alpha$ is a compound wfe</u> if and only if $\alpha$ is a wfe and $\alpha$ is not atomic.

Since we have already introduced the k<u>th</u> analogue to an atomic wfe, it remains only to do the same for compound wfe's.

D38. If $\tau_o,\ldots,\tau_r$ are simple types, $r>0$, $\eta,\alpha_1,\ldots,\alpha_r$ are wfe's of types $<\tau_1,\ldots,\tau_r,\tau_o>$, $\tau_1,\ldots,\tau_r$ respectively, k is a natural number, and $\eta^k$, $\alpha_1{}^k,\ldots,\alpha_r{}^k$ are the k<u>th</u> analogues to $\eta$, $\alpha_1,\ldots,\alpha_r$ respectively, then <u>the kth</u> analogue to $\eta^\frown\alpha_1^\frown\cdots^\frown\alpha_r$ is $\eta^{k\frown}\alpha_1{}^{k\frown}\cdots^\frown\alpha_r{}^k$.

The elevation of a type and the analogue to a wfe are related in the following simple manner.

T27. If $\alpha$ is a wfe of type $\tau$, then the type of the j<u>th</u> analogue to $\alpha$ is the j<u>th</u> elevation of $\tau$.

We can now introduce the bar notation $(\bar{\alpha})$ in a precise way. We sometimes speak of $\bar{\alpha}$ as, simply, <u>the</u> analogue to $\alpha$.

D39. If $\alpha$ is a wfe, then $\bar{\alpha}$ is the 1<u>st</u> analogue to $\alpha$

The rank of a wfe is now given in terms of its type.

D40. If $\alpha$ is a wfe, the rank of $\alpha$ is given by the following:

(1) if $\alpha$ is an atomic wfe of type $\tau$, then the rank of $\alpha$ is the rank of $\tau$.

(2) if $\eta, \beta_1, \ldots, \beta_r$ are atomic wfe's of types $<\tau_1, \ldots, \tau_r, \tau_o>$, $\tau_1, \ldots, \tau_r$ respectively, with ranks $k_o$, $k_1, \ldots, k_r$ respectively, and $r>0$, then the rank of $\eta^\frown\beta_1{}^\frown\cdots{}^\frown\beta_r$ is the maximum of: $k_o, \ldots, k_r$.

We now define wfe of $\mathcal{K}_n$ with reference to the notion of rank.

D41. $\alpha$ is a wfe of $\mathcal{K}_n$ if and only if $\alpha$ is a wfe and the rank of $\alpha$ is less than or equal to n.

The following theorems either follow immediately from the definitions or require a simple induction on the wfe's.

T28. Each wfe has a unique type.

T29. All compound wfe's are of simple type.

T30. If $\alpha$ is a wfe of $\mathcal{K}_n$, then $\bar{\alpha}$ is a wfe of $\mathcal{K}_{n+1}$.

T31. (1) There are wfe's of $\mathcal{K}_{n+1}$ of type t which are not wfe's of $\mathcal{K}_n$.

(2) All wfe's of type i are wfe's of $\mathcal{K}_o$.

T32. $\alpha$ is a wfe of $\mathcal{K}_n$ of type $\tau$ if and only if $\bar{\alpha}$ is a wfe of $\mathcal{K}_{n+1}$ of type $\tilde{\tilde{\tau}}$.

The following theorem illustrates the limited resources of the languages in our present hierarchy. All wfe's of the higher types have the form $\overline{\alpha}$.

T33. $\beta$ is a wfe of $\mathcal{K}_{n+1}$ of type $\tilde{\tilde{\tau}}$ if and only if there is a wfe $\alpha$ of $\mathcal{K}_n$ of type $\tau$ such that $\beta$ is $\overline{\alpha}$.

Proof by induction on $\beta$.

The following theorem provides a useful form of induction of the wfe's of $\mathcal{K}_{n+1}$.

T34. $\alpha$ is a wfe of $\mathcal{K}_{n+1}$ if and only if $\alpha$ satisfies one of the following mutually exclusive conditions:

(1) $\alpha$ is an atomic wfe of $\mathcal{K}_o$

(2) $\alpha$ is $\overline{\beta}$, for some atomic wfe $\beta$ of $\mathcal{K}_n$

(3) $\alpha$ is Id($\tilde{\tilde{\tau}}$), for some simple type $\tau$ of $\mathcal{K}_n$

(4) $\alpha$ is Delta($\tilde{\tilde{\tau}}$), for some simple type $\tau$ of $\mathcal{K}_n$

(5) $\alpha$ is Nec

(6) there are $\eta$, $\beta_1, \ldots, \beta_r$, $\tau_o, \ldots, \tau_r$ such that r>0, $\eta$, $\beta_1, \ldots, \beta_r$ are wfe's of $\mathcal{K}_{n+1}$ of types $<\tau_1, \ldots, \tau_r, \tau_o>$, $\tau_1, \ldots, \tau_r$ respectively, and $\alpha$ is $\eta{}^\frown\beta_1{}^\frown\cdots{}^\frown\beta_r$.

The following abbreviations provide a more familiar notation.

D42. If j is a natural number, then

(1) $(\phi \supset_j \psi) = \mathrm{Cond}_j{}^\frown\phi{}^\frown\psi$

(2) $\sim_j \phi = \mathrm{Neg}_j \phi$

(3) $(\phi \&_j \psi) = \sim_j(\phi \supset_j \sim_j \psi)$

(4) $(\phi \vee_j \psi) = (\sim_j \phi \supset_j \psi)$

(5) $(\phi \equiv_j \psi) = \mathrm{Id}_j(t)^\frown \phi^\frown \psi$

D43. If j is a natural number, and the type of both $\alpha$ and $\beta$ is the jth elevation of the simple type $\tau$, then $(\alpha =_j \beta) = \mathrm{Id}_j(\tau)^\frown \alpha^\frown \beta$

D44. If j is a natural number, the type of $\alpha_1$ is the j*th* elevation of the simple type $\tilde{\tau}$, and the type of $\beta$ is j*th* elevation of the simple type $\tau$, then $\Delta_j(\alpha_1, \beta) = \mathrm{Delta}_j(\tilde{\tau})^\frown \alpha_1^\frown \beta$.

D45. If j is a natural number, $N_j \phi = \mathrm{Nec}_j{}^\frown \phi$.

D46. If j is a natural number, $P_j = \mathrm{Pred}_j(0,0)$.

D47. If j is a natural number, and the type of $\phi_1$ is the j*th* elevation of $t_1$, then $\mathrm{Tr}_j \phi_1 = \Delta_j(\phi_1, (P_j \supset_j P_j))$.

We follow the general practice of omitting the subscript when it is "0". Hence we write "$(\alpha = \beta)$", "$(\phi \supset \psi)$", etc.

T35. If $\phi, \psi$ are wfe's of type t, $\phi_1$ is a wfe of type $t_1$, $\alpha$, $\beta$ are wfe's of type $\tau$, and $\alpha_1$ is a wfe of type $\tilde{\tau}$, then all of the following are wfe's of type t.

(1) $(\phi \supset \psi)$

(2) $\sim \phi$

(3) $(\phi \& \psi)$

(4) $(\phi \vee \psi)$

(5) $(\phi \equiv \psi)$

(6) $(\alpha = \beta)$

(7) $\triangle(\alpha_1, \beta)$

(8) $N\phi_1$

(9) $P$

(10) $Tr\phi_1$

T36. If $\phi$, $\psi$, $\phi_1$, $\alpha$, $\beta$, $\alpha_1$ are as in the hypothesis of T30, and their j<u>th</u> analogues are $\phi^j$, $\psi^j$, ${\phi_1}^j$, $\alpha^j$, $\beta^j$, ${\alpha_1}^j$ respectively, then each of the following is a wfe of type $t_j$ and the j<u>th</u> analogue to the corresponding wfe of T35:

(1) $(\phi^j \supset_j \psi^j)$

(2) $\sim_j \phi^j$

(3) $(\phi_j \&_j \psi^j)$

(4) $(\phi^j \vee_j \psi^j)$

(5) $(\phi^j \equiv_j \psi^j)$

(6) $(\alpha^j =_j \beta^j)$

(7) $\triangle_j({\alpha_1}^j, \beta^j)$

(8) $N_j {\phi_1}^j$

(9) $P_j$

(10) $Tr_j {\phi_1}^j$

The following theorem provides a useful form of induction over sentences of $\mathcal{K}_{n+1}$.

T37. $\phi$ is a wfe of $\mathcal{K}_{n+1}$ of type t if and only if $\phi$

satisfies one of the following mutually exclusive conditions:

(1) there are natural numbers m,p and wfe's $\alpha_1,\ldots,\alpha_p$ of type i and $\emptyset = \mathrm{Pred}(m,p)^\frown\alpha_1{}^\frown\cdots{}^\frown\alpha_p$

(2) there are wfe's $\Psi, \chi$ of $\mathcal{K}_{n+1}$ of type t and $\emptyset$ is one of the following: (a) $(\psi \supset \chi)$, (b) $\sim\Psi$, (c) $(\Psi \equiv \chi)$

(3) there are wfe's $\alpha,\beta$ of $\mathcal{K}_n$, both of simple type $\tau$, and and $\emptyset = (\overline{\alpha} = \overline{\beta})$

(4) there are wfe's $\alpha$ ,$\beta$ of $\mathcal{K}_0$, both of simple type i, and $\emptyset = (\ \alpha = \beta)$

(5) there are wfe's $\alpha$ ,$\beta$, both of simple type $\tau$, such that $\alpha$ is a wfe of $\mathcal{K}_n$, $\beta$ is a wfe of $\mathcal{K}_{n+1}$, and $\emptyset = \triangle(\overline{\alpha} ,\beta)$

(6) there is a wfe $\Psi$ of $\mathcal{K}_n$ of type t and $\emptyset = \mathrm{N}\,\Psi$

Proof using theorems 31 (2), 32-35.

## 17. The Interpretation of $\mathcal{K}_0$

We assume that we have available an infinite proper set consisting of all possible "individuals". The models for $\mathcal{K}_0$ will all draw their domains from this set. We can simplify the definition of <u>model for $\mathcal{K}_0$</u> by first introducing two subsidiary notions: <u>the universe of a type of $\mathcal{K}_0$ in a domain of individuals D</u>, and the <u>identity function on the set K.</u>

D48. If $\tau$ is a type of $\mathcal{K}_0$, and D is a set, then <u>the universe of $\tau$ in D</u> is given by the following:

(1) the universe of i is D

(2) the universe of t is $\{T,F\}$

(3) if $\tau_o,\ldots,\tau_r$ are simple types of $\mathcal{K}_o$, $u_o,\ldots,u_r$ are their respective universes, and $r>0$, then the universe of $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$ is the set of all function from $(u_1 x\cdots x u_r)$ into $u_o$.

D49. If K is a set, then <u>the identity function on K</u> is that function from $(K x K)$ into $\{T,F\}$ which assigns T to a pair $\langle xy\rangle$ if and only if $x = y$.

D50. <u>$\mathfrak{M}$ is a model for $\mathcal{K}_o$</u> ($\mathfrak{M} \in M_o$) if and only if there are D, R such that:

(1) $\mathfrak{M} = \langle DR\rangle$

(2) D is a non-empty set of individuals

(3) R is a function whose domain is the set of all atomic wfe's of $\mathcal{K}_o$ and which assigns to each such wfe of type $\tau$ an element of the universe of $\tau$ in D.

(4) if u, v are truth values, then

(a) R(Cond)(u,v) = T if and only if u = F or v = T

(b) R(Neg)(u) = T if and only if u = F

(5) if $\tau$ is a simple type of $\mathcal{K}_o$, then $R(Id(\tau))$ is the identity function on the universe of $\tau$ in D.

D51. If $\alpha$ is a wfe of $\mathcal{K}_o$, and $\mathfrak{M} = \langle DR\rangle$ is a model for $\mathcal{K}_o$, then <u>the value of $\alpha$ in $\mathfrak{M}$ with respect to $\mathcal{K}_o$</u> ($Val_{\mathfrak{M}}^{\mathcal{K}_o}(\alpha)$) is given by the following:

(1) if $\alpha$ is atomic, the value of $\alpha = R(\alpha)$

(2) if $\eta$, $\beta_1,\ldots,\beta_r$ are wfe's of $\mathcal{K}_o$ of types $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$, $\tau_1,\ldots,\tau_r$ respectively with values $f$, $x_1,\ldots,x_r$ respectively, then the value of $\eta^\frown\beta_1{}^\frown\cdots{}^\frown\beta_r = f(x_1,\ldots,x_r)$.

We can now introduce the universe of a type in a model.

D52. If $\tau$ is a type of $\mathcal{K}_o$, and $\mathcal{M} = \langle DR\rangle$ is a model for $\mathcal{K}_o$, then <u>the universe of $\tau$ in $\mathcal{M}$ with respect to $\mathcal{K}_o$</u> ($\mathcal{U}^{\mathcal{K}_o}_{\mathcal{M}}(\tau)$) is simply the universe of $\tau$ in D.

The partial adequacy of our definitions of <u>universe</u> and <u>value</u> are shown in the following theorem.

T38. If $\alpha$ is a wfe of $\mathcal{K}_o$ of type $\tau$, and $\mathcal{M} \in M_o$, then $\mathrm{Val}^{\mathcal{K}_o}_{\mathcal{M}}(\alpha) \in \mathcal{U}^{\mathcal{K}_o}_{\mathcal{M}}(\tau)$.

Proof by induction on $\alpha$.

## 18. The Syntactical Interpretation of $\mathcal{K}_n$

Following the pattern of section 14, we first introduce <u>the syntactical entity corresponding to the wfe $\alpha$ with respect to the language $\mathcal{K}_n$</u>.

D53. If $\alpha$ is a wfe of $\mathcal{K}_n$, then <u>the syntactical entity corresponding to $\alpha$ with respect to $\mathcal{K}_n$</u> ($\mathrm{Synt}^{\mathcal{K}_n}(\alpha)$) is given by the following:

(1) if $\alpha$ is of simple type, then $\mathrm{Synt}^{\mathcal{K}_n}(\alpha) = \alpha$

(2) if $\tau_o,\ldots,\tau_r$ are simple types of $\mathcal{K}_n$, $r>0$, $\eta$ is a wfe of $\mathcal{K}_n$ of complex type $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$, and

$w_o, \ldots, w_r$ are the sets of all wfe's of $\mathcal{K}_n$ of types $\tau_o, \ldots, \tau_r$ respectively, then $\mathrm{Synt}^{\mathcal{K}_n}(\eta)$ is that function from $(w_1 \times \cdots \times w_r)$ into $w_o$ which assigns to each $\langle \beta_1, \ldots, \beta_r \rangle$ in its domain, the wfe $\eta^\frown \beta_1^\frown \cdots ^\frown \beta_r$.

The first part of the following theorem is the main step in showing that the semantical relation which assigns to each wfe $\alpha$ of $\mathcal{K}_n$, $\mathrm{Synt}^{\mathcal{K}_n}(\alpha)$ yields a Fregean semantical system.

T39. If $\tau_o, \ldots, \tau_r$ are simple types of $\mathcal{K}_n$, $r>0$, and $\eta$, $\alpha_1, \ldots, \alpha_r$ are wfe's of $\mathcal{K}_n$ of types $\langle \tau_1, \ldots, \tau_r, \tau_o \rangle$, $\tau_1, \ldots, \tau_r$ respectively, then

(1) $\mathrm{Synt}^{\mathcal{K}_n}(\eta^\frown \alpha_1^\frown \cdots ^\frown \alpha_r) = \mathrm{Synt}^{\mathcal{K}_n}(\eta)(\mathrm{Synt}^{\mathcal{K}_n}(\alpha_1), \ldots,$
$\mathrm{Synt}^{\mathcal{K}_n}(\alpha_r))$

(2) $\eta^\frown \alpha_1^\frown \cdots ^\frown \alpha_r = \mathrm{Synt}^{\mathcal{K}_n}(\eta)(\alpha_1, \ldots, \alpha_r)$

The universes of the higher types consist of syntactical entities. The definition makes use of the cases of theorem 26.

D54. If $\tau$ is a type of $\mathcal{K}_n$, and $\mathcal{M} \in M_o$, then <u>the universe of $\tau$ in $\mathcal{M}$ with respect to the Syntactical interpretation of $\mathcal{K}_n$</u> $(\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_n}(\tau))$ is given by the following:

(1) if $n = 0$, then $\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_n}(\tau)) = \mathcal{U}_{\mathcal{M}}^{\mathcal{K}_o}(\tau)$

(2) if $n = m+1$, then

  (a) if $\tau$ is a simple type of $\mathcal{K}_o$, then $\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_n}(\tau) = \mathcal{U}_{\mathcal{M}}^{\mathcal{K}_o}(\tau)$

  (b) if $\tau$ is a simple type of $\mathcal{K}_m$, then $\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_n}(\tilde{\tau}) =$

the set of all wfe's of $\mathcal{K}_m$ of type $\tau$

(c) if $\tau_o,\ldots,\tau_r$ are simple types of $\mathcal{K}_n$, $r > o$, and $\tau = \langle\tau_1,\ldots,\tau_r,\tau_o\rangle$, then $\mathcal{U}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\tau)$ = the set of all functions from $(\mathcal{U}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\tau_1) \times \ldots \times \mathcal{U}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\tau_r))$ into $\mathcal{U}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\tau_o)$.

The universes of the higher types consist of non-contingent entities.

T40. If $\mathfrak{M},\mathfrak{N} \in M_o$, and $\tau$ is a type of $\mathcal{K}_n$, then $\mathcal{U}^{S,\mathcal{K}_{n+1}}_{\mathfrak{M}}(\tilde{\tau}) = \mathcal{U}^{S,\mathcal{K}_{n+1}}_{\mathfrak{N}}(\tilde{\tau})$.

Proof by the definition.

We now introduce the notion of value, the definition makes use of the cases of theorem 34.

D55. If $\alpha$ is a wfe of $\mathcal{K}_n$, and $\mathfrak{M} \in M_o$, then <u>the value of $\alpha$ in $\mathfrak{M}$ with respect to the syntactical interpretation of $\mathcal{K}_n$</u> ($\mathrm{Val}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\alpha)$) is given by the following:

(1) if $n = o$, then $\mathrm{Val}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\alpha) = \mathrm{Val}^{\mathcal{K}_o}_{\mathfrak{M}}(\alpha)$

(2) if $n = m+1$, then

(a) if $\alpha$ is an atomic wfe of $\mathcal{K}_o$, then $\mathrm{Val}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\alpha) = \mathrm{Val}^{\mathcal{K}_o}_{\mathfrak{M}}(\alpha)$

(b) if $\alpha$ is an atomic wfe of $\mathcal{K}_m$, then $\mathrm{Val}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\overline{\alpha}) = \mathrm{Synt}^{\mathcal{K}_m}_{\mathfrak{M}}(\alpha)$

(c) if $\tau$ is a simple type of $\mathcal{K}_m$, then $\mathrm{Val}^{S,\mathcal{K}_n}(\mathrm{Id}(\tilde{\tau}))$ = the identity function on $\mathcal{U}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\tilde{\tau})$

(d) if $\tau$ is a simple type of $\mathcal{K}_m$, then $\mathrm{Val}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\mathrm{Delta}(\tilde{\tau}))$ = the unique f such that:

(i) f is a function from $(\mathcal{U}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\tau) \times \mathcal{U}^{S,\mathcal{K}_n}_{\mathfrak{M}}(\tau))$

into $\{T,F\}$

(ii) if $<\beta, X>$ is in the domain of f, then $f(\beta, X) =$ T if and only if $Val^{S,\mathcal{K}_m}_{\mathcal{M}}(\beta) = X$

(e) $Val^{S,\mathcal{K}_n}_{\mathcal{M}}$ (Nec) = the unique f such that:

(i) f is a function from the set of all wfe's of $\mathcal{K}_m$ of type t into $\{T,F\}$

(ii) if $\emptyset$ is in the domain of f, then $f(\emptyset) = T$ if and only if for all $\eta \in M_o$, $Val^{S,\mathcal{K}_m}_{\eta}(\emptyset) =$ T

(f) if $\tau_o, \ldots, \tau_r$ are simple types of $\mathcal{K}_n$, $r > o$, and $\eta$, $\alpha_1, \ldots, \alpha_r$ are wfe's of $\mathcal{K}_n$ of types $<\tau_1, \ldots, \tau_r, \tau_o>$, $\tau_1, \ldots, \tau_r$ respectively, then $Val^{S,\mathcal{K}_n}_{\mathcal{M}}(\eta^\frown\alpha_1{}^\frown\ldots{}^\frown\alpha_r) =$ $Val^{S,\mathcal{K}_n}_{\mathcal{M}}(\eta)(Val^{S,\mathcal{K}_n}_{\mathcal{M}}(\alpha_1), \ldots, Val^{S,\mathcal{K}_n}_{\mathcal{M}}(\alpha_r))$.

T41. If $\alpha$ is a wfe of $\mathcal{K}_n$ of type $\tau$, and $\mathcal{M} \in M_o$, then $Val^{S,\mathcal{K}_n}_{\mathcal{M}}(\alpha) \in \mathcal{U}^{S,\mathcal{K}_n}_{\mathcal{M}}(\tau)$.

Proof by induction on $\alpha$.

The wfe $\overline{\alpha}$ provides a standard name of the syntactical entity corresponding to $\alpha$.

T42. If $\alpha$ is a wfe of $\mathcal{K}_n$, and $\mathcal{M} \in M_o$, then

(1) $Val^{S,\mathcal{K}_{n+1}}_{\mathcal{M}}(\overline{\alpha}) = Synt^{\mathcal{K}_n}(\alpha)$

(2) if $\alpha$ is of simple type, then $Val^{S,\mathcal{K}_{n+1}}_{\mathcal{M}}(\overline{\alpha}) = \alpha$

(3) if $\eta \in M_o$, then $Val^{S,\mathcal{K}_{n+1}}_{\mathcal{M}}(\overline{\alpha}) = Val^{S,\mathcal{K}_{n+1}}_{\eta}(\overline{\alpha})$

Proof: (1) by induction on $\alpha$ using T39 for the inductive step; (2) and (3) by (1).

T43. If $\alpha_1$ is a wfe of $\mathcal{K}_{n+1}$ of simple type $\tilde{\tau}$, and $\mathfrak{M} \in M_o$, then $\mathrm{Val}_{\mathfrak{M}}^{S,\mathcal{K}_{n+1}}(\alpha_1)$ is a wfe of $\mathcal{K}_n$ of simple type $\tau$.

Proof by theorem 41.

The following theorems relate the semantical notions for $\mathcal{K}_n$ under the syntactical interpretation to those for $\mathcal{K}_{n+1}$.

T44. (1) If $\alpha$ is a wfe of $\mathcal{K}_n$ of simple type, then
$\mathrm{Synt}^{\mathcal{K}_n}(\alpha) = \alpha = \mathrm{Synt}^{\mathcal{K}_{n+1}}(\alpha)$

(2) If $\eta$ is a wfe of $\mathcal{K}_n$ of complex type $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$, and $w_1(n),\ldots,w_r(n)$ are the sets of all wfe's of $\mathcal{K}_n$ of types $\tau_1,\ldots,\tau_r$ respectively, then
$\mathrm{Synt}^{\mathcal{K}_n}(\eta) = \mathrm{Synt}^{\mathcal{K}_{n+1}}(\eta) \upharpoonright (w_1(n)\times\cdots\times w_r(n))$[69]

(3) If $\eta$ is a wfe of $\mathcal{K}_n$ of complex type, then $\mathrm{Synt}^{\mathcal{K}_n}(\eta) \subseteq \mathrm{Synt}^{\mathcal{K}_{n+1}}(\eta)$

Proof: (1) and (2) by the definitions; (3) by (2).

T45. If $\mathfrak{M} \in M_o$, then

(1) if $\tau$ is a simple type of $\mathcal{K}_o$, $\mathcal{U}_{\mathfrak{M}}^{S,\mathcal{K}_n}(\tau) = \mathcal{U}_{\mathfrak{M}}^{S,\mathcal{K}_{n+1}}(\tau)$

(2) if $\tilde{\tau}$ is a simple type of $\mathcal{K}_n$, $\mathcal{U}_{\mathfrak{M}}^{S,\mathcal{K}_n}(\tilde{\tau}) \subseteq \mathcal{U}_{\mathfrak{M}}^{S,\mathcal{K}_{n+1}}(\tilde{\tau})$

(3) if $\tau_o,\ldots,\tau_r$ are simple types of $\mathcal{K}_n$, and $r>0$; then if $f \in \mathcal{U}_{\mathfrak{M}}^{S,\mathcal{K}_n}(\langle\tau_1,\ldots,\tau_r,\tau_o\rangle)$, then there is a $g \in \mathcal{U}_{\mathfrak{M}}^{S,\mathcal{K}_{n+1}}(\langle\tau_1,\ldots,\tau_r,\tau_o\rangle)$ such that $f \subseteq g$.

Proof: (1) by the definition, (2) by T30, (3) by (1) and (2).

T46. If $\mathcal{M} \in M_o$, then

(1) if $\alpha$ is a wfe of $\mathcal{K}_n$ of simple type, $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\alpha) = \mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_{n+1}}(\alpha)$

(2) if $\eta$ is a wfe of $\mathcal{K}_n$ of complex type, $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\eta) \subseteq \mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_{n+1}}(\eta)$

Proof by cases: $n = 0$, or $n = m + 1$ for some m. In the first case, the "$\subseteq$" of (2) can be strengthened to "$=$"; (2) is then proved by T29, D55(2)(a), and (1) is proved by induction on $\alpha$. In the second case, the proof is similar using the cases of T34.

## 19. The Intensional Interpretation of $\mathcal{K}_n$

We first introduce the general notion of a <u>$\tau$-concept</u>, where $\tau$ is a simple type. Our earlier notions of an individual-concept and a proposition will then correspond to i-concepts and t-concepts respectively.

D56. If $\tau$ is a simple type, then <u>the $\tau$-concepts</u> are given by the following:

(1) if $\tau$ is a simple type of $\mathcal{K}_o$, then f is a $\tau$-concept if and only if f is a function whose domain is $M_o$ and which assigns to each $\mathcal{M} \in M_o$ an element of $\mathcal{U}_{\mathcal{M}}^{\mathcal{K}_o}(\tau)$

(2) if $\tau$ is a simple type, of $\mathcal{K}_n$, then f is a $\tilde{\tau}$-concept if and only if f is a function whose domain is $M_o$ and which assigns to each $\mathcal{M} \in M_o$ a $\tau$-concept.

The universes for each type under the intensional interpretation are now introduced in the natural way, with concepts here playing the role played by syntactical entities under the syntactical interpretation. The definition makes use of the cases of theorem 26.

D57. If $\tau$ is a type of $\mathcal{K}_n$, and $\mathcal{M} \in M_o$, then the universe of $\tau$ in $\mathcal{M}$ with respect to the Intensional interpretation of $\mathcal{K}_n$ ($\mathcal{U}^{I,\mathcal{K}_n}_{\mathcal{M}}(\tau)$) is given by the following:

(1) if n = 0, then $\mathcal{U}^{I,\mathcal{K}_n}_{\mathcal{M}}(\tau) = \mathcal{U}^{\mathcal{K}_o}_{\mathcal{M}}(\tau)$

(2) if n = m + 1, then

(a) if $\tau$ is a simple type of $\mathcal{K}_o$, then $\mathcal{U}^{I,\mathcal{K}_n}_{\mathcal{M}}(\tau) = \mathcal{U}^{\mathcal{K}_o}_{\mathcal{M}}(\tau)$

(b) if $\tau$ is a simple type of $\mathcal{K}_m$, then $\mathcal{U}^{I,\mathcal{K}_n}_{\mathcal{M}}(\tilde{\tau}) =$ the set of all $\tau$-concepts

(c) if $\tau_o,\ldots,\tau_r$ are simple types of $\mathcal{K}_n$, r>0, then $\mathcal{U}^{I,\mathcal{K}_n}_{\mathcal{M}}(<\tau_1,\ldots,\tau_r, \tau_o>) =$ the set of all functions from $(\mathcal{U}^{I,\mathcal{K}_n}_{\mathcal{M}}(\tau_1)\times\cdots\times\mathcal{U}^{I,\mathcal{K}_n}_{\mathcal{M}}(\tau_r))$ into $\mathcal{U}^{I,\mathcal{K}_n}_{\mathcal{M}}(\tau_o)$.

The universes of the higher types consist of non-contingent entities.

T47. If $\mathcal{M}\mathcal{N} \in M_o$, and $\tau$ is a type of $\mathcal{K}_n$, then $\mathcal{U}^{I,\mathcal{K}_{n+1}}_{\mathcal{M}}(\tilde{\tau}) = \mathcal{U}^{I,\mathcal{K}_{n+1}}_{\mathcal{N}}(\tilde{\tau})$.

We now introduce the notion of value. The definition uses the cases of theorem 34.

D58. If $\alpha$ is a wfe of $\mathcal{K}_n$, and $\mathcal{M} \in M_o$, then the value of $\alpha$ in $\mathcal{M}$ with respect to the Intensional interpretation of $\mathcal{K}_n$ ($\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\alpha)$) is given by the following:

(1) if $n = 0$, then $\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\alpha) = \mathrm{Val}^{\mathcal{K}_o}_{\mathcal{M}}(\alpha)$

(2) if $n = m+1$, then

(a) if $\alpha$ is an atomic wfe of $\mathcal{K}_o$, then $\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\alpha) = \mathrm{Val}^{\mathcal{K}_o}_{\mathcal{M}}(\alpha)$

(b1) if $\alpha$ is an atomic wfe of $\mathcal{K}_m$ of simple type, then $\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\bar{\alpha})$ = the unique f such that

(i) f is a function whose domain is $M_o$

(ii) if $\mathcal{N} \in M_o$, then $f(\mathcal{N}) = \mathrm{Val}^{I,\mathcal{K}_m}_{\mathcal{N}}(\alpha)$

(b2) if $\tau_o,\ldots,\tau_r$ are simple types of $\mathcal{K}_m, r>0$, $\eta$ is a wfe of $\mathcal{K}_m$ of complex type $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$, and $c_o,\ldots,c_r$ are the sets of all $\tau_o$-concepts,$\ldots$,$\tau_r$-concepts respectively, then $\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\bar{\eta})$ = the unique f such that

(i) f is a function from $(c_1 x \cdots x c_r)$ into $c_o$

(ii) if $\mathcal{M} \in M_o$, and $\langle g_1,\ldots,g_r\rangle$ is in the domain of f, then $f(g_1,\ldots,g_r)(\mathcal{N}) = \mathrm{Val}^{I,\mathcal{K}_m}_{\mathcal{N}}(\eta)(g_1(\mathcal{N}),\ldots,g_r(\mathcal{N}))$.

(c) if $\tau$ is a simple type of $\mathcal{K}_m$, then $\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\mathrm{Id}(\tilde{\tau}))$ = the identity function on $\mathcal{U}^{I,\mathcal{K}_n}_{\mathcal{M}}(\tilde{\tau})$

(d) if $\tau$ is a simple type of $\mathcal{K}_m$, then $\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\mathrm{Delta}\ (\tilde{\tau}))$ = the unique f such that

(i) f is a function from $(\mathcal{U}^{I,\mathcal{K}_n}_{\mathcal{M}}(\tilde{\tau})\ x\, \mathcal{U}^{I,\mathcal{K}_n}_{\mathcal{M}}(\tau))$ into $\{T,F\}$

(ii) if $\langle g,x\rangle$ is in the domain of f, then $f(g,x) = T$ if and only if $g(\mathcal{M}) = x$

(e) $\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\mathrm{Nec})$ = the unique f such that

(i) f is a function from the set of all t-concepts into $\{T,F\}$

(ii) if g is a t-concept, then $f(g) = T$ if and only if for all $\eta \in M_o$, $g(\eta) = T$.

(f) if $\tau_o, \ldots, \tau_r$ are simple types of $\mathcal{K}_n$, $r>0$, and $\eta$, $\alpha_1, \ldots, \alpha_r$ are wfe's of $\mathcal{K}_n$ of types $<\tau_1, \ldots, \tau_r, \tau_o>$, $\tau_1, \ldots, \tau_r$ respectively, then
$\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\eta^\frown\alpha_1^\frown\cdots^\frown\alpha_r) =$
$\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\eta)\ (\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\alpha_1), \ldots, \mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{M}}(\alpha_n))$

The notion of <u>sense</u> can now be introduced. We will establish in theorem 49 that under the intensional interpretation the value of $\overline{\alpha}$ is the sense of $\alpha$.

D59. If $\alpha$ is a wfe of $\mathcal{K}_n$, then <u>the Sense of $\alpha$ with respect to $\mathcal{K}_n$</u> ($\mathrm{Sense}^{\mathcal{K}_n}(\alpha)$) is given by the following:

(1) if $\alpha$ is of simple type, then $\mathrm{Sense}^{\mathcal{K}_n}(\alpha)$ = the unique f such that

(a) f is a function whose domain is $M_o$

(b) if $\eta \in M_o$, then $f(\eta) = \mathrm{Val}^{I,\mathcal{K}_n}_{\eta}(\alpha)$

(2) if $\tau_o, \ldots, \tau_r$ are simple types of $\mathcal{K}_n$, $r>0$, $\eta$ is a wfe of $\mathcal{K}_n$ of complex type $<\tau_1, \ldots, \tau_r, \tau_o>$, and $c_o, \ldots, c_r$ are the sets of all $\tau_o$-concepts, ..., $\tau_r$-concepts respectively, then $\mathrm{Sense}^{\mathcal{K}_n}(\eta)$ = the unique f such that

(a) f is a function from $(c_1 \times \cdots \times c_r)$ into $c_o$

(b) if $\mathcal{N} \in M_o$, and $\langle g_1,\ldots,g_r\rangle$ is in the domain of f, then $f(g_1,\ldots,g_r)(\mathcal{N}) =$ $\mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{N}}(\eta)(g_1(\mathcal{N}),\ldots,g_r(\mathcal{N}))$.

The second part of the following theorem provides the main step in arguing that the semantical relation which assigns to each wfe $\alpha$ of $\mathcal{K}_n$, $\mathrm{Sense}^{\mathcal{K}_n}(\alpha)$ yields a Fregean semantical system.

T48. (1) If $\alpha$ is a wfe of $\mathcal{K}_n$ of simple type, and $\mathcal{N} \in M_o$, then $\mathrm{Sense}^{\mathcal{K}_n}(\alpha)(\mathcal{N}) = \mathrm{Val}^{I,\mathcal{K}_n}_{\mathcal{N}}(\alpha)$

(2) If $\tau_o,\ldots,\tau_r$ are simple types of $\mathcal{K}_n$, r>0, and $\eta$, $\alpha_1,\ldots,\alpha_r$ are wfe's of $\mathcal{K}_n$ of types $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$, $\tau_o,\ldots,\tau_r$, respectively, then $\mathrm{Sense}^{\mathcal{K}_n}(\eta)(\mathrm{Sense}^{\mathcal{K}_n}(\alpha_1), \ldots,\mathrm{Sense}^{\mathcal{K}_n}(\alpha_r)) = \mathrm{Sense}^{\mathcal{K}_n}(\eta^\frown\alpha_1{}^\frown\cdots{}^\frown\alpha_r)$

Proof: (1) by the definition, (2) by showing that both sides of the equality are $\tau_o$-concepts and then using (1) to show that they have the same value for a given $\mathcal{N} \in M_o$.

We now easily show the required result for the value of $\overline{\alpha}$.

T49. If $\alpha$ is a wfe of $\mathcal{K}_n$, and $\mathcal{M} \in M_o$, then $\mathrm{Val}^{I,\mathcal{K}_{n+1}}_{\mathcal{M}}(\overline{\alpha}) = \mathrm{Sense}^{\mathcal{K}_n}(\alpha)$.

Proof by induction on $\alpha$ using T48 (2) for the inductive step.

The values of wfe's are in the appropriate universe.

T50. If $\alpha$ is a wfe of $\mathcal{K}_n$ of type $\tau$ and $\mathcal{M} \in M_o$, then $Val_{\mathcal{M}}^{I,\mathcal{K}_n}(\alpha) \in \mathcal{U}_{\mathcal{M}}^{I,\mathcal{K}_n}(\tau)$.

Proof by induction on $\alpha$.

All wfe's of the higher simple types denote concepts (those of higher complex types denote representatives of natural-concepts).

T51. If $\alpha_1$ is a wfe of $\mathcal{K}_{n+1}$ of simple type $\tilde{\tau}$, and $\mathcal{M} \in M_o$, then $Val_{\mathcal{M}}^{I,\mathcal{K}_n}(\alpha_1)$ is a $\tau$-concept.

Proof by T50.

We now turn to theorems relating the semantical notion for $\mathcal{K}_n$ under the intensional interpretations to those for $\mathcal{K}_{n+1}$. It will be evident that none of the subtleties of the syntactical interpretation (for example, theorems 44(2), 45(2), 46(2))have counterparts under the present interpretation. In fact, theorems 52-54 suggest very strongly the natural construction of the language $\mathcal{K}_\omega$ of the following chapter.

T52. If $\tau$ is a type of $\mathcal{K}_n$, and $\mathcal{M} \in M_o$, then $\mathcal{U}_{\mathcal{M}}^{I,\mathcal{K}_n}(\tau) = \mathcal{U}_{\mathcal{M}}^{I,\mathcal{K}_{n+1}}(\tau)$

Proof by cases using T26.

T53. If $\alpha$ is a wfe of $\mathcal{K}_n$, and $\mathcal{M} \in M_o$, then $Val_{\mathcal{M}}^{I,\mathcal{K}_n}(\alpha) = Val_{\mathcal{M}}^{I,\mathcal{K}_{n+1}}(\alpha)$

Proof by induction on n. In the inductive step, proof is by induction on $\alpha$ using the cases of T34 and making use of T52.

T54. If $\alpha$ is a wfe of $\mathcal{K}_n$, then $\mathrm{Sense}^{\mathcal{K}_n}(\alpha) = \mathrm{Sense}^{\mathcal{K}_{n+1}}(\alpha)$

Proof by T53.

## 20 Some Adequacy Theorems

The following theorems indicate that the formal treatment of the logical signs of our languages is in accord with the informal exposition. Theorems 57-59 also provide a succinct comparison of the two interpretations. Theorems 55-58 follow directly from the definitions.

T55. If $\phi$ and $\psi$ are wfe's of $\mathcal{K}_n$ of type t, and $\mathcal{M} \in M_o$, then

(1) $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\phi \supset \psi) = T$ if and only if, if $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\phi) = T$ then $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\psi) = T$

(2) $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\phi \supset \psi) = T$ if and only if, if $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\phi) = T$ then $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\psi)$

(3) $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\sim\phi) = T$ if and only if, it is not the case that $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\phi) = T$

(4) $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\sim\phi) = T$ if and only if, it is not the case that $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\phi) = T$

(5) $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\phi \& \psi) = T$ if and only if, $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\phi) = T$ and $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\psi) = T$

(6) $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\phi \& \psi) = T$ if and only if, $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\phi) = T$ and $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\psi) = T$

(7) $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\phi \vee \psi) = T$ if and only if, $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\phi) = T$ or $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\psi) = T$

(8) $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\phi \vee \psi) = T$ if and only if, $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\phi) = T$

or $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\psi) = T$

(9) $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\phi \equiv \psi) = T$ if and only if, $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\phi) = T$ if and only if $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\psi) = T$

(10) $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\phi \equiv \psi) = T$ if and only if, $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\phi) = T$ if and only if $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\psi) = T$

T56. If $\alpha$ and $\beta$ are wfe's of $\mathcal{K}_n$ of the same simple type, and $\mathcal{M} \in M_o$, then

(1) $\mathrm{Val}^{S,\mathcal{K}_n}(\alpha = \beta) = T$ if and only if $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\alpha) = \mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\beta)$

(2) $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\alpha = \beta) = T$ if and only if $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\alpha) = \mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\beta)$

T57. If $\alpha_1$ and $\beta$ are wfe's of $\mathcal{K}_{n+1}$ of the simple types $\tau$ and $\tau$ respectively, and $\mathcal{M} \in M_o$, then

(1) $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_{n+1}}(\Delta(\alpha_1, \beta)) = T$ if and only if $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_{n+1}}(\alpha_1)) = \mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_{n+1}}(\beta)$

(2) $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_{n+1}}(\Delta(\alpha_1, \beta)) = T$ if and only if $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_{n+1}}(\alpha_1)(\mathcal{M}) = \mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_{n+1}}(\beta)$

T58. If $\phi_1$ is a wfe of $\mathcal{K}_{n+1}$ of type $t_1$, and $\mathcal{M} \in M_o$, then

(1) $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_{n+1}}(N\phi_1) = T$ if and only if for all $\mathcal{N} \in M_o$, $\mathrm{Val}_{\mathcal{N}}^{S,\mathcal{K}_n}(\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_{n+1}}(\phi_1)) = T$

(2) $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_{n+1}}(N\phi_1) = T$ if and only if for all $\mathcal{N} \in M_o$, $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_{n+1}}(\phi_1)(\mathcal{N}) = T$

T59. If $\phi_1$ is a wfe of $\mathcal{K}_{n+1}$ of type $t_1$, and $\mathcal{M} \in M_o$, then

(1) $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_{n+1}}(\mathrm{Tr}\phi_1) = T$ if and only if

$\mathrm{Val}_{\mathfrak{M}}^{S,\mathcal{K}_n}(\mathrm{Val}_{\mathfrak{M}}^{S,\mathcal{K}_{n+1}}(\phi_1)) = T$

(2) $\mathrm{Val}_{\mathfrak{M}}^{I,\mathcal{K}_{n+1}}(\mathrm{Tr}\phi_1) = T$ if and only if $\mathrm{Val}_{\mathfrak{M}}^{I,\mathcal{K}_{n+1}}(\phi_1)(\mathfrak{M}) = T$

Proof by D47, T57, T55(1) and (2).

T60. If $\alpha$ and $\beta$ are wfe's of $\mathcal{K}_n$ of the same simple type, and $\mathfrak{M} \in M_o$, then

(1) $\mathrm{Val}_{\mathfrak{M}}^{S,\mathcal{K}_{n+1}}(\Delta(\overline{\alpha},\beta)) = T$ if and only if $\mathrm{Val}_{\mathfrak{M}}^{S,\mathcal{K}_n}(\alpha) = \mathrm{Val}_{\mathfrak{M}}^{S,\mathcal{K}_n}(\beta)$

(2) $\mathrm{Val}_{\mathfrak{M}}^{I,\mathcal{K}_{n+1}}(\Delta(\overline{\alpha},\beta)) = T$ if and only if $\mathrm{Val}_{\mathfrak{M}}^{I,\mathcal{K}_n}(\alpha) = \mathrm{Val}_{\mathfrak{M}}^{I,\mathcal{K}_n}(\beta)$

Proof: (1) by T57(1), T42(2), T46(1);

(2) by T57(2), T49, T48(1), T53.

T61. If $\phi$ is a wfe of $\mathcal{K}_n$ of type t, and $\mathfrak{M} \in M_o$, then

(1) $\mathrm{Val}_{\mathfrak{M}}^{S,\mathcal{K}_{n+1}}(N\overline{\phi}) = T$ if and only if for all $\mathfrak{N} \in M_o$, $\mathrm{Val}_{\mathfrak{N}}^{S,\mathcal{K}_n}(\phi) = T$

(2) $\mathrm{Val}_{\mathfrak{M}}^{I,\mathcal{K}_{n+1}}(N\overline{\phi}) = T$ if and only if for all $\mathfrak{N} \in M_o$, $\mathrm{Val}_{\mathfrak{N}}^{I,\mathcal{K}_n}(\phi) = T$

Proof: (1) by T58(1), T42(2);

(2) by T58(2), T49, T48(1).

T62. If $\phi$ is a wfe of $\mathcal{K}_n$ of type t, and $\mathfrak{M} \in M_o$, then

(1) $\mathrm{Val}_{\mathfrak{M}}^{S,\mathcal{K}_{n+1}}(\mathrm{Tr}\overline{\phi}) = T$ if and only if $\mathrm{Val}_{\mathfrak{M}}^{S,\mathcal{K}_n}(\phi) = T$

(2) $\mathrm{Val}_{\mathfrak{M}}^{I,\mathcal{K}_{n+1}}(\mathrm{Tr}\overline{\phi}) = T$ if and only if $\mathrm{Val}_{\mathfrak{M}}^{I,\mathcal{K}_n}(\phi) = T$

Proof: (1) by T59(1), T42(2);

(2) by T59(2), T49, T48(1).

Note that in view of T33, theorems 57-59 represent no more general a case than do theorems 60-62. However, in the language $\mathcal{L}_{\omega}$, which contains variables, counterparts to all the preceding theorems (57-62) will still hold, whereas T33 will fail. Thus both sets (57-59, 60-62) were stated here as illustrative of what is to come. It is also interesting to note that only theorems 57-59 reveal the differences between the syntactical and intensional interpretations; theorems 60-62 reveal the similarities. In the latter connection, we add theorem 63 to emphasize a point made in note 62 on the two interpretations of truth. The following definitions of <u>validity</u> are introduced for the sake of theorem 63.

D60. If $\emptyset$ is a wfe of $\mathcal{K}_n$ of type t, then <u>$\emptyset$ is valid under the Syntactical interpretation of $\mathcal{K}_n$</u> ($\models_{S,\mathcal{K}_n} \emptyset$) if and only if for $\eta \in M_o$, $\mathrm{Val}_{\eta}^{S,\mathcal{K}_n}(\emptyset) = T$

D61. If $\emptyset$ is a wfe of $\mathcal{K}_n$ of type t, then <u>$\emptyset$ is valid under the Intensional interpretation of $\mathcal{K}_n$</u> ($\models_{I,\mathcal{K}_n} \emptyset$) if and only if for all $\eta \in M_o$, $\mathrm{Val}_{\eta}^{I,\mathcal{K}_n}(\emptyset) = T$

T63. If $\emptyset$ is a wfe of $\mathcal{K}_n$ of type t, then

(1) $\models_{S,\mathcal{K}_n} (\mathrm{Tr}\ \bar{\emptyset} \equiv \emptyset)$

(2) $\models_{I,\mathcal{K}_n} (\mathrm{Tr}\ \bar{\emptyset} \equiv \emptyset)$

Proof: (1) by T62(1), T55(9), T46(1)

(2) by T62(2), T55(10), T53

CHAPTER 4

# THE LANGUAGE $\mathcal{K}_\omega$

In the preceding chapter we constructed a hierarchy of languages $\mathcal{K}_n$. The purpose of $\mathcal{K}_1$, it will be recalled, was to provide a direct discourse treatment for singly oblique contexts of $\mathcal{K}_0$. Similarly, $\mathcal{K}_2$ can be used for doubly oblique contexts of $\mathcal{K}_0$, $\mathcal{K}_3$ for triply oblique contexts, and so on. Thus for contexts of $\mathcal{K}_0$ of any degree of obliquity, there is some language in the hierarchy within which a direct discourse treatment can be given. However, since a development in accordance with the method of indirect discourse allows contexts of arbitrary (finite) degree within a single language, it seems desirable to attempt the same for the direct discourse method. This is done by constructing the language $\mathcal{K}_\omega$.

## 21. The Language $\mathcal{K}_\omega$

Given the notions of type, wfe, etc. for each of the languages $\mathcal{K}_n$, and keeping in mind the fact that each language of the hierarchy is included in its successor, we could simply introduce the corresponding notions for $\mathcal{K}_\omega$ by taking unions. But recall that our method of introducing these notions for the language $\mathcal{K}_n$ was first to give a general formulation and then subdivide using the idea of rank. Thus the appropriate notions for $\mathcal{K}_\omega$ were already introduced in sections 16.1 and 16.2. The following theorems verify that the same results would obtain if we were to take unions.

T64. (1) $\tau$ is a type if and only if there is a natural number n, such that $\tau$ is a type of $\mathcal{K}_n$

(2) $\alpha$ is a wfe if and only if there is a natural number n, such that $\alpha$ is a wfe of $\mathcal{K}_n$.

Proof by the definitions, using the facts that every type has a finite rank, and every wfe has a type.

When we form the language $\mathcal{L}_\omega$, which contains variables, new wfe's will be introduced. For this reason, we now identify our unrestricted notion of well formed expression with that of well formed expression of $\mathcal{K}_\omega$. Since no new types are introduced in $\mathcal{L}_\omega$, there is no need for a similar restriction of the notion of type.

D62. <u>$\alpha$ is a wfe of $\mathcal{K}_\omega$</u> if and only if $\alpha$ is a wfe.

T65. $\alpha$ is a wfe of $\mathcal{K}_\omega$ of type $\tau$ if and only if there is a natural number n such that $\alpha$ is a wfe of $\mathcal{K}_n$ of type $\tau$.

Proof by D62, T64.

The following theorem shows that $\mathcal{K}_\omega$ is closed with respect to the bar function. That is, every wfe of $\mathcal{K}_\omega$ has an analogue in $\mathcal{K}_\omega$. This is one of the essential properties, lacking in the languages $\mathcal{K}_n$, required to provide a direct discourse treatment of arbitrary degrees of obliquity.

T66. $\alpha$ is a wfe of $\mathcal{K}_\omega$ of type $\tau$ if and only if $\overline{\alpha}$ is a wfe of $\mathcal{K}_\omega$ of type $\tilde{\tau}$.

Proof by T65, T32.

The restriction (which will no longer hold for the language $\mathcal{L}_\omega$) that all wfe's of types $\tilde{\tau}$ have the form $\overline{\alpha}$ applies to $\mathcal{K}_\omega$ as it does to the languages $\mathcal{K}_n$.

T67. $\beta$ is a wfe of $\mathcal{K}_\omega$ of type $\tilde{\tau}$ if and only if there is a wfe $\alpha$ of $\mathcal{K}_\omega$ of type $\tau$ such that $\beta$ is $\overline{\alpha}$.

Proof by T65, T33.

## 22. The Syntactical Interpretation of $\mathcal{K}_\omega$

In view of theorem 44, which links $\mathrm{Synt}^{\mathcal{K}_n}(\alpha)$ to $\mathrm{Synt}^{\mathcal{K}_{n+1}}(\alpha)$, the notion for $\mathcal{K}_\omega$ can be introduced in a simple fashion.

D63. If $\alpha$ is a wfe of $\mathcal{K}_\omega$ of rank n, then <u>the syntactical entity corresponding to $\alpha$ with respect to $\mathcal{K}_\omega$</u> ($\mathrm{Synt}^{\mathcal{K}_\omega}(\alpha)$) is given by the following:

(1) if $\alpha$ is of simple type, then $\mathrm{Synt}^{\mathcal{K}_\omega}(\alpha) = \alpha$

(2) if $\eta$ is of complex type, then $\mathrm{Synt}^{\mathcal{K}_\omega}(\eta) = \bigcup_{m>n} \mathrm{Synt}^{\mathcal{K}_m}(\eta)$ 70

Since by theorem 45, the universe of a simple type with respect to $\mathcal{K}_n$ is included in the corresponding universe with respect to $\mathcal{K}_{n+1}$, we can form the universe with respect to $\mathcal{K}_\omega$ simply by taking the union of the universes for each of the $\mathcal{K}_n$. However, the relationship between the universes of a complex type with respect to languages $\mathcal{K}_n$ and $\mathcal{K}_{n+1}$ is somewhat more complicated. In fact, if we build up the universe of a complex type with respect to $\mathcal{K}_\omega$ from the corresponding universes with respect to the $\mathcal{K}_n$ in the natural way, the result would be that the universe of $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$ would not consist of <u>all</u> functions from $(u_1 x \cdots x u_r)$ into $u_o$ (where $u_o,\ldots,u_r$ are the universes of $\tau_o,\ldots,\tau_r$ respectively) but only some of these functions.[71] We shall avoid these complications (which do not arise at all under the intensional interpretation) by introducing the universe of a complex type directly as the set of all such functions, rather than by way of the corresponding universes for the languages $\mathcal{K}_n$. Such a procedure can work no harm, so long as we exclude from our languages both variables of complex type and primitive descriptive constants of higher types. For the universes of the types are introduced primarily for heuristic reasons. We see what values a variable of the given type might take, if we had such variables; and we see what entities an arbitrary descriptive constant of the given type might denote, if we had such constants. Thus an insight is supplied into the ontological presuppositions of our languages, or, more properly, into certain natural extensions of our languages. But when, as in the present case, such extensions involve a considerable increase in complexity, we will satisfy ourselves with a simpler treatment adequate for the special case before us.

D64. If $\tau$ is a type of rank n, and $\mathcal{M} \in M_0$, then <u>the universe of $\tau$ in $\mathcal{M}$ with respect to the Syntactical interpretation of $\mathcal{K}_\omega$</u> ($\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_\omega}(\tau)$) is given by the following:

(1) if $\tau$ is a simple type, then $\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_\omega}(\tau) = \bigcup_{m \geq n} \mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_m}(\tau)$

(2) if $\tau_0, \ldots, \tau_1$ are simple types, r>0, and $\tau = \langle \tau_1, \ldots, \tau_r, \tau_0 \rangle$, then $\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_\omega}(\tau)$ = the set of all functions from $(\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_\omega}(\tau_1) \times \cdots \times \mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_\omega}(\tau_r))$ into $\mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_\omega}(\tau_0)$.

By theorem 46, the value of a wfe $\alpha$ of simple type remains constant for all languages $\mathcal{K}_m$, where m is greater than or equal to the rank of $\alpha$, that is, where $\alpha$ is a wfe of $\mathcal{K}_m$. The values of a wfe $\eta$ of complex type form an inclusion chain, that is, starting with the value of $\eta$ in $\mathcal{K}_n$ (where n is the rank of $\eta$) each value is included in that for the succeeding language. Thus we are led to the following definition.

D65. If $\alpha$ is a wfe of $\mathcal{K}_\omega$ of rank n and $\mathcal{M} \in M_0$, then <u>the value of $\alpha$ in $\mathcal{M}$ with respect to the Syntactical interpretation of $\mathcal{K}_\omega$</u> ($\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\alpha)$) is given by the following:

(1) if $\alpha$ is of simple type, then $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_\omega}(\alpha) = \mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_n}(\alpha)$

(2) if $\eta$ is of complex type, then $\mathrm{Val}_{\mathcal{M}}^{S,\mathcal{K}_\omega}(\eta) =$

$\bigcup_{m \geq n} Val_{\mathcal{M}}^{S,\mathcal{K}_m}(\eta)$

T68. If $\alpha$ is a wfe of $\mathcal{K}_\omega$ of type $\tau$, and $\mathcal{M} \in M_o$, then $Val_{\mathcal{M}}^{S,\mathcal{K}_\omega}(\alpha) \in \mathcal{U}_{\mathcal{M}}^{S,\mathcal{K}_\omega}(\tau)$

Proof by cases: (i) if $\tau$ is a simple type, proof is by D65 (1), D64 (1), T41; (ii) if $\tau$ is a complex type, proof is by D65 (2), D64 (2), T41, D54, and set theory.

T69. If $\alpha$ is a wfe of $\mathcal{K}_\omega$, and $\mathcal{M} \in M_o$, then $Val^{S,\mathcal{K}_\omega}(\overline{\alpha}) = Synt^{\mathcal{K}_\omega}(\alpha)$.

Proof by T64 (2), D62, D65, T42 (1), D53, D63.

## 23. The Intensional Interpretation of $\mathcal{K}_\omega$

The constancy, indicated in theorems 52-54, between the semantical notions for the intensional interpretation of $\mathcal{K}_n$ and those for the intensional interpretation of $\mathcal{K}_{n+1}$ allow us to frame simple definitions for the corresponding notions in $\mathcal{K}_\omega$. The difficulties alluded to in connection with D64 do not arise here. In fact, a definition which separates clauses in the manner of D64 would be equivalent to the following.

D66. If $\tau$ is a type of rank n, and $\mathcal{M} \in M_o$, then <u>the universe of $\tau$ in $\mathcal{M}$ with respect to the Intensional interpretation of $\mathcal{K}_\omega$ ($\mathcal{U}_{\mathcal{M}}^{I,\mathcal{K}_\omega}(\tau)$) is $\mathcal{U}_{\mathcal{M}}^{I,\mathcal{K}_n}(\tau)$</u>

D67. If $\alpha$ is a wfe of $\mathcal{K}_\omega$ of rank n, and $\mathcal{M} \in M_o$, then <u>the value of $\alpha$ in $\mathcal{M}$ with respect to the Intensional interpre-</u>

<u>tation of $\mathcal{K}_\omega$</u> ($\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_\omega}(\alpha)$) is $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_n}(\alpha)$.

D68. If $\alpha$ is a wfe of $\mathcal{K}_\omega$ of rank n, then <u>the Sense of $\alpha$ with respect to $\mathcal{K}_\omega$</u> ($\mathrm{Sense}^{\mathcal{K}_\omega}(\alpha)$) is $\mathrm{Sense}^{\mathcal{K}_n}(\alpha)$.

As explained earlier, the sense of a wfe of simple type, assigns to each model the value of the wfe in that model.

T70. If $\alpha$ is a wfe of $\mathcal{K}_\omega$ of simple type, and $\mathcal{N} \in M_o$, then $\mathrm{Sense}^{\mathcal{K}_\omega}(\alpha)(\mathcal{N}) = \mathrm{Val}_{\mathcal{N}}^{I,\mathcal{K}_\omega}(\alpha)$.

Proof by D68, D67, T48 (1).

We also obtain the appropriate theorems about the value of the analogue to a wfe and the relation between the values of wfe's and the universes of their types.

T71. If $\alpha$ is a wfe of $\mathcal{K}_\omega$, and $\mathcal{M} \in M_o$, then $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_\omega}(\overline{\alpha}) = \mathrm{Sense}^{\mathcal{K}_\omega}(\alpha)$.

Proof by D68, D67, T49.

T72. If $\alpha$ is a wfe of $\mathcal{K}_\omega$ of type $\tau$, and $\mathcal{M} \in M_o$, then $\mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_\omega}(\alpha) \in \mathcal{U}_{\mathcal{M}}^{\mathcal{K}_\omega}(\tau)$.

Proof by D67, D66, T50.

## 24. Logical Axioms for $\mathcal{K}_\omega$

Let us provisionally distinguish our logical signs into two groups. The signs from "extensional" logic, namely, Cond, Neg, Id(t), Id(i); and the "special" signs Delta ($\tau$), Id($\tilde{\tau}$), Nec, $\overline{\alpha}$. We first turn our attention to the former group which we will henceforth call the purely logical signs. All sentences of $K_\omega$ whose validity depends solely on the interpretation of these signs are derivable from two sets of axioms, the tautologies and the identity axioms. Tautologies are defined in terms of truth evaluations; the identity axioms are defined in terms of substitution.

D69. f is a truth evaluation for $K_\omega$ if and only if:

(1) f is a function from the set of all wfe's of $K_\omega$ of type t into $\{T, F\}$,

(2) if $\phi, \psi$ are in the domain of f, then

(a) f $(\phi \supset \psi)$ = T if and only if, if f($\phi$) = T then f($\psi$) = T

(b) f($\sim\phi$) = T if and only if, it is not the case that f($\phi$) = T

(c) f $(\phi \equiv \psi)$ = T if and only if, f($\phi$) = T if and only if f($\psi$) = T

D70. $\phi$ is a tautology of $K_\omega$ if and only if

(1) $\phi$ is a wfe of $K_\omega$ of type t

(2) if f is a truth evaluation for $K_\omega$, then f($\phi$) = T.

D71. If $\alpha$, $\beta$ are wfe's of $K_\omega$ of the same simple type, and $\gamma$, $\delta$ are wfe's of $K_\omega$ of the same simple type, then $\delta$ is a result of substituting $\alpha$ for $\beta$ at none or more places in $\gamma$ ($\mathrm{Sub}^{K_\omega}(\alpha, \beta, \gamma, \delta)$) if and only if:

(1) $\gamma = \beta$, and either $\delta = \alpha$ or $\delta = \beta$; or

(2) $\gamma \neq \beta$, and either

(a) $\gamma$ is atomic and $\delta = \gamma$, or

(b) there are r, and simple types, $\tau_0, \ldots, \tau_r$, $\eta$, $\mu_1, \ldots, \mu_r$, $\mu'_1, \ldots, \mu'_r$ such that $r > 0$, $\eta$ is a wfe of $\mathcal{K}_\omega$ of type $\langle \tau_1, \ldots, \tau_r, \tau_0 \rangle$, $\mu_1, \ldots, \mu_r$ are wfe's of types $\tau_1, \ldots, \tau_r$ respectively, if $1 \leq j \leq r$ then $\mathrm{Sub}^{\mathcal{K}_\omega}(\alpha, \beta, \mu_j, \mu'_j)$, $\gamma = \eta^\frown \mu_1^\frown \cdots ^\frown \mu_r$, and $\delta = \eta^\frown \mu'_1{}^\frown \cdots ^\frown \mu'_r$.

The identity axioms consist of all instances of Frege's Law; which incorporates Leibniz' Law, Euclid's Law, and interchange of material equivalents, and all instances of the Self-Identity Law.

D72. <u>$\phi$ is an instance of Frege's Law in $\mathcal{K}_\omega$</u> if and only if there are $\alpha$, $\beta$, $\gamma$, $\delta$ such that:

(1) $\alpha, \beta$ are wfe's of $\mathcal{K}_\omega$ of the same simple type

(2) $\gamma, \delta$ are wfe's of $\mathcal{K}_\omega$ of the same simple type

(3) $\mathrm{Sub}^{\mathcal{K}_\omega}(\alpha, \beta, \gamma, \delta)$

(4) $\phi$ is $((\alpha = \beta) \supset (\gamma = \delta))$.

D73. <u>$\phi$ is an instance of the Self-Identity Law in $\mathcal{K}_\omega$</u> if and only if there is an $\alpha$ such that:

(1) $\alpha$ is a wfe of $\mathcal{K}_\omega$ of simple type

(2) $\phi$ is $(\alpha = \alpha)$.

D74. <u>$\phi$ is a Logical Axiom of $\mathcal{K}_\omega$</u> if and only if

(1) $\phi$ is a tautology of $\mathcal{K}_\omega$, or

(2) $\emptyset$ is an instance of Frege's Law in $\mathcal{K}_\omega$, or

(3) $\emptyset$ is an instance of the Self-Identity Law in $\mathcal{K}_\omega$.

We now introduce the notion of validity under the two interpretations of $\mathcal{K}_\omega$, and show that all logical axioms are valid.

D75. <u>$\emptyset$ is valid under the Syntactical interpretation of $\mathcal{K}_\omega$</u> ($\models_{S,\mathcal{K}_\omega} \emptyset$) if and only if:

(1) $\emptyset$ is a wfe of $\mathcal{K}_\omega$ of type t

(2) for all $\mathcal{n} \in M_o$, $Val^{S,\mathcal{K}_\omega}_{\mathcal{n}}(\emptyset) = T$.

D76. <u>$\emptyset$ is valid under the Intensional interpretation of $\mathcal{K}_\omega$</u> ($\models_{I,\mathcal{K}_\omega} \emptyset$) if and only if:

(1) $\emptyset$ is a wfe of $\mathcal{K}_\omega$ of type t

(2) for all $\mathcal{n} \in M_o$, $Val^{I,\mathcal{K}_\omega}_{\mathcal{n}}(\emptyset) = T$.

T73. If $\emptyset, \Psi$ are wfe's of $\mathcal{K}_\omega$ of type t, and $\mathcal{m} \in M_o$, then

(1) $Val^{S,\mathcal{K}_\omega}_{\mathcal{m}}(\emptyset \supset \Psi) = T$ if and only if, if $Val^{S,\mathcal{K}_\omega}_{\mathcal{m}}(\emptyset) = T$ then $Val^{S,\mathcal{K}_\omega}_{\mathcal{m}}(\Psi) = T$

(2) $Val^{I,\mathcal{K}_\omega}_{\mathcal{m}}(\emptyset \supset \Psi) = T$ if and only if, if $Val^{I,\mathcal{K}_\omega}_{\mathcal{m}}(\emptyset) = T$ then $Val^{I,\mathcal{K}_\omega}_{\mathcal{m}}(\Psi) = T$

(3) $Val^{S,\mathcal{K}_\omega}_{\mathcal{m}}(\sim\emptyset) = T$ if and only if, it is not the case that $Val^{S,\mathcal{K}_\omega}_{\mathcal{m}}(\emptyset) = T$

(4) $Val^{I,\mathcal{K}_\omega}_{\mathcal{m}}(\sim\emptyset) = T$ if and only if, it is not the case that $Val^{I,\mathcal{K}_\omega}_{\mathcal{m}}(\emptyset) = T$

(5) $Val^{S,\mathcal{K}_\omega}_{\mathcal{m}}(\emptyset \& \Psi) = T$ if and only if, $Val^{S,\mathcal{K}_\omega}_{\mathcal{m}}(\emptyset) = T$ and $Val^{S,\mathcal{K}_\omega}_{\mathcal{m}}(\Psi) = T$

(6) $Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\phi \& \psi) = T$ if and only if, $Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\phi) = T$ and $Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\psi) = T$

(7) $Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\phi \vee \psi) = T$ if and only if, $Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\phi) = T$ or $Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\psi) = T$

(8) $Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\phi \vee \psi) = T$ if and only if $Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\phi) = T$ or $Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\psi) = T$

(9) $Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\phi \equiv \psi) = T$ if and only if $Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\phi) = T$ if and only if $Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\psi) = T$

(10) $Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\phi \equiv \psi) = T$ if and only if, $Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\phi) = T$ if and only if $Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\psi) = T$

Proof by D65, D67, T55.

T74. If $\phi$ is a tautology of $\mathcal{K}_\omega$, then

(1) $\vdash_{S,\mathcal{K}_\omega} \phi$

(2) $\vdash_{I,\mathcal{K}_\omega} \phi$

Proof: (1) by T73, T68 the function which assigns to each sentence $\phi$ of $\mathcal{K}_\omega$, $Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\phi)$, is a truth evaluation. Hence if $\phi$ is a tautology, $Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\phi) = T$ for all $\mathcal{M} \in M_o$; (2), similar using T73, T72.

T75. If $\alpha, \beta$ are wfe's of $\mathcal{K}_\omega$ of the same simple type, $\gamma, \delta$ are wfe's of $\mathcal{K}_\omega$ of the same simple type, $Sub^{\mathcal{K}_\omega}(\alpha, \beta, \gamma, \delta)$, and $\mathcal{M} \in M_o$, then

(1) if $Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\alpha) = Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\beta)$, then $Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\gamma) = Val^{S,\mathcal{K}_\omega}_{\mathcal{M}}(\delta)$

(2) if $Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\alpha) = Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\beta)$, then $Val^{I,\mathcal{K}_\omega}_{\mathcal{M}}(\gamma) =$

$\mathrm{Val}^{I,\mathcal{K}_\omega}_{\mathfrak{M}}(\delta)$.

Proof by induction on $\gamma$.

T76. If $\alpha, \beta$ are wfe's of $\mathcal{K}_\omega$ of the same simple type, and $\mathfrak{M} \in M_o$, then

(1) $\mathrm{Val}^{S,\mathcal{K}_\omega}_{\mathfrak{M}}(\alpha = \beta) = T$ if and only if $\mathrm{Val}^{S,\mathcal{K}_\omega}_{\mathfrak{M}}(\alpha) = \mathrm{Val}^{S,\mathcal{K}_\omega}_{\mathfrak{M}}(\beta)$

(2) $\mathrm{Val}^{I,\mathcal{K}_\omega}_{\mathfrak{M}}(\alpha = \beta) = T$ if and only if $\mathrm{Val}^{I,\mathcal{K}_\omega}_{\mathfrak{M}}(\alpha) = \mathrm{Val}^{I,\mathcal{K}_\omega}_{\mathfrak{M}}(\beta)$.

Proof by D65, D67, T56.

T77. If $\phi$ is an instance of Frege's Law in $\mathcal{K}_\omega$, then

(1) $\models_{S,\mathcal{K}_\omega} \phi$

(2) $\models_{I,\mathcal{K}_\omega} \phi$

Proof by T75, T73, T76, D75, D76.

T78. If $\phi$ is an instance of the Self-Identity Law in $\mathcal{K}_\omega$, then,

(1) $\models_{S,\mathcal{K}_\omega} \phi$

(2) $\models_{I,\mathcal{K}_\omega} \phi$

Proof by T76, D75, D76.

T79. If $\phi$ is a Logical Axiom of $\mathcal{K}_\omega$, then

(1) $\models_{S,\mathcal{K}_\omega} \phi$

(2) $\models_{I,\mathcal{K}_\omega} \phi$

Proof by D74, T75, T77, T78.

We now introduce the notion of a modus ponens consequence of a set of sentences A.

D77. <u>$\emptyset$ is a modus ponens consequence of A</u> ($\vdash_{MP,A} \emptyset$) if and only if:

(1) A is a set of wfe's of $\mathcal{K}_\omega$ of type t

(2) there are S and n such that:

(a) S is a finite sequence of length n+1

(b) $S_n = \emptyset$

(c) if $0 \leq m \leq n$, then either

(i) $S_m \in A$, or

(ii) there are $j,k < m$, and $S_j = (S_k \supset S_m)$

T80. If $\emptyset, \psi$ are wfe's of $\mathcal{K}_\omega$ of type t, then

(1) if $\models_{S,\mathcal{K}_\omega} (\emptyset \supset \psi)$ and $\models_{S,\mathcal{K}_\omega} \emptyset$, then $\models_{S,\mathcal{K}_\omega} \psi$

(2) if $\models_{I,\mathcal{K}_\omega} (\emptyset \supset \psi)$ and $\models_{I,\mathcal{K}_\omega} \emptyset$, then $\models_{I,\mathcal{K}_\omega} \psi$

Proof by T73, D75, D76.

T81. If A is the set of Logical Axioms of $\mathcal{K}_\omega$, and

$\vdash_{MP,A} \emptyset$, then

(1) $\models_{S,\mathcal{K}_\omega} \emptyset$

(2) $\models_{I,\mathcal{K}_\omega} \emptyset$

Proof by induction on the length of the proof of $\emptyset$ using T79, T80.

## 25. Completeness and Decidability of $\mathcal{K}_o$

We first introduce the notion of validity for formulas of $\mathcal{K}_o$, and show that it coincides with validity under both interpretations for such formulas.

D78. $\phi$ is valid in $\mathcal{K}_o$ ($\models_{\mathcal{K}_o} \phi$) if and only if:

(1) $\phi$ is a wfe of $\mathcal{K}_o$ of type t

(2) for all $\eta \in M_o$, $\mathrm{Val}_{\eta}^{\mathcal{K}_o}(\phi) = T$

T82. If $\phi$ is a wfe of $\mathcal{K}_o$, then

(1) $\models_{\mathcal{K}_o} \phi$ if and only if $\models_{S,\mathcal{K}_\omega} \phi$

(2) $\models_{\mathcal{K}_o} \phi$ if and only if $\models_{I,\mathcal{K}_\omega} \phi$

Proof by D65, D67.

The following theorem establishes that validity under the two interpretations agrees for formulas of $\mathcal{K}_o$. That this is not the case, in general, for arbitrary formulas of will be shown in theorem 95.

T83. If $\phi$ is a wfe of $\mathcal{K}_o$, then $\models_{S,\mathcal{K}_\omega} \phi$ if and only if $\models_{I,\mathcal{K}_\omega} \phi$.

Proof by T82.

The following two theorems are well known from the literature.[72] The second will be called <u>the completeness theorem for $\mathcal{K}_o$</u>.

T84. If $\phi$ is a wfe of $\mathcal{K}_o$, and A is the set of Logical

Axioms of $\mathcal{K}_\omega$, then $\vdash_{\mathcal{K}_0} \phi$ only if $\vdash_{MP,A} \phi$

T85. If $\phi$ is a wfe of $\mathcal{K}_0$, and A is the set of Logical Axioms of $\mathcal{K}_\omega$, then $\vdash_{\mathcal{K}_0} \phi$ if and only if $\vdash_{MP,A} \phi$.

Proof by T84, T81, T82.

The decidability of $\mathcal{K}_0$ is also well known,[72] but we will give an argument differing slightly from the familiar forms. Our argument depends primarily on T. First we must introduce the notion of an <u>instance</u> of a formula of $\mathcal{K}_0$. For this purpose we define some auxiliary notions.

D79. If $\delta$ is a wfe of $\mathcal{K}_0$ of simple type, and $\alpha_1,\ldots,\alpha_p$ are wfe's of $\mathcal{K}_0$ of type i, then $\underline{\delta[\alpha_1,\ldots,\alpha_p]}$ is the result of simultaneously replacing all occurrences of the <u>jth</u> individual constant (Opsymb(j,o)) in $\delta$ by $\alpha_j$, for all $1 \leq j \leq p$.

D80. <u>f is a substitution function for $\mathcal{K}_0$</u> if and only if:

(1) f is a function whose domain is the set of all predicates and operation symbols of $\mathcal{K}_0$

(2) f assigns a wfe of $\mathcal{K}_0$ of type i to each operation symbol of $\mathcal{K}_0$

(3) f assigns a wfe of $\mathcal{K}_0$ of type t to each predicate of $\mathcal{K}_0$

D81. <u>$\delta$ is a $\mathcal{K}_0$ instance of $\gamma$</u> if:

(1) $\gamma$ is a wfe of $\mathcal{K}_0$ of simple type

(2) there is a substitution function for $\mathcal{K}_o$, f, such that $\delta$ is the result of simultaneously replacing all occurrences of $\eta^\frown\alpha_1{}^\frown\cdots{}^\frown\alpha_p$ in $\gamma$ by $f(\eta)[\alpha_1,\ldots,\alpha_p]$, where $\eta$ is a p-place predicate or operation symbol and $\alpha_1,\ldots,\alpha_p$ are wfe's of $\mathcal{K}_o$ of type i.

T86. If $\phi$ is a wfe of $\mathcal{K}_o$ of type t, $\psi$ is a $\mathcal{K}_o$ instance of $\phi$, and $\models_{\mathcal{K}_o} \phi$, then $\models_{\mathcal{K}_o} \psi$.

Proof by induction on $\phi$.

D82. <u>$\phi$ is a contingent identity disjunction</u> if and only if $\phi$ is a disjunction each of whose disjuncts has the form $(\alpha = \beta)$, where $\alpha$, $\beta$ are distinct wfe's of $\mathcal{K}_o$ of type i.

T87. If $\phi$ is a contingent identity disjunction, then neither $\phi$ nor $\sim\phi$ is valid in $\mathcal{K}_o$.

Proof: $\sim\phi$ can not be valid since $\phi$ holds in any model $\langle DR\rangle$ where D contains only a single element. To show that $\phi$ is not valid, construct an $\mathfrak{N} \in M_o$ isomorphic to $\langle D,R\rangle$ where D is the set of all wfe's of $\mathcal{K}_o$ of type i and where $R(\mathrm{Opsymb}(m,p)) = \mathrm{Synt}^{\mathcal{K}_o}(\mathrm{Opsymb}(m,p))$. Then if $\alpha$ is a wfe of $\mathcal{K}_o$ of type i, $\mathrm{Val}^{\mathcal{K}_o}_{\langle D,R\rangle}(\alpha) = \alpha$, thus each disjunct in $\phi$ will be false, and hence so will $\phi$.

The following theorem is related to a result of Quines [73] to the effect that a formula of the predicate calculus has a contravalid instance just in case it is falsifiable in a model whose universe contains a single element.

T88. If $\phi$ is a wfe of $\mathcal{K}_o$ of type t, and $\phi$ is not valid in $\mathcal{K}_o$, then there are $\psi, \chi$ such that:

(1) $\psi$ is a $\mathcal{K}_o$ instance of $\phi$

(2) $\models_{\mathcal{K}_o} (\psi \supset \chi)$

(3) $\chi$ is a contingent identity disjunction

Proof: Since $\phi$ is not valid, it must fail in some $\langle D,R\rangle \in M_o$. In fact $\langle D,R\rangle$ can be chosen to satisfy the following conditions: (a) D is finite, (b) for each element $X \in D$, there is a wfe $\beta_X$ of $\mathcal{K}_o$ such that $\mathrm{Val}^{\mathcal{K}_o}_{\langle DR\rangle}(\beta_X) = X$. Form the instance $\psi$ of $\phi$ as follows. If $\eta$ is a p-place predicate occurring in $\phi$ and $R(\eta) = \{\langle x_1^1,\ldots,x_p^1\rangle,\ldots,\langle x_1^m,\ldots,x_p^m\rangle\}$, then replace $\eta^\frown\alpha_1^\frown\ldots^\frown\alpha_p$ by $(\,((\alpha_1 == \beta_{x_1^1})\&\ldots\&(\alpha_p == \beta_{x_p^1}))\vee\ldots\vee((\alpha_1 == \beta_{x_1^m})\&\ldots\&(\alpha_p == \beta_{x_p^m}))\,)$. If $\eta$ is a o-place predicate, replace it with either $(\delta == \delta)$ or $\sim(\delta == \delta)$ (where $\delta$ = Opsymb(o,o)) depending on whether $R(\eta)$ = T or F. This replacement does not affect the truth value of $\phi$, hence $\psi$ is also false in $\langle DR\rangle$. Form $\chi$ as follows. First put $\psi$ into conjunctive normal form. Since $\psi$ contains no predicates (other than identity) each conjunct will be a disjunction of wfe's of the form $(\alpha == \beta)$. Now any conjunct with a disjunct of the form $(\alpha == \alpha)$ will be true, hence since $\psi$ is false, one of the conjuncts must not contain any such disjunct. Let $\chi$ be the first such conjunct, then $\chi$ will be a contingent identity disjunction which is implied by $\psi$.

T89. If $\emptyset$ is a wfe of $\mathcal{K}_0$ of type t, and A is the set of logical axioms of $\mathcal{K}_\omega$, then $\emptyset$ is not valid in $\mathcal{K}_0$ if and only if there are $\psi, \chi$ such that:

(1) $\psi$ is a $\mathcal{K}_0$ instance of $\emptyset$

(2) $\vdash_{\overline{MP,A}} (\psi \supset \chi)$

(3) $\chi$ is a contingent identity disjunction.

Proof: from left to right by T88, T85; from right to left by T87, T85, T82, T80, T86.

T90. The set of all valid wfe's of $\mathcal{K}_0$ is decidable.

Proof: By T85, T89 we can enumerate both the given set and its complement within the class of wfe's of $\mathcal{K}_0$ of type t.

## 26. Additional Axioms for $\mathcal{K}_\omega$

We now turn to axioms (or, more exactly, axiom schemes) governing the special logical signs of our direct discourse languages. Namely, those introduced in the languages $\mathcal{K}_{n+1}$: Delta ($\tilde{\tilde{\tau}}$); Id ($\tilde{\tilde{\tau}}$), Nec, and all wfe's $\overline{\alpha}$. In the present section, we formulate the remaining axioms and establish that all theorems are valid.

Our first set of axioms links the delta predicates with the bar notation.

D83. <u>$\emptyset$ is a Delta Axiom of $\mathcal{K}_\omega$</u> if and only if there are $\alpha, \beta$ such that:

(1) $\alpha, \beta$ are wfe's of $\mathcal{K}_\omega$ of the same simple type

(2) $\emptyset$ is $(\triangle(\overline{\alpha},\beta) \equiv (\alpha = \beta))$.[74]

It follows from theorem 67, that formulas of $\mathcal{K}_\omega$ whose main connective is $\mathrm{Id}(\widetilde{\widetilde{\tau}})$, have the form, $(\overline{\alpha} = \overline{\beta})$. Thus, such formulas reflect our principle of individuation for the entities in the universes of the types $\widetilde{\widetilde{\tau}}$. We now introduce the axioms governed by these principles. Under the syntactical interpretation, the universe of the simple type $\widetilde{\widetilde{\tau}}$ consists of wfe's of type $\tau$. The denotation of $\overline{\alpha}$ is simply $\alpha$. Thus $\overline{\alpha}$ and $\overline{\beta}$ have the same denotation just in case $\alpha$ is the same wfe as $\beta$.

D84. <u>$\emptyset$ is a Syntactical Individuating Axiom of $\mathcal{K}_\omega$</u> if and only if there are $\alpha,\beta$ such that:

(1) $\alpha,\beta$ are wfe's of $\mathcal{K}_\omega$ of the same simple type

(2) if $\alpha = \beta$, then $\emptyset$ is $(\overline{\alpha} = \overline{\beta})$

(3) if $\alpha \neq \beta$, then $\emptyset$ is $\sim(\overline{\alpha} = \overline{\beta})$

Under the intensional interpretation, the universe of the simple type $\widetilde{\widetilde{\tau}}$ consists of concepts. The denotation of $\overline{\alpha}$ is now the sense of $\alpha$. Hence the problem of the identity of the denotations of $\overline{\alpha}$ and $\overline{\beta}$ reduces to that of the identity of the sense of $\alpha$ and the sense of $\beta$. Here it will be recalled (section 15.1) that we decided to equate the senses of $\alpha$ and $\beta$ when $\alpha$ and $\beta$ are themselves logically equivalent. That is, when $(\alpha = \beta)$ expresses a necessary proposition.

D85. <u>$\phi$ is an Intensional Individuating Axiom of $\mathcal{K}_\omega$</u> if and only if there are $\alpha, \beta$ such that:

(1) $\alpha, \beta$ are wfe's of $\mathcal{K}_\omega$ of the same simple type

(2) $\phi$ is $(N\overline{(\alpha = \beta)} \equiv (\overline{\alpha} = \overline{\beta}))$

We will introduce a number of axioms on necessity. The modal axioms of the first four kinds are the familiar principles of the modal system S5.[75] The last two kinds of modal axioms are required to prove of a contingent sentence (proposition) that it is not necessary.

D86. <u>$\phi$ is a Modal Axiom of $\mathcal{K}_\omega$</u> if and only if there are wfe's $\psi, \chi$ of $\mathcal{K}_\omega$ of type t and $\phi$ satisfies one of the following conditions:

(1) $\phi$ is $(N\overline{(\psi \supset \chi)} \supset (N\overline{\psi} \supset N\overline{\chi}))$

(2) $\phi$ is $(N\overline{\psi} \supset \psi)$

(3) $\phi$ is $(N\overline{\psi} \supset N\overline{N\overline{\psi}})$

(4) $\phi$ is $(\sim N\overline{\psi} \supset N\overline{\sim N\overline{\psi}})$

(5) $\chi$ is a $\mathcal{K}_0$ instance of $\psi$ and $\phi$ is $(N\overline{\psi} \supset N\overline{\chi})$

(6) $\psi$ is a contingent identity disjunction, and $\phi$ is $\sim N\overline{\psi}$

This completes our list of axioms for $\mathcal{K}_\omega$. Note that the axioms given for the two interpretations differ only with respect to the individuation axioms (D84, D85).[76] We will now establish that all of the axioms are valid under the appropriate interpretation of $\mathcal{K}_\omega$.

T91. If $\emptyset$ is a Delta Axiom of $\mathcal{K}_\omega$, then

(1) $\underset{S,\mathcal{K}_\omega}{\models} \emptyset$

(2) $\underset{I,\mathcal{K}_\omega}{\models} \emptyset$

Proof by D65, D67, T60, T73, T76.

An immediate consequence of the preceding theorem shows that a certain intuitive criterion of adequacy is fulfilled:

T92. If $\alpha$ is a wfe of $\mathcal{K}_\omega$ of simple type, then

(1) $\underset{S,\mathcal{K}_\omega}{\models} \triangle(\bar{\alpha}, \alpha)$

(2) $\underset{I,\mathcal{K}_\omega}{\models} \triangle(\bar{\alpha}, \alpha)$

Proof by T91, T73, T76.

The following theorem states three important properties of necessity: first that $N\emptyset$ is true just in case $\emptyset$ is valid, second that when $\emptyset$ is valid, $N\emptyset$ is valid, and third that a formula of the form $N\emptyset$ is logically determinate in the sense that either it or its negation is valid.

T93. If $\emptyset$ is a wfe of $\mathcal{K}_\omega$ of type t, then

(1) if $\mathcal{M} \in M_o$, $\mathrm{Val}^{S,\mathcal{K}_\omega}_{\mathcal{M}}(N\emptyset) = T$ if and only if $\underset{S,\mathcal{K}_\omega}{\models} \emptyset$

(2) if $\mathcal{M} \in M_o$, $\mathrm{Val}^{I,\mathcal{K}_\omega}_{\mathcal{M}}(N\emptyset) = T$ if and only if $\underset{I,\mathcal{K}_\omega}{\models} \emptyset$

(3) if $\underset{S,\mathcal{K}_\omega}{\models} \emptyset$, then $\underset{S,\mathcal{K}_\omega}{\models} N\emptyset$

(4) if $\underset{I,\mathcal{K}_\omega}{\models} \emptyset$, then $\underset{I,\mathcal{K}_\omega}{\models} N\emptyset$

(5) either $\models_{S,\mathcal{K}_\omega} N\emptyset$ or $\models_{S,\mathcal{K}_\omega} \sim N\emptyset$

(6) either $\models_{I,\mathcal{K}_\omega} N\emptyset$ or $\models_{I,\mathcal{K}_\omega} \sim N\emptyset$

Proof: (1), (2) by D65, D67, T61; (3), (4) by (1), (2); (5), (6) by (1), (2), T73.

Properties of Id($\tilde{\tau}$) which correspond to those of Nec given in the preceding theorem are given in the following.

T94. If $\alpha$, $\beta$ are wfe's of $\mathcal{K}_\omega$ of the same simple type, then

(1) if $\mathfrak{M} \in M_o$, $Val^{S,\mathcal{K}_\omega}(\bar{\alpha} = \bar{\beta}) = T$ if and only if $\alpha = \beta$

(2) if $\mathfrak{M} \in M_o$, $Val^{I,\mathcal{K}_\omega}(\bar{\alpha} = \bar{\beta}) = T$ if and only if $\models_{I,\mathcal{K}_\omega} (\alpha = \beta)$

(3) $\models_{S,\mathcal{K}_\omega} (\bar{\alpha} = \bar{\beta})$ if and only if $\alpha = \beta$

(4) $\models_{I,\mathcal{K}_\omega} (\bar{\alpha} = \bar{\beta})$ if and only if $\models_{I,\mathcal{K}_\omega} (\alpha = \beta)$

(5) either $\models_{S,\mathcal{K}_\omega} (\bar{\alpha} = \bar{\beta})$ or $\models_{S,\mathcal{K}_\omega} \sim(\bar{\alpha} = \bar{\beta})$

(6) either $\models_{I,\mathcal{K}_\omega} (\bar{\alpha} = \bar{\beta})$ or $\models_{I,\mathcal{K}_\omega} \sim(\bar{\alpha} = \bar{\beta})$

Proof: (1) by T76, T69, D63; (2) by T76, T71, T70; (3), (4) by (1), (2); (5), (6) by (1), (2), T73.

The preceding theorem provides a convenient proof of the fact, asserted in connection with theorem 83, that validity under the syntactical interpretation of $\mathcal{K}_\omega$ and validity under the intensional interpretation of $\mathcal{K}_\omega$ do not, in general, agree. In fact, neither implies the other.

T95. Let $\emptyset$ be Pred(0,0). Then

(1) $(\bar{\emptyset} = \overline{\sim\sim\emptyset})$ is valid under the intensional interpretation of $\mathcal{K}_\omega$ but not under the syntactical interpretation

(2) $\sim(\bar{\emptyset} = \overline{\sim\sim\emptyset})$ is valid under the syntactical interpretation of $\mathcal{K}_\omega$ but not under the intensional interpretation.

Proof: (1) by T94, T73, T76; (2) by (1), T94.

We can now establish the validity of the remaining axioms.

T96. (1) If $\emptyset$ is a Syntactical Individuating Axiom of $\mathcal{K}_\omega$, then $\models_{S,\mathcal{K}_\omega} \emptyset$.

(2) If $\emptyset$ is an Intensional Individuating Axiom of $\mathcal{K}_\omega$, then $\models_{I,\mathcal{K}_\omega} \emptyset$.

Proof: (1) by T94, T73; (2) by T93, T94, T73.

T97. If $\emptyset$ is a Modal Axiom of $\mathcal{K}_\omega$, then

(1) $\models_{S,\mathcal{K}_\omega} \emptyset$

(2) $\models_{I,\mathcal{K}_\omega} \emptyset$

Proof by the cases of D86: (1) by T93, T73, T80; (2), (3), (4) by T93, T73; (5) by T93, T82, T86, T73; (6) by T93, T82, T87, T73.

The <u>theorems of $\mathcal{K}_\omega$</u> are the consequences of the axioms by two rules of inference: modus ponens, and modal generalization. The latter leads from a theorem $\emptyset$ to the theorem $N\overline{\emptyset}$.[77] We next define our new notion of consequence.

D87. <u>$\emptyset$ is a consequence of A by modus ponens and modal generalization</u> if and only if

(1) A is a set of wfe's of $\mathcal{K}_\omega$ of type t

(2) there are S and n such that:

(a) S is a finite sequence of length n+1

(b) $S_n = \emptyset$

(c) if $0 \leq m \leq n$, then either

(i) $S_m \in' A$, or

(ii) there are $j,k < m$, and $S_j = (S_k \supset S_m)$, or

(iii) there is a $j<m$, and $S_m = N\overline{S_j}$.

D88. Let $A^s$ be the set of all $\emptyset$ satisfying one of the following conditions: $\emptyset$ is a Logical Axiom of $\mathcal{K}_\omega$, $\emptyset$ is a Delta Axiom of $\mathcal{K}_\omega$, $\emptyset$ is a Syntactical Individuating Axiom of $\mathcal{K}_\omega$, $\emptyset$ is a Modal Axiom of $\mathcal{K}_\omega$. Then <u>$\Psi$ is a theorem of the Syntactical interpretation of $\mathcal{K}_{\underline{\omega}}$</u> ($\vdash_{S,\mathcal{K}_\omega} \Psi$) if and only if $\Psi$ is a consequence of $A^s$ by modus ponens and modal generalization.

D89. Let $A^I$ be the set of all $\emptyset$ satisfying one of the following conditions: $\emptyset$ is a Logical Axiom of $\mathcal{K}_\omega$, $\emptyset$ is a Delta Axiom of $\mathcal{K}_\omega$, $\emptyset$ is an Intensional Individuating Axiom of $\mathcal{K}_\omega$, $\emptyset$ is a Modal Axiom of $\mathcal{K}_\omega$. Then <u>$\Psi$ is a theorem of the Intensional interpretation of $\mathcal{K}_{\underline{\omega}}$</u> ($\vdash_{I,\mathcal{K}_\omega} \Psi$) if and only if $\psi$ is a consequence of $A^I$ by modus ponens and modal generalization.

The following theorem, which asserts that all theorems are valid, provides half of the completeness argument which will be finished in the following section.

T98. (1) If $\vdash_{S,\mathcal{K}_\omega} \phi$, then $\vDash_{S,\mathcal{K}_\omega} \phi$

(2) If $\vdash_{I,\mathcal{K}_\omega} \phi$, then $\vDash_{I,\mathcal{K}_\omega} \phi$

Proof by induction on the length of the proof using cases (i), (ii), (iii) of D87: case (i) by T79, T91, T96, T97; case (ii) by T80; case (iii) by T93.

## 27. Completeness and Decidability of $\mathcal{K}_\omega$

The crucial lemmas for both the completeness and the decidability argument are theorems 102 and 103. These in turn depend on theorems 99 and 100. The latter indicate the general idea of the arguments. Each formula of rank n+1 is shown to be provably equivalent to a formula of rank n. By this process each formula of $\mathcal{K}_\omega$ is reduced to a formula of $\mathcal{K}_0$ where we can apply the results of section 25.

T99. Let n be any natural number such that all wfe's $\psi$ of $\mathcal{K}_n$ of type t satisfy both of the following conditions:

(1) if $\vDash_{S,\mathcal{K}_\omega} \psi$, then $\vdash_{S,\mathcal{K}_\omega} \psi$

(2) if it is not the case that $\vDash_{S,\mathcal{K}_\omega} \psi$, then $\vdash_{S,\mathcal{K}_\omega} \sim N\Psi$.

Then if $\phi$ is any wfe of $\mathcal{K}_{n+1}$ of type t, there is a $\Gamma$ such that:

(3) $\Gamma$ is a wfe of $\mathcal{K}_n$ of type t, and

(4) $\vdash_{S,\mathcal{K}_\omega} (\emptyset \equiv \Gamma)$

Proof: Assume the hypothesis and assume that $\emptyset$ is a wfe of $\mathcal{K}_{n+1}$ of type t. We show that there is an appropriate $\Gamma$ by induction on $\emptyset$ using the cases of T37.

Case (1): $\emptyset$ is a wfe of $\mathcal{K}_o$, hence of $\mathcal{K}_n$. Let $\Gamma = \emptyset$.

Case (2): by the inductive hypothesis and the logical axioms there is an equivalent $\Gamma$ of the same structure.

Case (3): subcase (i) $\alpha = \beta$: by the individuation axioms $\vdash_{S,\mathcal{K}_\omega} \emptyset$. Hence, let $\Gamma$ be any tautology.

subcase (ii) $\alpha \neq \beta$: by the individuation axioms $\vdash_{S,\mathcal{K}_\omega} \sim\emptyset$. Hence let $\Gamma$ be the negation of any tautology.

Case (4): argument as in Case (1).

Case (5): by the delta axioms $\vdash_{S,\mathcal{K}_\omega} (\emptyset \equiv (\alpha = \beta))$. Hence, using the logical axioms, it suffices to show that there is a $\psi$ provably equivalent to $(\alpha = \beta)$.

subcase (i) the type of $\alpha$ is i: argument as in Case (4).

subcase (ii) the type of $\alpha$ is $\tilde{\tilde{\tau}}$: argument as in Case (3).

subcase (iii) the type of $\alpha$ is t: argument as in Case (2).

Case (6): subcase (i) $\models_{S,\mathcal{K}_\omega} \psi$: by the hypothesis of the the theorem, $\vdash_{S,\mathcal{K}_\omega} \psi$. Thus by modal generalization $\vdash_{S,\mathcal{K}_\omega} N\psi$, that is $\vdash_{S,\mathcal{K}_\omega} \phi$. Hence, let $\Gamma$ be as in Case (3)(i).

subcase (ii) not $\models_{S,\mathcal{K}_\omega} \psi$: by the hypothesis of the theorem $\vdash_{S,\mathcal{K}_\omega} \sim N\psi$ that is, $\vdash_{S,\mathcal{K}_\omega} \sim\phi$. Hence, let $\Gamma$ be as in Case (3)(ii).

T100. If '$\models_{S,\mathcal{K}_\omega}$' and '$\vdash_{S,\mathcal{K}_\omega}$' are replaced in theorem 99 by '$\models_{I,\mathcal{K}_\omega}$' and '$\vdash_{I,\mathcal{K}_\omega}$' respectively, the result is also true.

Proof is as for T99 with exception of Case (3), which is as follows:

Case (3): subcase (i) $\models_{I,\mathcal{K}_\omega} (\alpha = \beta)$: by the hypothesis of the theorem, modal generalization, the individuation axioms, and the logical axioms $\vdash_{I,\mathcal{K}_\omega} \phi$. Hence let $\Gamma$ be any tautology.

subcase (ii) not $\models_{I,\mathcal{K}_\omega} (\alpha = \beta)$: by the hypothesis of the theorem, the individuation axioms, and the logical axioms $\vdash_{I,\mathcal{K}_\omega} \phi$. Hence, let $\Gamma$ be the negation of any tautology.

T101. If A is the set of logical axioms of $\mathcal{K}_\omega$, and $\vdash_{MP,A} \phi$, then

(1) $\vdash_{S,\mathcal{K}_\omega} \phi$

(2) $\vdash_{I,\mathcal{K}_\omega} \phi$

Proof by induction on the length of the proof of $\phi$.

T102. If n is any natural number, and $\Psi$ is a wfe of $\mathcal{K}_n$ of type t, then

(1) if $\vDash_{S,\mathcal{K}_\omega} \Psi$, then $\vdash_{S,\mathcal{K}_\omega} \Psi$

(2) if it is not the case that $\vDash_{S,\mathcal{K}_\omega} \Psi$, then $\vdash_{S,\mathcal{K}_\omega} \sim N\Psi$

Proof by induction on n. Case (i) n = 0: (1) by T82, T84, T101. To show (2), assume the antecedent of (2). Then by T82, T89, T101, and modal axioms of kind 5 and 6, there are $\Gamma, \chi$ such that $\vdash_{S,\mathcal{K}_\omega} (\Gamma \supset \chi)$, $\vdash_{S,\mathcal{K}_\omega} \sim N\chi$, $\vdash_{S,\mathcal{K}_\omega} (N\Psi \supset N\chi)$. But then by modal generalization on $(\Gamma \supset \chi)$, modal axioms of kind 1, and logical axioms $\vdash_{S,\mathcal{K}_\omega} \sim N\Psi$. Case (ii) n = m+1: assume the antecedent of the theorem. Then by the inductive hypothesis and T99 there is a $\Gamma$ such that $\Gamma$ is a wfe of $\mathcal{K}_m$ and (A), $\vdash_{S,\mathcal{K}_\omega} (\Psi \equiv \Gamma)$. To show (1), assume the antecedent of (1). Then by T98 and (A), $\vDash_{S,\mathcal{K}_\omega} \Gamma$. Hence, by the hypothesis of induction $\vdash_{S,\mathcal{K}_\omega} \Gamma$, therefore by (A) $\vdash_{S,\mathcal{K}_\omega} \Psi$. To show (2), assume the antecedent of (2). Again by T98 and (A), not $\vDash_{S,\mathcal{K}_\omega} \Gamma$. Thus by the hypothesis of induction $\vdash_{S,\mathcal{K}_\omega} \sim N\Gamma$. But by (A), the logical axioms, modal generalization, and modal axioms of kind 1, $\vdash_{S,\mathcal{K}_\omega} (N\Psi \supset N\Gamma)$. Hence, by the logical axioms $\vdash_{S,\mathcal{K}_\omega} \sim N\Psi$.

T103. If '$\vDash_{S,\mathcal{K}_\omega}$' and '$\vdash_{S,\mathcal{K}_\omega}$' are replaced in theorem 102 by

'$\models_{I,\mathcal{K}_\omega}$' and '$\vdash_{I,\mathcal{K}_\omega}$' respectively, the result is also true.

Proof is as for T102 but using T100 in place of T99.

The following is the completeness theorem for $\mathcal{K}_\omega$.

T104. (1) $\models_{S,\mathcal{K}_\omega} \emptyset$ if and only if $\vdash_{S,\mathcal{K}_\omega} \emptyset$

(2) $\models_{I,\mathcal{K}_\omega} \emptyset$ if and only if $\vdash_{I,\mathcal{K}_\omega} \emptyset$

Proof: (1) by T98 (1), T102 (1); (2) by T98 (2), T103 (1).

T105. (1) The set of all $\emptyset$ such that $\models_{S,\mathcal{K}_\omega} \emptyset$ is decidable.

(2) The set of all $\emptyset$ such that $\models_{I,\mathcal{K}_\omega} \emptyset$ is decidable.

Proof: By T104, T102 (2), T103 (2) we can enumerate both the given sets and their complements within the class of wfe's of $\mathcal{K}_\omega$ of type t.

## 28. Comparison With Indirect Discourse

For purposes of comparison, let us now consider a system of modal logic developed by the method of indirect discourse. We call the language, $\mathcal{K}_{i.d.}$. All well formed expressions of $\mathcal{K}_{i.d.}$ have type t. The atomic formulas are sentential constants, thus identity does not occur. We introduce the wfe's of $\mathcal{K}_{i.d.}$ by a recursive definition.

D90. $\emptyset$ is a wfe of $\mathcal{K}_{i.d.}$ if and only if $\emptyset$ satisfies one of the following conditions:

(1) there is a natural number m such that $\emptyset$ is Pred (m,o)

(2) there are wfe's of $\mathcal{K}_{1.d.}$ $\psi, \chi$ such that $\phi$ is $(\psi \supset \chi)$.

(3) there is a wfe of $\mathcal{K}_{1.d.}$ $\psi$ such that $\phi$ is $\sim\psi$.

(4) there is a wfe of $\mathcal{K}_{1.d.}$ $\psi$ such that $\phi$ is $N\psi$.

Our axioms will be such as to allow a natural interpretation of $\mathcal{K}_{1.d.}$. Donald Kalish has dubbed this system S13.[78]

Let us imagine that tautology of $\mathcal{K}_{1.d.}$ is defined on the pattern of D69 and D70. Then we can succinctly introduce the axioms of S13.

D91. $\phi$ is an axiom of S13 if and only if there are wfe's of $\mathcal{K}_{1.d.}$ $\psi, \chi$ such that $\phi$ satisfies one of the following conditions:

(1) $\phi$ is a tautology of $\mathcal{K}_{1.d.}$.

(2) $\phi$ is $(N(\psi \supset \chi) \supset (N\psi \supset N\chi))$.

(3) $\phi$ is $(N\psi \supset \psi)$.

(4) $\phi$ is $(N\psi \supset NN\psi)$.

(5) $\phi$ is $(\sim N\psi \supset N\sim N\psi)$.

(6) $\psi$ is also a wfe of $\mathcal{K}_{o}$, $\psi$ is not a tautology, and $\phi$ is $\sim N\psi$.

Axiom schemes (1)-(5) are the axioms of the more familiar Lewis system S5. Axiom scheme (6) adds $\sim N\psi$ for modal free non-tautologies $\psi$. Note that the first clause in (6) requires that $\psi$ does not contain Nec. The notion of a theorem of S13 (written '$\vdash_{S13}$') is defined with respect to the inference rules modus ponens and modal generalization, only here the latter rule leads from $\phi$ to $N\phi$ rather than to $N\bar{\phi}$. We now introduce the value of a wfe of $\mathcal{K}_{i.d.}$.

D92. If $\phi$ is a wfe of $\mathcal{K}_{i.d.}$, and $\mathcal{M} \in M_o$, then <u>the value of $\phi$ in $\mathcal{M}$ with respect to $\mathcal{K}_{i.d.}$</u> $(\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_{i.d.}}(\phi))$ is given by the following:

(1) $\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_{i.d.}}(\phi) \in \{T, F\}$

(2) if $\phi$ is $\mathrm{Pred}(m,o)$ then $\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_{i.d.}}(\phi) = \mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_o}(\phi)$

(3) if $\phi$ is $(\psi \supset \chi)$, then $\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_{i.d.}}(\phi) = T$ if and only if, if $\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_{i.d.}}(\psi) = T$ then $\mathrm{Val}^{\mathcal{K}_{i.d.}}(\chi) = T$

(4) if $\phi$ is $\sim\psi$, then $\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_{i.d.}}(\phi) = T$ if and only if it is not the case that $\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_{i.d.}}(\psi) = T$

(5) if $\phi$ is $N\psi$, then $\mathrm{Val}_{\mathcal{M}}^{\mathcal{K}_{i.d.}}(\phi) = T$ if and only if for all $\mathcal{N} \in M_o$, $\mathrm{Val}_{\mathcal{N}}^{\mathcal{K}_{i.d.}}(\psi) = T$.

We define <u>validity in $\mathcal{K}_{i.d.}$</u> (written '$\models_{\mathcal{K}_{i.d.}}$') in the usual way. We can now state the completeness theorem for $\mathcal{K}_{i.d.}$ which was first proved in Carnap [3].

T106. $\vdash_{S13} \phi$ if and only if $\models_{\mathcal{K}_{i.d.}} \phi$

If the sentences of $\mathcal{K}_{i.d.}$ are understood as denoting truth values, $N\phi$ is certainly an oblique context of $\phi$. Thus translation into $\mathcal{K}_\omega$ should just amount to replacing $\phi$ in such contexts by its analogue $\bar{\phi}$.

D93. If $\phi$ is a wfe of $\mathcal{K}_{i.d.}$, then <u>the translation of $\phi$ into direct discourse</u> $(\mathrm{Trans}(\phi))$ is given by the following.

(1) if m is a natural number, $\mathrm{Trans}(\mathrm{Pred}(m,o)) = \mathrm{Pred}(m,o)$

(2) if $\psi, \chi$ are wfe's of $\mathcal{K}_{1.d.}$, then Trans$(\psi \supset \chi)$ = (Trans$(\psi) \supset$ Trans$(\chi)$)

(3) if $\psi$ is a wfe of $\mathcal{K}_{1.d.}$, then Trans $(\sim\psi)$ = $\sim$(Trans$(\psi)$)

(4) if $\psi$ is a wfe of $\mathcal{K}_{1.d.}$, then Trans $(N\psi)$ = N $\overline{\text{Trans}(\psi)}$.

Every wfe of $\mathcal{K}_{1.d.}$ has as its translation a wfe of $\mathcal{K}_\omega$. The following theorem indicates that our translation is correct.

T107. If $\phi$ is a wfe of $\mathcal{K}_{1.d.}$, and $\mathcal{M} \in M_o$, then

(1) $\text{Val}_{\mathcal{M}}^{\mathcal{K}_{1.d.}}(\phi) = \text{Val}_{\mathcal{M}}^{S,\mathcal{K}_\omega}(\text{Trans}(\phi))$

(2) $\text{Val}_{\mathcal{M}}^{\mathcal{K}_{1.d.}}(\phi) = \text{Val}_{\mathcal{M}}^{I,\mathcal{K}_\omega}(\text{Trans}(\phi))$

(3) $\vdash_{\mathcal{K}_{1.d.}} \phi$ if and only if $\vdash_{S,\mathcal{K}_\omega} \text{Trans}(\phi)$

(4) $\vdash_{\mathcal{K}_{1.d.}} \phi$ if and only if $\vdash_{I,\mathcal{K}_\omega} \text{Trans}(\phi)$

(5) $\vdash_{S13} \phi$ if and only if $\vdash_{S,\mathcal{K}_\omega} \text{Trans}(\phi)$

(6) $\vdash_{S13} \phi$ if and only if $\vdash_{I,\mathcal{K}_\omega} \text{Trans}(\phi)$

Proof: (1), (2) by induction on $\phi$; (3), (4) by (1), (2); (5), (6) by (3), (4), T106, T104.

Theorem 107 provides us with another large class of sentences with respect to which our two interpretations of $\mathcal{K}_\omega$ agree.

T108. If $\phi$ = Trans$(\psi)$ for some wfe $\psi$ of $\mathcal{K}_{1.d.}$, and

$\mathfrak{M} \in M_0$, then $\mathrm{Val}_{\mathfrak{M}}^{S,\mathcal{K}_\omega}(\phi) = \mathrm{Val}_{\mathfrak{M}}^{I,\mathcal{K}_\omega}(\phi)$.

Proof by T107.

## CHAPTER 5

## THE LANGUAGE $\mathcal{L}_\omega$

A full development of the material of the preceding four chapters would provide a treatment of at least the following topics. A language $\mathcal{L}_\omega$, modeled on $\mathcal{K}_\omega$, but based on the full first order predicate calculus with identity and description, can be constructed. The comparison of $\mathcal{L}_\omega$ to an indirect discourse development of quantified modal logic raises a number of new and interesting questions. In particular, the problem of translating such languages of indirect discourse into $\mathcal{L}_\omega$ can be shown to involve far greater difficulties than those concerned with the translation of $\mathcal{K}_{i.d.}$ into $\mathcal{K}_\omega$. The translation problem leads one to consider some versions of essentialism. The notion of <u>the essence of an entity x</u> can be roughly explicated as that concept which is a concept of x in every possible state of affairs. Various essentialist interpretations of $\mathcal{L}_\omega$ are possible, with interesting relations to different systems of quantified modal logic. An investigation in this area reveals the exact extent of support for Quine's often repeated claim that quantified modal logic presupposes essentialism.

The full development of these topics exceeds certain limitations on the present work. However, the construction and intensional interpretation of $\mathcal{L}_\omega$ are so centrally related to the aim of providing a foundation for intensional logic that it seems important to include at least the following abstract of that development.

## 29. Syntactical Interpretations

The language $\mathcal{L}_\omega$ differs from $\mathcal{K}_\omega$ in the presence of variables of each simple type and the two variable binding operators: the universal quantifier, and the description operator. Thus the base language $\mathcal{L}_0$, upon which the hierarchy leading to $\mathcal{L}_\omega$ is built, is the full first order predicate calculus with identity and descriptions.

If the syntactical and intensional interpretations of the hierarchy of languages $\mathcal{K}_n$ are compared, it will be noted that the intensional interpretation offers several advantages by way of simplicity. Under the intensional interpretation Nec can be dropped as a primitive constant and reintroduced by definition as suggested in footnote 76. Under the intensional interpretation, the universes of a given type with respect to $\mathcal{K}_n$, $\mathcal{K}_{n+1}$, and $\mathcal{K}_\omega$ are all the same. Also, under the intensional interpretation, the value (denotation) and sense of a given wfe of $\mathcal{K}_n$ remain constant for

$\mathcal{K}_n$, $\mathcal{K}_{n+1}$, and $\mathcal{K}_\omega$.

The difficulties connected with the relation between the universes of a given type with respect to the syntactical interpretations of $\mathcal{K}_n$ and $\mathcal{K}_{n+1}$ would be multiplied with respect to $\mathcal{L}_\omega$, seeming to require not only a new hierarchy construction but the ramification of each higher simple type of $\mathcal{K}_\omega$ into orders. Note that whereas a variable of type $t_1$ in $\mathcal{L}_1$ takes as values any wfe of $\mathcal{L}_0$ of type t, a variable of the same type in $\mathcal{L}_2$ draws its values from the wider class of wfe's of $\mathcal{L}_1$ of type t. Hence the natural course seems to be to divide the type $t_1$ of $\mathcal{L}_\omega$ into the types $[t_1,0]$, $[t_1,1]$, $[t_1,2]$, etc., where $\mathcal{L}_{n+1}$ would contain variables of types $[t_1,0],\ldots,[t_1,n]$. The language $\mathcal{L}_\omega$ could then be, as before, the union of the languages $\mathcal{L}_n$. This problem does not arise under the intensional interpretation of $\mathcal{L}_\omega$. Further simplifications available to the intensional interpretation of $\mathcal{L}_\omega$ but denied to the syntactical interpretation concern the fact that if $\alpha$ and $\beta$ differ just by rewrite of a bound variable, $\overline{\alpha}$ and $\overline{\beta}$ must have distinct denotations under the syntactical interpretation but will have the same denotation under the intensional interpretation (since $\alpha$ and $\beta$, being logically equivalent, will have the same sense).

For the above reasons, and others, we leave the syntactical interpretation of a direct discourse language based on the full first order predicate calculus to future developments. For the present we content ourselves with an intensional interpretation of $\mathcal{L}_\omega$, hoping the contents of Chapters 2, 3, and 4 to have supplied sufficient justification for our earlier claim of an essential similarity between different direct discourse treatments as opposed to the fundamental differences between direct and indirect discourse treatments.

## 30. The Language $\mathcal{L}_\omega$

The type structure of $\mathcal{L}_\omega$ remains that of $\mathcal{K}_\omega$. The new atomic wfe's consist of an infinite supply of variables of each simple type. The variables of each type are ordered so that we can speak of <u>the $m$th variable of type $\tau$</u>. We also add two variable binding operators: <u>The Universal Quantifier</u>, and <u>The Description Operator</u>. The variable binding operators have no types, and thus are not wfe's. They are the only syncategorematic expressions of our language. Thus, the <u>atomic wfe's of $\mathcal{L}_\omega$</u> consist of all the atomic wfe's of $\mathcal{K}_\omega$ plus the variables.

The <u>well-formed expressions of $\mathcal{L}_\omega$</u> consist of the atomic wfe's of $\mathcal{L}_\omega$ plus <u>compound wfe's of $\mathcal{L}_\omega$</u> formed in one of the following three ways.

(1) If $\eta$, $\alpha_1, \ldots, \alpha_r$ are wfe's of $\mathcal{L}_\omega$ of types $\langle \tau_1, \ldots, \tau_r, \tau_0 \rangle, \tau_1, \ldots, \tau_r$ respectively, then $\eta ^\frown \alpha_1 ^\frown \ldots ^\frown \alpha_r$ is a wfe of $\mathcal{L}$ of type $\tau_0$.

(2) If $\alpha$ is a variable of $\mathcal{L}_\omega$, and $\phi$ is a wfe of $\mathcal{L}_\omega$ of type t, then The Universal Quantifier$^\frown \alpha ^\frown \phi$ is a wfe of

$\mathcal{L}_\omega$ of type t.

(3) If $\alpha$ is a variable of $\mathcal{L}_\omega$ of type $\tau$, and $\phi$ is a wfe of $\mathcal{L}_\omega$ of type t, then The Description Operator$^\frown\alpha^\frown\phi$ is a wfe of $\mathcal{L}_\omega$ of type $\tau$.

Note that all wfe's of $\mathcal{K}_\omega$ are also wfe's of $\mathcal{L}_\omega$.

We carry over all the abbreviations of D42-D47 and add the following:

(1) $\forall\alpha\phi$ = The Universal Quantifier$^\frown\alpha^\frown\phi$

(2) $\iota\alpha\phi$ = The Description Operator$^\frown\alpha^\frown\phi$

(3) $\exists\alpha\phi$ = $\sim\forall\alpha\sim\phi$

<u>The set of free variables of the wfe $\alpha$</u> is defined in the usual way. The order of the variables allows us to speak of <u>the $n^{th}$ variable of type $\tau$ not free in $\alpha$</u>. A wfe without free variables is said to be <u>closed</u>.

Our treatment of descriptions follows the method of Frege as modified by Carnap. For each simple type $\tau$ we choose <u>the distinguished constant of type $\tau$</u> in accord with the following.

(1) The distinguished constant of type $i_j$ is $\text{Opsymb}_j(o,o)$

(2) The distinguished constant of type $t_j$ is $\text{Pred}_j(o,o)$

An improper description of type $\tau$ will then have the same denotation as the distinguished constant of type $\tau$.

## 31. The Intensional Interpretation of $\mathcal{L}_\omega$

When $\mathfrak{M}\in M_o$, and $\tau$ is a type <u>the universe of $\tau$ in $\mathfrak{M}$ with respect to $\mathcal{L}_\omega$</u> ($\mathcal{U}_\mathfrak{M}(\tau)$) remains as it was for $\mathcal{K}_\omega$.

Only the closed wfe's of $\mathcal{L}_\omega$ can be said to have a denotation properly speaking. But for wfe's containing free variables we can introduce a kind of quasi-denotation relative to an assignment of values to the variables.

If $\mathcal{M} \in M_0$, then <u>$\mathcal{a}$ is an assignment to variables for $\mathcal{M}$</u> if and only if $\mathcal{a}$ is a function from the variables of $\mathcal{L}_\omega$, which assigns to each variable of type $\tau$ an element of the universe of $\tau$ in $\mathcal{M}$. If $\mathcal{a}$ is an assignment to variables for $\mathcal{M}$, $\alpha$ is a variable of type $\tau$ and x is an element of the universe of $\tau$ in $\mathcal{M}$, then $\mathcal{a}^{\alpha}_{x}$ is that assignment to variables which is exactly like $\mathcal{a}$ except possibly for $\alpha$, to which it assigns x.

When $\mathcal{M} \in M_0$, $\mathcal{a}$ is an assignment to variables for $\mathcal{M}$, and $\alpha$ is a wfe of $\mathcal{L}_\omega$, then <u>the value of $\alpha$ in $\mathcal{M}, \mathcal{a}$ with respect to the intensional interpretation of $\mathcal{L}_\omega$</u> ($\mathrm{Val}_{\mathcal{M}\mathcal{a}}(\alpha)$) is given by the following.

(1) If $\alpha$ is an atomic wfe of $\mathcal{L}_\omega$ other than a variable, then $\mathrm{Val}_{\mathcal{M}\mathcal{a}}(\alpha) = \mathrm{Val}_{\mathcal{M}}^{I,\mathcal{K}_\omega}(\alpha)$.

(2) If $\alpha$ is a variable of $\mathcal{L}_\omega$, then $\mathrm{Val}_{\mathcal{M}\mathcal{a}}(\alpha) = \mathcal{a}(\alpha)$.

(3) If $\eta, \alpha_1, \ldots, \alpha_r$ are wfe's of $\mathcal{L}_\omega$ of types $\langle\tau_1, \ldots, \tau_r, \tau_0\rangle, \tau_1, \ldots, \tau_r$ respectively, then $\mathrm{Val}_{\mathcal{M}\mathcal{a}}(\eta^\frown\alpha_1^\frown\ldots^\frown\alpha_r) = \mathrm{Val}_{\mathcal{M}\mathcal{a}}(\eta)(\mathrm{Val}_{\mathcal{M}\mathcal{a}}(\alpha_1), \ldots, \mathrm{Val}_{\mathcal{M}\mathcal{a}}(\alpha_r))$.

(4) If $\alpha$ is a variable of $\mathcal{L}_\omega$ of type $\tau$, and $\phi$ is a wfe of $\mathcal{L}_\omega$ of type t, then $\mathrm{Val}_{\mathcal{M}\mathcal{a}}(\forall\alpha\phi) = T$ if for all $x \in \mathcal{U}_{\mathcal{M}}(\tau)$, $\mathrm{Val}_{\mathcal{M}\mathcal{a}^{\alpha}_{x}}(\phi) = T$; otherwise $\mathrm{Val}_{\mathcal{M}\mathcal{a}}(\forall\alpha\phi) = F$

(5) If $\alpha$ is a variable of $\mathcal{L}_\omega$ of type $\tau$, and $\phi$ is a wfe of $\mathcal{L}_\omega$ of type t, then $\mathrm{Val}_{\mathfrak{M}\mathfrak{a}}(\iota\alpha\phi)$ = the unique $x \in \mathcal{U}_{\mathfrak{M}}(\tau)$ such that $\mathrm{Val}_{\mathfrak{M}\mathfrak{a}_x^\alpha}(\phi) = T$, if there is such a unique x; otherwise $\mathrm{Val}_{\mathfrak{M}\mathfrak{a}}(\iota\alpha\phi) = \mathrm{Val}_{\mathfrak{M}\mathfrak{a}}$(the distinguished constant of type $\tau$).

It follows easily that the value of a wfe is in the appropriate universe and that the values of the wfe's of $\mathcal{K}_\omega$ remain as they were.

## 32. Senses and Quasi-Senses

Just as we can only speak of closed wfe's as properly having denotations, we can only speak of closed wfe's as properly having senses. But on analogy to quasi-denotations we can assign quasi-senses to wfe's which contain free variables. Here we must relativize to an assignment of senses to variables. We need not introduce any new functions here since we can again make use of our assignments (of denotations) to variables. Let $\mathfrak{a}$ be an assignment to variables, then we may think of $\mathfrak{a}$ as assigning senses to variables in the following way. If $\alpha$ is the $m^{\underline{th}}$ variable of type $\tau$, and $\beta$ is the $m^{\underline{th}}$ variable of type $\tilde{\tau}$, then $\mathfrak{a}$ assigns as sense to $\alpha$, the concept $\mathfrak{a}(\beta)$.

The matter can be put more easily if we introduce one of the clause in the definition of the bar function for $\mathcal{L}_\omega$. If $\alpha$ is the $m^{\underline{th}}$ variable of type $\tau$, then $\bar{\alpha}$ is the $m^{\underline{th}}$ variable of type $\tilde{\tau}$. Now let $\mathfrak{a}$ be any assignment to variables and $\mathfrak{N}$ be any model in $M_0$. Then <u>the reduction of $\mathfrak{a}$ to $\mathfrak{N}$</u>

$(\mathcal{A}_\eta)$ is that function on variables which assigns to each variable $\alpha$, $\mathcal{A}(\bar{\alpha})(\eta)$. Note that $\mathcal{A}_\eta$ is an assignment (of denotations) to variables for $\eta$. The (quasi)-sense of a variable $\alpha$ relative to an assignment $\mathcal{A}$ will be that function which assigns to each $\eta \in M_o$, $\mathcal{A}_\eta(\alpha)$. Following this pattern, we can introduce the relativized notion of sense (or quasi-sense) as follows. If $\alpha$ is a wfe of $\mathcal{L}_\omega$ of simple type, and $\mathcal{A}$ is an assignment to variables, then <u>the sense of $\alpha$ relative to $\mathcal{A}$</u> ($\text{Sense}_{\mathcal{A}}(\alpha)$) is that function on $M_o$, which assigns to each $\eta \in M_o$, $\text{Val}_{\eta\,\mathcal{A}_\eta}(\alpha)$. If $\eta$ is a wfe of $\mathcal{L}_\omega$ of complex type, then $\eta$ must be an atomic wfe of $\mathcal{K}_\omega$. Hence $\eta$ cannot contain any free variables, and the sense of $\eta$ can be as it was for $\mathcal{K}_\omega$.

If $\alpha$ is a closed wfe of $\mathcal{L}_\omega$, its value in a model is unaffected by the choice of assignment and similarly its quasi-sense is unaffected by the choice of assignment. Thus for such wfe's we will sometimes speak simply of their sense (rather than quasi-sense). The above definitions imply that the senses of all wfe's of $\mathcal{K}_\omega$ remain as they were.

## 33. The Bar Function in $\mathcal{L}_\omega$

We now extend the definition of the bar function to cover all wfe's of $\mathcal{L}_\omega$. The possibility of defining $\overline{\forall \alpha \phi}$ and $\overline{\iota \alpha \phi}$ without introducing two whole hierarchies of variable binding operators (namely, the $j^{\text{th}}$ analogue to The Universal Quantifier, and the $j^{\text{th}}$ analogue to The Description Operator, for each $j$) very greatly simplifies the structure of $\mathcal{L}_\omega$.[79] A few fundamental theorems are stated without proof.

The following defines <u>the analogue to a wfe $\alpha$ of $\mathcal{L}_\omega$</u> ($\overline{\alpha}$).

(1) If $\alpha$ is an atomic wfe of $\mathcal{K}_\omega$, $\overline{\alpha}$ remains as before.

(2) If $\alpha$ is the $m^{\underline{th}}$ variable of $\mathcal{L}_\omega$ of type $\tau$, then $\overline{\alpha}$ is the $m^{\underline{th}}$ variable of $\mathcal{L}_\omega$ of type $\tilde{\tau}$.

(3) If $\eta^\frown\alpha_1{}^\frown\ldots{}^\frown\alpha_r$ are wfe's of $\mathcal{L}_\omega$ of types $<\tau_1,\ldots,\tau_r,\tau_0>$, $\tau_1,\ldots,\tau_r$ respectively, then $\overline{\eta^\frown\alpha_1{}^\frown\ldots{}^\frown\alpha_r}$ is $\overline{\eta}^\frown\overline{\alpha}_1{}^\frown\ldots{}^\frown\overline{\alpha}_r$.

(4) If $\alpha$ is a variable of $\mathcal{L}_\omega$, and $\emptyset$ is a wfe of $\mathcal{L}_\omega$ of type t, then

$$\overline{\forall\alpha\emptyset} \text{ is } \iota\beta_1(\forall\overline{\alpha}N(\beta_1\supset_1\overline{\emptyset})\ \& \\ \forall\gamma_1(\forall\overline{\alpha}N(\gamma_1\supset_1\overline{\emptyset})\supset N(\gamma_1\supset_1\beta_1))\ ),$$

where $\beta_1$ and $\gamma_1$ are respectively the first and second variables of type $t_1$ which are not free in $\emptyset$ and which are different from $\overline{\alpha}$ (if $\overline{\alpha}$ happens to be of type $t_1$).

(5) If $\alpha$ is a variable of $\mathcal{L}_\omega$ of type $\tau$, and $\emptyset$ is a wfe of $\mathcal{L}_\omega$ of type t, then

$\overline{\iota\alpha\emptyset}$ is

$$\iota\beta N(\overline{\forall\alpha(\emptyset\equiv(\alpha=\beta))\vee(\sim\exists\beta\forall\alpha(\emptyset\equiv(\alpha=\beta))\ \&\ (\beta=\gamma))}\ ),$$

where $\beta$ is the first variable of type $\tau$ which is not free in $\emptyset$ and which is different from $\alpha$, and $\gamma$ is the distinguished constant of type $\tau$.

It is beyond the scope of this abstract to give a detailed proof of the adequacy of clauses (4) and (5) above, but some insight into the idea involved may be gained from the following remarks. The argument, of course, is by induction. The right hand side of (4) may be read as follows: "the proposition which is first, true only in models for which $\overline{\emptyset}$ denotes a true proposition under every assignment to $\overline{\alpha}$, and second, the strongest proposition satisfying the first condition". Also, from the inductive hypothesis, it can be shown that the wfe $\forall \alpha \emptyset$ is true in $\mathcal{M}$ if and only if for all assignments of a concept of the appropriate kind to $\overline{\alpha}$, $\overline{\emptyset}$ denotes a proposition which is true in $\mathcal{M}$. The right hand side of (5) is formed by taking the wfe $(\iota \alpha \emptyset = \beta)$ and eliminating the description. This results in an equivalant wfe $\Gamma$. Hence, $\iota \overline{\beta} N \overline{\Gamma}$ (the right hand side of (5)) is equivalent to $\iota \beta N(\overline{\iota \alpha \emptyset = \beta})$, which by the individuating principle for intensions is equivalent to $\iota \overline{\beta}\ (\overline{\iota \alpha \emptyset} = \overline{\beta})$, which is equivalent to $\overline{\iota \alpha \emptyset}$. A more systematic development leads to the following fundamental theorem on the adequacy of $\mathcal{L}_\omega$.

T109. If $\alpha$ is a wfe of $\mathcal{L}_\omega$, $\mathcal{M} \in M_0$, and $\mathcal{Q}$ is an assignment to variables for $\mathcal{M}$, then $\mathrm{Val}_{\mathcal{M}\mathcal{Q}}(\overline{\alpha}) = \mathrm{Sense}_{\mathcal{Q}}(\alpha)$.

Note also that although $\mathcal{L}_\omega$ is closed under the bar function, that is, every wfe of $\mathcal{L}_\omega$ has an analogue in $\mathcal{L}_\omega$; $\mathcal{L}_\omega$ does not satisfy the condition of theorem 67 with respect to $\mathcal{K}_\omega$. In fact, not only does $\mathcal{L}_\omega$ contain wfe's of type $\tilde{\tau}$ which do not have the form $\overline{\alpha}$, it even contains wfe's of type $\tilde{\tau}$ which are not equivalent to any wfe of the form $\overline{\alpha}$.

If we recall that $\overline{\alpha}$ is a standard name of the concept expressed by $\alpha$, we may call the wfe's alluded to above <u>contingent names of concepts</u>. For example, let P be Pred(o,o) and let $p_1$ be a variable of type $t_1$. Then $\iota p_1((P \& (p_1 \equiv \overline{(P \supset P)}))\vee(\sim P \& (p_1 \equiv \overline{\sim(P \supset P)})))$ denotes the necessary proposition if P is true, and the impossible proposition if P is false. If we call the wfe in the example "$\phi_1$", we see that $N\phi_1$ is equivalent to P. Hence, in contrast to theorem 93 for $\mathcal{K}_\omega$, neither $N\phi_1$ nor $\sim N\phi_1$ is valid. The availability of such wfe's as $\phi_1$ in $\mathcal{L}_\omega$ allows us to now treat informal arguments closely related to that given in section 8 in connection with the comparison of direct and indirect discourse treatments of obliquity.

## 34. The Incompleteness of $\mathcal{L}_\omega$

The valid wfe's of $\mathcal{L}_\omega$ under the intensional interpretation do not form a recursively enumerable set. Thus, according to the usual notions of axiomatization, they are not axiomatizable. The proof of non-axiomatizability is simple. If $\phi$ is a sentence of the first order predicate calculus (or, more exactly, the sublanguage of $\mathcal{L}_\omega$, $\mathcal{L}_o$) and $\phi$ is not valid, then $\sim N\overline{\phi}$ is valid under the intensional interpretation of $\mathcal{L}_\omega$ (see theorem 114(8) below). Therefore a complete axiomatization would allow us to enumerate both the validities and invalidities of first order logic, thereby providing a decision procedure contrary to Church's theorem.

## 35. Some Valid Formulas of $\mathcal{L}_\omega$

All the usual logical principles of the first order predicate calculus with identity and descriptions are available in $\mathcal{L}_\omega$, subject only to the restriction that the formulas in question be well formed according to our type rules. Leibniz' Law (in fact, Frege's Law), existential generalization, universal instantiation (sometimes called "specification"), are all valid irrespective of the occurrence or placement of modal signs. In addition, if $\phi$ is a valid formula of $\mathcal{L}_\omega$, so are $N\phi$ and $\forall\alpha\,\phi$. Thus modal generalization and universal generalization are both validity preserving inference rules. Modus ponens, of course, remains a truth preserving inference rule. For those who treat extensionality as a property of theories which obey Frege's Law (in its many forms), we can show that $\mathcal{L}_\omega$, under the intensional interpretation, is fully "extensional".

Corresponding to the set of Delta Axioms of $\mathcal{K}_\omega$, we now state two separate schemes. The complication is due to the fact that a wfe may contain free variables. If $\alpha$ is a variable, $\overline{\alpha}$ is simply another variable. No special requirement is placed upon assignments to variables which would make $\Delta(\overline{\alpha},\alpha)$ valid in such cases. Thus, the counterpart to the Delta Axioms requires an additional hypothesis. As before, we will use the symbol "$\models_{I,\mathcal{L}_\omega}$" to indicate validity under the intensional interpretation of $\mathcal{L}_\omega$.

T110. If $\alpha$, $\beta$ are wfe's of $\mathcal{L}_\omega$ of the same simple type, then

(1) if the free variables of $\alpha$ are $\delta_1, \ldots, \delta_n$, then
$\vdash_{I,\mathcal{L}_\omega} (\Delta(\overline{\delta_1}, \delta_1) \& \ldots \& \Delta(\overline{\delta_n}, \delta_n) \supset (\Delta(\overline{\alpha}, \beta) \equiv (\alpha = \beta)))$

(2) if $\alpha$ is closed, then $\vdash_{I,\mathcal{L}_\omega} (\Delta(\overline{\alpha}, \beta) \equiv (\alpha = \beta))$

The loss of theorem 67 requires separate statement of the principles that (1) Delta($\widetilde{\tau}$) denotes a function, (2) there are no empty concepts, and (3) everything falls under at least one concept.[80]

T111. If $\beta, \gamma$ are variables of $\mathcal{L}_\omega$ of type $\tau$, and $\alpha_1$ is a variable of $\mathcal{L}_\omega$ of type $\widetilde{\tau}$, then

(1) $\vdash_{I,\mathcal{L}_\omega} ((\Delta(\alpha_1, \beta) \& \Delta(\alpha_1, \gamma)) \supset (\beta = \gamma))$

(2) $\vdash_{I,\mathcal{L}_\omega} \forall \alpha_1 \exists \beta \Delta(\alpha_1, \beta)$

(3) $\vdash_{I,\mathcal{L}_\omega} \forall \beta \exists \alpha_1 \Delta(\alpha_1, \beta)$

The principle of individuation for concepts can be expressed in $\mathcal{L}_\omega$ in the same form as in $\mathcal{K}_\omega$.

T112. If $\alpha, \beta$ are variables of $\mathcal{L}_\omega$ of the same type, then
$\vdash_{I,\mathcal{L}_\omega} (N(\overline{\alpha = \beta}) \equiv (\overline{\alpha} = \overline{\beta}))$

In connection with the preceding theorem note that $(\overline{\alpha = \beta})$ is simply $(\overline{\alpha} =_1 \overline{\beta})$, and that the whole wfe can be universally generalized with respect to the variables $\overline{\alpha}, \overline{\beta}$.

As indicated in footnote 76, Nec could be introduced in $\mathcal{L}_\omega$ by an axiom of definitional form. This fact is verified in the following theorem.

T113. Let $p_1$ be the first variable of type $t_1$, and let P be Pred(o,o), then $\vdash_{I,\mathcal{L}_\omega} (Np_1 \equiv (p_1 = (\overline{P \supset P})))$

Of the six kinds of modal axioms of $\mathcal{K}_\omega$, only the first can be generalized to $\mathcal{L}_\omega$ without adding some restrictive hypothesis (for example, that the wfe's are closed).

T114. If $\phi, \psi$ are wfe's of $\mathcal{L}_\omega$ of type t, then

(1) $\models_{I,\mathcal{L}_\omega} (N(\overline{\phi \supset \psi}) \supset (N\overline{\phi} \supset N\overline{\psi}))$

(2) if $\phi$ is closed, then $\models_{I,\mathcal{L}_\omega} (N\overline{\phi} \supset \phi)$

(3) if the free variables of $\phi$ are $\delta_1, \ldots, \delta_n$, then $\models_{I,\mathcal{L}_\omega} ((\Delta(\overline{\delta}_1, \delta_1) \& \ldots \& \Delta(\overline{\delta}_n, \delta_n)) \supset (N\overline{\phi} \supset \phi))$

(4) $\models_{I,\mathcal{L}_\omega} (N\overline{\phi} \supset Tr\ \overline{\phi})$

(5) if $\phi$ is closed, then $\models_{I,\mathcal{L}_\omega} (N\overline{\phi} \supset N\overline{N\overline{\phi}})$

(6) if $\phi$ is closed, then $\models_{I,\mathcal{L}_\omega} (\sim N\overline{\phi} \supset N\overline{\sim N\overline{\phi}})$

(7) if $\phi$ is closed, and $\psi$ is an instance $\phi$ obtained by proper substitution on predicates and operation symbols of $\mathcal{K}_0$, then $\models_{I,\mathcal{L}_\omega} (N\overline{\phi} \supset N\overline{\psi})$

(8) if $\phi$ is closed, and not $\models_{I,\mathcal{L}_\omega} \phi$, then $\models_{I,\mathcal{L}_\omega} \sim N\overline{\phi}$.

Finally, we come to some principles which combine modality with quantification. Theorem 115 asserts the validity of some principles whose indirect discourse counterparts have frequently been assumed by modal logicians.

T115. If $\phi$ is a wfe of $\mathcal{L}_\omega$ of type t, and $\alpha$ is a variable of $\mathcal{L}_\omega$, then

(1) $\models_{I,\mathcal{L}_\omega} (\forall\overline{\alpha}\, N\overline{\phi} \equiv N\overline{\forall\alpha\phi})$

(2) $\models_{I,\mathcal{L}_\omega} (\exists\overline{\alpha}\, N\overline{\phi} \supset N\overline{\exists\alpha\phi})$

The following theorem asserts the validity of a principle to which, it seems, no previous modal logician has given assent.[81]

T116. If $\emptyset$ is a wfe of $\mathcal{L}_\omega$ of type t, and $\alpha$ is a variable of $\mathcal{L}_\omega$, then $\vdash_{I,\mathcal{L}_\omega} (N\exists\alpha\emptyset \supset \exists\bar{\alpha}N\bar{\emptyset})$

The proof of theorem 116 goes roughly as follows. Assume that $N\exists\alpha\emptyset$ is true, then $\exists\alpha\emptyset$ is true in every model. Hence, for each model $\mathcal{N}$ there is some entity $\mathcal{N}_\alpha$ which when taken as the value of the variable $\alpha$ makes $\emptyset$ true in $\mathcal{N}$. Let f be that function on $M_o$ which assigns to each $\mathcal{N} \in M_o$, $\mathcal{N}_\alpha$. Note that f is a concept; in fact, f is in the universe of the type of $\bar{\alpha}$. But now if f is taken as the value of $\bar{\alpha}$, $\bar{\emptyset}$ will denote the necessary proposition; hence $N\bar{\emptyset}$ will be true, and therefore $\exists\bar{\alpha}N\bar{\emptyset}$ is true.

Let us now look at the kind of argument that might be given for rejecting the principle of theorem 116. One might argue that although in every model $\mathcal{N}$ there is <u>something</u> which when taken as the value of $\alpha$ makes $\emptyset$ true in $\mathcal{N}$, that thing might vary from model to model. Therefore it might still be the case that no <u>single thing</u> would be such that in every model it had the property expressed by $\emptyset$. When put in this way the presupposition of the argument comes clearly to light. The variable $\bar{\alpha}$ is to range only over concepts which pick out the same thing in each model. Thus the universes of concepts are to contain only <u>essences</u>. This is essentialism.

NOTES

1. We will use the following expressions in speaking about functions. Let f be a function, then the set of entities to which f is applicable is called the domain of f, the set of values obtained by applying f to an element of its domain is called the range of f. If f is a two-place function which is applicable to any pair of entities <z,y> when z is an element of A and y is an element of B, then A is called the first domain of f and B is called the second domain of f. Functions of more than two places are treated similarly. Alternatively, we may use the notation "(AxB)" for the set of all couples <z,y> where z is an element of A and y is an element of B, and say of a two-place function whose first domain is A and whose second domain is B that its domain is (simply) (AxB). If f is any function whose domain is A and whose range is included in C (that is, every element of the range of f is an element of C although C may not coincide with the range of f), we sometimes say that f is a function from A into C. Thus, if f is that formation rule which assigns to every pair of sentences their conjunction, and A is the set of all sentences, then we would say that f is a function from (AxA) into A. A function is said to be 1-1 if it always assigns distinct elements of its range to distinct elements of its domain.

2. The term "construction" is taken from the abstract Curry [1].

3. Church [7], p. 32.

4. In general, if we identify a type with the set of wfe's having that type, E, T and A are determined by F alone as follows: E = the union of the domains of the elements of F, $T(\alpha)$ = the domain of any element of F which has $\alpha$ in its domain, $A = E^*$ minus the union of the ranges of elements of F. The only requirement seems to be that every wfe occur as a part of some other wfe.

5. In particular, for certain metamathematical purposes and in the field of pragmatics, it is often natural to impose additional recursivity conditions, for example, that there be an effective way of determining the type of each wfe.

6. We adopt the practice of using "part" for elements with respect to the structure with which we are primarily concerned, and "piece" for elements with respect to other possible structures. Hence, what is a part from one point of view is merely a piece from another. Those pieces of expressions which are semantical elements, we call parts and we imagine the formation rules of a language to be constructed so as to reflect these ideas. Thus expressions which are thought of as syncategeramatic would not occur as wfe's, hence not as a part of any wfe though of course they may be pieces of wfe's.

7. Note that we can define the relevant substitution notion as follows. Let C be a construction of $\beta$. Then S (which we call the structure of C) is to be a sequence of the same length as C and such that if $C_n$ is atomic, $S_n = n$; if $C_m$ is the result of applying a formation rule f to $\langle C_{m_1}, \ldots, C_{m_j} \rangle$, $S_n = \langle f, m_1, \ldots, m_j \rangle$ (our earlier requirements assure that S is unique). $\delta$ is now said to be the result of substituting one or more occurrences of $\gamma$ for $\alpha$ in $\beta$, just in case $\delta$ can

be obtained as the last element of some sequence D where: D has the same length as C; if $S_n = n$ and $C_n \neq \alpha$, $D_n = S_n$; if $S_n = \langle f, m_1, \ldots, m_j \rangle$ and $C_n \neq \alpha$, $D_n = f(D_{m_1}, \ldots, D_{m_j})$; if $S_n = \alpha$, $D_n = \alpha$ or $D_n = \gamma$; for some k such that $C_k = \alpha$, $D_k = \gamma$.

This notion of substitution, given in terms of the structure of a wfe, has certain advantages over more familiar forms in that it automatically takes account of idiosyncracies in the design of the wfe's. For example, a formation rule f which when applied to a two-place predicate and two terms yields the corresponding formula may have the feature that f('F' 'x' 'y') = 'F(xy)' and f('=' 'x' 'y') = '(x = y)'. It is clear that the most useful substitution notion would have the substitution of 'F' for '=' in '(x = y)' be 'F(xy)' (that is, substitution of 'F' for '=' in f('F', 'x', 'y') is f('F', 'x', 'y')) rather than '(xFy)'.

8. Carnap [4], pp. 121, 122.

9. The best expositions known to us of Frege's semantical ideas are to be found in Carnap [4], especially pp. 118-133, and Church [7], especially pp. 3-31. In the latter, the inchangeability principles are mentioned on pp. 8, 9. The present discussion involves a considerable generalization and development of what is explicitly found in Frege. In addition, we deviate from Frege in a few points, especially with regard to his notions of an unsaturated (ungsättigt) expression and a function.

10. Our general notions of a language and a Fregean semantical system can be given so as to apply to languages containing variables and arbitrary variable binding operators. However, certain additional subtleties are thereby required so that it seems best

for purposes of the present informal discussion to think of our languages as not containing variables.

11. The first precise development of semantics occurs in Tarski [1] where the notion of satisfaction is introduced and shown to be a fundamental concept of semantics.

12. This point is due to Frege [3].

13. It should be noted that even the restriction suggested in footnote 10, that we exclude variables from the languages presently under consideration, does not prohibit us from introducing a constant desc, such that if $\alpha, \beta$ are wfe of the type name and $\emptyset$ a wfe of the type sentence, then $g(\alpha, \beta, \emptyset) = \mathrm{desc}^\frown\text{'(}\ulcorner\alpha^\frown\beta^\frown\emptyset^\frown\text{')'}$, $g(\alpha, \beta, \emptyset)$ is a wfe of the type name, and $\mathrm{rden}(g(\alpha, \beta, \emptyset))$ = the unique x such that either $\emptyset$ is true and $x = \mathrm{rden}\ (\alpha)$ or $\emptyset$ is false and $x = \mathrm{rden}\ (\beta)$. Expressions of this form will play an important role in our constructions. Many expressions of English might plausibly be claimed to have this form. For example, statements about the future where there are only two possibilities ('The next president' uttered after the nominations), statements of the form 'the other ...', etc.

14. Note that C1 still allows other possibilities. For example, we may divide the sentences into four equivalence classes: true sentences with names as parts ($T^+$), true sentences without names as parts ($T^-$), false sentences with names as parts ($F^+$), false sentences without names as parts ($F^-$). Corresponding to such formation rules as cond ($\mathrm{cond}(\emptyset, \psi) = \ulcorner(\emptyset \supset \psi)\urcorner$) we would then take the semantical operation with the truth table

| $R(\phi)$ | $R(\psi)$ | cond* $(R(\phi)R(\psi))$ |
|---|---|---|
| $T^+$ | $T^+$ | $T^+$ |
| $T^+$ | $T^-$ | $T^+$ |
| $T^+$ | $F^+$ | $F^+$ |
| $T^+$ | $F^-$ | $F^+$ |
| $T^-$ | $T^+$ | $T^+$ |
| $T^-$ | $T^-$ | $T^-$ |
| $T^-$ | $F^+$ | $F^+$ |
| $T^-$ | $F^-$ | $F^-$ |
| $F^+$ | $T^+$ | $T^+$ |
| $F^+$ | $T^-$ | $T^+$ |
| $F^+$ | $F^+$ | $T^+$ |
| $F^+$ | $F^-$ | $T^+$ |
| $F^-$ | $T^+$ | $T^+$ |
| $F^-$ | $T^-$ | $T^-$ |
| $F^-$ | $F^+$ | $T^+$ |
| $F^-$ | $F^+$ | $T^-$ |

15. The argument given above that sentences of differing truth value can not have the same denotation (in languages of appropriate complexity) uses a construction from Church [1]. That the language contains no oblique contexts is implicitly assumed therein.

16. Frege seems to have at times believed (see, for example, Frege [1]) that in an expression like 'Fa' the denotation of 'F' must be something incomplete or "unsaturated" which when placed in proximity with the object denoted by 'a' immediately absorbs it and forms a new object (here, a truth value). One of his main arguments (given at the end of Frege [2]) was that if 'F' in 'Fa' simply denoted an object (for example, a set of ordered couples) then there is nothing to hold the compound expression 'Fa' together, that is, 'Fa' would not be a <u>sentence</u> with a single denotation but rather just a string of expressions each with its own

denotation like 'John, Fred, Bob'. A similar argument is given in Church [7], pp. 32-35, against the possibility of eliminating all connectives (but compare his footnotes 87 and 90). Here we see again the difficulties involved in attempting a semantical analysis simply with reference to the expressions themselves and ignoring what we have taken as the essential ingredient in a language, namely the structure on the expressions. If the compound expression 'Fa' is simply thought of as a token obtained by pushing the token 'F' next to the token 'a', the unity of 'Fa' is lost. Similarly, once one realizes that the expression 'Fa' is a new abstract object, related to the abstract objects 'F' and 'a' by certain formation operations, there seems no longer any reason for the denotation of 'Fa' to be obtained simply by pushing the denotations of 'F' and 'a' together. Rather, we would expect the denotation of 'Fa' to be a new object related to the denotations of 'F' and 'a' by certain regular semantical operations.

17. Interestingly enough, for languages of appropriate complexity an extension along the latter lines would again give us the relation of denotation. However, this symmetry does not hold in general for extensions in accord with C1 and C2.

18. We gave C1 and C2 in application to the extension to sentences of the particular semantical relation restricted-denotation. However, the general form of our criteria for an arbitrary relation being extended to an arbitrary type of wfe is clear.

19. Church, although he apparently conceives of the matter much as we do, adopts the practice of speaking of sentences as names. He writes, "An important advantage of regarding sentences as names is that all the ideas

and explanations of §§01-03 can then be taken over at once and applied to sentences, and related matters, as a special case. Else we should have to develop independently a theory of the meaning of sentences; and in the course of this, it seems, the developments of these three sections would be so closely paralleled that in the end the identification of sentences as a kind of names (though not demonstrated) would be very forcefully suggested as a means of simplifying and unifying the theory. In particular we shall require variables for which sentences may be substituted, forms which become sentences upon replacing their free variables by appropriate constants, and associated functions of such forms--things which, on the theory of sentences as names, fit naturally into their proper place in the scheme set forth in §§02-03. (Church [7], p. 24; underlining added.) It is our feeling that such a practice, even after the underlined remark, tends to provide an already unfamiliar theory with an unnecessarily exotic flavor which is not likely to facilitate its acceptance.

20. On Frege's terminology and its English forms see Carnap [4], p. 118, footnote 21. Also, in this connection, see Church [4], p. 47, lines 11-14, on the rendering of 'Gedanke' as 'proposition' rather than the misleading 'thought' adopted in Frege [4].

21. Thus obliquity is indicated by a failure of extensionality.

22. For an early source see Quine [3]. For one of the most recent and richest sources see Quine [6].

23. For example, the attitude of Quine [2] (1947) "When modal logic is extended to include quantification theory, . . . serious obstacles to interpretation are encountered" is echoed in Quine [7] (1961) "confusion

of use and mention . . . seems to be a sustaining force [for modal logic], engendering an illusion of understanding."

24. It is often said that in the development of mathematics no non-extensional contexts arise. On the contrary, the fundamentally important incompleteness result of Gödel [2] turns on his discovery of a precise treatment of such contexts as ⌜the sentence S says that $\phi$⌝. Numerous "philosophical interpretations" of his result have foundered on the non-extensionality of this context and the subtlety of Gödel's (extensional) treatment.

25. Carnap [4], p. 141. Here as in other formulations (for example, Carnap [1], §67) there is a proviso to the effect that only non-extensional systems useful for scientific purposes are included. For the view that there are no such systems see Quine [6], especially §45.

26. This insistence may take one of the following forms:
   (a) We may insist that since planets have no syllables, (1) and (2) both denote zero.
   (b) We may insist that although (1) and (2) are grammatically correct, as a matter of fact, they have no denotation. (Thus, assimilating (1) and (2) to: "the number of feathers in the wings of Pegasus".)
   (c) We may insist that (1) and (2) are grammatically ill-formed, and hence meaningless.

27. Perhaps Frege was led to this position by his strong insistence on Frege's principle. Indeed, something close to that requirement seems almost to be a precondition of any semantical treatment that we would call an "analysis".

28. Note that in view of the possibility of ambiguities, we must specify more exactly our use of such terms as "oblique", "extensional." In general we use these terms with respect to the ordinary denotation of expressions. Thus we shall continue to speak of (1) and (2) as indicating that the context is oblique.

29. It is interesting to note that in Carnap [1] (1934, especially §§63-71) we already find a fairly extensive analysis of oblique contexts in terms of ambiguity, here considering the specific case of an expression denoting itself. The possibility of analyzing all oblique contexts in this way is made the basis for the thesis of extensionality. There follows an illuminating discussion of indirect and direct discourse (referred to by Carnap as the "quasi-syntactical" and "syntactical" methods) with special reference to the logic of modalities.

30. Another method of treating oblique contexts is just to eliminate them as contexts (as in the parenthetical remark about 'Hesperus' not being a part of 'Hesperus$_1$'). We may preserve the symbolic form, but avoid the difficulties by restricting the part-whole relation. However, since such a method completely eliminates the piece from the field of any transformation rules, it does not in general provide an adequate treatment.

31. This argument is due essentially to Church [6], footnote 23. But there, as in commentaries such as Myhill [1], p. 79, it is not clearly emphasized that direct versus indirect discourse is at the heart of the matter.

32. The numerals enclosed in parentheses are introduced as abbreviations to refer to the expressions they enumerate. Thus, (9) = 'John's favorite sentence is

necessary'. The displayed expressions function autonomously.

33. See, for example, Quine [7], p. 329, in which Quine admits (and corrects) the error, but incorrectly attributes it to Church.

34. For an example of the latter kind see the assertion in Smullyan [1], p. 31, that the truth of "There is an x, such that x numbers the planets and it is necessary that x is greater than 6." is simply a matter of "brute fact".

35. For example, in Church [5]. The distinction between direct and indirect discourse seems to have been little noticed, the partisans of each taking their method for granted.

36. Such systems were first developed in Carnap [3], [4] and Barcan [1].

37. The terms "complex" and "compound" are not used interchangeably. Well formed expressions are either atomic (that is, without parts) or <u>compound</u> (that is, having parts). Types are either simple or <u>complex</u>. Compound wfe's are constructed by combining a wfe of complex type with one or more wfe's of simple type. The wfe's of complex type include p-place predicates and operation symbols where p>0, and sentential connectives. All wfe's of complex type will be atomic.

38. Thus, the simple types are just numbers. But the notation 'i' and 't' seems somehow less confusing than the use of numerals.

39. We will later introduce further atomic wfe's, for example, $\mathrm{Opsymb}_j(m,p)$ for arbitrary natural numbers j. There, as here, we tacitly make the natural assumptions about distinctness of atomic wfe's.

40. The identity sign for truth values is simply the material biconditional, that is, the familiar two-place sentential connective often written '$\equiv$'. It is clearly natural to have an identity predicate for each universe of entities. Our method of assigning a type to each constant (including logical signs) requires that we use distinct identity signs for distinct universes.

41. Thus we exclude such expressions as desc of note 13, which would have type $\langle i,i,t,i\rangle$.

42. We write 'T' for 'Truth' and 'F' for 'Falsehood'.

43. Another method would assign the type i also to the analogues of wfe's of types i and t, and introduce two one-place predicates, say "$T^{i_1}$" and "$T^{t_1}$", applicable to all wfe's of type i, which would mark the difference otherwise indicated by the types $i_1$ and $t_1$. In this way we would obtain a first order theory. (Such reductions of many sorted theories to first order theories have been discussed in Montague [1], Quine [5], and Wang [1]). However, in the present case, "$T^{i_1}$" and "$T^{i_2}$" would have to be treated as logical signs, and thus we could not be assured of one of the main benefits of such formalization, the completeness theorem of Gödel [1] (at least not under our intended interpretations). Also, since the type distinctions in $\mathcal{K}_1$ reflect certain intuitive ontological distinctions in the subject matter, the form of language in which wfe's of types i, $i_1$, and $t_1$ are distinguished provides a useful device for first investigating the foundations of intensional logic.

44. The same purpose could be achieved by having $\bar{\eta}$ denote $\eta$. But this would involve introducing semantical operations of other forms than the familiar application

of function to argument. It also violates our simple understanding of wfe's of complex type as denoting functions, or alternatively it leads us to introduce $\eta$ as having a type distinct from but parallel to the type $\langle\tau_1,\ldots,\tau_r,\tau_0\rangle$ and to introduce new formation rules to accommodate such wfe's.

45. Tarski [3], p. 156.

46. Church suggests how we may partially express the principle in the language $\mathcal{K}_1$. Let $\alpha$ and $\beta$ be wfe's of $\mathcal{K}_0$ of the same simple type $\tau$ (that is, both of type i or both of type t). Let us write '$(\alpha = \beta)$' for '$\mathrm{Id}(\tau)^\frown\alpha^\frown\beta$'. To say that $\alpha$ and $\beta$ are logically equivalent is to say that $(\alpha = \beta)$ is valid. This is to say that the proposition expressed by $(\alpha = \beta)$, namely the denotation of $\overline{(\alpha = \beta)}$ under the intensional interpretation of $\mathcal{K}_1$, is the necessary proposition. In other words, writing '$\mathrm{N}\phi_1$' for $\mathrm{Nec}^\frown\phi_1$', $\mathrm{N}\overline{(\alpha = \beta)}$ is true. We can easily express the fact that $\alpha$ and $\beta$ have the same sense, by $(\overline{\alpha} = \overline{\beta})$. Hence, writing '$(\phi \equiv \psi)$' for '$\mathrm{Id}(t)^\frown\phi^\frown\psi$', we can express the principle of individuation by:

$$(\mathrm{N}\overline{(\alpha = \beta)} \equiv (\overline{\alpha} = \overline{\beta}))$$

In view of our definition of the bar function, $\overline{(\alpha = \beta)} = \mathrm{Id}_1(\tau)^\frown\overline{\alpha}^\frown\overline{\beta}$. Therefore the conditional from right to left is merely an instance of Leibniz' Law plus the principle: $\mathrm{N}\overline{(\alpha = \alpha)}$. Hence, writing '$(\phi \supset \psi)$' for '$\mathrm{Cond}^\frown\phi^\frown\psi$', the essence of our principle of individuation is:

$$(\mathrm{N}\overline{(\alpha = \beta)} \supset (\overline{\alpha} = \overline{\beta}))$$

47. An interesting discussion of this point may be found in Carnap [6], §9, VIII.

48. The method would be to narrow the relation of logical equivalence by admitting "logically impossible" models,

that is, models which, for example, assign variant truth functions to Neg so that $\phi$ and $\mathrm{Neg}^\frown\mathrm{Neg}^\frown\phi$ no longer have the same value in every model, and hence are no longer logically equivalent.

49. If we were to enrich our language by the addition of variables and quantifiers, we could express the fact that $\bar{\alpha}$ denotes an empty concept as follows:

$$\sim \exists x \Delta(\bar{\alpha}, x)$$

where we write '$\sim\phi$' for '$\mathrm{Neg}^\frown\phi$' and '$\Delta(\bar{\alpha}, \beta)$' for Delta $(i_1)^\frown\bar{\alpha}^\frown\beta$.

This possibility immediately suggests a treatment of "existence". Namely, to translate a sentence of the form, "Pegasus exists." by $(\exists x)\Delta(\overline{\text{"Pegasus"}}\ x)$, and "Pegasus doesn't exist." by $\sim(\exists x)\Delta(\overline{\text{"Pegasus"}}\ x)$. Thus, we consider the context $\ulcorner\alpha$ exists$\urcorner$ as oblique. "Pegasus" might then be considered to be denotationless, and similarly for any compound expressions of which it (though not, of course, $\overline{\text{"Pegasus"}}$) is a part. Hence if $\alpha$ is a name, the sentence $(\exists x)(\alpha = x)$ will be either logically true or denotationless according as $(\exists x)\Delta(\bar{\alpha} x)$ is true or false. Such a treatment makes "existence" a predicate, but of concepts rather than individuals.

50. According to alternative (1:5), if $\alpha$ is any name of type i and x is a variable of the same type

(3) $\quad \exists x\ (x = \alpha)$

is valid. But if $y_1$ is a variable of type $i_1$,

(4) $\quad \forall y_1 \exists x \Delta(y_1, x)$

is not valid. Note that according to proposal (1), (3) may fail for names of type i; and according to proposal (2), (4) is valid.

51. In Carnap [4] (1947), p. 181.

52. Carnap defines a proposition as a set of states of affairs. Our notion may be thought of as the characteristic function of a proposition in his sense. The present notion fits more immediately into a unified treatment.

53. Another aim, secured by adopting the method of direct discourse, is that the "intensional" object languages should be completely extensional (in the sense of section 6).

54. In Tarski [3] (1950) the notions of arithmetical class and arithmetical function are introduced. These notions, which correspond to propositions and properties which are expressible in first-order languages, have proved quite fruitful in the investigation of mathematical systems.

55. Other analyses are possible. For example, we might identify states of affairs with ordered couples consisting of a number and a model. This would avoid the intuitive objection to the identification of states of affairs with models, namely, that intuitively distinct states might determine the same model of $\mathcal{K}_0$. Thus, a proposition which is not expressible in $\mathcal{K}_0$ might hold in one state and fail in another, both of which determine the same model of $\mathcal{K}_0$. Actually, our method is at least partially immune to this objection. For suppose, what seems reasonable, that in accordance with Leibniz' principle of identity of indiscernibles we identify "completely isomorphic" states (in some intuitive sense of "completely isomorphic" adequate to Leibniz' principle). Then each state of affairs determines not a unique model $\mathfrak{M}$ of $\mathcal{K}_0$ but rather an isomorphism class M of such models. Hence, since we do not identify isomorphic models, our method can be thought of as representing each state of affairs by a

unique element of the isomorphism class which it determines. Accordingly, we will admit propositions (and other intensional entities) which have different values for isomorphic models.

56. Church [6], p. 22.

57. In particular, there has recently appeared some interesting work interpreting necessity in terms of relations between models. These relations determine which models are "possible" relative to others. The first work along these lines seems to have been reported by Montague in a talk delivered to the Annual Spring Conference in Philosophy at the University of California, Los Angeles, in 1955 (later published as Montague [2]). More recently, Hintikka [1] and Kripke [1] have reported interesting results relating different relations between models to the familiar modal calculi S1-S5 of C. I. Lewis.

58. See, for example, Carnap [4].

59. This method of assuring that notions to be introduced determine legitimate set theoretical entities was used in Tarski [3], p. 706, footnote 3.

60. The possibility should also be considered of forming a language closer to ordinary metalanguages by combining the features of both interpretations of $\mathcal{K}_1$. We simply duplicate the new wfe's which are added to $\mathcal{K}_0$ and assign the syntactical interpretation to one set of new expressions and the intensional interpretation to the other. Suppose we assign the types $i_1^S$, $t_1^S$, etc., to wfe's receiving the syntactical interpretation, and the types $i_1^I$, $t_1^I$, etc., to wfe's receiving the intensional interpretation. We could then introduce an operation symbol S of type $\langle t_1^S, t_1^I \rangle$ where the wfe $(S(\alpha) = \beta)$ would be true just in case $\beta$ denoted

the sense of the sentence denoted by $\alpha$. Languages (or more properly speaking, semantical systems) which contain certain wfe's with a syntactical interpretation and other wfe's with an intensional interpretation, we call dual languages. Such languages seem worthy of further study. However, their structure becomes rather intricate as we move to languages capable of treating doubly, and triply oblique contexts.

61. The exact use of corners ($\ulcorner, \urcorner$) here may be made clear by an example. If $\beta$ = 'John', and $\alpha = \overline{\beta}$ = '$\text{John}_1$', then: $\ulcorner$the name $\alpha$ denotes the individual $\beta\urcorner$ = 'the name $\text{John}_1$ denotes the individual John' and $\ulcorner$the individual concept $\alpha$ is a concept of the individual $\beta\urcorner$ = 'the individual concept $\text{John}_1$ is a concept of the individual John'.

62. The two interpretations of $Tr(\emptyset)$ are of interest with respect to the controversy over whether truth is a property of sentences or propositions. The precise definition of truth as a property of sentences (for sentences of a wide class of languages) was first given in Tarski [1] (1936). We are unaware of any precise definition of truth as a property of propositions before the present work. Most of the polemics in this controversy seem to have come from the defenders of truth as a property of propositions, perhaps in reaction to the success of Tarski's theory (see, for example, Kneale and Kneale [1], Strawson [1], and many others). The importance imputed to the "proper" solution of this "controversy" seems to us surprising in view of the essential isomorphism between the two notions. For example, all sentences of the form $(Tr(\overline{\emptyset}) \equiv \emptyset)$ are valid under both interpretations. Further comparisons of the two notions

of truth occur in Chapter 3.

63. The exceptions are logically determinate sentences and standard names (in the sense introduced in section 14.3).

64. Church has used "determines" in this way in Church [7], p. 6; he also uses "characterizes" in a similar way. The temptation to simply use "denotation" for this relation, also, is quite strong; and Church [3], [6] does use the suggestive expression "$\Delta$" for this relation. Carnap also sometimes uses the same expression for both denotation and Determination. In Carnap [6], section 9, III he writes, "the properties P and Q . . . have the same extension" although his main use of "extension" is in the context "extension of a designator" rather than "extension of a property".

65. This, in fact, was the sense in which "denotation" was originally used in section 4.

66. Church [6], footnote 13.

67. See, for example, Tarski [2], section 9.

68. Note the difference between the identity sign "=" (read "short equals") which is used to define "$=\!=$" (read "long equals"). The former is simply a shorthand in English for "is identical with"; the latter is a special sign of our theory used to denote a certain formation rule of our constructed languages.

69. When f is a function and A is a set included in its domain, we write "$f \upharpoonright A$" (read "f restricted to A") for that function g whose domain is A and which is such that for $x \in A$, $g(x) = f(x)$.

70. When g(n) is a set for each n such that Fn, the union of g(n) for n such that F(n) ($\bigcup_{F(n)} g(n)$) is the least set containing all such g(n).

71. There is an exception, this is not true when $\langle\tau_1,\ldots,\tau_r,\tau_o\rangle$ is a complex type of $\mathcal{K}_o$.

72. For example, Ackerman [1].

73. Quine [4].

74. The validity of the Delta Axioms suggests that we might have introduced '$\Delta(\bar{\alpha},\beta)$' by definition (as we did with 'Tr($\bar{\phi}$)', '($\phi$ & $\psi$)', etc.), thus eliminating all the atomic wfe's Delta ($\tilde{\tau}$). We could indeed have taken such a course successfully in $\mathcal{K}\omega$, but this is due only to the poverty of this language as expressed in theorem 67. When we turn to the language $\mathcal{L}\omega$, which contains variables of all simple types, wfe's of the form $\Delta(\alpha_1,\beta)$, where $\alpha_1$ and $\beta$ are variables of the appropriate types, will not be equivalent to any Delta-free wfe.

75. S5 is the strongest of a number of systems of modal logic developed in Lewis and Langford [1], from suggestions in Lewis [1].

76. Not all of the axiom schemes, as given, are independent. In fact, all modal axioms of kinds 1-4 are derivable from the remaining axiom schemes (which are independent) using the intensional individuating axioms, and modal axioms of kinds 2-4 are derivable from the remaining axiom schemes (which are independent) using the syntactical individuating axioms. The stronger result from the intensional individuating axioms depends on the fact that those axioms allow one to prove ($N\phi \equiv (\phi = \overline{(P \supset P)})$ ), where $\phi$ is any wfe of $\mathcal{K}\omega$ of type t and P is Pred(o,o).

This result shows that under the intensional interpretation, '$N\phi$' could have been introduced by definition (as were 'Tr($\bar{\phi}$)', '($\phi$ & $\psi$)', etc.). In fact, the following general definition (not dependent on theorem 33) could be given.

Definition If $\emptyset_1$ is a wfe of type $t_1$, then

$$N\emptyset_1 = (\emptyset_1 = \overline{(P\supset P)}).$$

Just this method is used to introduce necessity in Church [6]. But such a course would not provide the desired properties of necessity under the syntactical interpretation.

77. The inference rule of modal generalization has a slightly different character from that of modus ponens. Whereas modus ponens is truth preserving (that is, when applied to true sentences its result is always true), modal generalization is only validity preserving. If it is desired to avoid inference rules which are not truth preserving, an equivalent system can be obtained by dropping the rule of modal generalization, replacing each axiom $\emptyset$ by the new axiom $N\overline{\emptyset}$, and retaining modal axiom (2) $(N\overline{\Psi}\supset\Psi)$.

78. In a talk, "Modal Logic Explored Semantically," to the UCLA Logic Colloquium (1959). A slightly different formulation of an equivalent system MPC was given in Carnap [3]. Both Carnap and Kalish provided completeness theorems and decision procedures.

79. It is interesting to note that in a development of intensional logic based on the simple theory of types (such as in Church [6]) with the types $\langle\tau_0 \ldots \tau_r\rangle$, $\langle\tilde{\tau}_0 \ldots \tilde{\tau}_r\rangle$ distinguished, a similar introduction of the bar function is possible. To be specific, if the language contains (as in Church [6]) the single variable binding operator '$\lambda$' plus function symbols $U$, D (to be applied to lambda expressions) corresponding to the universal quantifier and description operator, the bar function can be adequately defined without introducing new operators '$\lambda_1$', '$\lambda_2$', etc.,

or new logical constants $U_{(j)}$, $D_{(j)}$.

80. The principles of theorem 110 correspond to axioms 11-15 of Church [6]. The principle of theorem 111 (1) corresponds to Church's axiom 17. Church's system contains no counterparts to the principles of theorem 111 (2)(which he explicitly rejects) or theorem 111 (3)(an oversight?).

81. We have separated T116 from T115, although the principle of T116 is simply the converse of T115 (2). In this we follow a tradition of attending to the latter and ignoring the former.

With respect to the system S2 of Carnap [4], the indirect discourse counterpart to the principle of T115 (2):

(1) $$(\exists \alpha N\emptyset \supset N \exists \alpha \emptyset)$$

is asserted to be valid (p. 186). But it is not noted that when $\emptyset$ contains no modal signs, the indirect discourse counterpart to the principle of T116:

(2) $$(N\exists \alpha \emptyset \supset \exists \alpha N\emptyset)$$

is also valid in S2.

It is somewhat surprising that (2) has not at least received some discussion. For it is virtually equivalent to the rejection of the indirect discourse principle:

(3) $$\forall \alpha \forall \beta((\alpha = \beta) \supset N(\alpha = \beta))$$

where $\alpha$, $\beta$ are any variables; and the latter principle has received a good deal of attention.

To argue fully the claim of equivalence would take us beyond our present scope. But we remark that the addition of both (2) and (3) to a quantified modal logic (in indirect discourse) whose other axioms are simply the normal principles of quantification plus the axioms of Lewis' S5, results in the theorem:

(4) $(\sim N\forall\alpha\forall\beta(\alpha = \beta) \supset (N\phi \equiv \phi))$.

We may read (4) as asserting that if it is not necessary that there is exactly one thing (a plausible assumption), then any sentence is necessary just in case it is true. While the consequent of (4) is not an outright contradiction, it is equally disastrous for modal logic.

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