LOGIC, LOGICISM, AND INTUITIONS IN MATHEMATICS

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CHAPTER 1

INTRODUCTION

My subject is logicism. Logicism is the combination of these two subtheses:

(i) Concepts of mathematics can be given logical definitions

(ii) Truths of mathematics can be derived by using logical definitions and inference rules.

I refer to these two as, respectively, the descriptive and deductive aims of logicism. In the case of success in attaining them, logicism is thought to provide both a semantical and an epistemological foundation to mathematics, i.e. mathematical truth will be shown to be logical truth, and mathematical knowledge will be shown to be logical knowledge. I am going to consider to what extent the descriptive and deductive aims of logicism are reachable. I am also going to consider the basic motivations behind formulating such aims. My own motivation, in this work, can be summarized with Russell's words in his [1919]:

> So much of modern mathematical work is obviously on the borderline of logic [if there is such a borderline], so much of modern logic is symbolic and formal that the very close relationship of logic and mathematics has become obvious to every instructed student. (p. 194)

Before giving a brief analysis of contents of what I shall be concerned with in the present work, I should point out that I adopt a general view of logicism. Instead of analyzing the differences between various kinds of logicism, I take "logicism" as an umbrella term, and basically deal with the common features of different logicisms.[1] I do not argue for logicism against the other big schools of philosophy of mathematics, namely

[1] G. Frege's [1884], R. Dedekind's [1888], B. Russell & A. N. Whitehead's *Principia Mathematica*, L. Wittgenstein's [1921] 6-6.3, F. P. Ramsey's "The Foundations of Mathematics", A. Church's [1956] §55, D. Bostock's [1974], Field's [1984] are examples of important works that can be studied under the title "logicism".

formalism and intuitionism, in detail. But I put my reasons to believe those two views are not plausible, briefly as follows: The formalist view, which argues mathematics to be a meaningless formal game is not plausible because the rules of meaningless games are arbitrary whereas the rules we follow in mathematics are inherent in life and nature, such that we cannot make arbitrary changes in them. Intuitionism, is not plausible either. For the agnostic viewpoint of intuitionism, concerning existential claims about non-constructable entities, force mathematicians to give up many fruitful methods of mathematics.[2]

In the following chapters I mainly deal with two problems: (i) whether intuition is dispensable in mathematics, (ii) whether the logic of logicism can be counted as logic. These two problems determine the division of chapters in the present work. In Chapter 2, I consider to what extent intuition is dispensable in mathematics. Section 2.1 briefly analyzes the concept of topic-neutrality, and what we should understand by the topic-neutrality of mathematics. This is crucial in making my standpoint clear whenever I say that mathematics is topic-neutral. For calling mathematics topic-neutral is counter-intuitive at first sight – as long as mathematics has its own subject matter. Thus I hold a different sense of topic-neutrality, in terms of *fair application* to other topics.

In section 2.2, I try to shed light on the dispensability of intuition in mathematics, since the main logicist motivation can be put as dispensing with intuition completely. I find two aspects in mathematics: activity and knowledge – a distinction faithful to Wittgenstein [1953], p. 227[e] – and suggest that intuition is indispensable for the activity, but not for the knowledge.

In section 2.3, I present Frege's theory of number as a case study, which is the first attempt to reduce arithmetic to logical principles. I consider Frege's basic definitions and proofs, and whether his project can be accepted as a successful piece of logicism.

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[2] For the basics of intuitionism, see Heyting [1956].

In Chapter 3, I discuss some arguments against the view that higher-order logic should not be counted as logic. Against what is known as the *first-order thesis*, I present some arguments for the view that higher-order logic is logic. In section 3.1, I argue for the superiority of the higher-order over the first-order, and consider an argument, which puts forth the important role of higher-order logic in the communication of mathematical language. By the way, mathematics has a logic only if it is communicable. In section 3.2, I shall briefly note the deficiency of higher-order logic and its implication on the reaches of the logicist philosophy of mathematics.

In Chapter 4, the conclusion part, I present my conclusions concerning sections 2.1-3.2, and sum it up with a general conclusion for the whole work. In the light of my considerations, I aim to reach at a conclusion that mathematics and logic are, in fact, parts of the same subject.[3]

[3] This is faithful to Church [1956], §55.

CHAPTER 2

DISPENSING WITH INTUITION

2.1 A Note on the topic-neutrality of mathematics

Roughly, logicism, in the form defended by the early logicists, "hoped to show...that mathematics did not have any 'subject matter', but dealt with pure relations among concepts..." (Benacerraf [1964], p.11). This well summarises the basic inclination of the logicist. But this rough way of putting it is in need of clarification. Especially, what is meant by *pure relations among concepts* must be made clear. It may be insightful only if the blur in it is worked on hard. For there appears immediately the danger of falling into the depthless Kantian half-truth that *concepts are empty without intuitions*.[4] Holding, for instance, that mathematics and logic are one, as the study of pure, contentless concept forms, the knowledge that is gained from logic (and hence mathematics) will really be a colorless knowledge. Surely, relations among those forms will also be pure forms of some sort. But if they are all pure, how is it possible that they are distinguishable; and how are they many, rather than one, if they are not distinguishable? Such a conception of logic (and hence mathematics) as a pure attributeless realm – even in case we have good reasons to believe in its existence as the mysterious origin of the all knowledge – cannot be shown to serve as the origin, for particular cases, and for particular symbolisms. The chief proposition of logic can be attributed to Wittgenstein, i.e. "[Are we] getting closer and closer to saying that logic cannot be described?" (Wittgenstein, [1969], §501). But this will not help us here. Logicist inclinations may be of three kinds: mystical, logical, and mathematical inclinations. In the sense that it cannot be described, logic (and hence mathematics) is a concealed mystery, whereas in many uses of the word it is not.

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[4] It is well known that the other half of the Kantian truth is that *intuitions are blind without concepts*.

It is known as the topic-neutrality view of logic that logic is applicable to any subject matter without committing to the existence of any special entity that belongs to the subject. Hence, logic taken as topic-neutral and mathematics as a topic, logicism immediately fails. For what it promises is the existence of mathematical objects, i.e. numbers[5], by purely logical means. To want that logic both have a mathematical content, and be topic-neutral at the same time is not acceptable. However, what the logicist aims at seems to be a contentful topic-neutrality, which is self-contradictory; when it is argued that the seemingly mathematical content was actually an empty content, hence a topic-neutral one, the problem will be solved. For

> ... for logic (and hence mathematics) to deal with the relations among concepts is not for [the logicist] to have a special subject matter – in the way, say, that living organisms constitute the subject matter of biology. (Benacerraf, [1964], p.11, note 5)

Neverthless, this is not much enlightening. For when we say that pure relations among concepts constitute the subject matter of mathematics, it will seem that mathematics has a special subject matter; unless *pure relations among concepts* are exemplified as being not special, but general. In case what we talk about in mathematics is not a special subject matter, but a general one, the special thing about it is that it is applicable to the special subject matters. To put it in Wittgenstein's words, "...the generality in mathematics is not accidental generality." (Wittgenstein, [1921], 6.03) A simple example is an arithmetical one. 3+1=4 can be seen as a general law for ordinary objects. 3 apples and 1 carrot make 4 objects. 3 carrots and 1 apple also make 4 objects. It is obvious that the statement "3+1=4" is in a sense, topic neutral at least to all topics which include numerable things. "The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only actual, not only the intuitable, but everything thinkable." (Frege, [1884], §14). One

[5] In particular Frege's logicism. See Frege [1884].

may object here that everything thinkable is not numerable, and one can argue that what is numerable is not the widest domain of all. But this has something to do with how we prefer to interpret the word *thinkable*. Leaving it aside, it is sufficiently clear that the numerable is a very wide domain, even if not the widest. To quote from Wittgenstein,

> 50. When does one say, I know that ....x....=....? When one has checked the calculation.
> 51. What sort of proposition is "What could a mistake here be like?"? It would have to be a logical proposition. But it is a logic that is not used, because what it tells us is not learned through propositions. – It is a logical proposition; for it does describe the conceptual (linguistic) situation.
> 52. This situation is not the same for a proposition like "At this distance from the sun there is a planet" and "Here is a hand" (namely my own hand). (Wittgenstein, [1969], §§50-52)

Maybe a little distorting what Wittgenstein means by the *logical*, we can conclude that our point is well summarised in the above words – logic (and hence mathematics) in the view of logicism is the tool to analyse the pure relations among formal concepts.

2.2 A remark on the role of intuition in mathematics

The logicist philosophy of mathematics can be traced back to the works of Leibniz and Bolzano. One quotation for each will suffice to see that nothing seems different in motivation between them and the father of the logicist school of mathematics, Gottlob Frege. Leibniz says:

> I have ... been urging ... the importance of demonstrating all the secondary axioms which we ordinarily use, by bringing them back to axioms which are primary, i.e. immediate and demonstrable; they are the ones which ... I have been calling 'identities'. (Leibniz [1765], 408)

To put forth the similar view of Bolzano, we quote from Alberto Coffa [1982]:

> Bolzano's problem was to prove that a continuous real function that takes values above and below zero, must also take a zero value somewhere in between.... Bolzano's problem looks like a problem only to someone who has already understood that intuition is not an indispensable aid to mathematical knowledge, but rather a cancer that has to be extirpated in order to make mathematical progress possible. ... If Kant had known about Bolzano's paper there can be little doubt that he would have regarded it as a philosophically incoherent effort to prove the obvious. The paper[6] was, instead, one of the landmarks of nineteenth-century mathematics. (pp. 36-37)

It is clear that Leibniz seems to be more radical, in getting rid of intuition in mathematics – by defending the view that we can even arrive at tautologies ("identities" in Leibniz's terminology) – than Bolzano, who just sees intuition to be an illness to be cured. Now, although the apparent argument for logicism in Coffa's words is an appealing to authority, it provides the basic logicist motivation. For it would be impossible to believe that logicism will succeed one day, without being impressed by such examples; since no argument other than what *we* call the logical analysis of mathematics guarantees its conclusion.

Frege's logicist work is usually mentioned as a supplement to the mathematical works of Bolzano and his followers, such as Cauchy, Weierstrass, Dedekind, and Cantor.[7] It is, in sum, to go to the farthest possible in eliminating intuition from mathematics. §1 of Frege [1884] summarizes that the recent work of Frege's times had shown that mathematics prefers rigour. This is where Frege starts from. He says: "... in mathematics a mere conviction, supported by a mass of successful applications, is not good enough. Proof is now demanded of many things that formerly passed as self-evident." (Frege [1884], §1).

That the logicist is inclined to prevent intuition as much as possible in mathematics, is clear. Frege hopes to have claimed, in his [1884] §87, that he had shown that it was at least probable that arithmetical laws were analytic, by examining to the extent we can dispense

[6] This is the paper that Bolzano published in 1817. For more information see Coffa [1982].

[7] See Coffa [1982] and Demopoulos [1994] for more detail.

with intuition in arithmetic. On the other hand, if it is possible to dispense with intuition until we reach mere tautologies, then even a logicist may be unhappy with this result. Having realized such a possibility, Frege gives an argument to those unhappy, for not to worry about the tautologies. In §16, Frege says:

> And how do the empty forms of logic come to digorge so rich a content? ... Everyone who uses words or mathematical symbols makes the claim that they mean something and no one will expect any sense to emerge from empty symbols. But it is possible for a mathematician to perform quite lengthy calculations without understanding by his symbols anything intuitable.... And that does not mean that the symbols have no sense; we still distinguish between the symbols themselves and their content, even though it may be that the content can only be grasped by their aid.

This argument is against the views of those intuitionists[8] like Poincaré. Poincaré finds logical inferences too colorless to be a part of mathematical reasoning. Michael Detlefsen put it as "Using Poincaré's own figure, the 'logician' is like a writer who is well-versed in grammar, but has no ideas." (Detlefsen [1992], p.350). The point of logicism should be to express mathematics by appealing to its content. For without mathematical content logic is just logic, and nothing more. It will teach us nothing mathematical, unless we know something mathematical. Without appealing to mathematical content, logical representation of mathematics would be like a translation into an unknown language, in which we cannot see what it is being talked about. However, holding the mathematical worries of the logicist, there is nothing wrong in appealing to *content*. Thus, it cannot be just the grammar that the logicist prefers. What he prefers is grammar as a supplement to the ideas. Here, there is a problem of regress, concerning this way of dispensing with intuition. For it will just be dispensing with an intuition *A* and passing to another intuition *B*. The argument

[8] We use the word "intuitionist" roughly here, and not in a definitive sense.

needed for the logicist, to handle the regress, is the one which distinguishes logic from non-logic.

One of the three principles that Frege mentions in the introduction of Frege [1884] viz., to have had kept, during his enquiry, the distinction between the logical and the psychological, may show us the way to understand the basic logicist motivation, which also promises the needed argument that distinguishes logic from non-logic. Frege says that he had kept seperate the psychological from the logical, and the subjective from the objective (Ibid., p.x). This contention leads us to the very important debate on the nature and content of logic that took place in the 19th century.

Many logicians believed logic to be strongly related with cognition and psychology. Frege's [1879] was an attempt to save logic from the subjective muddle. Husserl was also a supporter of the anti-psychologistic position. Both Frege and Husserl held that logic had nothing to do with psychology. Husserl's argument against psychologism is that to argue that logic is bound to psychology is self-refuting; for this very view should rest on a psychological argument and is thus only relatively true (Pulkkinen [1994], pp. 54-55). On the other hand, there were many logicians who advocated these relative truths. Cristoph Sigwart, Wilhelm Wundt, Benno Erdman, and Theodor Lipps are some representatives of the psychologistic position (see Pulkkinen [1994]).

In §27 and §28 of his [1884], Frege argues that anti-psychologism holds also in arithmetic. He argues number to be not an object of psychology, but something objective. This view is something that can hardly be rejected. For mathematics is the same everywhere. On the other hand, what is psychological differs from person to person, from time to time and from place to place for one person. This argument precludes the uses of number words in mystical contexts, e.g. when it is said that 19 is divine. Frege's unhappiness with the strong intuitions of the Indians shows itself at the beginning of the [1884]. In §1, he says:

> In arithmetic, if only because many of its methods and concepts originated in India, it has been the tradition to reason less strictly than in geometry, which was in the main developed by the Greeks.

Here, I think Frege's anti-psychologistic position has to be criticized. To quote from somewhere else:

> [T]he laws of logic ought to be guiding principles for thought in the attainment of truth, yet this is only too easily forgotten, and here what is fatal is the double meaning of the word "law". In one sense a law asserts what is; on the other hand it prescribes what ought to be. Only in the latter sense can the laws of logic be called 'laws of thought'; so far as they stipulate the way in which one ought to think. (Frege, [1903], p.12)

This is the view that logic is not the physics – where psychology can be counted as the physics of thought – but the ethics of thought (see Pulkkinen [1994], pp. 41-57). Not quite being sure about, but being convinced in a certain sense that Frege prefers that mathematics should be done rigorously; I mention here that I am not in complete agreement with him.

I hold a distinction between *mathematical knowledge* and *mathematical activity*. Concerning logicism, thus, it seems better to argue that rigorous mathematics is necessary, but on the other hand, what is done without rigor is not something to be forbidden. Otherwise what is called mathematical discovery will be in danger.

Take the number 1729, as an example. The Indian mathematician Ramanujan said once that it was the smallest number expressible as the sum of two cubes in two different ways. (Hardy [1959], p. 12) For sure it is not a proposition of the same kind as another one he held, viz., that "God is the number 0, the attributeless." (Ranganathan [1967], p. 82 and p. 101) The truth value of the former depends on a couple of operations, in doing which we are not free to do whatever we like. On the other hand, the truth-value of the latter proposition is a matter of some subjective presuppositions. But the interesting thing is that a mathematician can see this interesting property of the number 1729 with some sort of inspiration. The story of 1729 is below, from the mouth of G.H. Hardy:

> I remember going to see him [Ramanujan] once he was lying ill in Putney. I had ridden a taxi-cab No1729, and remarked that the number seemed to me rather a dull one, one that I hoped it was not an unfavorable omen. "No" he replied, "It is the smallest number expressible as the sum of two cubes in two different ways" [1729 = $12^3 + 1^3$, 1729 = $10^3 + 9^3$] I asked him naturally whether he could tell me the situation of the corresponding problem for the fourth powers; and he replied after a moments thought, that he knows no obvious example, and supposed that the first such number must be very large. (Hardy [1959], p. 12)

It is obvious that Ramanujan's immediate reply to Hardy is far away from a rigorous thinking, and it is also obvious that such quick intuitions are indispensable in mathematics. But still we can argue, after seeing what Ramanujan said was true, that whoever tries to object to Ramanujan's words about the number 1729 will fail on logical grounds. On the other hand, there are many mystical views possible on psychological grounds, on whether God is the number 0, or the number 1, or the number Ω.[9] Thus, the logicist would conclude that Ramanujan's words about the number 1729 are logically decidable and certain. He would also conclude that Ramanujan's words about God being the number 0 are logically undecidable, and subjective. One of the main aims of logicism i.e. to put forth what is psychological is irrelevant to mathematics, is now clearer.

Hitherto I have mentioned two basic inclinations of logicism. One is to separate the logical from the psychological, and the other is to purge the psychological (intuition) out of mathematical knowledge. To understand these two aims better, I am going to consider L.E.J. Brouwer, the founder of the intuitionist school in the philosophy of mathematics. But

[9] Ω as the last one of the transfinites.

first, what is it to be logically decidable, and what is it to fail on logical grounds? Speaking metaphorically, to fail on logical grounds is like falling down on logical grounds, i.e. I fell down while I was running on the logical grounds, because I met an obstacle, say a contradiction – a stone, and my face kissed the ground. Unfortunately, one's falling or surviving is not guaranteed for every mathematical proposition. Kurt Gödel's incompleteness results showed that there are propositions of arithmetic, which can neither be proved, nor disproved. Granted this result, it is not guaranteed that every proposition of mathematics is logically decidable.

It is evident that a logically decidable proposition must not be something accepted under subjective constraints. For example, $3 + 2 = 5$, not because the number 3 is male, and the number 2 is female, and the number 5 is marriage. For the number 4 can also be male for somebody, but we wrongly conclude that $4 + 2 = 5$, in such a case. It is not difficult to distinguish a psychologically accepted proposition from a logically accepted one. It is at least psychologically easy. Nevertheless, a logical proposition might have been accepted psychologically first, and then seen to be an objective truth. That kind of discovery creates no problem for the logicist aims. For although mathematics seems to be a logical field, we humans are fortunately psychological beings.

I now move on to another problem. How will the logicist show that mathematical propositions are logical propositions? No doubt what we call logical laws must be strictly logical. Otherwise, the logicist motivation would lose its meaning. However, still, the question "What is logic?" is not an easy question. Roughly, the job of the logicist is to write mathematics with the logic that he considers to be the true logic. But, which logic? Frege, for instance, used his conceptual notation that he introduced in Frege [1879]. Suppose he had chosen the appropriate logic for the logicist project. Then to show that mathematics is logic, he had to make the necessary translations from the mathematical concepts to the logical ones, that is, to define the concepts of mathematics in the language of logic. To be sure about the truth-values of mathematical propositions, he had to derive the propositions of mathematics from the definitions he had madetogether with the inference rules of the logic. Frege tried to do so for arithmetic.[10] Through rigor, by giving sharp definitions, he tried to show that objects of arithmetic were determinate and fixed.

In order to understand the logical/psychological distinction better, we shall next consider the case of Brouwer. Brouwer had opposite intentions to those of Frege's. His position can be summarized as the view that logic is mathematics. He has two theses. He defends the view that the objects of mathematics are mental constructions, and secondly, the view that language and logic cannot provide security for mathematical certainty, for they are imperfect translations of what we have in our minds (Placek [1999], p. 2). The problem with Brouwer's theses is the communication problem of the mental constructions. For how can one hope to communicate his own mental constructions with other people? Moreover, how can one hope to communicate them by using the so called – called by Brouwer himself – imperfect language? There appears the danger of solipsism for Brouwer's view. Brouwer *no doubt* had a subjective view of mathematics in *his* mind, whereas Frege argued for strictly the opposite; i.e. that mathematical knowledge is objective. "[It] is in the nature of mathematics to prefer proofs where proofs are possible." (Frege [1884], §2). Brouwer would welcome this. However, what Frege means here is one step further always. If possible, the logical proof, which is independent of intuition, is what mathematics prefers. I emphasize this as the very characteristic of logicism. Yet, what can be a better ground than logical proof? Remember Ramanujan's words. Suppose someone intuits the fact that 1729 is the smallest number expressible in two different ways as the sum of two cubes? How will he tell his intuition to someone else? By way of simpler

[10] Not for the whole of mathematics, because Frege believed that geometry was not purely logical. See Frege [1884], §§13-14.

intuitions, one may suggest. However, the logicist suggestion goes further; we must fill in the logical gaps in mathematics. Thereafter everybody will, in principle, be able to judge their appropriateness.

We observe here that the basic differences between the intuitionist Brouwer and the logicist Frege are parallel to the differences between the nature of mathematical activity and the nature of mathematical knowledge. Brouwer seems to be interested in *how* we do mathematics; on the other hand Frege seems to be interested in *what* we do in mathematics. In a letter to his supervisor, Brouwer put his intentions as follows: "…what I brought you now, exclusively treats *how mathematics roots in life*, and how, therefore, the points of departure of the theory ought to be…" (Van Dalen [1999], p. 90). In my view the way mathematics roots in life does not imply any departure of the theory. But also in my view that does not mean the way mathematics roots in life is irrelevant to mathematics. I am here trying to adapt a Wittgensteinean remark on the double aspect of the situation, i.e. mathematics is an activity as much as it is knowledge (Wittgenstein [1953], p. 227). We neither have a right to say mathematics is all along mental construction nor to say that it is all along logical. Suppose one has a mathematical theorem at hand to prove, of which he did not see the proof before. How would he approach the problem? Unless he had a standard method to follow, that is, unless he knew how that proof might be classified, he will try to fit the theorem to be proved, to the methods that he already knew; or he will make some changes in the form of the theorem, and make it something that he is more acquainted with; or else he will see the way to the proof with a sudden flash in his mind. Otherwise, he will hopelessly and blindly be looking at the theorem for a while. Probably, after the informal struggle with the theorem, if one succeeded, he would be able to give a rigorous formalisation for what he had done.

Mathematics, contrary to Frege's point, seems to have something to do with psychology. Mathematical reasoning makes use of metaphors, analogies, conjectures, etc. On the other hand the *real* nature of mathematical knowledge has really nothing to do with anything psychological, if we believe that mathematics is objective all over or at least has a transcommunal validity. To conclude from my discussion, both views – intuitionism and logicism, and Brouwer's and Frege's varitiesin particular – seem to miss a thing that the other is aware of.

A similar consideration to the one mentioned here, is made by Feferman [1981] on Lakatos' philosophy of mathematics.[11] There we see the distinction between some sort of mathematical justification and mathematical discovery. Feferman accuses Lakatos to misseen the fruitful consequences of the logical analysis, and the logical structure of a mathematical theory. According to Feferman,

> [Lakatos] plays only one tune on a single instrument – admittingly with a number of satisfying variations – where what is wanted is much greater melodic variety and the resources of a symphonic orchestra. (Feferman [1981], p.78)

Neither a hard-nosed logicism can account for the activity aspect of mathematics, nor a radical subjectivist view can account for the knowledge aspect of mathematics adequately. It is a fact that the mathematician is like an unconscious painter. He tries to paint on a canvas, but he observes that there is more than his actual work on the canvas, i.e. the canvas seems to be painted before his will; before his paint. The mathematician moves his brush in a certain area on the canvas, however, some other non-brushed parts are also painted. Who painted those is a meaningful question for the mathematician. After all, it is a quite well painted picture. Thus, one is more inclined to say "I didn't, but He did it.", as

[11] Lakatos argues that since a mathematical theory can be formally axiomatized only after the creative work is done, logicism is not a philosophy of mathematics, but a philosophy of dead mathematics. (Musgrave [1977], p199)

Bach said for his compositions. This metaphorical depiction of the state of the mathematician is more against Brouwerian motivations than those of Frege. Brouwer, in the rejected[12] but philosophically valuable parts of his dissertation, points out that

> [M]an creates much more regularity in nature, than originally occurred spontaneously in it. He desires that regularity because it makes him stronger in the struggle for existence. (quoted by Van Dalen [1999], p. 91)

And on p. 82 of his dissertation, he says, "The intellectual consideration of the world widens its scope…." (ibid.). On the other hand, a simple observation of the mathematical activity suggests that the world is already maximally wide, and thus every novel intellectual finding is rather like a discovery than a creation or an invention. Any logicist who does not claim that mathematics is a logico-mental construction of some special sort, seems to be accepting the world as maximally wide.

2.3 A Case Study of Frege [1884] §§68-83

Intuitions are indispensable in mathematical activity. But they may be dispensable in mathematical knowledge. For the fact that one can intuit a simple theorem without being aware of its provability by means of a number of axioms, proves the dispensability of intuitions, at least in favor of simpler intuitions. The question here is to what extent one can dispense iteratively with simpler intuitions.

It can be observed that human mathematical activity is simply not logical. It may be almost irrational. An active mind is apt to follow some rules uncontrollably fast, and is subject to sudden transitions, which can have a sense only after retrospection. Let us call this aspect of mathematics *hot* mathematics. On the other hand, what is gained as a textual knowledge has undeniably a force upon our thinking and rule following. Somehow,

[12] Some parts of Brouwer's dissertation had been found too philosophical for a thesis in mathematics and rejected by the examining committee. See Van Dalen [1999] for detail.

mathematical language gives us the *necessary* criteria to do mathematics. How is this possible? Let us call the body of mathematical (and textual) knowledge *frozen* mathematics. Here, hotness represents the subjective constraints and the uncontrollable transition through intuitions. Frozenness represents the static, objective body of mathematical knowledge. Assuming the existence of frozen mathematics, human work in mathematics is analogous to the process of freezing; and the search for the logic of mathematics is analogous to the search for a melting point of the melting process. What I am trying to understand here is the relationship between mathematical activity and the corresponding knowledge.

Who managed to go to the nearest possible to the melting point of arithmetic is Gottlob Frege, where the very melting point itself is determined as the foundation, where intuitions are completely superfluous, i.e. where arithmetic is analytic. In Frege [1884] §§68-83, the search is outlined. Frege gives a brief outline of how the logical definitions of the basic concepts of arithmetic can be carried out.

Frege used second-order logic in giving some of his definitions. In fact, there have been serious attacks against second-order logic as logic. Arguments against the second-order logic come especially from Benacerraf [1960][13], and Quine [1970]. The idea is that second, and generally, higher-order logic is not logic, but set theory; thus it is mathematics rather than logic. Quine, for instance, argues that higher-order logic commits itself to the existence of classes, since we quantify over predicates, functions, etc. in the higher-order, and that its predicates are class-like entities. Moreover, there is the Russell Paradox against Frege's definition of numbers as extensions. Concerning the difficulties led by the Russell Paradox and Quine-Benacerrraf argument, we shall mention some ways out in Frege's

[13] I had no first hand access to Benacerraf [1960] (*Logicism: Some Considerations*, Ph. D. Dissertation, Princeton University). The source of my information about it is Boolos [1996], and Steiner [1975].

system. As we shall see later on, whether we choose first-order or higher-order logic, we will not be able to hold both of the needed meta-properties of completeness and categoricity.[14] For completeness precludes categoricity, and categoricity precludes completeness. Hence, the chosen logic will be inadequate in either describing or deriving mathematics.

Secondly, it is a well-known consequence of Gödel's incompleteness results that there are true but unprovable propositions in mathematics. So, never mind logic, that mathematics itself has gaps of its own is proved by Gödel. Those gaps cannot be filled in with logic, hence intuition gets in. I am going to deal with the difficulties for logicism more closely in chapter 3. Now, turning back to Frege, he was not aware of the problems when he wrote his [1884]. He had certain objectives, and whether a logicist philosophy is possible with such objectives, is what I am investigating.

One of the most important observations of Frege in his book was that numbers assert something about concepts. What does this mean? The answer lies in Frege's distinction between concept and object. According to this distinction, concepts are unsaturated (incomplete) Platonic entities under which saturated things, namely objects fall. To give a Fregean example, take the concept "moon of Jupiter" ([1884], §57). When we ask the question "How many moons of Jupiter are there?", we answer 4; that is, 4 things fall under the concept "moon of Jupiter". We use a predicate letter, e.g. *F*, to represent the concept, and say that the number of things falling under *F* is 4, or equivalently say that there are 4 *F*s. In fact the question "How many?" is not a meaningful question, by itself. For we wonder how many what. We need a concept in order to give a number which applies to the given concept. By observing the logical form of statements like "The number of the moons

[1414]A logic is complete iff all its true sentences are provable; and a logic is categorical iff it has categorical sets of sentences, that is to say, it has sets of sentences all models of which are isomorphic.

of Jupiter is 4”, we can conclude, as Frege does, that numbers are self subsistent objects which assert information about concepts. Starting from the concept/object distinction, Frege tries to fix the sense of numerical identity. In brief, what Frege does is to investigate the nature of numbers as objects, by considering the concepts that they belong to. A summary of what is outlined in Frege [1884], §§62-84 (definitions and their symbolisations) is as follows:

Definition 1:

> *The number which belongs to the concept F* is the extension of the concept “[equinumerous] to the concept F” (§68).

Definition 1, sym: (Definition 1 in symbolic form)

$$\#F = \text{ext}[H\text{: } H \approx F],$$

where $\#F$ is the number which belongs to the concept F, and $\text{ext}[H\text{: } H \approx F]$ is the extension of the concept “equinumerous to F”. To fix the sense of this numerical identity statement, the next thing to be done is to define the equinumerousity of concepts.

Definition 2:

> *“the concept F is [equinumerous] to the concept G”* [iff] “there exists a relation ϕ which correlates one to one the objects falling under the concept F with the objects falling under the concept G” (§72).

Definition 2, sym:

$F \approx G$ iff $\exists\phi(\phi$ correlates F and G & ϕ is one-one).

To fill in the gaps of Definition 2, Frege defines,

Definition 3:

> …every object which falls under the concept *F* stands in the relation ϕ to an object falling under the concept *G*, and if to every object which falls under *G* there stands in the relation ϕ an object falling under *F*, …[iff] *the objects falling under **F** and **G** are correlated with each other by the relation* $\boldsymbol{\phi}$. (§71)

Definition 3, sym:

> $\boldsymbol{\phi}$ *correlates* ***F*** and ***G*** iff $\forall x[Fx \rightarrow \exists y(Gy \,\&\, x\phi y)] \,\&\, \forall y[Gy \rightarrow \exists x(Fx \,\&\, y\phi x)]$

There is still a gap to be filled in, i.e. the definition of "ϕ is a one to one relation".

Definition 4:

> [$\boldsymbol{\phi}$ *is a one to one relation* iff] 1. If *d* stands in relation ϕ to *a*, and if *d* stands in the relation ϕ to *e*, then generally, whatever *d*, *a* and *e* may be, *a* is the same as *e*. 2. If *d* stands in the relation ϕ to *a*, if *b* stands in the relation ϕ to *a*, then generally, whatever *d*, *b*, and *a* may be, *d* is the same as *b*. (§72)

Definition 4, sym:

$\boldsymbol{\phi}$ *is a one to one relation* iff $((d\phi a \,\&\, d\phi e) \rightarrow a = e) \,\&\, ((d\phi a \,\&\, b\phi a) \rightarrow d = b))$

We can write Definition 4,sym as $\forall x \forall y \forall z([(x\phi z \,\&\, x\phi y) \rightarrow y = z] \,\&\, [(y\phi x \,\&\, z\phi x) \rightarrow y = z])$. So we have fixed the sense of $F \approx G$. The next thing to be fixed is the sense of $\#F$. For in Definition 1, sym, left hand side of the identity statement is $\#F$, and it is essential to fix its sense in order to fix the sense of the whole identity.

Definition 5:

"***n*** *is a number*" [iff] "there exists a concept such that n is the Number which belongs to it". (§72)

Since Frege defines the Number which belongs to the concept F in §68 (in Definition 1 of the present work), Definition 5 is not circular.

Definition 5, sym:

n *is a number* iff $\exists F(\#F = n)$

Definition 6:

…the Number which belongs to the concept *F* is identical with the Number which belongs to the concept *G* if[f] the concept F is [equinumerous] to the concept G. (§73)

Definition 6 is given in the light of a quotation from David Hume, by Frege. It is known as *Hume's Principle*.

Definition 6, sym:

$\#F = \#G \leftrightarrow F \approx G$

Definition 6 is not sufficient to fix the concept of number. It just gives us information about what it is for two numbers to be identical. It is not sufficient to fix the concept of number, because there is still a gap which has not yet been filled in. That gap traces back to Definition 1. What is meant by the *extension* of a concept, i.e. ext[H: $H \approx F$], has not been fixed. Nevertheless, without its being fixed we can proceed with the help of Definition 6 and define some other arithmetical concepts. As it is widely known, the gap, that I am talking about, in Definition 1 is tried to be filled in by Frege [1903], by introducing an axiom, whose inconsistency was shown by Russell (more about it later).

Definition 7:

0 is the Number which belongs to the concept "not identical to itself" (§74)

Definition 7, sym:

$0 = \#[x: x \neq x]$

That is to say 0 is the number of Fs, such that things that are not identical with themselves fall under F, i.e. $0 = \#F$ where $F = [x: x \neq x]$.

Definition 8:

"***n** follows in the series of natural numbers directly after **m***" [iff] "there exists a concept F, and an object falling under it x, such that the number which belongs to the concept F is n and the Number which belongs to the concept 'falling under F but not identical with x' is m". (§74)

Definition 8, sym:

$m\boldsymbol{P}n$ iff $\exists F \exists x(Fx \ \& \ \#F = n \ \& \ \#[y\!: Fy \ \& \ y \neq x] = m)$

where $m\boldsymbol{P}n$ stands for the expression **"**n follows in the series of natural numbers directly after m", or "*m* precedes *n*" (*P* stands for "precedes"). The purpose of Definition 7 and 8 is to show that the objects of arithmetic, i.e. numbers, can *all* be logically defined.

Definition 9:

1 is the Number which belongs to the concept "identical with 0" (§77)

Definition 9, sym:

$1 = \#[x\!: x = 0]$

Definition 10:

"*y* follows in the ϕ-series [the series of natural numbers for the special case] after *x*" [iff] "if every object to which *x* stands in the relation ϕ falls under the concept *F*, and if from the proposition that *d* falls under the concept *F* it follows universally whatever *d* may be, that every object to which *d* stands in the relation ϕ falls under the concept *F*, then *y* falls under the concept *F*, whatever concept *F* may be". (§79)

Definition 10, sym:

y *follows in the* ϕ*-series* x iff $\forall F([\forall u(x\phi u \rightarrow Fu)$ & $\forall z\forall w(Fz$ & $z\phi w \rightarrow Fw)] \rightarrow Fy]$

where z is the substitute for d in Definition 10, and for the special case, of the series of natural numbers, we can modify Definition 10, sym as $y > x$ iff $\forall F([\forall u(xPu \rightarrow Fu)$ & $\forall z\forall w(Fz$ & $zPw \rightarrow Fw)] \rightarrow Fy]$, where the symbol '>' stands for the relation "larger than", and P stands for "precedes". Frege's main aim is to prove that there is a number n, for every number m, in the series of natural numbers, which directly follows m. The use of Definition 10 is to realize this aim. For we need Definition 10 to define the concept to which the number whose existence is aimed to be proved, belongs to.

Definition 11:

…a is a member of the series of natural numbers ending with n, iff n either follows in the series of natural numbers after a or is identical with a. (§81)

Definition 11, sym:

$y \geq x$ iff $(y > x) \vee (y = x)$

where y is the substitute for n, and x is the substitute for a. Definition 11 defines the needed concept, such that the number which belongs to this concept will be the directly follower of n in the series of natural numbers. Now the informal outline of Frege, which I have tried to make more explicit with the symbolizations, presents a picture of how the foundations for arithmetic, i.e. defining each individual belonging to the series of natural numbers and showing them to be infinite in number, can be handled.[15] Concerning the proof of the infinity – that there is no last member – of the series of natural numbers, I have not introduced anything yet. Frege gives an informal sketch of the proof in §82, but I shall rather appeal to the formal proofs given in the appendix of Boolos [1990]. But before that, one point must be examined – that there is still a gap in Frege's outline.

We followed Frege [1884] through §§68-81, and saw, successively, that the definitions are given in a logical fashion, by refraining from leaving any gap for intuition to get in. However, one notion remains to be defined viz., the extension of a concept (see Definition 1). Actually, Frege tries to fill in the gap in his [1903] in §20, by introducing his Basic Law V, being unaware of its inconsistency. On the other hand, in [1884] §69 he supposes that what is meant by the extension of a concept is clear (see §69, note 1). Although Frege uses, informally, the basic idea of the Basic Law V in §73 of [1884], he does not give a precise the law. But beside this, the proof of the infinity of the series of natural numbers can be handled by using Hume's Principle as a contextual definition[16] of number. However, in formulating Hume's Principle (Definition 6, in the present work) we use the expression 'number of'. Frege probably thought that he should give an explicit definition of number, in order to give a non-circular contextual definition as a true foundation. For had he used Hume's Principle as a contextual definition of the notion of number, he would have offered an explanation of numbers in terms of numbers themselves. Frege was aware of the circularity, hence to prevent it he defines the number of *F*s as the extension of the concept *F*, in §68. Frege thought, in [1903], that he gave a more logical principle than Hume's

[15] $0 = \#[x: x \neq x]$, $1 = \#[x: x = 0]$. Since $\sim([x: x \neq x] \approx [x: x = 0])$, $0 \neq 1$ by Definition 6, sym. Then we define $2 = \#[x: x = 0 \vee x = 1]$, and so on for 3,4,…. The thing left to be done is to show that there are infinitely many such numbers. This will complete the derivation of the natural number system of Frege.

[1616]What is essential in a contextual definition is to fix the sense of the uses of the term to be defined. A contextual definition, thus, is not an explicit definition.

Principle, namely the Basic Law V, which was actually an inconsistent proposition. Basic Law V, to put it informally, says that the extension of a concept F is identical with the extension of a concept G, iff F and G are coextensive, i.e. all and only the objects falling under F fall under G. It can be symbolised as $\text{ext}(F) = \text{ext}(G) \leftrightarrow \forall x(Fx \leftrightarrow Gx)$.[17] The important purpose of the Basic Law V is to derive Hume's Principle.[18] In that way the problem with Hume's Principle was thought to be solved, but in fact, it was fatally inconsistent. [19]

To go back to our main concern, we give below the proofs that are sufficient to conclude that there are infinitely many natural numbers. I follow Boolos [1990] in giving the proofs. I use the following definitions in the proofs:

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[17] Basic Law V is defined for functions, in general (see Frege [1903], §20). Informally, it states that the courses of values of a function f is the same as the courses of values of a function g, iff f and g map every object to the same value. Symbolically, $\text{ext}(F) = \text{ext}(G) \leftrightarrow \forall \text{x}(f(x) \leftrightarrow g(x))$.

[18] Hume's Principle is $\#F = \#G \leftrightarrow F \approx G$. To prove, first, assume that $\#F = \#G$. Then by Definition 1, it follows that $\text{ext}[H\text{: } H \approx F] = \text{ext}[H\text{: } H \approx G]$. From this and the Basic Law V it follows that$\forall H([H\text{: } H \approx F](H) \leftrightarrow [H\text{: } H \approx G](H))$. Then we can say that $\forall H(H \approx F \leftrightarrow H \approx G)$. By universal instantiation it follows that $F \approx G$. *Secondly*, assume that $F \approx G$. By the transitivity of the relation "$\approx$", it follows that $\forall H(H \approx F \leftrightarrow H \approx G)$. From this we can say that $\forall H([H\text{: } H \approx F](H) \leftrightarrow [\ H\text{: } H \approx G](H))$. From this and the Basic Law V, it follows that $\text{ext}[H\text{: } H \approx F] = \text{ext}[H\text{: } H \approx G]$. Then by Definition 1 $\#F = \#G$.

[19] Russell obtained the contradiction with the concept 'extension of a concept which does not fall under the concept it is the extension of'. Frege's notation for this concept is as follows: $[x\text{: } x = \text{ext}(F)\ \&\ {\sim}Fx]$. Let G be the abbreviated name of this cocept. *First* assume that $\text{ext}(G)$ falls under Russell's concept G. That is $[x\text{: } x = \text{ext}(F)\ \&\ {\sim}Fx]\text{ext}(G)$. From this, we can say that $\exists F(\text{ext}(G) = \text{ext}(F)\ \&\ {\sim}F(\text{ext}(G))]$. By existential instantiation, we can get $\text{ext}(G) = \text{ext}(F)\ \&\ {\sim}F(\text{ext}(G))$. Then given the Basic Law V, it follows from $\text{ext}(G) = \text{ext}(F)$, that $\forall x(Gx \leftrightarrow Fx)$, by modus ponens. Since $\text{ext}(G)$ does not fall under F, it cannot fall under G either. However, this contradicts our assumption that it does so. *Secondly*, assume that $\text{ext}(G)$ does not fall under G. This time we shall have ${\sim}\exists F(\text{ext}(G) = \text{ext}(F)\ \&\ {\sim}F(\text{ext}(G))$. That is logically equivalent to $\forall F(\text{ext}(G) = \text{ext}(F) \rightarrow F(\text{ext}(G)))$. By universal instantiation we have $\text{ext}(G)$ fall under G. Again we have a contradiction.

Hume's Principle (HP): $\#F = \#G \leftrightarrow F \approx G$ (§73)

DefB1: $0 = \#[x\colon x \neq x]$ (§74)

DefB2: mPn iff $\exists F \exists y(Fy \mathbin{\&} \#F = n \mathbin{\&} \#[x\colon Fx \mathbin{\&} x \neq y] = m)$ (§76)

DefB3: xR^*y iff $\forall F(\forall a \forall b([(a = x \vee Fa) \mathbin{\&} aRb] \rightarrow Fb) \rightarrow Fy)$ (§79)

DefB4: $m \leq n$ iff $mP^*n \vee m = n$

DefB5: Finite n iff $0 \leq n$

Here P stands for "immediately preceding", and P^* stands for "preceding". (R and R^* stand for the generalization of these two, which are known as the hereditary relations (see Frege [1879], §§23-31). Observe that DefB1-DefB4 serve the same purpose and say the same thing as the previous definitions I have considered, i.e. DefB1 ≡ Definition 7, sym; DefB2 ≡ Definition 8, sym (when x in DefB2 is substituted with y, and *vice versa*); DefB3 ≡ Definition 10, sym (when R^* is substituted with *follows in the ϕ-series*, and a with z, and b with w, in DefB3); DefB4 ≡ Definition 11, sym (when m is substituted with x, and n with y, in DefB4). I give the list of the theorems to be proved below:

Thm1: $\#F = 0 \leftrightarrow \forall x{\sim}Fx$ (§75)
Thm2: $(mPn \mathbin{\&} m'Pn') \rightarrow (m = m' \leftrightarrow n = n')$ (§75)
Thm3: ${\sim}\exists x(xP0)$ (§78)
Thm4: $xRy \rightarrow xR^*y$ (Frege [1879], prop. 91)
Thm5: $(xR^*y \mathbin{\&} yR^*z) \rightarrow xR^*z$ (Frege [1879], prop. 98)
Thm 6: $xP^*n \rightarrow (\exists m(mPn) \mathbin{\&} \forall m(mPn \rightarrow (xP^*m \vee x = m)))$
Thm7: $0P^*n \rightarrow {\sim}nP^*n$ (§83)
Thm8: $(mPn \mathbin{\&} 0P^*n) \rightarrow \forall x(x \leq m \leftrightarrow (x \leq n \mathbin{\&} x \neq n))$ (§83)
Thm9: $(mPn \mathbin{\&} 0P^*n) \rightarrow \#[x\colon x \leq m]\ P\ \#[x\colon x \leq n]$
Thm10: $mPn \rightarrow ((0 \leq m \mathbin{\&} mP\#[x\colon x \leq m])$
$\rightarrow (0 \leq n \mathbin{\&} nP\#[x\colon x \leq n]))$ (§82)
Thm11: $0P\#[x\colon x \leq 0]$ (§82)
Thm12: $0 \leq n \rightarrow 0 \leq n \mathbin{\&} nP\#[x\colon x \leq n]$
Thm13: Finite $n \rightarrow nP\#[x\colon x \leq n]$

I also give below the Peano postulates. The reader may find it useful to observe the links between Thm1-Thm13 and the Peano postulates. [20]

(1) 0 is a natural number. (DefB1)

(2) If m is a natural number, and m precedes n, then n is a natural number. (DefB2)

(3) If m is a natural number and immediately precedes n and n', then $n = n'$. (Thm2)

(4) If m is a natural number, then there is a natural number n which is preceded by m. (Thm13)

(5) There is not a natural number which precedes 0. (Thm3)

(6) If m and m' are natural numbers and both precedes n, then $m = m'$. (Thm2)

(7) If 0 has a property F, and for any number m having the property F, it follows that the number n which is preceded by m also has the property F, then all natural numbers have that property F. (DefB3)

<u>Thm1:</u> $\#F = 0 \leftrightarrow \forall x{\sim}Fx$

1. $0 = \#[x: x \neq x]$ (DefB1)
2. $\#F = 0 \leftrightarrow F \approx [x: x \neq x]$ (1, HP)
3. ${\sim}\exists x(x \neq x)$
4. $F \approx [x: x \neq x] \leftrightarrow {\sim}\exists xFx$ (3)
5. $\#F = 0 \leftrightarrow \forall x{\sim}Fx$ (4, DefB1, HP, QN)

<u>Thm2:</u> $(mPn \;\&\; \mathrm{m'Pn'}) \rightarrow (m = m' \leftrightarrow n = n')$

[20] See Boolos [1996] for a complete analysis of this. The list of postulates, which are taken from Boolos [1996], pp. 145-146, does not include the operations of addition and multiplication. The two operations can be defined as follows: *Df(Addition):* $\forall x({\sim}\exists x(Fx \;\&Gx) \rightarrow \#F + \#G = \#[x: Fx \vee Gx])$; *Df(multiplication):* $\#F \times \#G = \#[(\mathrm{x, y}): Fx \;\&\; Gy]$.

I give the proof of Thm2 for the two directions of $m = m' \leftrightarrow n = n'$, separately.

<u>Thm2a:</u> $(mPn \ \& \ \text{m}'\text{Pn}') \rightarrow (m = m' \rightarrow n = n')$

1. mPn (assumption)
2. m′Pn′ (assumption)
3. m = m′ (assumption)
4. $\exists F \exists y(Fy \ \& \ \#F = n \ \& \ \#[x: Fx \ \& \ x \neq y] = m)$ (1, DefB2)
5. $Fy \ \& \ \#F = n \ \& \ \#[x: Fx \ \& \ x \neq y] = m$ (4, EI)
6. $\exists F \exists y(Fy \ \& \ \#F = n' \ \& \ \#[x: Fx \ \& \ x \neq y] = m')$ (2, DefB2)
7. $F'y' \ \& \ \#F' = n' \ \& \ \#[x: F'x \ \& \ x \neq y'] = m'$ (6, EI)
8. Fy (5, S)
9. $\#F = n$ (5, S)
10. $\#[x: Fx \ \& \ x \neq y] = m$ (5, S)
11. $F'y'$ (7, S)
12. $\#F' = n'$ (7, S)
13. $\#[x: F'x \ \& \ x \neq y'] = m'$ (7, S)
14. $\#[x: Fx \ \& \ x \neq y] = \#[x: F'x \ \& \ x \neq y']$ (3, 10, 13)
15. $[x: Fx \ \& \ x \neq y] \approx_{\varphi} [x: F'x \ \& \ x \neq y']$ (14, HP)
16. $\varphi' = \varphi \cup \{<y, y'>\}$
17. $F \approx_{\varphi'} F'$ (8, 11, 16)
18. $\#F = \#F'$ (16, HP)
19. $n = n'$ (9,12, 19)

<u>Thm2b:</u> $(mPn \ \& \ \text{m}'\text{Pn}') \rightarrow (n = n' \rightarrow m = m')$

1. mPn (assumption)
2. m′Pn′ (assumption)
3. $n = n'$ (assumption)
4. $\exists F \exists y(Fy \ \& \ \#F = n \ \& \ \#[x: Fx \ \& \ x \neq y] = m)$ (1, DefB2)
5. $Fy \ \& \ \#F = n \ \& \ \#[x: Fx \ \& \ x \neq y] = m$ (4, EI)
6. $\exists F \exists y(Fy \ \& \ \#F = n' \ \& \ \#[x: Fx \ \& \ x \neq y] = m')$ (2, DefB2)
7. $F'y' \ \& \ \#F' = n' \ \& \ \#[x: F'x \ \& \ x \neq y'] = m'$ (6, EI)
8. Fy (5, S)
9. $\#F = n$ (5, S)
10. $\#[x: Fx \ \& \ x \neq y] = m$ (5, S)
11. $F'y'$ (7, S)
12. $\#F' = n'$ (7, S)
13. $\#[x: F'x \ \& \ x \neq y'] = m'$ (7, S)
14. $\#F = \#F'$ (9, 12, 3)
15. $F \approx_{\psi} F'$ (14, HP)
16. $\exists! x(x \psi y')$ (15)
17. $\exists! x'(y \psi x')$ (15)
18. $\varphi = ((\psi - \{<\text{x}, \text{y}'>, <\text{y}, \text{x}'>\}) \cup \{<\text{x}, \text{x}'>\}) - \{<\text{y}, \text{y}'>\}$
19. $[x: Fx \ \& \ x \neq y] \approx_{\varphi} [x: F'x \ \& \ x \neq y']$ (18)

20. $m = m'$ (19, 10, 13, HP)

Thm3: $\sim\exists x(xP0)$

1. $\exists x(xP0)$ (assumption)
2. $aP0$ (1, UI)
3. $\exists F \exists y(Fy \,\&\, \#F = 0 \,\&\, \#[x\text{: } Fx \,\&\, x \neq y] = a)$ (DefB2, 2)
4. $Fb \,\&\, \#F = 0 \,\&\, \#[x\text{: } Fx \,\&\, x \neq b] = a$ (3, EI)
5. Fb (4, S)
6. $\#F = 0$ (4, S)
7. $\#[x\text{: } Fx \,\&\, x \neq b] = a$ (4, S)
8. $\#F = 0 \leftrightarrow \forall x \sim Fx$ (Thm1)
9. $\forall x \sim Fx$ (6, 8, MP)
10. $\sim Fb$ (9, UI)
11. $Fb \,\&\, \sim Fb$ (5, 10, Add)
12. $\sim\exists x(xP0)$ (11, 1, RA)

Thm4: $xRy \rightarrow xR^*y$

1. xRy (assumption)
2. $\forall a \forall b(((a{=}x \vee Fa) \,\&\, aRb) \rightarrow Fb)$ (assumption)
3. $((x = x \vee Fx) \,\&\, xRy) \rightarrow Fy$ (2, UI)
4. $(x = x \vee Fx) \rightarrow (xRy \rightarrow Fy)$ (3)
5. $x = x$
6. $x = x \vee Fx$ (5, Adj)
7. $xRy \rightarrow Fy$ (4, 6, MP)
8. Fy (1, 7, MP)
9. xR^*y (DefB3, 2, 8)

Thm5: $(xR^*y \,\&\, yR^*z) \rightarrow xR^*z$

1. xR^*y (assumption)
2. yR^*z (assumption)
3. $\forall a \forall b(((a{=}x \vee Fa) \,\&\, aRb) \rightarrow Fb)$ (assumption)
4. $((x = x \vee Fx) \,\&\, xRy) \rightarrow Fy$ (3, UI)
5. $((y = x \vee Fy) \,\&\, yRz) \rightarrow Fz$ (3, UI)
6. $((x = x \vee Fx) \,\&\, xR^*y) \rightarrow Fy$ (4, Thm4)
7. $((y = x \vee Fy) \,\&\, yR^*z) \rightarrow Fz$ (5, Thm4)
8. $((x = x \vee Fx) \rightarrow (xR^*y \rightarrow Fy)$ (6)
9. $x = x$
10. $x = x \vee Fx$ (9, Adj)
11. $xR^*y \rightarrow Fy$ (8, 10, MP)
12. Fy (1, 11, MP)

13. $(y = x \vee Fy) \rightarrow (yR^*z \rightarrow Fz)$ (7)
14. $Fy \vee y = x$ (12, Adj)
15. $yR^*z \rightarrow Fz$ (13, 14, MP)
16. Fz (2, 15, MP)
17. xR^*z (3, 16, DefB3)

Thm 6: $xP^*n \rightarrow (\exists m(mPn) \;\&\; \forall m(mPn \rightarrow (xP^*m \vee x= m)))$

1. xP^*n (assumption)
2. $\forall F(\forall a\forall b([(a = x \vee Fa) \;\&\; aPb]\rightarrow Fb) \rightarrow Fn)$ (1, DefB3)
3. $\forall a\forall b([(a = x \vee Fa) \;\&\; aPb]\rightarrow Fb) \rightarrow Fn)$ (2, UI)
3′. $F = [\text{z}: \exists m(mPz) \;\&\; \forall m(mPz \rightarrow (xP^*m \vee x = m))]$
4. $a = x \vee Fa$ (3, UI, assumption)
5. aPb (3, UI, assumption)
6. $\exists m(mPb)$ (5, EG)
7. mPb (6, EI)
8. $a = m$ (5, 7, Thm2)
9. $a = x \rightarrow x = m$ (4, 8)
10. Fa (assumption)
11. $\exists m'(m'Pa) \;\&\; (xP^*m' \vee x = m'))$ (3′,10)
12. $\exists m'(m'Pm) \;\&\; (xP^*m' \vee x =m'))$ (8, 10, 11, m/a)
13. $\exists m'(m'Pm)$ (12, S)
14. $m'\text{P}m$ (13, EI)
15. $xP^*m' \vee x = m'$ (12, S)
16. $m'P^*m$ (14, Thm4)
17. $xP^*m' \rightarrow xP^*m$ (15, 16, Thm5)
18. $x = m' \rightarrow xP^*m$ (15, 16, x/m′, Thm4)
19. xP^*m (15, 17, 18)
20. $\exists m(mPb) \;\&\; \forall m(mPb \rightarrow (xP^*m \vee x = m))$ (6, 7, 19)
21. Fb (20, 3′)
22. Fn (3, 4, 5, 20)
23. $\exists m(mPn) \;\&\; \forall m(mPn \rightarrow (xP^*m \vee x = m))$ (22, 3′)

Thm7: $0P^*n \rightarrow {\sim}nP^*n$

1. $0P^*n$ (assumption)
2. $\forall F(\forall a\forall b([(a = 0 \vee Fa) \;\&\; aPb]\rightarrow Fb) \rightarrow Fn)$ (1, DefB3)
3. $\forall a\forall b([(a = 0 \vee Fa) \;\&\; aPb]\rightarrow Fb) \rightarrow Fn)$ (2, UI)
3′. $F = [z: {\sim}zP^*z]$
4. $((a = 0 \vee Fa) \;\&\; aPb)\rightarrow Fb) \rightarrow Fn$ (3, UI)
5. $a = 0 \vee Fa$ (assumption)
6. aPb (assumption)
7. bP^*b (assumption) (Indirect)
7′. ${\sim}Fb$ (7, 3′)

8. aPb & $(aPb \rightarrow (bP^*a \vee b = a))$ (7, Thm6)
9. $bP^*a \vee b = a$ (8)
10. aP^*b (6, Thm4)
11. aP^*a (10, 9 , Thm5)
12. $\sim Fa$ (11, 3′)
13. $a = 0$ (5, 12, MTP)
14. $0P^*0$ (11, 13, $0/a$)
15. $\exists m(mP0)$ (14, Thm6)
16. $mP0$ (15, EI)
17. $\sim mP0$ (Thm3)
18. Fb (16, 17, 7′)
19. Fn (5, 6, 18, 4)
20. $\sim nP^*n$ (19, 3′)

Thm8: $(mPn \;\&\; 0P^*n) \rightarrow \forall x(x \leq m \leftrightarrow (x \leq n \;\&\; x \neq n))$

1. mPn (assumption)
2. $0P^*n$ (assumption)
3. mP^*n (1, Thm4)
4. $xP^*m \rightarrow xP^*n$ (3, Thm5)
5. $xP^*m \vee x = m \rightarrow xP^*n$ (4)
6. $\sim(nP^*n)$ (2, Thm7)
7. $x = n \rightarrow \sim(xP^*n)$ (6)
8. $xP^*n \rightarrow x \neq n$ (7)
9. $(x \leq n \;\&\; x \neq n) \rightarrow xP^*n$ (DefB4)
10. $xP^*n \rightarrow x \leq m$ (1, Thm6)
11. $x \leq m \rightarrow x \leq n$ (4, DefB4)
12. $(xP^*m \vee x = m) \rightarrow x \neq n$ (5, 8)
13. $x \leq m \rightarrow x \neq n$ (DefB4, 12)
14. $x \leq m \rightarrow (x \leq n \;\&\; x \neq n)$ (11, 13)
15. $(x \leq n \;\&\; x \neq n) \rightarrow x \leq m$ (9, 10)
16. $x \leq m \leftrightarrow (x \leq n \;\&\; x \neq n)$ (12, 13)

Thm9: $(mPn \;\&\; 0P^*n) \rightarrow \#[x: x \leq m]\; P\; \#[x: x \leq n]$

1. mPn (assumption)
2. $0P^*n$ (assumption)
3. $\forall x(x \leq m \leftrightarrow (x \leq n \;\&\; x \neq n))$ (1, 2, Thm8)
4. $[x: x \leq m] \approx [x: x \leq n \;\&\; x \neq n]$ (3)
5. $\#[x: x \leq m] = \#[x: x \leq n \;\&\; x \neq n]$ (4, HP)
6. $\exists F \exists y(Fy \;\&\; \#F = \#[x: x \leq n] \;\&\; \#[x: Fx \;\&\; x \neq y] = \#[x: x \leq m]$
   $\rightarrow \#[x: x \leq m]P\#[x: x \leq n]$ (DefB2)
7. $n \leq n \;\&\; \#[x: x \leq n] = \#[x: x \leq n] \;\&\; \#[x: x \leq n \;\&\; x \neq n] = \#[x: x \leq m]$
   $\rightarrow \#[x: x \leq m]P\#[x: x \leq n]$ (6, n/y, $[x: x \leq n]/F$)
8. $n \leq n$ (DefB4)

9. $\#[x: x \leq n] = \#[x: x \leq n]$
10. $\#[x: x \leq m]P\#[x: x \leq n]$ (8, 9, 5, 7)

<u>Thm10:</u> $mPn \rightarrow ((0 \leq m \ \& \ mP\#[x: x \leq m]) \rightarrow (0 \leq n \ \& \ nP\#[x: x \leq n]))$

1. mPn (assumption)
2. $0 \leq m$ (assumption)
3. $0P^*m \vee 0 = m$ (2, DefB4)
4. mP^*n (1, Thm4)
5. $0P^*m \rightarrow 0P^*n$ (4, 3, Thm5)
6. $0 = m \rightarrow 0P^*n$ (4, 0/m)
7. $0P^*n$ (3, 5. 6)
8. $mP[x: x \leq m]$ (assumption)
9. $\#[x: x \leq m] = n$ (8, DefB2)
10. $nP[x: x \leq n]$ (1, 7, 9, Thm4)

<u>Thm11:</u> $0P\#[x: x \leq 0]$

1. $0 \leq 0$ (DefB4)
2. $F = [x: x \leq 0]$ (assumption)
3. $F0$ (1, 2)
4. $xP^*0 \rightarrow \exists m(mP0)$ (Thm6)
5. $\sim(mP0)$ (Thm3)
6. $\exists x(Fx \ \& \ x \neq 0)$ (assumption)
7. $x' \leq 0$ (6, EI, S)
8. $x' \neq 0$ (6, EI, S)
9. $x'P^*0 \vee x' = 0$ (7, DefB4)
10. $x'P^*0$ (8, 9, MTP)
11. $\sim(mP^*0)$ (5, Thm4)
12. $\sim\exists x(Fx \ \& \ x \neq 0)$ (6, 10, 11)
13. $\#[x: Fx \ \& \ x \neq 0] = 0$ (12, Thm1)
14. $\exists F\exists y(Fy \ \& \ \#F = \#[x: x \leq 0] \ \& \ \#[x: Fx \ \& \ x \neq 0] = 0)$ (2, 3, 14, EG)
15. $0P\#[x: x \leq 0]$ (15, DefB2)

<u>Thm12:</u> $0 \leq n \rightarrow 0 \leq n \ \& \ nP\#[x: x \leq n]$

1. $0 = n$ (assumption)
2. $0 \leq 0 \ \& \ 0P\#[x: x \leq 0]$ (1, Thm11)
3. $0P^*n$ (assumption)
4. $\forall F(\forall a\forall b([(a = 0 \vee Fa) \ \& \ aPb] \rightarrow Fb) \rightarrow Fn)$ (3, DefB3)
5. $\forall a\forall b([(a = 0 \vee Fa) \ \& \ aPb] \rightarrow Fb) \rightarrow Fn$ (4, EI)

5′. $F = [z: 0 \leq z \ \& \ zP\#[\mathrm{x}: x \leq z]]$

6. $(m = 0 \vee Fm)$ & $mPn] \rightarrow Fn) \rightarrow Fn$ (5, UI)
7. $m = 0 \vee Fm$ (assumption)
8. mPn (assumption)
9. $m \neq 0 \rightarrow Fm$ (7)
10. $m = 0 \rightarrow Fm$ (5′, Thm11)
11. Fm (9, 10)
12. Fn (8, 11, Thm10)
13. $0 \leq n$ & $nP\#[x: x \leq n]$ (5′, 12)

Thm13:   Finite $n \rightarrow nP\#[x: x \leq n]$

1. Finite n (assumption)
2. $0 \leq n$ (1, DefB5)
3. $nP\#[x: x \leq n]$ (2, Thm12)

Hence that there is no last member of the series of natural numbers is proved. Having shown how the proofs, which are sufficient to conclude that there are infinitely many numbers can be carried out, we see that dispensing with intuition in arithmetic can be traced back to one basic axiom, namely Hume's Principle. Frege could, without mentioning the necessity to prove them, give intuitive definitions. For instance, instead of Definition 10, something like the one below could be given:

> If starting from *x* we transfer our attention continually from one object to another to which it stands in the relation ϕ, and if by this procedure we can finally reach *y*, then we say that *y* follows in the ϕ-series after *x*. (Frege [1884], §80)

However, this would not be appropriate for the logicist aims. Although it may well describe a process of discovery that *y* follows *x*, it is not a true justification. Moreover, it does not define what is meant by *y*'s following *x* (ibid, prg 3). Frege continues,

> Whether, as our attention shifts [as it is pointed out in the above definition], we reach *y* may depend on all sorts of subjective contributory factors, for example on the amount of time at our disposal or on the extent of our familiarity with the things concerned.

But we are trying to get rid of the subjective contributory factors in logicism. It is an interesting fact that when we apply this criticism to what we consider to be logical, i.e. to our way of discovering the logical laws, we will have a similar difficulty concerning the logical definitions. For they would be apt to be subjective, contributory factors in case of such a criticism. Hence the so-called logical definitions rest on non-logical factors. But, without appealing to intuition in the definitions, we can at least reach some definitive conceptions, which can be considered as a melting point of the pure logical (frozen) realm – if there is such a realm. We are able to make, thus, a distinction between the logical and the psychological.

My next concern is to judge how much appeal to intuition is involved in Hume's Principle. I am no more dealing with how much intuition there is in formulating Hume's Principle; because the obvious answer is that there is some. I am rather dealing with what Frege puts forward as his aim, viz., to show that arithmetic is analytic. Thus a question that needs to be asked is whether Hume's Principle is analytic. In order to count Hume's Principle as an analytic truth we must be able to prove it by means of logical laws and definitions (ibid, §3). Or else, we need a good reason to call Hume's Principle something that neither needs nor admits of proof, i.e. a primitive logical law. Both of these are problematic. For the former to be the case, every time we make a foundational definition, implies proofs to have no end., whereas at the end we need to stop at a contextual definition as the foundation of arithmetic. On the other hand, a good reason to count Hume's Principle as a primitive logical law is hard to find, because Hume's Principle asserts something about the special function "the number of", such that this assertion leads to the difficulty concerning the applicability of it to all contexts in which we talk about numbers. Hume's Principle does not, for instance, enable us to judge whether #(moons of Jupiter) = Julius Caesar, or not. (Ibid, §56) That is to say, Hume's Principle is not sufficient to fix the referents of number words, though it fixes the sense of the expression "the number of". This problem was thought to be solved, through the introduction of numbers as extensions of concepts. (Ibid, §68) For Julius Caesar refers to the human being, whereas the extension of a concept is so described that it refers to the corresponding non-spatio-temporal object. Here, although it still may be argued that the problem with Julius Caesar remains for the extensions also, we are not going to go into it.[21] For it is indeed well known that extensions of concepts led to the Russell Paradox. But what about Hume's Principle? What if we derive a contradiction from it?

> Yet it must be borne in mind that the rigor of the proof remains an illusion, eventhough no link be missing in the chain of our deductions, so long as the definitions are justified only as an afterthought, by our failing to come across any contradiction. (Ibid, p. ix, para 3)

Hence we must make sure that we will not come across any contradiction with Hume's Principle. This will presumably be a gain concerning the value of Hume's Principle for logicism. In fact, Hume's Principle is model-theoretically consistent, i.e. it is true in models with infinite domain. However, it is false in models with finite domains[22], hence it is logically invalid. Thus, in this sense, Hume's Principle cannot be analytic.

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[21] See Hodes [1984], pp. 136-139, and Heck [1997] for further discussion.

[22] Take a two element-domain D ={0, 1}. We can find two concepts F and G, such that the number of Fs is equal to the number of Gs, but F and G are not one-to-one correlated. In model theory concepts are members of the power-domain. Thus F and G will be taken from the members of the power-domain $P(D)$ = { {}, {0}, {1}, {0, 1}}. The function "#" will be a function from $P(D)$ to D. Here when F = {0}, we say #F =1. Similarly #{0, 1} has to be either 0 or 1, because the function "#" is defined as from $P(D)$ to D. But then #{0, 1} will unavoidably be equal to either one of the #{}, or #{0}, or #{1}, whereas obviously none of {}, {0}, or {1} is one to one correlated with {0, 1}. On the other hand, we can define the function "#" from P(D) to D, where D is infinite, in such a way that Hume's Principle holds. For example, it can be defined as that for all finite Fs #F = n + 1, and #F = 0. (See Boolos [1996], p. 145)

Suppose that we have found a logical principle from which arithmetic can be derived. If this amounts to saying that there is a finite list of logical (analytic) axioms, which suffice to prove all arithmetical truths, then there must be a countable list of truths of arithmetic. Nevertheless, there is no such list of truths of arithmetic as we know by Gödel incompleteness. Therefore, there cannot be such a list for a system of logic strong enough to derive all truths of arithmetic either. This may mean two things. It may be the case that arithmetic is not analytic all over. It may also be the case that there are uncountably many truths of logic (see Benacerraf [1981], pp.65-66). Frege would not like any of the two. For the former goes against Frege's basic thesis that truths of arithmetic are analytic in the sense of logical provability. Similarly, the latter goes against Fregean rigor. For rigor prefers "[t]he further we pursue [the] enquiries, the fewer become the primitive truths to which we reduce everything." (Frege [1884], §2). However, a mystically oriented account of logicism may claim that there are in fact uncountably many truths of logic and mathematics. That kind of approach would be strongly in need of a revolutionary account of logical and mathematical proof, hence it is little creditable for the time being.

Beside the counter arguments to logicist inclinations, it is quite appreciating that a single principle such as Hume's Principle is sufficient to generate all the arithmetic that can be done with the Peano axioms. In this sense, "[t]he further we pursue[d] the fewer bec[a]me the primitive truths to which we reduce everything" (ibid), where the meaning of the term "primitive truth" is a little bit distorted.[23]

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[23] If we add the modal possibility operator at the beginning of Hume's Principle, then it will be considered as logically valid, i.e.$\Diamond\forall F\forall G(\#F = \#G \leftrightarrow F \approx G)$, (see Hodes [1984]). However, this would open a new debate on whether modalities can be counted as logical constants. Thus we quit it here and defer similar discussions to Chapter 3.

CHAPTER 3

ON THE LOGIC OF LOGICISM

3.1 Arguments against the view that higher-order logic is not logic

The argument which comes from Benacerraf [1960] and Quine [1970] – I shall call it the *first-order thesis*, in accordance with Barwise [1985] – defends the view that "the logic capable of encompassing [the reduction of arithmetic to logic is] logic inclusive of set theory";[24] therefore, cannot be counted as logic, but mathematics. The first-order thesis maintains a distinction between mathematics and logic. The distinction is not refutable at first sight. For without starting from such a distinction, even the basic logicist claim that mathematics is logic turns out to be a triviality. We can *conclude* that the apparent distinction between mathematics and logic is a lie-distinction. But we cannot start with the very lie that there is no distinction at all.

Whether the concepts "class" and "membership, as they are used in set theory, are logical concepts is disputed. It is a matter of demarcating the logical from the non-logical, i.e. "why do logicians count 'is' logical and 'eats' non-logical?" (Musgrave [1977], p. 103). Probably, because of the much wider applicability of "is" than "eats". For everything *is* something, but not everything *eats* something. Whether the case of the verb "eats" is similar to that of the membership relation "$\in$" is the problem here. Yet from the logicist standpoint, concepts such as "class" and "membership" belong to logic in the sense all mathematical concepts, at the end, *belong* to logic.

Without quantification over predicates, functions etc. the logicist view is hopeless. In 2.3 we have considered the case of Frege, and saw that the basic principle of Frege's system, viz. Hume's Principle, was a second-order formula quantifying over predicates.[25]

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[24] Quine [1970], p. 66. Due to Boolos [1996], Benacerraf, similar to Quine, argues in his *Logicism: Some Considerations* (Ph.D. Dissertation, Princeton, 1960) that a system strong enough to give us arithmetic should not be counted as logic.

[25] For sure this is just an example, and not conclusive about the inadequacy of the first-order logic in formulating mathematics. We shall consider some conclusive arguments

My argument here is for the view that higher-order logic is logic. Quine, on the contrary, argues that higher-order logic is not logic, but mathematics. The main point of his argument is as follows:

> Consider first some ordinary quantifications: '∃*x* (*x* walks)', '∀*x* (*x* walks)', '∃*x* (*x* is prime)' The open sentence after the quantifier shows '*x*' in a position where a name could stand; a name of a walker, for instance, or of a prime number. The quantifications do not mean that names walk or are prime; what are said to walk or to be a prime are things that could be named *by* names in those positions. To put the predicate letter '*F*' in a quantifier, then, is to treat predicate positions suddenly as name positions, and hence to treat predicates as names of entities of some sort. The quantifier '∃*F*' or '∀*F*' says not that some or all predicates are thus and so, but that some or all entities of the sort named by predicates are thus and so. (Quine [1970], pp. 66-67)

Why Quine is uncomfortable with quantification over predicates is clear. He is not happy with the entities whose names are *F*, walking, primeness, wisdom etc. In Putnam's words,

> [He] is not likely to say:
> (A) "For all classes *S*, *M*, *P*: if all *S* are *M* and all *M* are *P*, then all *S* are *P*.
> He is more likely to write:
> (B) The following turns into a true sentence no matter what words or phrases of the appropriate kind one may substitute for the letters *S*, *M*, *P*: 'if all *S* are *M* and all *M* are *P*, then all *S* are *P*'."
> …[He] does not really believe that classes exist; so he avoids formulation (A). In contrast to classes, "sentences" and "words" seem relatively "concrete" so he employs formulation (B). (Putnam [1971], pp. 9-10)

Here, as Putnam puts it, "we must face the fact what is meant [by the words or phrases of the appropriate kind in formulation (B)] is all *possible* words and phrases of some kind or other, and that *possible words and phrases* are no more 'concrete' than classes are." (Ibid, p. 10). One may avoid this difficulty by defining a formal language and determining the possible substitution instances of *S*, *M*, and *P* in that formal language. This time, there

presently.

will be a problem concerning the concept of validity. For we shall inescapably define 'valid' in different languages seperately, as being true of all substitution instances in each defined language. This plurality of definitions of validity cannot be reduced to one general definition of validity as being true of all substitution instances in every language. For the concept "*all languages*" is "less 'concrete' than the notion of a 'class'." (Ibid, pp.10-11). Putnam's conclusion, with which we agree, is that "reference to 'classes' [or quantification over classes, propositions, etc.], or something equally 'non-physical' is indispensable to the science of logic. …at least today." (Ibid, p. 23)

Quine, on the other hand, argues that we need not say "for all classes" in the quoted formulation (A) of Putnam. The symbolization "$\forall$x ([(Sx $\rightarrow$ Mx) & (Mx $\rightarrow$ Px)] $\rightarrow$ (Sx $\rightarrow$ Px))" will suffice, and we will not be in need of referring to classes in such a symbolisation of (A). *S*, *M*, and *P* are, for Quine, schematic letters standing for any predicate you like (ibid, p. 27; Quine [1970], p. 66). Putnam seems to be right here in criticising the use of the phrase 'any predicate you like' as if there were no philosophical (i.e. referential) problem with it. From these considerations, we at least conclude that the meta-theory of first-order logic is bound to commit into the existence of non-concrete entities such as classes. But this is not to argue that set theory is logic. I have no "tendency to see set theory as logic" and to "overestimate the kinship between membership and predication". (Quine [1970], p.66) For the indispensability of referring to abstract entities in logic does not imply that an axiomatic theory of those entities is to be counted as logic. The argument I am here trying to present is for the logical status of higher-order logic. As I have mentioned at the beginning of this section, whether membership and class – in their set theoretical uses – are logical concepts is a disputed issue, and my investigation is away from that dispute. However, we know that there are essential differences between higher-order logic and set theory.[26] Those who argue

[26] See Boolos [1975] for a detailed analysis of the similarities and differences between higher-order logic and set theory. An example of difference is the loss of validity of some second-order formulas in set theoretical notation viz., $\exists F \forall x\, Fx$ is logically valid, but $\exists\alpha\forall$x

that higher-order logic is logic distinguish logical class and set theoretical class.[27] Shapiro, for instance, puts, in his [1985], that

> …understanding the second-order quantifiers of a given theory is not the same as grasping the set theoretical hierarchy. In a given theory, the quantifier "all subsets" ranges over the collection of subsets of a *fixed domain*. In general there is no powerset operator to be iterated… The set theoretic hierarchy, on the other hand, is a proper class that contains the result of iterating the powerset operator into the transfinite. (Ibid, p. 721)

The indispensable use of class-like entities in logic may be interpreted as that logic cannot be counted as topic-neutral. The view that topic-neutrality is a characteristic of logic is used to support the first-order thesis. Quine and others try to preclude logic from any special subject matter, and let it be transparent. The way we understand topic-neutrality is crucial in understanding whether the occurrence of classes and class-like entities in the logical language leads logic to be a topic-loaded area. For example, as Boolos says in his [1975], one may argue that logic is not topic-neutral as it is thought to be. One can say that logic, even in the form defended by many logicians to be topic-neutral, is about certain linguistic entities, i.e. negation, disjunction, conjunction, quantifiers, etc. (Boolos [1975], p. 517). But the way I understand topic-neutrality differs from this, and I do not conclude that logic is not topic-neutral just because it is about negation, conjunction and the like. Logic is topic-neutral because it is fairly applicable to many special subjects, and is a guide in our argumentations in those subjects. Thus classes and the like are those entities acceptable as constituents of a topic-neutral logic, as long as they do not distort *this* sense of topic-

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$x \in \alpha$ is not. Another example is that $\forall Y(Yx \rightarrow Yz)$ implies $x = z$, but $\forall \alpha(x \in \alpha \rightarrow z \in \alpha)$ does not.

[27] Granted that we quantify over class-like entities in logic, we shall be led to a circularity concerning the logical description of set theory as one of the branches of mathematics, as long as we cannot make a clear cut distinction between the logical and mathematical conception of class.

neutrality. I have briefly considered a similar point concerning the topic-neutrality of mathematics in 2.1.

An argument against the use of class-like entities in logic may be formulated by referring to the dilemma introduced by Benacerraf [1973]. Benacerraf, in his challenging paper, pointed out that satisfactory accounts of mathematical truth fail in explaining mathematical knowledge, and others which are successful in accounting for mathematical knowledge fail in explaining mathematical truth. Accepting Tarski semantics, we become realists in ontology of mathematics, but fall into the difficulty of explaining how the knowledge of a realm of abstract mathematical entities is attainable for us. Alternatively, being anti-realists we make mathematical knowledge attainable, however, purely syntactic procedures fail in giving a full account of mathematical truth against Gödel's incompleteness results.

In this way, I seem to have carried the dilemma concerning mathematical knowledge and truth to the logical knowledge and truth, by agreeing with Putnam in that class-like entities are indispensable in logic. The criticism is completely sound, for the logicist claims provide full satisfaction only if they throw light onto both the semantical and the epistemological problems concerning the philosophy of mathematics. At present we lack such an account. Hence, we have to face with the dilemmaic situation. I start from the observation that we already have mathematical knowledge, somehow. I may, for instance, buy the view that mathematics needs no foundation. This would be close to some sort of *methodological Platonism*.[28] In that case, our arguments for the logicist philosophy can have no epistemological claim, i.e. logicism cannot provide epistemological support for mathematics. By holding this view, I am just trying to push the argument for logicism, to the extent it is possible, where mathematics and logic will be taken as parts of the same

[28] This term is said to belong to M. Resnik. See Shapiro [1985], p.715. A similar view viz., *neutral realism*, is held by Shapiro (ibid).

subject.[29] On the other hand, if logicism had succeeded in what Frege dreamt it to do – i.e. if the whole of arithmetic could be derived from analytic truths of logic – then it would have been a solution both to the semantical and the epistemological problems.

We had marked that first-order logicism is hopeless. We are now going to consider an argument for why this is so.[30] The argument is due to Shapiro [1985]. Shapiro distinguishes between two purposes of axiomatization of a mathematical theory. In Shapiro's words, "[o]ne purpose ... is to organize and systematically present the truths and correct inferences of the [theory]," and the second purpose "is to *describe* a particular structure, an intended interpretation of a [mathematical theory]" (ibid, pp. 716-717). Concerning the former purpose our goal is completeness, and concerning the latter, the goal is categoricity.[31] Since I argue for the higher-order logic to be logic, I adopt Shapiro's argument in favor of my logicist line.

Due to the incompleteness of some theories of mathematics, i.e. of arithmetic, the former goal is impossible to achieve. This creates a problem for a complete deductive success of logicism too, but logic, as I prefer to advocate here, needs not have a deductive power which mathematics lacks. Indeed I do not argue that logic can do more than mathematics can. On the other hand, to handle what mathematics can, categoricity is a key concept to be worked on, and is a needed property for any system of logic the aim of which is at least to describe mathematical structures. To quote from Shapiro, "...if an

[29] This view is faithful to Church [1956], §55, Putnam [1971] Ch. IV, and Shapiro [1985].

[30] The inadequacy of first-order logic in formulating some essential mathematical concepts such as finitude, mathematical induction is one important aspect of the matter. I am here interested in a more fundamental argument that investigates the importance of the concept of categoricity in mathematical practice. For an outline of comparisons between first and second-order logics, see Shapiro [1985].

[31] "[A]n axiomatization – language and deductive system – is *complete* iff it has as theorems all (and only) the truths of that branch. …[A]n axiomatization – language and *semantics* – is *categorical* iff any two of its models are isomorphic." (Shapiro [1985], pp. 716-717)

axiomatization correctly describes a structure, then it also describes any isomorphic structure, ...for the purpose of description, a categorical axiomatization is the best we can do." (Ibid, p. 717) The important role that categoricity plays concerns the communication of mathematical language, i.e. "[e]ven if two mathematicians agree on an axiomatization of, say, arithmetic..., they cannot be sure that they have in mind the same ...interpretations of their agreed-on axiomatization." (Ibid, p. 719) In order to make sure that two mathematicians are talking about the same *realm*, we need to know that the axiomatization they both work on is a categorical one. Shapiro puts it, remarkably, as follows:

> One could, I suppose, postulate a faculty of mental telepathy between mathematicians to account for the communication of structures; but, without this, all communication is mediated by language. This is where categoricity is important. (Ibid, p. 720)

The fact that "Löwenheim-Skolem theorems imply that no set of sentences in a first-order language can be a categorical description of an infinite structure" (Ibid, p. 718), establishes the inadequacy of first-order attempts to describe mathematical structures.[32] This also establishes the superiority of second-order over first-order, for there are categorical sets of second-order sentences with infinite domains.[33]

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[32] Löwenheim's theorem says that "[A] first-order wff is valid in a countably infinite domain, [iff] it is valid in every non-empty domain.[414]" (Church [1956], p. 238) The generalization of this theorem, by Skolem, is seen needless to be given here. It is evident that Lövenheim's theorem suffices to see that we can find two non-isomorphic models of an infinite structure, which is described by using first-order logic; i.e. in Church's words "…[I]f a [first-order] wff is valid in the domain of natural numbers, it is also valid in [uncountably] infinite domains. If a [first-order] wff is satisfiable in [an uncountably] infinite domain, then it is satisfiable also in the domain of natural numbers." (Ibid, note 414)

[33] A simple example is the second-order induction principle, i.e. $\forall F(F0 \;\&\; \forall y(\mathrm{F}y \rightarrow \mathrm{Fs}(y)) \rightarrow \forall x\, \mathrm{F}x)$. For the proof of that the natural number structure can be described categorically see Dedekind [1888]. Note that it is presupposed above, and in any categorical set of second-order sentence, that second-order logic is unambiguously understood. We have no immediate reply to this very difficulty.

Up to here, in this section, I have argued against the view that higher-order logic is not logic. We saw that we are indispensably referring to abstract entities in logic and this gives logic a descriptive superiority – beside its epistemological and deductive weakness – over the logic that first-order thesis suggests. Now, admitting abstract entities, of high generality, in logic brings the conclusion that there is nothing wrong in admitting also what are referred as logics embodying mathematical concepts.[34] Barwise, in his [1985], asks the question

> ...given a particular mathematical property (like being a finite, infinite, countable, uncountable, or open set, or being a well-ordering or a continuous function, or having probability greater than some real number *r*), what is the logic implicit in the mathematician's use of the property? (Barwise [1985], p. 3)

We now seek to understand the logic of the use of mathematical concepts then. A logic which make use of some mathematical concepts "consists of a collection of mathematical structures, a collection of formal expressions, and a relation of satisfaction between the two." (Ibid, p.4) This conception of logic rests on a naive argument against the first-order thesis. Barwise calls it the mathematician-in-the-street conception of logic viz., logic as the study, in a specialized manner, of the valid forms of reasoning in mathematics. Barwise argues, against the first-order thesis, that

> [F]irst-order logic is just an artificial language constructed to help investigate logic much as the telescope is a tool constructed to help study heavenly bodies. From the perspective of the mathematician in the street, first-order thesis is like the claim that astronomy is the study of the telescope. (Ibid, p. 6)

[34] See Barwise [1985] for an outline of the study of such logics. Logics embodying mathematical objects use generalized quantifiers such as finitely many, countably many and like, and infinitely long formulas etc. See also Bostock [1974] for a logicism that treats numbers as generalized cardinality quantifiers. In such an account we say "for four x…", just like we say "for all x…" in ordinary quantification.

To a certain extend I adopt a similar conception of logic. That this conception may let us construct descriptively powerful logics can be the main argument which I can give in addition to my view that higher-order logic is logic.[35] But I am not going to go over it here.[36] In the next section, I am going to discuss the implications of the deductive deficiencies of higher-order logic concerning the logicist philosophy of mathematics.

3.2 A short note on the deductive aim of logicism

A descriptively strong system of logic enables us to fix the sense of a mathematical strucrure; as we saw in 3.1. However, descriptive strength brings a deficiency. It is that descriptively strong logics lack completeness, i.e. they are no use helpful in proving every described truth. N. Tennant, in his [2000], studies the interplay between the two crucial properties, namely completeness and categoricity, of logical systems. Tennant points out that a system of logic cannot have full *expressive* (descriptive) and *deductive* powers at the same time. He formulates this under the name "noncompossibility theorem", and gives a proof of it. (Tennant [2000], p. 272) What does this teach us? Concerning our case, since we permit higher-order quantification in logic, for the time being, our main problem is the lack of deductive power, i.e. we can never hope to prove everything that we describe with our logic. For Barwise, this is something concerning the complexity of our system of logic that have to be learnt to live with. (Barwise [1985], p. 7) However, to suggest the logicist to live with it is to tell him to give up the deductive aim (almost the half) of his main thesis. Yet the logicist has no alternative. He has to be satisfied with limited power. Optimizing this negative result, we may argue that the deductive ideal of logicism was a consequence

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[35] Aiming at better descriptions by means of logic, we are close to the later Wittgenstein's conception of logic, concerning *mathematical* games. "…[E]verything descriptive of a language game is part of logic." (Wittgenstein [1969], §56)

[36] See Read [1998]. Read argues for logicism with logics embodying mathematical concepts. He suggests an ongoing, an unaccomplished project of logicism.

of misconception about the nature of mathematical truth, and we can ascribe the misconception to the early logicists, not to ourselves. But what then can the use of the logicist philosophy of mathematics be? I already pointed, in 3.1, that logicism cannot solve the epistemological problem. For even if all mathematics could be shown to be a totality of tautologies, the referential access problem concerning the constituent parts of tautologies would remain.[37] Hence the conclusion that we are inclined to infer – that mathematics and logic, contrary to appearance, are not distinct subjects – may just be in favor of *rigorous descriptions* of mathematics; and may neither be in favor of epistemological foundations nor deductive success. This gives us nothing but insight about the logical structure of mathematics.

I aim to conclude that every attempt of description of mathematical theories is potentially a describable piece of logic (and hence mathematics), and also that "... [E]verything descriptive of a [mathematical] game is part of logic." (Wittgenstein [1969], §56)

[37] According to Tarski's definition of truth, we say that the sentence "2 + 1 = 3" is true iff 2 + 1 = 3. Since there is no 2 + 1 = 3 in nature, we let it be out of space and time. Similarly we can say that the sentence "p ∨ ~p" is true iff p ∨ ~p. By the same reasoning this is not a spatio-temporal fact either, it seems.

CHAPTER 4

CONCLUSIONS

In section 2.1, where my considerations started, I dealt with topic-neutrality. I have a simple claim that there is some sort of topic-neutrality in mathematics. Its source is the undeniable width of applicability and generality of mathematics. Once again appealing to Wittgenstein:

> In life it is never a mathematical proposition which we need, but we use mathematical propositions *only* in order to infer from propositions which do not belong to mathematics to others which equally do not belong to mathematics. (Wittgenstein [1921], 6.211)

In section 2.1, I considered whether one can give up intuition in mathematics. I divided the problem into two as whether one can give it up in *doing* mathematics and whether one can give it up in the outcome knowledge of mathematics. I argued that intuition is not dispensable for the former, bu for the latter. Intuition is dispensable at least for simpler intuitions, and this is what is called rigor. To support our arguments, we considered the logical/psychological distinction, which has a crucial importance in understanding the nature of mathematical knowledge. Despite the fact that we lack a clear-cut distinction of logic and psychology – psychology as the science of cognitive states and processes of the human mind – I found mathematics to be much more closer to logic than to psychology.

In section 2.2, I studied Frege's [1884], §§68-81, where Frege gives an outline of his logicist theory of number. I presented a compact analysis of Fregean definitions and proofs, and came to the conclusion that Frege's system of arithmetic can be based on one consistent axiom, namely Hume's Principle. I briefly discussed the logical value of Hume's Principle, and although it seems not to be an analytic truth, I concluded that it is a success of logical rigor to base all arithmetic that can be done with Peano axioms on a single principle.

A general conclusion to Chapter 2 is that we can, to a considerable extent, dispense with intuition in mathematical knowledge, which may be, for a philosopher of mathematical activity, a needless task to urge.

In section 3.1, I argued against Quine-Benacerraf view that higher-order logic should not be counted as logic. I agreed with Putnam in that abstract entities are indispensably used, quantified over, and referred to in logic. I then considered Shapiro's argument for the superiority of higher-order over the first-order, concerning the problem of how people understand others' use of mathematical language. However, I noted, there is an important deficiency of logic with abstract entities, in relation with the challenging paper of Benacerraf ([1973]). This deficiency is that logicism with abstract entities cannot provide an epistemological foundation for mathematics. For we being concrete beings – if we do not want to turn our problem into a problem concerning the nature of human mind – lack causal relations with abstract beings. One other deficiency is shortly considered in section 3.2, in relation with N. Tennant's [2000]. We faced the fact that what Tennant refers to as the noncompossibility theorem states that both of the descriptive and deductive aims of logicism cannot be attained at the same time. I concluded from this that our conception of logic which is pregnant to descriptively strong logics cannot give us a complete deductive power. This means, as a conclusion to Chapter 3, that what we can argue for is a descriptive logicism, and thus it is possible to describe a logical picture of mathematics, but not to paint the very picture itself.

As a general conclusion to the whole of my considerations, I put forward that the possibility of rigorous descriptions of mathematical theories suggest that logic and mathematics are too close areas of study. They are similar in both their efficiencies and deficiencies.[38] Thus, at the end of the considerations, I am in accordance with the words of B. Russell, which we quoted on page 1, and I end up with an appeal to authority: "...[L]ogic and mathematics should be characterized, not as different subjects, but elementary and advanced parts of the same subject." (Church [1956], p. 332)

[38] What the descriptive and deductive powers of logics interplay is analogous to the interplay between mathematical truth and mathematical proof.

REFERENCES

Barwise, J., [1985], "Model Theoretic Logics: Background and Aims", in J. Barwise, and S. Feferman (eds), *Model theoretic Logics*, Springer Verlag, 1985.

Benacerraf, P., [1964], "Introduction", in P. Benacerraf and H. Putnam (eds), *Philosophy of Mathematics*, Cambridge University Press, 1983.

Benacerraf, P., [1973], "Mathematical Truth", in P. Benacerraf and H. Putnam (eds), *Philosophy of Mathematics*, Cambridge University Press, 1983.

Benacerraf, P., [1981], "Frege: The Last Logicist", in W. Demopoulos (ed), *Frege's Philosophy of Mathematics*, Harvard University Press, 1995.

Boolos, G., [1975], "On Second-Order Logic", *Journal of Philosophy*,**72**: 509-527.

Boolos, G., [1981], "The Standard Equality of Numbers", in W. Demopoulos (ed), *Frege's Philosophy of Mathematics*, Harvard University Press, 1995.

Boolos, G., [1996], "On the Proof of Frege's Theorem", in A. Morton and S.P. Stich (eds), *Benacerraf and His Critics*, Coutridge, Mass: Blackwell, 1996.

Bostock, D., [1974], *Logic and Arithmetic vol 1*, Oxford University Press, 1974.

Church, A., [1956], *Introduction to Mathematical Logic* vol 1, Princeton University Press, 1956.

Coffa, A., [1982], "Kant, Bolzano and the Emergence of Logicism", in W. Demopoulos (ed), *Frege's Philosophy of Mathematics*, Harvard University Press, 1995.

Dedekind, R., [1888], "The Nature and Meaning of Numbers", in R. Dedekind, *Essays on the Theory of Numbers*, The Open Court Publishing Company, 1924.

Demopoulos, W., [1994], "Frege and the Rigorization of Analysis", in W. Demopoulos (ed), *Frege's Philosophy of Mathematics*, Harvard University Press, 1995.

Detlefsen, M., [1992], "Poincare Against the Logicians", *Synthese*, **90**: 349-378.

Feferman, S., [1981], "The Logic of Mathematical Discovery vs. The Logical Structure of Mathematics", in S. Feferman, *In the Light of Logic*, Oxford University Press, 1998.

Field, H., [1984], "Is Mathematical Knowledge Just Logical Knowledge?", *Philosophical Review* **93**: 509-552.

Frege, G., [1879], "Begriffsscrift: A Formula Language, modeled upon that of Arithmetic, for Pure Thought", in J. Van Heijnoort (ed), *From Frege to Gödel: A Sourcebook in Mathematical Logic, 1879-1931*, Harvard University Press, 1967.

Frege, G., [1884], *The Foundations of Arithmetic*, Northwestern University Press, 1980.

Frege, G., [1903], *The Basic Laws of Arithmetic*, Berkeley: University of California Press, 1964.

Hardy, G.H., [1959], "The Indian Mathematician Ramanujan", in G.H. Hardy *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*, Chelsea Publishing Company, New York, 1959.

Heck, R. Jr., [1997], "The Julius Caesar Objection", in R. Heck Jr. (ed), *Language, Thought, and Logic*, Oxford University Press, 1997.

Heyting, A., [1956], *Intuitionism: An Introduction*, North-Holland Publishing Company, 1966.

Hodes, H., [1984], "Logicism and the Ontological Commitments of Arithmrtic", *Journal of Philosophy*, **81**: 123-149.

Leibniz, G. W., [1765], *New Essays on Human Understanding*, P. Remnant and J. Bennett (eds), Cambridge University Press, 1996.

Musgrave, [1977], "Logicism Revisited", *British Journal for the Philosophy of Science*, **28**: 99-127.

Placek, T., [1999], *Mathematical Intuitionism and Intersubjectivity*, Kluwer Academic Publishers, 1999.

Pulkkinen, J., [1994], *The Threat of Logical Mathematism: A Study of on the Critique of Mathematical Logic in Germany at the turn of the 20th Century*, Peter Lang Gmb H, 1994.

Putnam, H., [1971], *Philosophy of Logic*, George Allen & Unwin Ltd, London, 1972.

Quine, W.O., [1970], *Philosophy of Logic*, Harvard University Press, 1986.

Ranganathan, S. R., [1967], *Ramanujan: The Man and the Mathematician*, Asia Publishing House, 1967.

Read, S., [1997], "Completeness and Categoricity: Frege, Gödel, and Model Theory", *History and Philosophy of Logic*, **18**: 79-93.

Russell, B., [1919], *Introduction to Mathematical Philosophy*, George Allen and Unwin Ltd, London, 1963.

Shapiro, S., [1985], "Second-order Languages and Mathematical Practice", *The Journal of Symbolic Logic*, **50**: 714-742.

Steiner, M., [1975], *Mathematical Knowledge*, Cornell University Press, Ithaca and London, 1975.

Tennant, N., [2000], "Deductive versus Expressive Power: A Pre-Gödelian Predicament", *Journal of Philosophy*, **97**: 257-277.

Van Dalen, D., [1999], *Mystic, Geometer, and Intuitionist: The Life of L.E.J. Brouwer*, Clarendon Press, Oxford, 1999.

Wittgenstein, L., [1921], *Tractatus Logico Philosophicus*, Routledge & Kegan Paul Ltd, 1960.

Wittgenstein, L., [1953], *Philosophical Investigations*, Blackwell Publishers, 1997.

Wittgenstein, L., [1959], *On Certainty*, Basil Blackwell, 1969.