Hyperintensional Foundations of Mathematical Platonism

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Abstract

This paper aims to provide hyperintensional foundations for mathematical platonism. I examine Hale and Wright's (2009) objections to the merits and need, in the defense of mathematical platonism and its epistemology, of the thesis of Necessitism. In response to Hale and Wright's objections to the role of epistemic and metaphysical modalities in providing justification for both the truth of abstraction principles and the success of mathematical predicate reference, I examine the Necessitist commitments of the abundant conception of properties endorsed by Hale and Wright and examined in Hale (2013), and demonstrate how a two-dimensional approach to the epistemology of mathematics is consistent with Hale and Wright's notion of there being non-evidential epistemic entitlement rationally to trust that abstraction principles are true. A choice point that I flag is that between availing of intensional or hyperintensional semantics. The hyperintensional semantic approach that I advance is a topic-sensitive epistemic two-dimensional truthmaker semantics. I countenance a hyperintensional semantics for novel epistemic abstractionist modalities. I suggest that observational type theory can be applied to first-order abstraction principles in order to make abstraction principles recursively enumerable. Epistemic and metaphysical states and possibilities may thus be shown to play a constitutive role in vindicating the reality of mathematical objects and truth, and in providing a conceivability-based route to the truth of abstraction principles as well as other axioms and propositions in mathematics.

1 Introduction

Modal notions have been availed of, in order to argue in favor of nominalist approaches to mathematical ontology. Field (1989) argues, for example, that mathematical modality can be treated as a logical consistency operator on a

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set of formulas comprising an empirical theory, such as Newtonian mechanics, in which the mathematical vocabulary has been translated into the vocabulary of physical geometry.[1] Hellman (1993) argues that intensional models both of first- and second-order arithmetic and of set theory motivate an eliminativist approach to mathematical ontology. On this approach, reference to mathematical objects can be eschewed, and possibly the mathematical structures at issue are nothing.[2]

This essay aims to provide modal foundations for mathematical platonism, i.e., the proposal that mathematical terms for sets; functions; and the natural, rational, real, and complex numbers refer to abstract – necessarily non-concrete – objects. Intensional constructions of arithmetic and set theory have been proposed by, inter alia, Putnam (1967a), Fine (1981); Parsons (1983); the authors in Shapiro (1985); Myhill (1985); Reinhardt (1988); Chihara (1990); Nolan (2002); Linnebo (2013; 2018a); and Studd (2013; 2019). Williamson (2013) emphasizes that mathematical languages are extensional, although in Williamson (2016) he argues that Orey sentences, such as the generalized continuum hypothesis – $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ – which are currently undecidable relative to the axioms of the language of Zermelo-Fraenkel Set Theory with choice as augmented by large cardinal axioms, are yet possibly decidable.[3] Hamkins and Löwe (2007; 2013) and Hamkins and Linnebo (2022) argue that the modal logic of set-forcing extensions of ground models satisfies at least S4.2, i.e., axioms K [$\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$]; T ($\Box\phi \rightarrow \phi$); 4 ($\Box\phi \rightarrow\Box\Box\phi$); and G ($\Diamond\Box\phi \rightarrow \Box\Diamond\phi$). While the foregoing approaches are consistent with realism about mathematical objects, they are nevertheless not direct arguments thereof. The aim of this essay is to redress the foregoing lacuna, and thus to avail of the resources of modal ontology and epistemology in order to argue for the reality of mathematical entities and truth.

In Section **2**, I outline the bearing of a novel, topic-sensitive epistemic two-dimensional truthmaker semantics on the epistemology of modality and philosophical methodology; in particular, conceivability-based approaches to targeting the metaphysical profiles of abstraction principles and undecidable sentences in the epistemology of mathematics.

In Section **3**, I outline the elements of the abstractionist foundations of mathematics.

In Section **4**, I examine Hale and Wright (2009)'s objections to the merits and need, in the defense of mathematical platonism and its epistemology, of the thesis of Necessitism, underlying the thought that whatever can exist actually does so. The Necessitist thesis is a consequence of the Barcan formula (cf. Barcan, 1946; 1947), which states that possibly if there is something which satisfies a condition, then there is something such that it possibly satisfies that

[1] For a generalization of Field's nominalist translation scheme to the differential equations in the theory of General Relativity, see Arntzenius and Dorr (2012).

[2] For further discussion of modal approaches to nominalism, see Burgess and Rosen (1997: II, B-C) and Leng (2007; 2010: 258).

[3] Compare Reinhardt (1974) on the imaginative exercises taking the form of counterfactuals concerning the truth of undecidable formulas. See Maddy (1988b), for critical discussion.

condition: $\Diamond\exists x\phi x \rightarrow \exists x\Diamond\phi x$. The Necessitist thesis is then that $\Box\forall x\Box\exists y(x = y)$, i.e. that necessarily everything is necessarily something. I argue that Hale and Wright's objections to Necessitism as a requirement on admissible abstraction can be answered; and I examine the role of the higher-order Necessitist proposal in their endorsement of an abundant conception of properties.

In Section **5**, I provide an account of the role of epistemic and metaphysical modality or hyperintensional states in explaining the prima facie justification to believe the truth of admissible abstraction principles, and demonstrate how it converges with both Hale and Wright's (op. cit.) and Wright's (2012b; 2014) preferred theory of non-evidential default entitlement rationally to trust the truth of admissible abstraction. I advance a novel, topic-sensitive epistemic two-dimensional truthmaker semantics, which I account for in detail.

Section **6** provides concluding remarks.

2 Conceivability, Topic-Sensitive Epistemic Two-Dimensional Truthmaker Semantics, and the Epistemology of Mathematics

In this paper, I advance a novel, topic-sensitive epistemic two-dimensional truthmaker semantics. The significance of topic-sensitive epistemic two-dimensional truthmaker semantics to the epistemology of modality and philosophical methodology is as follows. By dealing with hyperintensional truthmakers, i.e. epistemic states comprising a state space which compose via mereological fusion, topic-sensitive epistemic two-dimensional truthmaker semantics can be availed of in conceivability arguments which concern hyperintensional phenomena. For example, Elohim (2018) provides a ground-theoretic regimentation of the proposals in the metaphysics of consciousness, and argues that the intensional two-dimensional conceivability argument for the negation of the entailment from physical truths to phenomenal truths is no longer valid once phenomenal truths are hyperintensionally construed. However, topic-sensitive epistemic two-dimensional truthmaker semantics provides the resources to respond to this argument by countenancing two-dimensional hyperintensions, such that formulas would be defined relative to two parameters: The value of the formula would be defined relative to the first parameter ranging over epistemic states, which would determine the value of the formula relative to a second parameter ranging over metaphysical states. Thus, topic-sensitive epistemic two-dimensional truthmaker semantics can model conceivability arguments concerning hyperintensional phenomena in the philosophy of mind and mathematics.

In the philosophy of mathematics, topic-sensitive epistemic two-dimensional truthmaker semantics can further avoid the situation in intensional semantics according to which all necessary formulas express the same proposition because they are true at all possible worlds. As Williamson (2016: 244) writes: '...if one follows Robert Stalnaker in treating a proposition as the set of (metaphysically) possible worlds at which it is true, then all true mathematical formulas literally express the same proposition, the set of all possible worlds, since all true mathematical formulas literally express necessary truths. It is therefore trivial that if one true mathematical proposition is absolutely provable, they all are. Indeed, if you already know one true mathematical proposition (that 2 + 2 = 4, for example), you thereby already know them all. Stalnaker suggests that what mathematicians really learn are in effect new contingent truths about which mathematical formulas we use to express the one necessary truth, but his view faces grave internal problems, and the conception of the content of mathematical knowledge as contingent and metalinguistic is in any case grossly implausible.'

Two-dimensional hyperintensions can further be availed of in order for the epistemic profiles of abstraction principles, large cardinal axioms, and Orey sentences such as the Continuum Hypothesis (CH) to be a guide to their metaphysical profiles.[4]

Regarding e.g. CH, Williamson (2016) examines the extension of the metaphysically modal profile of mathematical truths to the question of absolute decidability. A statement is decidable if and only if there is an algorithm which proves it or its negation. Statements are absolutely undecidable if and only if they are 'undecidable relative to any set of axioms that are justified' rather than just relative to a system (Koellner, 2006: 153), and they are absolutely decidable if and only if they are not absolutely undecidable.

Williamson proceeds by suggesting the following line of thought. Suppose that A is a true interpreted mathematical formula which eludes present human techniques of provability; e.g. the continuum hypothesis (op. cit.). Williamson argues that mathematical truths are metaphysically necessary (op. cit.). Williamson then enjoins one to consider the following scenario: It is metaphysically possible that there is a species which finds A primitively compelling in virtue of their brain states and the evolutionary history thereof. Further, the species 'could not easily have come to believe ¬A or any other falsehood in a relevantly similar way'. He writes: 'In current epistemological terms, their knowledge of A meets the condition of safety: they could not easily have been wrong in a relevantly similar case. Here the relevantly similar cases include cases in which the creatures are presented with sentences that are similar to, but still discriminably different from, A, and express different and false propositions; by hypothesis, the creatures refuse to accept such other sentences, although they may also refuse to accept their negations ... Therefore A is absolutely provable, because the creatures can prove it in one line' (11). Williamson writes then that:

[4] A provisional definition of large cardinal axioms is as follows.

$\exists x\Phi$ is a large large cardinal axiom, because:

(i) Φx is a Σ_2-formula, where 'a sentence ϕ is a Σ_2-sentence if it is of the form: There exists an ordinal α such that $V_\alpha \Vdash \psi$, for some sentence ψ' (Woodin, 2019);

(ii) if κ is a cardinal, such that $V \models \Phi(\kappa)$, then κ is strongly inaccessible, where a cardinal κ is regular if the cofinality of κ – comprised of the unions of sets with cardinality less than κ – is identical to κ, and a strongly inaccessible cardinal is regular and has a strong limit, such that if $\lambda < \kappa$, then $2^\lambda < \kappa$ (see Kanamori, 2012: 360); and

(iii) for all generic partial orders $\mathbb{P} \in V_\kappa$, and all V-generics $G \subseteq \mathbb{P}$, $V[G] \models \Phi x$ (Koellner, 2006: 180).

'The claim is not just that A *would* be absolutely provable *if* there were such creatures. The point is the stronger one that A is absolutely *provable* because there *could* in principle be such creatures.'

Williamson's scenario evinces one issue for the 'back-tracking' approach to modal epistemology, at least as it might be applied to the issue of possible mathematical knowledge. On the back-tracking approach, the method of modal epistemology is taken to proceed by first discerning the metaphysical modal truths and then working backward to the exigent incompleteness of an individual's epistemic states concerning such truths (see Stalnaker, 2003).

The issue for the back-tracking method that Williamson's scenario illuminates is that the metaphysical mathematical possibility that CH is absolutely decidable must in some way converge with the epistemic possibility thereof. The normal mathematical techniques that Williamson specifies – i.e. proof and forcing – have both an epistemic and a metaphysical dimension. As with large cardinals, e.g., lack of inconsistency is a guide to metaphysical possibility. Woodin (2010) provides and discusses results with regard to the maximality of an inner model for one supercompact cardinal, and avails of such results as evidence for the claim that the set-theoretic universe, V, is Ultimate-L. The axiom for V = Ultimate-L implies the truth of CH, and states that '(i) There is a proper class of Woodin cardinals, and (ii) For each Σ_2-sentence ϕ, if ϕ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that $\mathrm{HOD}^{L(A,\mathbb{R})} \Vdash \phi$, where a set is universally Baire if for all topological spaces Ω and for all continuous functions $\pi : \Omega \rightarrow \mathbb{R}^n$, the preimage of A by π has the property of Baire in the space Ω' (Woodin, 2019). Such evidence might comprise an epistemic possibility with regard to the truth of CH, which can thus be a guide to its metaphysical mathematical possibility.

Leitgeb (2009) endeavors similarly to argue for the convergence between the notion of informal provability – countenanced as an epistemic modal operator, K – and mathematical truth. Availing of Hilbert's (1923/1996: ¶18-42) epsilon terms for propositions, such that, for an arbitrary predicate, $\mathbf{C}$(x), with x a propositional variable, the term 'ϵp.$\mathbf{C}$(p)' is intuitively interpreted as stating that 'there is a proposition, x(/p), s.t. the formula, that p satisfies $\mathbf{C}$, obtains' (op. cit.: 290). Leitgeb purports to demonstrate that $\forall$p(p $\rightarrow$ Kp), i.e. that informal provability is absolute; i.e. truth and provability are co-extensive. He argues as follows. Let Q(p) abbreviate the formula 'p $\wedge$ $\neg$K(p)', i.e., that the proposition, p, is true while yet being unprovable. Let K be the informal provability operator reflecting knowability or epistemic necessity, with $\langle$K$\rangle$ its dual. Then:

1. $\exists$p(p $\wedge$ $\neg$Kp) $\iff$ ϵp.Q(p) $\wedge$ $\neg$Kϵp.Q(p).

By necessitation,

2. K[$\exists$p(p $\wedge$ $\neg$Kp)] $\iff$ K[ϵp.Q(p) $\wedge$ $\neg$Kϵp.Q(p)].

Applying modal axioms, KT, to (1), however,

3. $\neg$K[ϵp.Q(p) $\wedge$ $\neg$Kϵp.Q(p)].

Thus,

4. $\neg$K$\exists$p(p $\wedge$ $\neg$Kp).

Leitgeb suggests that (4) be rewritten

5. $\langle K\rangle\forall p(p \rightarrow Kp)$.

Abbreviate $\forall p(p \rightarrow Kp)$ by $\mathtt{B}$. By existential introduction and modal axiom K, both

6. $\mathtt{B} \rightarrow \exists p[K(p \rightarrow \mathtt{B}) \vee K(p \rightarrow \neg\mathtt{B}) \wedge p]$, and

7. $\neg\mathtt{B} \rightarrow \exists p[K(p \rightarrow \mathtt{B}) \vee K(p \rightarrow \neg\mathtt{B}) \wedge p]$.

Thus,

8. $\exists p[K(p \rightarrow \mathtt{B}) \vee K(p \rightarrow \neg\mathtt{B}) \wedge p]$.

Abbreviate (8) by $\mathtt{C}(p)$. Introducing epsilon notation,

9. $[K(\epsilon p.\mathtt{C}(p) \rightarrow \mathtt{B}) \vee K(\epsilon p.\mathtt{C}(p) \rightarrow \neg\mathtt{B})] \wedge \epsilon p.\mathtt{C}(p)$.

By K,

10. $[K(\epsilon p.\mathtt{C}(p) \rightarrow K\mathtt{B}) \vee K(\epsilon p.\mathtt{C}(p) \rightarrow K\neg\mathtt{B})]$.

From (9) and necessitation, one can further derive

11. $K\epsilon p.\mathtt{C}(p)$.

By (10) and (11),

12. $K\mathtt{B} \vee K\neg\mathtt{B}$.

From (5), (12), and K, Leitgeb derives

13. $K\mathtt{B}$.

By, then, the T axiom,

14. $\forall p(p \rightarrow Kp)$ (291-292).

Rather than accounting for the coextensiveness of epistemic provability and truth, Leitgeb interprets the foregoing result as cause for pessimism with regard to whether the formulas countenanced in epistemic logic and via epsilon terms are genuinely logical truths if true at all (292). By contrast to this response, Leitgeb's proof might be thought to provide independent justification in favor of the epistemic two-dimensional approach pursued in this paper, according to which the epistemic possibility or verification of abstraction principles, large cardinal axioms and Orey sentences such as the Continuum Hypothesis is a guide to the metaphysical possibility thereof. The notion of epistemic possibility at issue converges with Leitgeb's notion of informal provability according to which it has semantic and intuitive aspects (274) and is not exhaustively determined by syntax and logic (268). Epistemic states and possibilities comprise what is conceivable, where what is conceivable might best be countenanced by what Azzouni (2013: 73) refers to as 'inference packages'. Azzouni (op. cit.) defines inference packages as follows: 'Inference packages are topic-specific, bundled, sets of principles naturally applied to certain areas: various visualization capabilities, language-manipulation capacities, kinesthetic abilities, and so on'. If epistemic states and possibilities are countenanced via inference packages, then the relevant notion of conceivability would be prima facie, non-ideal conceivability. Ideal conceivability targets the limit of apriori reflection unconstrained by finite limitations, whereas non-ideal conceivability is hostage to the feasibility of computability and the psychological limitations of finite knowers.[5]

One question is whether Orey sentences have a determinate epistemic intension given that there are currently models in which CH is true and models in

[5]For the distinction between ideal and prima facie (i.e. non-ideal) conceivability, see Chalmers (2002). For more on the relation between epistemic states and possibilities and Azzouni's notion of inference packages, see Elohim (ms).

which CH is false, such that it is not determinate which epistemic possibility is actual. The epistemic intensions of Orey sentences are arguably indeterminate for non-ideal reasoners, yet determinate for ideal ones. This optimism about the determinate truth of CH is corroborated in work by Woodin (2019), who demonstrates that the Ultimate-L conjecture is an existential number theoretic statement, such that it 'must be either true or false; it cannot be meaningless' (op. cit.).

With regard to abstraction principles, topic-sensitive epistemic two-dimensional truthmaker semantics is consistent with the appeal to the notion of 'expected epistemic utility' as a desideratum which must be satisfied in order for there to be non-evidential entitlement rationally to trust that abstraction principles are true. This consistency will be examined in further detail in Section **5** below.

3 The Abstractionist Foundations of Mathematics

The abstractionist foundations of mathematics are inspired by Frege's (1884/1980; 1893/2013) proposal that cardinal numbers can be explained by specifying an equivalence relation, expressible in the signature of second-order logic, on first- or higher-order entities. At first-order, in Frege (1884/1980: 64) the direction of the line, a, is identical to the direction of the line, b, if and only if lines a and b are parallel. At second-order, in Frege (op. cit.: 68) and Wright (1983: 104-105), the cardinal number of the concept, **A**, is identical to the cardinal number of the concept, **B**, if and only if there is a one-to-one correspondence between **A** and **B**, i.e., there is an injective and surjective (bijective) mapping, R, from **A** to **B**. With Nx: a numerical term-forming operator,

- $\forall \mathbf{A} \forall \mathbf{B}[\text{Nx: } \mathbf{A} = \text{Nx: } \mathbf{B} \equiv \exists R[\forall x[\mathbf{A}x \to \exists y(\mathbf{B}y \wedge Rxy \wedge \forall z(\mathbf{B}z \wedge Rxz \to y = z))] \wedge \forall y[\mathbf{B}y \to \exists x(\mathbf{A}x \wedge Rxy \wedge \forall z(\mathbf{A}z \wedge Rzy \to x = z))]]]$.

The foregoing is referred to as 'Hume's Principle'.[6] Frege's Theorem states that the Dedekind-Peano axioms for the language of arithmetic can be derived from Hume's Principle, as augmented to the signature of second-order logic.[7]

[6] Frege (1884/1980: 68) writes: 'the Number which belongs to the concept F is the extension of the concept [equinumerous] to the concept F' (cf. op. cit.: 72-73). Boolos (1987/1998: 186) coins the name, 'Hume's Principle', for Frege's abstraction principle for cardinals, because Frege (op. cit.: 63) attributes equinumerosity as a condition on the concept of number to Hume (1739-1740/2007: Book 1, Part 3, Sec. 1, SB71), who writes: 'When two numbers are so combin'd, as that the one has always an unite answering to every unite of the other, we pronounce them equal . . . '. Frege notes that identity of number via bijections is anticipated by the mathematicians, Ernst Schröder and Ernst Kossak, as well Cantor (1883/1996: Sec. 1), the last of whom writes: '[E]very well-defined set has a determinate power; two sets have the same power if they can be, element for element, correlated with one another reciprocally and one-to-one', where the power of a set corresponds to its cardinality (see Cantor, 1895/2007: 481).

[7] Cf. Dedekind (1888/1996) and Peano (1889/1967). See Wright (1983: 154-169) for a proof sketch of Frege's theorem; Boolos (1987) for the formal proof thereof; and Parsons (1964) for an initial conjecture of the theorem's validity.

Abstraction principles have further been specified both for the real numbers (cf. Hale, 2000a; Shapiro, 2000; and Wright, 2000), and for sets (cf. Wright, 1997; Shapiro and Weir, 1999; Hale, 2000b; and Walsh, 2016).

The philosophical significance of the abstractionist program consists primarily in its provision of a neo-logicist foundation for classical mathematics, and in its further providing a setting in which to examine constraints on the identity conditions constitutive of mathematical concept possession.[8] The philosophical significance of the abstractionist program consists, furthermore, in its circumvention of Benacerraf's (1973) challenge to the effect that our knowledge of mathematical truths is in potential jeopardy, because of the absence of naturalistic, in particular causal, conditions thereon. Both Wright (1983: 13-15) and Hale (1987: 10-15) argue that the abstraction principles are epistemically tractable, only if (i) the surface syntax of the principles – e.g., the term-forming operators referring to objects – are a perspicuous guide to their logical form; and (ii) the principles satisfy Frege's (1884/1980: X) context principle, such that the truth of the principles is secured prior to the reference of the terms figuring therein.

4 Abstraction and Necessitism

4.1 Hale and Wright's Arguments against Necessitism

One crucial objection to the abstractionist program is that – while abstraction principles might provide necessary and sufficient truth-conditions for the concepts of mathematical objects – an explanation of the actual truth of the principles has yet to be advanced (cf. Eklund, 2006; 2016). In response, Hale and Wright (2009: 197-198) proffer a tentative endorsement of an 'abundant' conception of properties, according to which fixing the sense of a predicate will be sufficient for predicate reference.[9] Eklund (2006: 102) suggests, by contrast,

[8]Shapiro and Linnebo (2015) prove that Heyting arithmetic can be recovered from Frege's Theorem. Criteria for consistent abstraction principles are examined in, inter alia, Hodes (1984a); Hazen (1985); Boolos (1990/1998); Heck (1992); Fine (2002); Weir (2003); Cook and Ebert (2005); Linnebo and Uzquiano (2009); Linnebo (2010); and Walsh (op. cit.).

[9]Hale and Wright (2009) and Wright (2012a) extend the abundant conception of properties to objects, although this extension is orthogonal to the discussion in this paper. The aim of this and the following section is to examine the Necessitist commitments of the abundant conception of properties, especially as exploited by Hale (2013a,b). For the sake of completeness, however, the abundant conception of objects can be characterized as follows. Hale and Wright argue that, in the case of objects, the senses of singular terms are not sufficient for reference, but rather the following must be satisfied: the truth of the context, viz. the right-hand-side of abstraction principles, by way of which singular terms for the objects on the left-hand-sides can be defined. This figures as an Aristotelian constraint to the effect that those contexts are objective truths occurring on the side of the World given that sense alone is not sufficient for reference. Hale and Wright claim: 'As with the abundant conception of properties, there is no additional gap to cross which requires "hitting off" something on the other side by virtue of its fit with relevant specified conditions, as the property of being composed of the element with atomic number 79 is hit off (or so let's suppose) by the combination of conditions that control our unsophisticated use of "gold". But nor is it the case that reference is bestowed by the possession of sense alone' (207). And they continue: 'The abstractionist conception of

that one way for the truth of the abstraction principles to be explained is by presupposing what he refers to as a 'Maximalist' position concerning the target ontology.[10] According to the ontological Maximalist position, if it is possible that a term has an extension, then actually the term does have the extension.

Hale and Wright (op. cit.) raise two issues for the ontological Maximalist proposal. The first is that ontological Maximalism is committed to a proposal that they take to be independently objectionable, namely ontological Necessitism (185). They write: 'Most obviously, maximalism denies the possibility of contingent non-existence, to which there are obvious objections' (op. cit.) Hale and Wright (op. cit.) raise a similar contention to the effect that actual, and not *merely possible*, reference is what the abstractionist program intends to target; and that Maximalism and Necessitism, so construed, are purportedly silent on the status of ascertaining when the possibilities at issue are actual.

The second issue that Hale and Wright find with Maximalism is that it misconstrues the demands that the abstractionist program is required to address. The abstractionist program is supposed to be committed to ontological Maximalism, because the possibility that a term has an extension will otherwise not be sufficient for the success of the term's reference. It is further thought that, without an appeal to Maximalism, and despite the actuality of successful mathematical reference, there are yet possible situations in which the mathematical terms still do not refer (193). In response, they note that no 'collateral metaphysical assistance' – such as ontological Maximalism would be intended to provide – is necessary in order to explain the truth of abstraction principles (op. cit.). Rather, there is prima facie, default entitlement rationally to trust that the abstraction principles are actually true, and such entitlement is sufficient to foreclose upon the risk that possibly the mathematical terms therein do not refer (192).

In the remainder of this section, I will argue that Hale and Wright's objections to Necessitism and the ontological Maximalist approach to admissible abstraction both can be answered, and that the proposals are in any case implicit in their endorsement of the abundant conception of properties.

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the truth of the right-hand sides of instances of good abstractions as conceptually sufficient for the truth of the left-hand sides precisely takes the terms in question out of the market for "hitting off" reference to things whose metaphysical nature is broadly comparable to that of sparse properties, and assigns to them instead a referential role relevantly comparable to that of predicates as viewed by the abundant Aristotelian' (208). For similar comments, see Wright (2012: 132): 'In contrast with any Meinongian view, we need the truth of the right-hand side kind of context before we can claim existence. It is not enough that the abstract terms have a sense. Appropriate (atomic) statements containing them have to be true. But those truths can be objective. And the truth of the left-hand sides of instances of abstraction principles will be an objective matter just if that of their right-hand side counterparts is, because that is given as a necessary and sufficient condition. Thus where it is objectively so that a pair of properties are one-one correspondent, it will correspondingly be objectively so that some one number is the number of them both. But there will be no metaphysical hostage, no "fishing", in drawing this conclusion about their number. The reason is that numbers, like all abstracts, are to be compared to abundant Aristotelian properties: entities knowledge of which is fully grounded in knowledge of the truth of atomic predications and identity statements, respectively, and embodies no further conjecture about the nature of the World'.

[10] For further discussion of ontological Maximalism, see Hawley (2007) and Sider (2007: IV).

The principle of the necessary necessity of being (NNE) can be derived from the Barcan formula.[11] NNE states that necessarily all objects are such that necessarily there is something to which each is identical; $\Box\forall x\Box\exists y(x = y)$. Informally, necessarily everything has necessary being, i.e. necessarily everything is necessarily something, even if contingently non-concrete. Applied to entities at higher-order, NNE can be formalized as follows: $\Box\forall X\Box\exists Y\Box\forall x(Xx \iff Yx)$ (op. cit.: 264). Williamson (2013: 6.1-6.4) targets issues for the comprehension principle for identity properties of individuals, i.e. haecceities, if the negations of the Barcan formula and NNE are true at first-order, and thus for objects. With regard to properties and relations at higher-order, Williamson's arguments have targeted closure conditions, given a modalized interpretation of comprehension principles (op. cit.). The latter take the form, $\text{Comp}_M := \exists X\Box\forall x(Xx \iff A)$, with x an individual variable which may occur free in A and X a monadic first-order predicate variable which does not occur free in A (262). The Contingentist, by contrast, can countenance only 'intra-world' comprehension principles in which the modal operators and iterations thereof take scope over the entire formula; e.g. $\Diamond\exists X\forall x(Xx \iff A)$ (cf. Sider, 2016: 686). Williamson targets, in particular, a higher-order modal completeness property for a quasi-reflexive [for all $x,y \in R$, $Rxy \rightarrow \Box(Rxx \wedge Ryy)$], anti-symmetric [$(Rxy \wedge Ryx) \rightarrow x = y$], and transitive [$(Rxy \wedge Ryz) \rightarrow Rxz$] relation, $\leq$. The relation codifies upper bounds and least upper bounds, as well as modalized versions thereof, where '[t]o be an upper bound of a property is to be at least as great (in the sense of the ordering) as everything that has the property. To be a least upper bound of the property is to be an upper bound of the property that every upper bound of the property is at least as great as' (Williamson, 2013: 286). The claim that 'any possible property that can have a modal upper bound can have a modal least upper bound' is recorded by 'prefixing every universal quantifier with a necessity operator and the other quantifiers and the ordering symbol itself with a possibility operator. Formally: [i] $\Box\forall X[\Diamond\exists y\Box\forall x(Xx \rightarrow \Diamond x \leq y) \rightarrow \Diamond\exists y[\Box\forall x(Xx \rightarrow \Diamond x \leq y) \wedge \Box\forall z[\Box\forall x(Xx \rightarrow \Diamond x \leq z) \rightarrow \Diamond y \leq z]]]$' (287). Williamson notes that to apply this formula, one replaces Xx with the formula A, where x can be free in A but neither y nor z can be, in order to obtain the following: [ii] '$\Diamond\exists y\Box\forall x(A \rightarrow \Diamond x \leq y) \rightarrow \Diamond\exists y[\Box\forall x(A \rightarrow \Diamond x \leq y) \wedge \Box\forall z[\Box\forall x(A \rightarrow \Diamond x \leq z) \rightarrow \Diamond y \leq z]]]$'. However, one needs Comp_M in order to derive [ii] from [i], because Comp_M 'provide[s] a property over which the second-order quantifier [in [i]] ranges necessarily coextensive with A' (op. cit.). By rejecting Comp_M, Contingentists cannot preclude cases in which the parameters in A might be incompossible, such that there would be no property which is necessarily coextensive with A (op. cit.). The foregoing provides prima facie abductive support for the requirement of Necessitism in mathematics. The constitutive role of the Necessitist modal comprehension scheme in characterizing the relation between modal upper and least upper bounds answers Hale and Wright's first contention against the Necessitist commitments of ontological Maximalism.

Williamson refers to the assignments for models in the metaphysical setting

[11]See Williamson (2013: 38).

as universal interpretations (59). The analogue for logical truth occurs when a truth is metaphysically universal, i.e., if and only if its second-order universal generalization is true on the intended interpretation of the metalanguage (200). The connection between truth-in-a-model and truth simpliciter is then that – as Williamson puts it laconically – when 'the framework at least delivers a condition for a modal sentence to be true in a universal interpretation, we can derive the condition for it to be true in the intended universal interpretation, which is the condition for it to be true simpliciter' (op. cit.).

One of the crucial interests of the metaphysical universality of propositions is that the models in the class need not be pointed, in order to countenance the actuality of the possible propositions defined therein.[12] Rather, the class of true propositions generated by the metaphysically universal propositions is sufficient for the propositions actually to be true (268-269).[13] Williamson writes that 'since whatever is is, whatever is actually is: if there is something, then there actually is such a thing' (23). Thus, the foregoing characterization of actuality can explain why the metaphysically universal propositions which are true simpliciter are actual.

This account of actuality answers Hale and Wright's contention that the interaction between the possible and actual truth of sentences such as abstraction principles cannot be accounted for.[14]

4.2 Hale on the Necessary Being of Purely General Properties and Objects

Note, further, that the abundant conception of properties endorsed by Hale and Wright depends upon the Necessitist Thesis, and the truth of ontological Maximalism thereby. Hale writes: '[I]t is sufficient for the *actual* existence of a property or relation that there *could* be a predicate with appropriate satisfaction conditions ... *purely general* properties and relations exist as a matter of (absolute) *necessity*', where a property is purely general if and only if there could be a predicate for which, and it embeds no singular terms (Hale, 2013b: 133, 135; see also 2013a: 99-100).[15]

Hale argues for the necessary necessity of being for properties and propositions as follows (op. cit.: 135; 2013b: 167). Suppose that p refers to the proposition that a property exists, and that q refers to the proposition that

[12]A model is pointed if it includes a designated element. That the models are unpointed is noted in Williamson (2013: 100).

[13]Thanks here to xx for discussion.

[14]Cook (2016: 398) demonstrates how formally to define modal operators within Hume's Principle, i.e. the consistent abstraction principle for cardinal numbers. Necessitist Hume's Principle takes the form: $\Box\forall X,Y[\#(X) =_{\Box} \#(Y) \iff X \approx Y]$, where X and Y are second-order variables, # is a numerical term-forming operator, ≈ is a bijection, and for variables, x,y, of arbitrary type '$x =_{\Box} y \iff \exists z[z = x \land z = y \land \Box\exists w(w = z)]$'. See Cook (op. cit.) for further discussion.

[15]Cook (op. cit.: 388) notes the requirement of Necessitism in the abundant conception of properties, and discusses one point at which Williamson's and Hale's Necessitist proposals might be inconsistent. The points of divergence between the two variations on the proposal are examined in some detail below.

a predicate for the property exists. Let the necessity operator be defined as a counterfactual with an unrestricted, universally quantified antecedent, such that, for all propositions, ψ: $[\Box\psi \iff \forall\phi(\phi \Box\!\!\rightarrow \psi)]$ (135).[16] On the abundant conception of properties, $\Box[p \iff \Diamond q]$. Intuitively: Necessarily, there is a property if and only if possibly there is a predicate for that property. Given the counterfactual analysis of the modal operator: For all propositions about a property, if there were a proposition specifying a predicate s.t. the property is in the predicate's extension, then there would be that property.

From '$\Box[p \iff \Diamond q]$', one can derive both '$p \iff \Diamond q$', and – by the rule, RK – the necessitation thereof, '$\Box p \iff \Box\Diamond q$' (op. cit.). By the 5 axiom in S5, $\Diamond q \iff \Box\Diamond q$ (op. cit.). So, '$\Box\Diamond q \iff \Diamond q$'; '$\Diamond q \iff p$'; and '$\Box\Diamond q \iff \Box p$'. Thus – by transitivity – '$p \iff \Box p$' (op. cit.); i.e., all propositions about properties are necessarily true, such that the corresponding properties have necessary being. By the 4 axiom in S5, $\Box p \iff \Box\Box p$; so, the necessary being of properties and propositions is itself necessary. Given the endorsement of the abundant conception of properties – Hale and Wright are thus committed to higher-order Necessitism, i.e., the necessary necessity of being.

Hale (2013b) endeavors to block the ontological commitments of the Barcan formula and its converse by endorsing a negative free logic. Thus, in the derivation:

Assumption,

1. $\Box\forall x[F(x)]$.

By $\Box$-elimination,

2. $\forall x[F(x)]$.

By $\forall$-elimination,

3. $F(x)$.

By $\Box$-introduction,

4. $\Box[F(x)]$.

By $\forall$-introduction,

5. $\forall\Box[F(x)]$.

By $\rightarrow$-introduction,

6. $\Box\forall x[F(x) \rightarrow \forall x\Box[F(x)]$,

Hale imposes an existence-entailing assumption in the inference from lines (2) to (3), i.e.

'(Free$\forall$-Elimination) From $\forall x[A(x)]$, together with an existence-entailing premise $F(t)$, we may infer $A(t)$ where t can be any term' (op. cit.: 208-209).

Because the concept of, e.g., cardinal number is defined by abstraction principles which are purely general because they embed no singular terms, the properties of numbers are argued to have necessary being. The necessary being of the essential properties of number – i.e., higher-order Necessitism about purely general properties – along with the necessary existence of second-level functions in Hume's Principle are argued then to explain in virtue of what abstract objects such as numbers have themselves necessary being. As Hale writes: 'This enables

[16] Proponents of the translation from modal operators into counterfactual form include Stalnaker (1968/1975), McFetridge (1990: 138), and Williamson (2007).

the essentialist to give a simple and straightforward explanation of the necessary existence of cardinal numbers. There necessarily exist cardinal numbers because they are the values of the pure function Nu...u...for a certain range of arguments – pure first-level sortal properties – and both that function and those arguments to it exist necessarily. In short, certain objects – the cardinal numbers – exist necessarily because their existence is a consequence of the existence of a certain function and certain properties which themselves exist necessarily' (176-177).

By contrast, essential properties defined by theoretical identity statements, which if true are necessarily so, do embed singular terms and are thus not purely general. So, the essential nature of water, i.e., the property 'being comprised of one oxygen and two hydrogen molecules', has contingent being, explaining in virtue of what samples of water have contingent being (216-217).

One objection to the foregoing concerns the necessary being of different types of numbers. While an abstraction principle for cardinal numbers can be specified using only purely general predicates – i.e., Hume's Principle – abstraction principles for imaginary and complex numbers have yet to be specified. Shapiro (2000) provides an abstraction principle for the concepts of the reals by simulating Dedekind cuts, where abstraction principles are provided for the concepts of the cardinals, natural numbers, integers, and rational numbers, from which the reals are thence defined: Letting F, G, and R denote rational numbers, $\forall$F,G[$\mathbf{C}$(F) = $\mathbf{C}$(G) $\iff$ $\forall$R(F$\leq$R $\iff$ G$\leq$R)].[17] Hale's (2000a/2001) own definition of the concept of the reals is provided relative to a domain of quantities. The quantities are themselves taken to be abstract, rather than physical, entities (409). The quantitative domain can thus be comprised of both rational numbers as well as the abstracts for lengths, masses, and points.[18] The reals are then argued not to be numbers, but rather quantities defined via an abstraction principle which states that a set of rational numbers in one quantitative domain is identical to a set of rational numbers in a second quantitative domain if and only if the two domains are isomorphic (407).[19] Hale argues, then, that it is innocuous for the real abstraction principle to be conditional on the existence of at least one quantitative domain, because the rational numbers can be defined, similarly as on Shapiro's approach, via cut-abstractions and abstractions on the integers, naturals, and cardinals. Thus, the reals can be treated as abstracts derived from purely general abstraction principles, and are thus possessed of

[17]See Dedekind (1872/1996: Sec. 4), for the cut method for the definition of the reals. Concepts of rational numbers can themselves be obtained via an abstraction principle in which they are identified with quotients of integers – [$\mathbf{Q}\langle m,n\rangle = \mathbf{Q}\langle p,q\rangle \iff n = 0 \wedge q = 0 \vee n \neq 0 \wedge q \neq 0 \wedge m \times q = n \times p$]; concepts of the integers are obtained via an abstraction principle in which they are identified with differences of natural numbers – [$\mathbf{Diff}(\langle x,y\rangle) = \mathbf{Diff}(\langle z,w\rangle) \iff x + w = y + z$]; and concepts of the naturals are obtained via an abstraction principle in which they are identified with pairs of finite cardinals – '$\forall x,y,z,w[\langle x,y\rangle(=\mathbf{P}) = \langle z,w\rangle(=\mathbf{P}) \iff x = z \wedge y = w]$.

[18]An abstraction principle for lengths, based on the equivalence property of congruence relations on intervals of a line, or regions of a space, is defined in Shapiro and Hellman (2015: 5, 9). Shapiro and Hellman provide, further, an abstraction principle for points, defined as comprising, respectively, the left- and right-ends of intervals (op. cit.: 5, 10-12).

[19]See Hale (op. cit.: 406-407), for the further conditions that the domains are required to satisfy.

necessary being.

However, abstraction principles for imaginary numbers such as $i = \sqrt{-1}$, and complex numbers which are defined as the sum of a real number and a second real multiplied by i, have yet to be accounted for. The provision of an abstraction principle for complex numbers would, in any case, leave open a host of questions concerning the applicability of the numbers, violating what is referred to as Frege's constraint. Frege's constraint is satisfied when the application for a concept of number figures in its definition.[20] Such questions might include the inquiry into how, e.g., a complex-valued wave function might interact with physical ontology; e.g., how a lower-(3)-dimensional real-valued configuration space for particles might relate to the higher-($3n$)-dimensional, complex-valued wave function (cf. Simons, 2016; Ney, 2013; Maudlin, 2013). Another question concerns how purely general properties – denoted by predicates which take no singular terms as arguments – might interact with the satisfaction of Frege's constraint in the abstraction principles for complex numbers.

The modality in the Barcan-induced Necessitist proposal at first- and higher-order is, as noted, interpreted metaphysically, and incurs no similar issues with regard to the interaction between purely general properties and Frege's constraint. Further, because true on its second-order universal generalization on its intended, metaphysical interpretation, the possible truth-in-a-model of the relevant class of propositions is, as discussed in Section **4.1**, thus sufficient for entraining the actual truth of the relevant propositions.

5 Cardinality and Intensionality

An interesting residual question concerns the status of the worlds, upon the translation of modal first-order logic into the non-modal first-order language. However, whether objects satisfy the predicate can vary from point to point, in the non-modal first-order class of points.[21] Another issue is that modal propositional logic is equivalent only to the bisimulation-invariant fragment of first-order logic, rather than to the full variant of the logic (cf. van Benthem, 1983; Janin and Walukiewicz, 1996). Thus, there cannot be a faithful translation from each modal operator in modal propositional logic into a predicate of full first-order logic.[22]

One way to mitigate the foregoing issues might be by arguing that the language satisfies real-world rather than general validity, such that necessarily the predicate will be satisfied only at a designated point in a model – intuitively, the analogue of the concrete rather than some merely possible world, simulating thereby the translation from possibilist to actualist discourse (cf. Fine, op. cit.: 211,135-136, 139-140, 154, 166-168, 170-171) – by contrast to holding of

[20]For discussion, see e.g. Wright (2000).

[21]Suppose that the model is defined over the language of second-order arithmetic, such that the points in the model are the ordinals. A uniquely designated point might then be a cardinal number whose height is accordingly indexed by the ordinals.

[22]For further discussion of the standard translation between propositional modal and first-order non-modal logics, see Blackburn et al. (2001: 84).

necessity as interpreted as satisfaction at every point in the model. The reply would be consistent with what Williamson refers to as 'chunky-style Necessitism' which validates the following theorems: where the predicate C(x) denotes the property of being grounded in the concrete and P(x) is an arbitrary predicate, (a) '$\forall x \Diamond C(x)$', yet (b) '$\Box \forall x_1, ..., x_n P(x_1, \ldots, x_n) \rightarrow (Cx_1, \ldots, Cx_n)]$' (325-332). Williamson (33, fn.5) argues, however, in favor of general, rather than real-world validity. A second issue for the reply is that principle (b), in the foregoing, is inconsistent with Williamson's protracted defense of the 'being constraint', according to which $\Box \forall x_1, \ldots, x_n \Box [P(x_1, \ldots, x_n) \rightarrow \exists y_1, ..., y_n (x_1 = y_1, \ldots, x_n = y_n)]$, i.e. if $x_1, \ldots, x_n$ satisfy a predicate, then $x_1, \ldots, x_n$ are each something, even if possibly non-concrete (148).

A related issue concerns the translation of modalized, variable-binding, generalized quantifiers of the form:

'there are n objects such that ...',

'there are countably infinite objects such that ...',

'there are uncountably infinite objects such that ...' (Fritz and Goodman, 2017).

The generalized quantifiers at issue are modalized and consistent with first-order Necessitism, because the quantifier domains include all possible – including contingently non-concrete – objects. It might be argued that the translation is not of immediate pertinence to the ontology of mathematics, because the foregoing first-order quantifiers can be restricted such that they range over only uncountably infinite *necessarily* non-concrete objects – i.e. abstracta – by contrast to ranging unrestrictedly over all modal objects, including the contingently non-concrete entities induced via the Barcan formula – i.e., the 'mere possibilia' that are non-concrete as a matter of contingency. However, the Necessitist thesis can be valid even in the quantifier domain of a first-order language restricted to necessarily non-concrete entities. If, e.g., a mathematician takes, despite iterated applications of set-forming operations, the cumulative hierarchy of sets to have a fixed cardinal height, then the first-order Necessitist thesis will still be valid, because all possible objects will actually be still something.

The first-order Necessitist proposal engendered by taking the height of the cumulative hierarchy to be fixed is further consistent with the addition to the first-order language of additional intensional operators – such as those introduced by Vlach (1973) – in order to characterize the indefinite extensibility of the concept of set; i.e., that despite unrestricted universal quantification over all of the entities in a domain, another entity can be defined with reference to, and yet beyond the scope of, that totality, over which the quantifier would have further to range.[23] First-order Necessitism is further consistent with the relatively expanding domains induced by Bernays' (1942) Theorem. Bernays' Theorem states that class-valued functions from classes to sub-classes are not onto, where classes are non-sets (cf. Uzquiano, 2015a: 186-187). So, the car-

[23]The concept of indefinite extensibility is introduced by Dummett (1963/1978), in the setting of a discussion of the philosophical significance of Gödel's (1931) first incompleteness theorem. See the essays in Rayo and Uzquiano (2006); Studd (op. cit.). See Elohim (2024) for further discussion.

dinality of a class will always be less than the cardinality of its sub-classes. Suppose that that there is a generalization of Bernays' theorem, such that the non-sets are interpreted as possible objects. Thus, the cardinality of the class of possible objects will always be less than the cardinality of the sub-classes in the image of its mapping. Given iterated applications of Bernays' theorem, the cardinality of a domain of non-sets is purported then not to have a fixed height.

In both cases, however, the addition of Vlach's intensional operators permits there to be multiple-indexing in the array of parameters relative to which a cardinal can be defined. So, both the intensional characterization of indefinite extensibility and the generalization of Bernays' Theorem to possible objects are consistent with the first-order Necessitist proposal that all possible objects are actual, and so the cardinality of the target universe is fixed.[24]

Fritz and Goodman suggest that a necessary condition on the equivalence of propositions is that they define the same class of models (op. cit.: 1.4). The proposed translation of the modalized generalized quantifiers would be Contingentist, by taking (NNE) to be invalid, such that the domain in the translated model would be comprised of only possible concrete objects, rather than the non-concrete objects as well (op. cit.).

Because of the existence of non-standard models, the generalized quantifier that 'there are countably infinitely many possible ... cannot be defined in first-order logic. Fritz and Goodman note that generalized quantifiers ranging over countably infinite objects can yet be simulated by enriching one's first-order language with countably infinite conjunctions. On the latter approach, finitary existential and universal quantifiers can be defined as the countably infinite conjunction of formulas stating that, for all natural numbers n, 'there are n possible ...' (2.3).

Crucially, however, there are some modalized generalized quantifiers that cannot be similarly paraphrased – e.g., 'there are uncountably infinite possible objects s.t. ... – and there are some modalized generalized quantifiers that cannot even be defined in first-order languages – e.g. '*most* objects s.t. ...' (2.4-2.5)

In non-modal first-order logic, it is possible to define generalized quantifiers which range over an uncountably infinite domain of objects, by augmenting finitary existential and universal quantifiers with an uncountably infinite stock of variables and an uncountably infinite stock of conjunctions of formulas (2.4). Fritz and Goodman note, however, that the foregoing would require that the quantifiers bind the uncountable variables 'at once', s.t. they must have the

[24]Note that the proposal that the cardinality of the cumulative hierarchy of sets is fixed, despite continued iterated applications of set-forming operations, is anticipated by Cantor (1883/1996: Endnote [1]). Cantor writes: 'I have no doubt that, as we pursue this path ever further, we shall never reach a boundary that cannot be crossed, but that we shall also never achieve even an approximate conception of the absolute [...] The absolutely infinite sequence of numbers thus seems to me to be an appropriate symbol of the absolute; in contrast the infinity of the first number-class (I) [i.e., the countable infinity comprising the class of natural numbers, $\aleph_0$ – D.E.], which has hitherto sufficed, because I consider it to be a graspable idea (not a representation), seems to me to dwindle into nothingness by comparison' (op. cit.; cf. Cantor, 1899/1967).

same scope. The issue with the proposal is that, in the setting of modalized existential quantification over an uncountably infinite domain, the Contingentist paraphrase requires that bound variables take different scopes, in order to countenance the different possible sets that can be defined in virtue of the indefinite extensibility of cardinal number (op. cit.).

In order to induce the Contingentist paraphrase, Fritz and Goodman suggest defining 'strings of infinitely many existential and universal quantifiers', such that a modalized, i.e. Necessitist, generalized quantifier of the form, 'there are uncountably infinite possible ... can be redefined by an uncountably infinite sequence of finitary quantifiers with infinite variables and conjunction symbols of the form:

'Possibly for some x_1, possibly for some x_2, *etc.*: x_1,x_2,*etc.* are pairwise distinct and are each possibly ...',

where *etc.* denotes an uncountable sequence of, respectively, 'an uncountable string of interwoven possibility operators and existential quantifiers', and an 'uncountable string of variables' (op. cit.).

An argument against the proposed translation of the quantifier for there being uncountably infinite possible objects is that it is contentious whether an uncountable sequence of operators or quantifiers has a definite meaning [cf. Williamson (2013: 7.7)]. Thus, e.g., while negation can have a determinate truth condition which specifies its meaning, a string of uncountably infinite negation operators will similarly have determinate truth conditions and yet not have an intuitive, definite meaning (357). One can also define a positive or negative integer, x, such that sx is interpreted as the successor function, x+1, and px is interpreted as the inverse function, x-1. However, an infinitary expression consisting in uncountable, alternating iterations of the successor and inverse functions – *spsps*...x – will similarly not have a definite meaning (op. cit.). Finally, one can define an operator O_i mapping truth conditions for an arbitrary formula A to the truth condition, p, of the formula $\Diamond\exists x_i(Cx_i \wedge A)$, with Cx being the predicate for being concrete (258). Let the operators commute, such that O_iO_j iff O_jO_i (op. cit.). A total ordering of truth conditions defined by an infinite sequence of the operators can be defined, such that the relation is reflexive, anti-symmetric, transitive, and connected [$\forall$x,y(x$\leq$y $\vee$ y$\leq$x)] (op. cit.). However, total orders need not have a least upper bound; and the sequence, $O_iO_iO_i...(p)$, would thus not have a non-arbitrary, unique value (op. cit.). The foregoing might sufficiently adduce against Fritz and Goodman's Contingentist paraphrase of the uncountable infinitary modalized quantifier.

The philosophical significance of the barrier to a faithful translation from modal first-order to extensional full first-order languages, as well as a faithful translation from modalized, i.e. Necessitist, generalized quantifiers to Contingentist quantification, is arguably that the modal resources availed of in the abstractionist program might then be ineliminable.

6 Epistemic Hyperintensionality, Epistemic Utility, and Entitlement

In this section, I address, finally, Hale and Wright's second issue with regard to the role of modality in guaranteeing that the possible truth of abstraction principles provides warrant for the belief in their actual truth. While Necessitism is not immediately pertinent to the default entitlement to trust that abstraction principles are true, I will argue that epistemic modalities or hyperintensional states are yet relevant to Wright's application of the notions of entitlement and 'expected epistemic utility' to abstraction principles. As noted, Hale and Wright argue that there is non-evidential entitlement rationally to trust that acceptable abstraction principles are true, and thus that the terms defined therein actually refer. In response, I will proceed by targeting the explanation in virtue of which there is such epistemic, default entitlement. I will outline two proposals concerning the foregoing grounding claim – advanced, respectively, in Elohim (2024), above, and by Wright (2012b; 2014) – and I will argue that the approaches converge.

Wright's elaboration of the notion of rational trust, which is intended to subserve epistemic entitlement, appeals to a notion of expected epistemic utility in the setting of decision theory (2014: 226, 241). In order better to understand this notion of expected epistemic utility, we must be more precise.

There are two, major interpretations of (classical) expected utility. A model of decision theory is a tuple $\langle$A,O,K,V$\rangle$, where A is a set of acts; O is a set of outcomes; K encodes a set of counterfactual conditionals, where an act from A figures in the antecedent of the conditional and O figures in the conditional's consequent; and V is a function assigning a real number to each outcome. The real number is a representation of the value of the outcome. In evidential decision theory, the expected utility of an outcome is calculated as the product of the agent's credence, conditional on her action (where the action might be a mental action of trusting the truth of the relevant proposition), multiplied by the utility that she assigns to the outcome of the proposition in which she's placing her rational trust. In causal decision theory, the expected utility of an outcome is calculated as the product of the agent's credence, conditional on both her action and the causal efficacy thereof, by the utility of the outcome.

First, because the causal efficacy of one's choice of acts is presumably orthogonal to the non-evidential rational trust to believe that mathematical abstraction principles are true, I will assume that the notion of expected epistemic utility theory that Wright (op. cit.) avails of relies only on the subjective credence of the agent conditional on her action, multiplied by the utility that she assigns to the outcome of the proposition in which she's placing her rational trust. Thus expected epistemic utility in the setting of decision-theory will be calculated within the (so-called) evidential, rather than causal, interpretation of the latter.

Second, there are two, major interpretations concerning how to measure the subjective credences of an agent. The philosophical significance of this choice point is that it bears directly on the very notion of the *epistemic utility* that an agent's beliefs will possess.

The epistemic utility associated with the pragmatic approach is, generally, utility maximization. By contrast to the pragmatic approach, the epistemic approach to measuring the accuracy of one's beliefs is grounded in the notion of dominance (cf. Joyce, 1998; 2009). According to the epistemic approach, there is an ideal, or vindicated, probability concerning a proposition's obtaining, and if an agent's subjective probability measure does not satisfy the Kolmogorov axioms, then one can prove that it will always be dominated by a distinct measure; i.e. it will always be the case that a distinct subjective probability measure will be closer to the vindicated world than one's own. The epistemic utility associated with the epistemic approach is thus the minimization of inaccuracy (cf. Pettigrew, 2014).[25]

Wright notes that the rational trust subserving epistemic entitlement will be pragmatic, and makes the intriguing point that 'pragmatic reasons are not a special genre of reason, to be contrasted with e.g. epistemic, prudential, and moral reasons' (2012b: 484). He provides an example according to which one might be impelled to prefer the 'alleviation of Third world suffering' to one's own 'eternal bliss' (op. cit.); and so presumably has the pragmatic approach to expected utility in mind. The intriguing point to note, however, is that epistemic utility is variegated; one's epistemic utility might consist, e.g., in both the reduction of epistemic inaccuracy and in the satisfaction of one's preferences. Wright concludes that there is thus 'no good cause to deny certain kinds of pragmatic reason the title 'epistemic'. This will be the case where, in the slot in the structure of the reasons for an action that is to be filled by the desires of the agent, the relevant desires are focused on epistemic goods and goals' (op. cit.).

Third, and most crucially: The very idea of expected epistemic utility in the setting of decision theory makes implicit appeal to the notion of possible worlds. The full and partial beliefs of an agent will have to be defined on a probability distribution, i.e. a set of epistemically possible worlds. The philosophical significance of this point is that it demonstrates how Hale and Wright's appeal to default, rational entitlement to trust that abstraction principles are true converges with the modal approach to the epistemology of mathematics advanced in Elohim (2024). The latter proceeds by examining undecidable sentences via the epistemic interpretation of two-dimensional semantics. The latter can be understood as recording the thought that the semantic value of a proposition

[25]The distinction between the epistemic (also referred to as the alethic) and the pragmatic approaches to epistemic utility is anticipated by Clifford (1877) and James (1896), with Clifford endorsing the epistemic approach, and James the pragmatic. The distance measures comprising the scoring rules for the minimization of inaccuracy are examined in, inter alia, Fitelson (2001); Leitgeb and Pettigrew (2010); and Moss (2011). A generalization of Joyce's argument for probabilism to models of non-classical logic is examined in Paris (2001) and Williams (2012). A dominance-based approach to decision theory is examined in Easwaran (2014), and a dominance-based approach to the notion of coherence – which can accommodate phenomena such as the preface paradox, and is thus weaker than the notion of consistency in an agent's belief set – is examined in Easwaran and Fitelson (2015).

relative to a first parameter which ranges over epistemically possible worlds, will constrain the semantic value of the proposition relative to a second parameter which ranges over metaphysically possible worlds. The formal clauses for epistemic and metaphysical mathematical modalities are as follows:

Let C denote a set of epistemically possibilities, such that $[\![\phi]\!] \subseteq$ C;

(ϕ is a formula encoding a state of information at an epistemically possible world).

$\texttt{-pri}(x) = \lambda c.[\![x]\!]^{c,c}$;

(This is an epistemic intension, such that the two parameters relative to which x – a propositional variable – obtains its value are epistemically possible worlds).

$\texttt{-sec}(x) = \lambda w.[\![x]\!]^{w,w}$

(This is a metaphysical intension, such that the two parameters relative to which x obtains its value are metaphysically possible worlds).

Then:

- **Epistemic Mathematical Necessity**

  $[\![\blacksquare\phi]\!]^{c,w} = 1 \Longleftrightarrow \forall c'[\![\phi]\!]^{c',c'} = 1$

  (ϕ is true at all points in epistemic modal space).

- **Epistemic Mathematical Possibility**

  $[\![\blacklozenge\phi]\!] \neq \emptyset \iff [\![\neg\blacksquare\neg\phi]\!] = 1$

  (ϕ might be true if and only if it is not epistemically necessary for ϕ to be false).

Epistemic mathematical modality is constrained by consistency, and the formal techniques of provability and forcing. A mathematical formula is metaphysically impossible, if it can be disproved or induces inconsistency in a model.

In epistemic two-dimensional semantics, the value of a formula or term relative to a first parameter ranging over epistemic scenarios determines the value of the formula or term relative to a second parameter ranging over metaphysically possible worlds. The dependence is recorded by 2D-intensions. Chalmers (2006: 102) provides a conditional analysis of 2D-intensions to characterize the dependence: 'Here, in effect, a term's subjunctive intension depends on which epistemic possibility turns out to be actual. / This can be seen as a mapping from scenarios to subjunctive intensions, or equivalently as a mapping from (scenario, world) pairs to extensions. We can say: the two-dimensional intension of a statement S is true at (V, W) if V verifies the claim that W satisfies S. If $[A]_1$ and $[A]_2$ are canonical descriptions of V and W, we say that the two-dimensional intension is true at (V, W) if $[A]_1$ epistemically necessitates that $[A]_2$ subjunctively necessitates S. A good heuristic here is to ask "If $[A]_1$ is the case, then if $[A]_2$ had been the case, would S have been the case?". Formally, we can say that the two-dimensional intension is true at (V, W) iff '$\Box_1([A]_1 \rightarrow \Box_2([A]_2 \rightarrow S))$' is true, where '$\Box_1$' and '$\Box_2$' express epistemic and subjunctive necessity respectively'.

-$\mathtt{2D}(x) = \lambda c \lambda w [\![\mathrm{x}]\!]^{c,w} = 1$.

(This is a 2D intension. The intension determines a semantic value relative to two parameters, the first ranges over worlds from a first space and the second ranges over worlds from a distinct, second space. The value of the formula relative to the first parameter determines the value of the formula relative to the second.)

According, then, to the latter, the possibility of deciding mathematical propositions which are currently undecidable relative to a background mathematical language such as ZFC should be two-dimensional. The epistemic possibility of deciding Orey sentences can thus be a guide to the metaphysical possibility thereof.[26] Further, both the numerical term-forming operator, Nx, in abstraction principles, as well as entire abstraction principles themselves, can receive a two-dimensional treatment, such that the value of numerical terms relative to epistemic possibilities or topic-sensitive truthmakers considered as actual can determine the value of numerical terms relative to metaphysical possibilities or topic-sensitive truthmakers, and the epistemic possibility or topic-sensitive epistemic verification of an abstraction principle's truth can determine the metaphysical possibility or topic-sensitive metaphysical verification thereof. My multi-hyperintensional semantics includes structured hyperintensions for subsentential expressions of propositions, where structured hyperintensions are functions from subsentential expressions verified by topic-sensitive truthmakers to extensions (Chalmers, 2006b: 3.5). Two-dimensional hyperintensions are functions from a sentence verified by a topic-sensitive epistemic hyperintensional state, which determines the value of the sentence as verified by a topic-sensitive metaphysical hyperintensional state, to the sentence's extension. An abstraction principle modeled on Voevodsky's Univalence Axiom and function type equivalence is countenanced in chapters **2** and **3**. The truth of the first-order abstraction principle for hyperintensions is supposed to secure the existence of hyperintensions, and its truth is grounded in its possibly being recursively enumerable i.e. Turing machine computable, owing to results in observational type theory (see p. 71ftn.11, for further discussion).[27]

The convergence between Wright's and my approaches consists, then, in that – on both approaches – there is a set of epistemically possible worlds. In the former case, the epistemically possible worlds subserve the preference rankings for the definability of expected epistemic utility. Epistemic mathematical modality is thus constitutive of the notion of rational entitlement to which Hale and Wright appeal, and – in virtue of its convergence with the two-dimensional

[26]See Kanamori (2008) and Woodin (2010), for further discussion of the mathematical properties at issue.

[27]After writing this paper in 2015, a remark about conceivability-based, thought not two-dimensional, approaches to abstraction principles was published in 2020 by Bob Hale. Hale (2020: 270) writes: 'If Hume's Principle is true – that is, if conceivability implies possibility – then we should insist upon the more guarded description', with the description pertaining to whether conceiving that ϕ ought to be understood 'in terms of imagining (our) finding out or discovering [that ϕ] or in more guarded terms as imagining (our) having compelling evidence or good reason to believe' that ϕ. This remark is made in the context of a discussion of the counterconceivability of the essentiality of origins.

semantics here proffered – epistemically possible worlds can serve as a guide to the metaphysical mathematical possibility that mathematical propositions, such as abstraction principles for cardinals, reals, and sets, are true.

Novel epistemic abstractionist modalities can, further, be countenanced.

Linnebo writes: 'Let us add to our language the modal operators $\Box$ and $\Diamond$. We may think of '$\Box\phi$' as meaning "no matter what abstraction steps we carry out, it will remain the case that ϕ", and '$\Diamond\phi$' as "we can abstract so as to make it the case that ϕ". Obviously, this interpretation of the modal operators is different from the more familiar one in terms of metaphysical modality. In the useful terminology of (Fine, 2006), the present interpretation is "interpretational" rather than "circumstantial"; that is, it is concerned with how the language is interpreted, not with how reality is. In particular, every interpretational possibility is compatible with the metaphysically actual world' (2018: 61-62).

I propose to treat Linnebo's interpretational abstractionist modalities as epistemic modalities.

Both Linnebo's and my operators are 'transcendental', or 'extended' in the sense outlined by Fine (2005: 324, 326; 2020). Transcendental modality is 'true regardless of the circumstances [i.e., not 'in', but 'at' or 'of' all possible worlds], for we can recognize it to be true on the basis of its logical form alone and without regard to the circumstances' (Fine, 2005: 324, 326), by contrast to necessity which is an 'unextended' modality, and which can be interpreted as truth 'in' a possible world or truth 'in' all possible worlds (326-327). A 'superextended' modality applies to hybrid sentences which use worldly and unworldly expressions (op. cit.). Worldly entities have necessary existence as depending on the kinds of 'existents', and unworldly entities have transcendental existence as expressed by logical terms, like quantifiers and identity, and as expressed by predicates which pick out essential properties like being rational (324, 350-351). Necessary existence is 'object-driven', and two types can be distinguished depending on whether the objects' existence is indexed to times. Transcendental existence is an 'ontic notion of existence', 'domain-driven', and tied to the notion of being (351). 'For something to exist in this sense is simply for there to be something that it is. This is the sense of existence that is tied to our understanding of the quantifier; where '$\exists$y' is the unrestricted quantifier, x will exist in this sense if $\exists$y(x = y)' (350). Necessary existence is defined via worldly existents, i.e. 'the character of the object in question' (351) and 'the kind of thing that exists' (354), whereas transcendental existence is 'ontic' and tied to 'ontology' (351), the notion of being, and is expressed by unworldly, logical expressions like quantifiers and identity (350).

My epistemic abstractionist box operator is a transcendental modality, a weak modality, and satisfies the condition of 'general validity'. ϕ is weakly necessary if and only if it is not false in all possible worlds and converges with the notion of general validity, which is necessity interpreted as 'falsity at no world in a model' (Davies and Humberstone, 1979: 1). The epistemic abstractionist box operator contrasts with the epistemic abstractionist diamond operator, where the latter records necessity interpreted as real world validity. ϕ is real world valid 'if for no model is it false at the actual world of the model' (op. cit.).[28] The relation between the interpretation of the diamond operator as an abstraction yielding ϕ and real world validity might be captured by analogy to forcing, i.e. a possible extension of a ground model, where forcing can too be interpreted as validity as, for example, in the semantics for modal logic.[29]

If one prefers hyperintensional semantics to possible worlds semantics – in order e.g. to avoid the situation in intensional semantics according to which all necessary formulas express the same proposition because they are true at all possible worlds – one can avail of the following topic-sensitive epistemic two-dimensional truthmaker semantics, which specifies a notion of exact verification in a state space and where states are parts of whole worlds. According to truthmaker semantics for epistemic logic, a modalized state space model is a tuple $\langle$S, P, $\leq$, v$\rangle$, where S is a non-empty set of states, P is the subspace of possible states where states s and t comprise a fusion when s $\sqcup$ t$\in$P, $\leq$ is a partial order, and v: Prop $\rightarrow$ (2^S x 2^S) assigns a bilateral proposition $\langle p^+, p^-\rangle$ to each atom p$\in$Prop with p^+ and p^- incompatible (Fine 2017a,b; Hawke and Özgün, forthcoming: 10-11). Exact verification ($\vdash$) and exact falsification ($\dashv$) are recursively defined as follows (Fine, 2017a: 19; Hawke and Özgün, forthcoming: 11):

s $\vdash$ p if s$\in[\![p]\!]^+$

(s verifies p, if s is a truthmaker for p i.e. if s is in p's extension);

s $\dashv$ p if s$\in[\![p]\!]^-$

(s falsifies p, if s is a falsifier for p i.e. if s is in p's anti-extension);

s $\vdash \neg$p if s $\dashv$ p

(s verifies not p, if s falsifies p);

s $\dashv \neg$p if s $\vdash$ p

(s falsifies not p, if s verifies p);

s $\vdash$ p $\wedge$ q if $\exists$v,u, v $\vdash$ p, u $\vdash$ q, and s = v $\sqcup$ u

(s verifies p and q, if s is the fusion of states, v and u, v verifies p, and u verifies q);

s $\dashv$ p $\wedge$ q if s $\dashv$ p or s $\dashv$ q

(s falsifies p and q, if s falsifies p or s falsifies q);

s $\vdash$ p $\vee$ q if s $\vdash$ p or s $\vdash$ q

(s verifies p or q, if s verifies p or s verifies q);

s $\dashv$ p $\vee$ q if $\exists$v,u, v $\dashv$ p, u $\dashv$ q, and s = v $\sqcup$ u

(s falsifies p or q, if s is the fusion of the states v and u, v falsifies p, and u falsifies q);

s $\vdash \forall$xϕ(x) if $\exists s_1, \ldots, s_n$, with $s_1 \vdash \phi(a_1), \ldots, s_n \vdash \phi(a_n)$, and $s = s_1 \sqcup \ldots \sqcup s_n$

[s verifies $\forall$xϕ(x) "if it is the fusion of verifiers of its instances $\phi(a_1), \ldots, \phi(a_n)$" (Fine, 2017c)];

[28]Kuhn (2020) develops a semantics for transcendental modalities which makes them necessity operators applied to unworldly truths, although does not distinguish between general and real world validity in his semantics.

[29]See Avigad, 2004, for historical discussion of the interpretation of validity as forcing.

s $\dashv$ ∀xϕ(x) if s $\dashv$ ϕ(a) for some individual a in a domain of individuals (op. cit.)

[s falsifies ∀xϕ(x) "if it falsifies one of its instances" (op. cit.)];

s $\vdash$ ∃xϕ(x) if s $\vdash$ ϕ(a) for some individual a in a domain of individuals (op. cit.)

[s verifies ∃xϕ(x) "if it verifies one of its instances $\phi(a_1)$, ..., $\phi(a_n)$" (op. cit.)];

s $\dashv$ ∃xϕ(x) if $\exists s_1, \ldots, s_n$, with $s_1 \dashv \phi(a_1), \ldots, s_n \dashv \phi(a_n)$, and $s = s_1 \sqcup \ldots \sqcup s_n$ (op. cit.)

[s falsifies ∃xϕ(x) "if it is the fusion of falsifiers of its instances" (op. cit.)];

s exactly verifies p if and only if s $\vdash$ p if s∈⟦p⟧;

s inexactly verifies p if and only if s ⊳ p if ∃s'≤S, s' $\vdash$ p; and

s loosely verifies p if and only if, ∀v, s.t. s ⊔ v $\vdash$ p (35-36);

s $\vdash$ Aϕ if and only if for all u∈P there is a u'∈P such that u' ⊔ u∈P and u' $\vdash$ ϕ, where Aϕ denotes the apriority of ϕ[30]; and

s $\dashv$ Aϕ if and only if there is a v∈P such that for all u∈P either v ⊔ u∉P or u $\dashv$ ϕ[31];

s $\vdash$ A(Aϕ) if and only if for all u∈P there is a u'∈P such that u' ⊔ u ∈P and u' $\vdash$ ϕ and there is a u"∈P such that u' ⊔ u"∈P and u" $\vdash$ ϕ;

s $\vdash$ A(∀xϕ(x)) if and only if for all u∈P there is a u'∈P such that u $\vdash$ [u' $\vdash$ $\exists s_1, \ldots, s_n$, with $s_1 \vdash \phi(a_1), \ldots, s_n \vdash \phi(a_n)$, and $u' = s_1 \sqcup \ldots \sqcup s_n$];

s $\vdash$ A(∃xϕ(x)) if and only if or all u∈P there is a u'∈P such that u $\vdash$ [u' $\vdash$ ϕ(a)] for some individual a in a domain of individuals (op. cit.).

In order to account for two-dimensional indexing, we augment the model, M, with a second state space, S*, on which we define both a new parthood relation, ≤*, and partial function, V*, which serves to map propositions in a domain, D, to pairs of subsets of S*, {1,0}, i.e. the verifier and falsifier of p,

[30]In epistemic two-dimensional semantics, epistemic possibility is defined as the dual of apriority or epistemic necessity, i.e. as not being ruled-out apriori (¬■¬), and follows Chalmers (2011: 66). Apriority receives, however, different operators depending on whether it is defined in truthmaker semantics or possible worlds semantics. Both operators are admissible, and the definition in terms of truthmakers is here taken to be more fundamental. The definition of apriority here differs from that of DeRose (1991: 593-594) – who defines the epistemic possibility of P as being true iff "(1) no member of the relevant community knows that P is false and (2) there is no relevant way by which members of the relevant community can come to know that P is false" – by defining epistemic possibility in terms of apriority rather than knowledge. It differs from that of Huemer (2007: 129) – who defines the epistemic possibility of P as it not being the case that P is epistemically impossible, where P is epistemically impossible iff P is false, the subject has justification for ¬P "adequate for dismissing P", and the justification is "Gettier-proof" – by not availing of impossibilities, and rather availing of the duality between apriority as epistemic necessity and epistemic possibility.

[31]A more natural clause for apriority in truthmaker semantics might perhaps be thought to be 's $\vdash$ A(ϕ) iff there is a t∈P such that for all t'∈P t'∈P and t' $\vdash$ ϕ', because the latter echoes the clause for the necessity operator according to which necessity is truth at all accessible worlds, 'M,w ⊩ □(ϕ) iff ∀w'[If R(w,w'), then M,w' ⊩ ϕ]'. However, appealing to a single state that comprises a fusion with all possible states and is a necessary verifier is arguably preferable to the claim that necessity be recorded by there being all states comprising a fusion with a first state serving to verify a proposition p, because the latter claim is silent about whether the corresponding verifier of p in the fusion of all of those states is necessary. Thanks here to Peter Hawke.

such that $[\![p]\!]^{+} = 1$ and $[\![p]\!]^{-} = 0$. Thus, M = ⟨S, S*, D, ≤, ≤*, V, V*⟩. The two-dimensional hyperintensional profile of propositions may then be recorded by defining the value of p relative to two parameters, c,i: c ranges over subsets of S, and i ranges over subsets of S*.

(*) M,s∈S,s*∈S* ⊢ p iff:
(i) $\exists c_s [\![p]\!]^{c,c} = 1$ if $s \in [\![p]\!]^{+}$; and
(ii) $\exists i_{s*} [\![p]\!]^{c,i} = 1$ if $s^* \in [\![p]\!]^{+}$

(Distinct states, s,s*, from distinct state spaces, S,S*, provide a multi-dimensional verification for a proposition, p, if the value of p is provided a truthmaker by s. The value of p as verified by s determines the value of p as verified by s*).

We say that p is hyper-rigid iff:

(**) M,s∈S,s*∈S* ⊢ p iff:
(i) $\forall c'_s [\![p]\!]^{c,c'} = 1$ if $s \in [\![p]\!]^{+}$; and
(ii) $\forall i_{s*} [\![p]\!]^{c,i} = 1$ if $s^* \in [\![p]\!]^{+}$

The epistemic abstractionist box operator can be defined as an epistemically necessary truthmaker, and the epistemic abstractionist diamond operator can be defined as an epistemic truthmaker. The foregoing yields the first account of transcendental hyperintensionality in the literature.

The application of truthmaker semantics to abstraction principles might coincide with Cameron (2008)'s suggestion that truthmaker theory be appealed to in order to account for the truth of abstraction principles rather than the prior existence of objects to which the quantifiers in the principles are ontologically committed. Hale and Wright (2009: 186, ftn. 19) object to this maneuver that the target conception of ontological commitment is necessary for understanding how truth-conditions are fixed, and so ought not to be eschewed. In response, the role of ontological commitment in satisfying the truth-conditions of abstraction principles appears to be consistent with a truthmaker conception of hyperintensional states which verify the principles (both epistemically and metaphysically on the view proffered in this paper). Cameron (op. cit.: 11) notes that: 'Whether or not we are ontologically committed to numbers depends solely on whether we need them as truthmakers', so truthmaker theory itself does not entirely adduce against the requirement that the prior existence of objects in a quantifier domain is necessary in order to fix truth-conditions for a target sentence. The truthmaker approach is also consistent with a predicative conception of abstraction principles, as advanced by Linnebo (2018a), according to which objects are introduced via the principle and iterations thereof rather than there being a totality of objects prior to the stipulation of the principle.

Epistemic (primary), subjunctive (secondary), and 2D hyperintensions can be defined as follows, where hyperintensions are functions from states to extensions, and intensions are functions from worlds to extensions. Epistemic two-dimensional truthmaker semantics receives substantial motivation by its capacity (i) to model conceivability arguments involving hyperintensional metaphysics, and (ii) to avoid the problem of mathematical omniscience entrained by intensionalism about propositions[32]:

- Epistemic Hyperintension:

  $\mathtt{pri}(x) = \lambda s.[\![x]\!]^{s,s}$, with s a state in the epistemic state space S

- Subjunctive Hyperintension:

  $\mathtt{sec}_{v_@}(x) = \lambda w.[\![x]\!]^{v_@,w}$, with w a state in metaphysical state space W

- 2D-Hyperintension:

  $\mathtt{2D}(x) = \lambda s \lambda w [\![\mathrm{x}]\!]^{s,w} = 1$.

If a formula is two-dimensional and the two parameters for the formula range over distinct spaces, then there won't be only one subject matter for the formula, because total subject matters are construed as sets of verifiers and falsifiers and there will be distinct verifiers and falsifiers relative to each space over which each parameter ranges. This is especially clear if one space is interpreted epistemically and another is interpreted metaphysically. Availing of topics, i.e. subject matters, however, and assigning the same topics to each of the states from the distinct spaces relative to which the formula gets its value is one way of ensuring that the two-dimensional formula has a single subject matter.

Following the presentation of topic models in Berto (2018; 2019), Canavotto et al (2020), and Berto and Hawke (2021), atomic topics comprising a set of topics, T, record the hyperintensional intentional content of atomic formulas, i.e. what the atomic formulas are about at a hyperintensional level. Topic fusion is a binary operation, such that for all x, y, z∈T, the following properties are satisfied: idempotence ($x \oplus x = x$), commutativity ($x \oplus y = y \oplus x$), and associativity [$(x \oplus y) \oplus z = x \oplus (y \oplus z)$] (Berto, 2018: 5). Topic parthood is a partial order, ≤, defined as $\forall x,y \in T(x \leq y \iff x \oplus y = y)$ (op. cit.: 5-6). Atomic topics are defined as follows: $\mathrm{Atom}(x) \iff \neg\exists y < x$, with < a strict order. Topic parthood is thus a partial ordering such that, for all x, y, z∈T, the following properties are satisfied: reflexivity ($x \leq x$), antisymmetry ($x \leq y \wedge y \leq x \rightarrow x = y$), and transitivity ($x \leq y \wedge y \leq z \rightarrow x \leq z$) (6). A topic frame can then be defined as {W, R, T, ⊕, t}, with t a function assigning atomic topics to atomic formulas. For formulas, ϕ, atomic formulas, p, q, r (p_1, p_2, ...), and a set of atomic topics, $\mathrm{Ut}\phi = \{p_1, \ldots\ p_n\}$, the topic of ϕ, $t(\phi) = \oplus \mathrm{Ut}\phi = t(p_1) \oplus \ldots \oplus t(p_n)$ (op. cit.). Topics are hyperintensional, though not as fine-grained as syntax. Thus $t(\phi) = t(\neg\neg\phi)$, $t\phi = t(\neg\phi)$, $t(\phi \wedge \psi) = t(\phi) \oplus t(\psi) = t(\phi \vee \psi)$ (op. cit.).

The diamond and box operators can then be defined relative to topics:

$\langle M,w \rangle \Vdash \Diamond^t\phi$ iff $\langle R_{w,t} \rangle(\phi)$

$\langle M,w \rangle \Vdash \Box^t\phi$ iff $[R_{w,t}](\phi)$, with

[32] See Author (ms_1) through (ms_n) for further discussion.

$\langle R_{w,t}\rangle(\phi) := \{w'\in W t'\in T \mid R_{w,t}[w', t'] \cap \phi \neq \emptyset$ and $t'(\phi) \leq t(\phi)$
$[R_{w,t}](\phi) := \{w'\in W t'\in T \mid R_{w,t}[w', t'] \subseteq \phi$ and $t'(\phi) \leq t(\phi)$.

We can then combine topics with truthmakers rather than worlds, thus countenancing doubly hyperintensional semantics, i.e. topic-sensitive epistemic two-dimensional truthmaker semantics:

- Topic-Sensitive Epistemic Hyperintension:

  $\mathtt{pri}_t(x) = \lambda s \lambda t.[\![x]\!]^{s\cap t, s\cap t}$, with s a truthmaker from an epistemic state space.

- Topic-Sensitive Subjunctive Hyperintension:

  $\mathtt{sec}_{v_@\cap t}(x) = \lambda w \lambda t.[\![x]\!]^{v_@\cap t, w\cap t}$, with w a truthmaker from a metaphysical state space.

- Topic-Sensitive 2D-Hyperintension:

  $\mathtt{2D}(x) = \lambda s \lambda w \lambda t[\![x]\!]^{s\cap t, w\cap t} = 1$.

  Topic-sensitve two-dimensional hyperintensions can be defined, for numerical term-forming operators, N:x, in abstraction principles, and for entire abstraction principles. The truth of a first-order abstraction principle for hyperintensions is supposed to secure the existence of hyperintensions, and its truth is grounded in its possibly being recursively enumerable i.e. Turing machine computable, owing to results in observational type theory.

7 An Abstraction Principle for Epistemic (Hyper-)Intensions

In this section, I define the homotopic abstraction principle for epistemic (hyper)intensions. Intensional isomorphism, as a jointly necessary and sufficient condition for the identity of intensions, is first proposed in Carnap (1947: §14). The isomorphism of two intensional structures is argued to consist in their logical, or L-, equivalence, where logical equivalence is co-extensive with the notions of both analyticity (§2) and synonymy (§15). Carnap writes that: '[A]n expression in S is L-equivalent to an expression in S' if and only if the semantical rules of S and S' together, without the use of any knowledge about (extra-linguistic) facts, suffice to show that the two have the same extension' (p. 56), where semantical rules specify the intended interpretation of the constants and predicates of the languages (4).[33] The current approach differs from Carnap's by basing the equivalence relation necessary for an abstraction principle for epistemic intensions on Voevodsky's (2006) Univalence Axiom, which collapses identity with isomorphism in the setting of intensional type theory.[34]

[33]For criticism of Carnap's account of intensional isomorphism, based on Carnap's (1937: 17) 'Principle of Tolerance' to the effect that pragmatic desiderata are a permissible constraint on one's choice of logic, see Church (1954: 66-67).

[34]Note further that, by contrast to Carnap's approach, epistemic hyperintensions are here distinguished from linguistic intensions. See Chalmers (2006), for further discussion of the distinction between (i) epistemic, and (ii) contextual including linguistic, intensions.

Topological Semantics

In the topological semantics for modal logic, a frame is comprised of a set of points in topological space, X, and an accessibility relation, R:
F = ⟨X,R⟩;
X = $(X_x)_{x \in X}$; and
R = $(Rxy)_{x,y \in X}$ iff $R_x \subseteq X_x$ x X_x, s.t. if Rxy, then ∃o⊆X, with x∈o s.t. ∀y∈o(Rxy),
where the set of points accessible from a privileged node in the space is said to be open.[35] A model defined over the frame is a tuple, M = ⟨F,V⟩, with V a valuation function from subsets of points in F to propositional variables taking the values 0 or 1. Necessity is interpreted as an interiority operator on the space:
M,x ⊩ $\Box\phi$ iff ∃o⊆X, with x∈o, such that ∀y∈o M,y ⊩ ϕ.

Homotopy Theory

Homotopy Theory countenances the following identity, inversion, and concatenation morphisms, which are identified as continuous paths in the topology. The formal clauses, in the remainder of this section, evince how homotopic morphisms satisfy the properties of an equivalence relation.[36]

Reflexivity

∀x,y:A∀p(p : x $=_A$ y) : τ(x,y,p), with A and τ designating types, 'x:A' interpreted as 'x is a token of type A', p • q is the concatenation of p and q, $\mathtt{refl}_x$: x $=_A$ x for any x:A is a reflexivity element, and e:$\prod_{x:A}\tau(a,a,\mathtt{refl}_\alpha)$ is a dependent function[37]:
∀α:A∃e(α) : $\tau(\alpha,\alpha,\mathtt{refl}_\alpha)$;
p,q : (x $=_A$ y)
∃r∈e : p $=_{(x=_A y)}$ q
∃μ : r $=_{(p=_{(x=_A y)} q)}$ s.

Symmetry

∀A∀x,y:A∃H_Σ(x=y → y=x)
H_Σ := p ↦ p^{-1}, such that
∀x:A($\mathtt{refl}_x \equiv {\mathtt{refl}_x}^{-1}$).

[35] In order to ensure that the Kripke semantics matches the topological semantics, X must further be Alexandrov; i.e., closed under arbitrary unions and intersections. Thanks here to Peter Milne.
[36] The definitions and proofs at issue can be found in the Univalent Foundations Program (2013: 2.0-2.1). A homotopy is a continuous mapping or path between a pair of functions.
[37] A dependent function is a function type 'whose codomain type can vary depending on the element of the domain to which the function is applied' (Univalent Foundations Program (op. cit.: §1.4).

Transitivity

$\forall$A$\forall$x,y:A$\exists$$H_T$(x=y $\rightarrow$ y=z $\rightarrow$ x=z)
H_T := p $\mapsto$ q $\mapsto$ p $\bullet$ q, such that
$\forall$x:A[$\texttt{refl}_x \bullet \texttt{refl}_x \equiv \texttt{refl}_x$].

Homotopic Abstraction

$\prod_{x:A}$B(x) is a dependent function type. For all type families A,B, there is a homotopy:

H := [(f $\sim$ g) :$\equiv$ $\prod_{x:A}$(f(x) = g(x)], where
$\prod_{f:A\rightarrow B}$[(f $\sim$ f) $\wedge$ (f $\sim$ g $\rightarrow$ g $\sim$ f) $\wedge$ (f $\sim$ g $\rightarrow$ g $\sim$ h $\rightarrow$ f $\sim$ h)],
such that, via Voevodsky's (op. cit.) Univalence Axiom, for all type families A,B:U, there is a function:
$\texttt{idtoeqv}$: (A $=_U$ B) $\rightarrow$ (A $\simeq$ B),
which is itself an equivalence relation:
(A $=_U$ B) $\simeq$ (A $\simeq$ B).

Epistemic hyperintensions take the form,
$\texttt{pri}(x) = \lambda c.[\![x]\!]^{c,c}$,
with c an topic-sensitive epistemic truthmaker.

Abstraction principles for epistemic hyperintensions take, then, the form of function type equivalence:

- $\exists$f,g[f(x) = g(x)] $\simeq$ [f(x) $\simeq$ g(x)].[38]

8 Concluding Remarks

In this essay, I have endeavored to provide an account of the modal foundations of mathematical platonism. Hale and Wright's objections to the idea that

[38] Observational type theory countenances 'structure identity principles' which are type equivalences between identification types, and the theory is said to be observational because the type formation rules satisfy structure-preserving definitional equality. Higher observational type theory holds for propositional equality. 'The idea of higher observational type theory is to make these and analogous structural characterizations of identification types be part of their definitional inference rules, thus building the structure identity principle right into the rewrite rules of the type theory' (2023: https://ncatlab.org/nlab/show/higher+observational+type+theory). Shulman (2022) argues that higher observational type theory is one way to make the Univalence Axiom computable. Wright (2012c: 120) defines Hume's Principle as a pair of inference rules, and higher observational type theory might be one way to make first-order abstraction principles defined via inference rules, although not higher-order abstraction principles, computable. The Burali-Forti paradox could be circumvented, because the target abstraction principles would not be based on isomorphism like the Univalence Axiom. See Burali-Forti (1897/1967). Hodes (1984) and Hazen (1985) note that abstraction principles based on isomorphism with unrestricted comprehension entrain the paradox. I avoid the Burali-Forti paradox in my abstraction principle for two-dimensional hyperintensions because the definition is not augmented to second-order logic like in the abstractionist foundations of mathematics, is instead taken in isolation, and the definition defines functions from sets of epistemic states taken as actual to sets of metaphysical states to extensions.

Necessitism cannot account for how possibility and actuality might converge were shown to be readily answered. In response, further, to Hale and Wright's objections to the role of modalities in countenancing the truth of abstraction principles and the success of mathematical predicate reference, I demonstrated how my two-dimensional intensional and hyperintensional approaches to the epistemology of mathematics are consistent with Hale and Wright's conception of the epistemic entitlement rationally to trust that abstraction principles are true. Epistemic and metaphysical states and possibilities may thus be shown to play a constitutive role in vindicating the reality of mathematical objects and truth, and in explaining our possible knowledge thereof.

References

Arntzenius, F., and C. Dorr. 2012. Calculus as Geometry. In Arntzenius (2012), *Space, Time, and Stuff*. Oxford University Press.

Avigad, J. 2004. Forcing in Proof Theory. *Bulletin of Symbolic Logic*, 10:3.

Azzouni, J. 2013. The Compulsion to Believe: Logical Inference and Normativity. In G. Preyer and G. Peter (eds.), *Philosophy of Mathematics: Set Theory, Measuring Theories, and Nominalism*. De Gruyter.

Barcan, R. 1946. A Functional Calculus of First Order Based on Strict Implication. *Journal of Symbolic Logic*, 11.

Barcan, R. 1947. The Identity of Individuals in a Strict Functional Calculus of Second Order. *Journal of Symbolic Logic*, 12.

Benacerraf, P. 1973. Mathematical Truth. *Journal of Philosophy*, 70.

Berto, F. 2018. Aboutness in Imagination. *Philosophical Studies*, 175.

Berto, F. 2019. The Theory of Topic-Sensitive Intentional Modals. In I. Sedlar and M. Blicha (eds.), The Logica Yearbook, 2018. College Publications.

Berto, F., and P. Hawke. 2021. Knowledge relative to Information. Mind, 130: 517.

Blackburn, P., M. de Rijke, and Y. Venema. 2001. *Modal Logic*. Cambridge University Press.

Boolos, G. 1987/1998. The Consistency of Frege's *Foundations of Arithmetic*. In J.J. Thomson (ed.), *On Being and Saying*. MIT Press. Reprinted in Boolos, *Logic, Logic, and Logic*, ed. R. Jeffrey. Harvard University Press.

Boolos, G. 1990/1998. The Standard of Equality of Numbers. In Boolos (ed.), *Meaning and Method*. Cambridge University Press. Reprinted in Boolos (1998).

Burali-Forti, C. 1897/1967. A Question on Transfinite Numbers (tr. J. van Heijenoort). In van Heijenoort (ed.), *From Frege to Gödel*. Harvard University Press.

Burgess, J., and G. Rosen. 1997. *A Subject with No Object*. Oxford University Press.

Cameron, R. 2008. Truthmakers and Ontological Commitment. *Philosophical Studies*, 140.

Canavotto, I., F. Berto, and A. Giordani. 2020. Voluntary Imagination: a Fine-grained Analysis. *Review of Symbolic Logic*.

Cantor, G. 1883/1996. *Foundations of a General Theory of Manifolds*, tr. W.B. Ewald. In Ewald (ed.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics, Vol. II*. Oxford University Press.

Cantor, G. 1895/2007. *Contributions to the Founding of the Theory of Transfinite Numbers*, tr. P. Jourdain. Cosimo.

Cantor, G. 1899/1967. 'Letter to Dedekind', tr.S. Bauer-Mengelberg and J. van Heijenoort. In van Heijenoort (ed.), *From Frege to Gödel*. Harvard University Press.

Carlson, T. 2016. Collapsing Knowledge and Epistemic Church's Thesis. In L. Horsten and P. Welch (eds.), *Gödel's Disjunction*. Oxford University Press.

Carnap, R. 1937. *The Logical Syntax of Language*, tr. A. Smeaton. Routledge.

Carnap, R. 1947. *Meaning and Necessity*. University of Chicago Press.

Chalmers, D. 1996. *The Conscious Mind*. Oxford University Press.

Chalmers, D. 2002. Does Conceivability Entail Possibility? In T. Gendler and J. Hawthorne (eds.), *Conceivability and Possibility*. Oxford University Press.

Chalmers, D. 2004. Epistemic Two-dimensional Semantics. *Philosophical Studies*, 118.

Chalmers, D. 2006. The Foundations of Two-dimensional Semantics. In M. Garcia-Carpintero and J. Macia (eds.), *Two-dimensional Semantics*. Oxford University Press.

Chihara, C. 1990. *Constructibility and Mathematical Existence*. Oxford University Press.

Church, A. 1954. Intensional Isomorphism and the Identity of Belief. *Philosophical Studies*, 5:5.

Clifford, W.K. 1877. The Ethics of Belief. *Contemporary Review*, 29.

Cook, R. 2014. Possible Predicates and Actual Properties. *Synthese*, DOI 10.1007/s11229-014-0592-1.

Cook, R. 2016. Necessity, Necessitism, and Numbers. *Philosphical Forum*, 47:3-4.

Cook, R., and P. Ebert. 2005. Abstraction and Identity. *Dialectica*, 59:2.

Davies, M., and L. Humberstone. 1980. Two Notions of Necessity. *Philosophical Studies*, 38.

Dedekind, R. 1872/1996. 'Continuity and Irrational Numbers', tr. W.B. Ewald. In Ewald (ed.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics, Vol. II*. Oxford University Press.

Dedekind, R. 1888/1996. *Was sind und was sollen die Zahlen?*, tr. Ewald. In Ewald (1996).

Dever, J. ms. Quantity without Quantities.

Dummett, M. 1963/1978. The Philosophical Significance of Gödel's Theorem. In Dummett (1978), *Truth and Other Enigmas*. Harvard University Press.

Dummett, M. 1991. *Frege: Philosophy of Mathematics*. Duckworth.

Easwaran, K. 2014. Decision Theory without Representation Theorems. *Philosophers' Imprint*, 14:27.

Easwaran, K., and B. Fitelson. 2015. Accuracy, Coherence, and Evidence. In T. Gendler and J. Hawthorne (eds.), *Oxford Studies in Epistemology, Volume 5*. Oxford University Press.

Eklund, M. 2006. Neo-Fregean Ontology. *Philosophical Perspectives*, 20.

Eklund, M. 2016. Hale and Wright on the Metaontology of Neo-Fregeanism. In P. Ebert and M. Rossberg (eds.), *Abstractionism: Essays in Philosophy of Mathematics*. Oxford University Press.

Elohim, D. 2018 Grounding, Conceivability, and the Mind-Body Problem. *Synthese*, 195:2.

Elohim, D. 2024. *Epistemic Modality and Hyperintensionality in Mathematics*. Ph.D. Dissertation, Arché, University of St Andrews. Amazon.

Field, H. 1989. *Realism, Mathematics, and Modality*. Blackwell Publishing.

Fine, K. 1981. First-order Modal Theories I – Sets. *Nous*, 15:2.

Fine, K. 2002. *The Limits of Abstraction*. Oxford University Press.

Fine, K. 2005. *Modality and Tense*. Oxford University Press.

Fine, K. 2006. Relatively Unrestricted Quantification. In A. Rayo and G. Uzquiano (eds.), *Absolute Generality*. Oxford University Press.

Fine, K. 2017a. A Theory of Truthmaker Content I: Conjunction, Disjunction, and Negation. *Journal of Philosophical Logic*, 46:6.

Fine, K. 2017b. A Theory of Truthmaker Content II: Subject-matter, Common Content, Remainder, and Ground. *Journal of Philosophical Logic*, 46:6.

Fine, K. 2017c. Truthmaker Semantics. In B. Hale, C. Wright, and A. Miller (eds.), *A Companion to Philosophy of Language*. Blackwell.

Fine, K. 2020. Comments on Steven T. Kuhn's "Necessary, Transcendental, and Universal Truth", in M. Dumitru (ed.), *Metaphysics, Meaning, and Modality*. Oxford University Press.

Fitelson, B. 2001. *Studies in Confirmation Theory*. Ph.D. Dissertation, University of Wisconsin - Madison.

Frege, G. 1884/1980. *The Foundations of Arithmetic*, 2nd ed., tr. J.L. Austin. Northwestern University Press.

Frege, G. 1891/1997. 'Function and Concept', tr. P. Geach. In M. Beaney (ed.), *The Frege Reader*. Blackwell Publishing.

Frege, G. 1893/2013. *Basic Laws of Arithmetic, Vol. I-II*, tr. and ed. P. Ebert, M. Rossberg, C. Wright, and R. Cook. Oxford University Press.

Frege, G. 1897/1997. 'Logic', tr. P. Long and R. White. In Beaney, *The Frege Reader*.

Fritz, P. 2016. First-order Modal Logic in the Necessary Framework of Objects. *Canadian Journal of Philosophy*, 46:4-5.

Fritz, P., and J. Goodman. 2017. Counting Incompossibles. *Mind*, 126:504.

Gödel, K. 1931. On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems I. In Gödel (1986), *Collected Works, Volume I*, eds. S. Feferman, J. Dawson, S. Kleene, G. Moore, R. Solovay, and J. van Heijenoort. Oxford University Press.

Gödel, K. 1951. Some Basic Theorems on the Foundations of Mathematics and their Implications. In Gödel (1995), *Collected Works, Volume III*, eds. S. Feferman, J. Dawson, W. Goldfarb, C. Parsons, and R. Solovay. Oxford University Press.

Goldblatt, R. 1993. *Mathematics of Modality*. CSLI Publications.

Goldblatt, R. 2003. Questions of Canonicity. In V. Hendricks and J. Malinowski (eds.), *Trends in Logic – 50 Years of Studia Logica*. Kluwer.

Hale, B. 1987. *Abstract Objects*. Basil Blackwell.

Hale, B. 2000a. Reals by Abstraction. *Philosophia Mathematica*, 3:8. Reprinted in Hale and Wright (2001).

Hale, B. 2000b. Abstraction and Set Theory. *Notre Dame Journal of Formal Logic*, 41:4.

Hale, B. 2013a. *Necessary Beings*. Oxford University Press.

Hale, B. 2013b. Properties and the Interpretation of Second-order Logic. *Philosophia Mathematica*, 3:21.

Hale, B. 2020. CCCP. In A. Miller (eds.), *Logic, Language, and Mathematics: Themes from the Philosophy of Crispin Wright*. Oxford University Press.

Hale, B., and C. Wright. 2001. *The Reason's Proper Study*. Oxford University Press.

Hale, B., and C. Wright. 2009. The Metaontology of Abstraction. In D. Chalmers, D. Manley, and R. Wasserman (eds.), *Metametaphysics*. Oxford University Press.

Hamkins, J., and B. Löwe. 2008. The Modal Logic of Forcing. *Transactions of the American Mathematical Society*, 360:4.

Hamkins, J., and B. Löwe. 2013. Moving Up and Down in the Generic Multiverse. In K. Lodaya (ed.), *Logic and Its Applications*. Springer.

Hamkins, J., and Ø. Linnebo. 2022. The Modal Logic of Set-theoretic Potentialism and the Potentialist Maximality Principles. *Review of Symbolic Logic*, 15:1.

Hawke, P., and A. Özgün. Forthcoming. Truthmaker Semantics for Epistemic Logic. In F. Faroldi and F van de Putte (eds.), *Outstanding Contributions to Logic: Kit Fine*.

Hawley, K. 2007. Neo-Fregeanism and Quantifier Variance. *Proceedings of the Aristotelian Society Supplementary Volume*, 81.

Hazen, A. 1985. Review of Crispin Wright's *Frege's Conception of Numbers as Objects*. *Australasian Journal of Philosophy*, 63:2.

Heck, R. 1992. On the Consistency of Second-order Contextual Definitions. *Nous*, 26.

Hellman, G. 1993. *Mathematics without Numbers*. Oxford University Press.

Hilbert, D. 1923/1996. 'The Logical Foundations of Mathematics', tr. W.B. Ewald. In Ewald (ed.), *From Kant to Hilbert: A Source Book in the Foundations of Mathematics, Vol. II.*

Hjortland, O. 2016. Anti-exceptionalism about Logic. *Synthese*, DOI 10.1007/s11098-016-0701-8.

Hodes, H. 1984. Logicism and the Ontological Commitments of Arithmetic. *Journal of Philosophy*, 81:3.

Hume, D. 1739-1740/2007. *A Treatise of Human Nature*, eds. D.F. Norton and M.J. Norton. Oxford University Press.

James, W. 1896. The Will to Believe. *The New World*, 5.

Janin, D., and I. Walukiewicz. 1996. On the Expressive Completeness of the Propositional μ-Calculus with respect to Monadic Second-order Logic. In U. Montanari and V. Sasone (eds.), *Proceedings CONCUR, '96*. Springer.

Jeffrey, R. 1965. *The Logic of Decision*. University of Chicago Press.

Joyce, J. 1998. A Non-pragmatic Vindication of Probabilism. *Philosophy of Science*, 65:4.

Joyce, J. 2009. Accuracy and Coherence. In F. Huber and C. Schmidt-Petri (eds.), *Degrees of Belief*. Springer.

Kanamori, A. 2008. *The Higher Infinite*, 2nd ed. Springer-Verlag.

Kanamori, A. 2012. Large Cardinals with Forcing. In D. Gabbay, A. Kanamori, and J. Woods (eds.), *Handbook of the History of Logic: Sets and Extensions in the Twentieth Century*. Elsevier.

Koellner, P. 2006. On the Question of Absolute Undecidability. *Philosophia Mathematica*, (III), 14.

Koellner, P., and W.H. Woodin. 2010. Large Cardinals from Determinacy. In M. Foreman and A. Kanamori (eds.), *Handbook of Set Theory, Volume 3*. Springer.

Kuhn. S. 2020. Necessary, Transcendental, and Universal Truth. In M. Dumitru (ed.), *Metaphysics, Meaning, and Modality*. Oxford University Press.

Leitgeb, H. 2009. On Formal and Informal Provability. In O. Bueno and Ø. Linnebo (eds.), *New Waves in Philosophy of Mathematics*. Palgrave Macmillan.

Leitgeb, H., and R. Pettigrew. 2010. An Objective Justification of Bayesianism I: Measuring Inaccuracy. *Philosophy of Science*, 77.

Lemmon, E.J. (with D. Scott). 1966/1977. Intensional Logic/*An Introduction to Modal Logic*, ed. K. Segerberg. In N. Rescher (ed.), *American Philosophical Quarterly Monograph Series*, Vol. 11. Basil Blackwell.

Leng, M. 2007. What's There to Know? In Leng, A. Paseau, and M. Potter (eds.), *Mathematical Knowledge*. Oxford University Press.

Leng, M. 2010. *Mathematics and Reality*. Oxford University Press.

Linnebo, Ø. 2010. Some Criteria on Acceptable Abstraction. *Notre Dame Journal of Formal Logic*, 52:3.

Linnebo, Ø. 2009. Bad Company Tamed. *Synthese*, 170.

Linnebo, Ø. 2013. The Potential Hierarchy of Sets. *Review of Symbolic Logic*, 6:2.

Linnebo, Ø. 2018. *Thin Objects: An Abstractionist Account*. Oxford University Press.

Linnebo, Ø., and G. Uzquiano. 2009. What Abstraction Principles Are Acceptable? Some Limitative Results. *British Journal for the Philosophy of Science*, 60.

Maddy, P. 1988. Believing the Axioms II. *Journal of Symbolic Logic*, 53:3.

Marfori, M.A., and L. Horsten. 2016. Epistemic Church's Thesis and Absolute Undecidability. In L. Horsten and P. Welch (eds.), *Gödel's Disjunction*. Oxford University Press.

Maudlin, T. 2013. The Nature of the Quantum State. In A. Ney and D. Albert (eds.), *The Wave Function*. Oxford University Press.

McFetridge, I. 1990. *Logical Necessity*, eds. J. Haldane and R. Scruton. Aristotelian Society Series, Volume 11.

McGee, V. 1996. Logical Operations. *Journal of Philosophical Logic*, 25:6.

Moss, S. 2011. Scoring Rules and Epistemic Compromise. *Mind*, 120:480.

Myhill, J. Intensional Set Theory. In Shapiro (ed.), *Intensional Mathematics*. Elsevier Science Publishers.

Ney, A. 2013. Ontological Reduction and the Wave Function Ontology. In Ney and D. Albert (eds.), *The Wave Function*. Oxford University Press.

Nolan, D. 2002. *Topics in the Philosophy of Possible Worlds*. Routledge.

Paris, J.B. 2001. A Note on the Dutch Book Method. In *Proceedings of the Second International Symposium on Imprecise Probabilities and Their Applications, ISIPTA*. Shaker.

Parsons, C. 1964. Frege's Theory of Number. In M. Black (ed.), *Philosophy in America*. Allen and Unwin.

Parsons, C. 1983. *Mathematics in Philosophy*. Cornell University Press.

Peano, G. 1889/1967. The Principles of Arithmetic, Presented by a New Method (tr. J. van Heijenoort). In van Heijenoort (ed.), *From Frege to Gödel*. Harvard University Press.

Pettigrew, R. 2014. Accuracy and Evidence. *Dialectica*, 67:4.

Putnam, H. 1967. Mathematics without Foundations. *Journal of Philosophy*, 64.

Ramsey, F.P. 1926. Truth and Probability. In Ramsey (1960), *The Foundations of Mathematics*, ed. R. Braithwaite. Littlefield, Adams, and Co.

Rayo, A., and G. Uzquiano (eds.). 2006. *Absolute Generality*. Oxford University Press.

Reinhardt, W. 1974. Remarks on Reflection Principles, Large Cardinals, and Elementary Embeddings. In T. Jech (ed.), *Proceedings of Symposia in Pure Mathematics, Vol. 13, Part 2: Axiomatic Set Theory*. American Mathematical Society.

Reinhardt, W. 1988. Epistemic Set Theory. *Notre Dame Journal of Formal Logic*, 29:2.

Savage, L. 1954. *The Foundations of Statistics*. Dover.

Segerberg, K. 1971. *An Essay in Classical Modal Logic*. *Filosofiska Studier*, Vol. 13. Uppsala Universitet.

Shapiro, S. 1985. Epistemic and Intuitionistic Arithmetic. In Shapiro (ed.), *Intensional Mathematics*. Elsevier Science Publishers.

Shapiro, S. 2000. Frege meets Dedekind: A Neologicist Treatment of Real Analysis. *Notre Dame Journal of Formal Logic*, 41:4.

Shapiro, S., and G. Hellman. 2015. Frege meets Aristotle: Points as Abstracts. *Philosophia Mathematica*, doi:10.1093/philmat/nkv021.

Shapiro, S., and Ø. Linnebo. 2015. Frege meets Brouwer (or Heyting or Dummett). *Review of Symbolic Logic*, doi:10.1017/S1755020315000039.

Shapiro, S., and A. Weir. 1999. New V, ZF, and Abstraction. *Philosophia Mathematica*, 3:7.

Shulman, M. 2022. Towards third generation HOTT. slides.

Sider, T. 2007. Neo-Fregeanism and Quantifier Variance. *Proceedings of the Aristotelian Society Supplementary Volume*, 81.

Sider, T. 2016. On Williamson and Simplicity in Modal Logic. *Canadian Journal of Philosophy*, 46:4-5.

Simons, P. 2016. Applications of Complex Numbers and Quaternions. In P. Ebert and M. Rossberg (eds.), *Abstractionism: Essays in Philosophy of Mathematics*. Oxford University Press.

Stalnaker, R. 1968/1975. A Theory of Conditionals. In J. Cornman (ed.), *Studies in Logical Theory*. Blackwell Publishing.

Stalnaker, R. 1978. Assertion. In P. Cole (ed.), *Syntax and Semantics, Vol. 9*. Academic Press.

Stalnaker, R. 2003. *Ways a World Might Be*. Oxford University Press.

Studd, J. 2013. The Iterative Conception of Set: a (Bi-)Modal Characterisation. *Journal of Philosophical Logic*, 42.

tudd, J. 2019. *Everything, More or Less*. Oxford University Press.

The Univalent Foundations Program. 2013. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study.

Voevodsky, V. 2006. A Very Short Note on the Homotopy λ-Calculus. Unpublished.

Walsh, S. 2016. Fragments of Frege's *Grundgesetze* and Gödel's Constructible Universe. *Journal of Symbolic Logic*, 81:2.

Weir, A. 2003. Neo-Fregeanism: An Embarrassment of Riches. *Notre Dame Journal of Formal Logic*, 44.

Williams, R. 2012. Gradational Accuracy and Nonclassical Semantics. *Review of Symbolic Logic*, 5:4.

Williamson, T. 2007. *The Philosophy of Philosophy.* Oxford University Press.

Williamson, T. 2013. *Modal Logic as Metaphysics.* Oxford University Press.

Williamson, T. 2016. Absolute Provability and Safe Knowledge of Axioms. In L. Horsten and P. Welch (eds.), *Gödel's Disjunction.* Oxford University Press.

Woodin, W.H. 2010. Strong Axioms of Infinity and the Search for V. *Proceedings of the International Congress of Mathematicians.*

Woodin, W.H. 2019. The Continuum Hypothesis (slides)

Wright, C. 1983. *Frege's Conception of Numbers as Objects.* Aberdeen University Press.

Wright, C. 1997. The Philosophical Significance of Frege's Theorem. In R. Heck (ed.), *Language, Thought, and Logic.* Oxford University Press. Reprinted in Hale and Wright (2001).

Wright, C. 2000. Neo-Fregean Foundations for Real Analysis. *Notre Dame Jouranl in Formal Logic*, 41:4.

Wright, C. 2001. Is Hume's Principle Analytic? In Hale and Wright (2001).

Wright, C. 2012a. Frege and Benacerraf's Problem. In R. DiSalle, M. Frappier, and D.Brown (eds.), *Analysis and Interpretation in the Exact Sciences: Essays in Honour of William Demopoulos.* Springer.

Wright, C. 2012b. Replies, Part IV: Warrant Transmission and Entitlement. In A. Coliva (ed.), *Mind, Meaning and Knowledge.* Oxford University Press.

Wright, C. 2014. On Epistemic Entitlement II. In D. Dodd and E. Zardini (eds.), *Scepticism and Perceptual Justification.* Oxford University Press.