Fixed Points in the Epistemic Hyperintensional μ-Calculus and the KK Principle

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Abstract

This essay provides a novel account of iterated epistemic states. The essay argues that states of epistemic determinacy might be secured by countenancing iterated epistemic states on the model of fixed points in the modal μ-calculus. Despite the epistemic indeterminacy witnessed by the invalidation of modal axiom 4 in the sorites paradox – i.e. the KK principle: $\Box\phi \rightarrow \Box\Box\phi$ – a epistemic hyperintensional μ-automaton permits fixed points to entrain a principled means by which to iterate epistemic states and account thereby for necessary conditions on self-knowledge. The epistemic hyperintensional μ-calculus is applied to the iteration of the epistemic states of a single agent instead of the common knowledge of a group of agents, and is thus a novel contribution to the literature.

This essay provides a novel account of self-knowledge, which avoids the epistemic indeterminacy witnessed by the invalidation of modal axiom 4 in epistemic logic; i.e. the KK principle: $\Box\phi \rightarrow \Box\Box\phi$. The essay argues, by contrast, that – despite the invalidation of modal axiom 4 on its epistemic interpretation – states of epistemic determinacy might yet be secured by countenancing self-knowledge on the model of fixed points in monadic second-order modal logic, i.e. the modal μ-calculus.

Counterinstances to modal axiom 4 – which records the property of transitivity in labeled transition systems[1] – have been argued to occur within various interpretations of the sorites paradox. Suppose, e.g., that a subject is presented with a bounded continuum, the incipient point of which bears a red color hue and the terminal point of which bears an orange color hue. Suppose, then, that the cut-off points between the points ranging from red to orange are indiscriminable, such that the initial point, a, is determinately red, and matches the next apparent point, b; b matches the next apparent point, c; and thus – by transitivity – a matches c. Similarly, if b matches c, and c matches d, then b matches d. The sorites paradox consists in that iterations of transitivity would

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[1]See Kripke (1963).

entail that the initial and terminal points in the bounded continuum are phenomenally indistinguishable. However, if one takes transitivity to be the culprit in the sorites, then eschewing the principle would entail a rejection of the corresponding modal axiom (4), which records the iterative nature of the relation.[2] Given the epistemic interpretation of the axiom – namely, that knowledge that a point has a color hue entails knowing that one knows that the point has that color hue – a resolution of the paradox which proceeds by invalidating axiom 4 subsequently entrains the result that one can know that one of the points has a color hue, and yet not know that they know that the point has that color hue (Williamson, 1990: 107-108; 1994: 223-244; 2001: chs. 4-5).

The non-transitivity of phenomenal indistinguishability corresponds to the non-transitivity of epistemic accessibility. As Williamson (1994: 242) writes: "The example began with the non-transitive indiscriminability of days in the height of the tree, and moved on to a similar phenomenon for worlds. It seems that this can always be done. Whatever x, y and z are, if x is indiscriminable from y, and y from z, but x is discriminable from z, then one can construct miniature worlds w_x, w_y and w_z in which the subject is presented with x, y and z respectively, everything else being relevantly similar. The indiscriminability of the objects is equivalent to the indiscriminability of the corresponding worlds, and therefore to their accessibility. The latter is therefore a non-transitive relation too." The foregoing result holds, furthermore, in the probabilistic setting, such that the evidential probability that a proposition has a particular value may be certain – i.e., be equal to 1 – while the iteration of the evidential probability operator – recording the evidence with regard to that evidence – is yet equal to 0. Thus, one may be certain on the basis of one's evidence that a proposition has a particular value, while the higher-order evidence with regard to one's evidence adduces entirely against that valuation (Williamson, 2014).

In the foregoing argument, 'safety' figures as a necessary condition on knowledge, and is codified by margin-for-error principles of the form: $\forall x \forall \phi[K^{m+1}\phi(x) \rightarrow K^{m}\phi(x+1)]$, with m a natural number (Williamson, 2001: 128; Gómez-Torrente, 2002: 114). Intuitively, the safety condition ensures that if one knows that a predicate is satisfied, then one knows that the predicate is satisfied in relevantly similar worlds. Williamson targets the inconsistency of margin-for-error principles, the luminosity principle ['$\forall x \forall \phi[\phi(x) \rightarrow K\phi(x)$'], and the characterization of the sorites as occurring when an initial state satisfies a condition, e.g. being red, and a terminal state satisfies a distinct condition, e.g. being orange. As Srinivisan (2013: 4) writes: 'By [the luminosity principle], if C obtains in α_0, then S knows that C obtains in α_0. By [margin-for-error principles], if S knows that C obtains in α_0, then C obtains in α_1. By [the characterization of the sorites], C does obtain in α_0; therefore, C obtains in α_1. Similarly, we can establish that C also obtains in α_2, α_3, α_4, ..., α_n. But according to [the characterization of the sorites] C doesn't obtain in α_n. Thus we arrive at a contradiction'. The triad evinces that the luminosity principle is false, given the plausibility of margin-for-error principles and the characterization of the sorites.

[2]For more on non-transitivist approaches to the sorites, see Zardini (2019).

In cases, further, in which conditions on knowledge are satisfied, epistemic indeterminacy is supposed to issue from the non-transitivity of the accessibility relation on worlds (1994: 242).

The anti-luminosity argument can be availed of to argue against the KK principle. If states are not luminous, then knowing that ϕ will not entail that one knows that one knows that ϕ. A different argument is presented, as well, in Williamson (2001: ch. 5, p. 115-116). Suppose the following:

(1_i) If K that x is i+1 inches tall, then ¬K¬x is i inches tall

(If an agent knows that some object is i+1 inches tall, then for all the agent knows the object is i inches tall); and

(C) 'If p and all members of the set X are pertinent propositions, p is a logical consequence of X, and [an agent] knows each member of X, then he knows p' (op. cit.: 116).

Suppose that:

(2_i) An agent knows that the object is not i inches tall.

By the KK principle, (3_i) follows form (2_i).

(3_i) An agent knows that she knows that an object is not i inches tall.

Suppose a proposition (q) which states that the object is i+1 inches tall. By (1), then the agent knows that ¬(2_i). However, if (3_i), then the agent knows (2_i). Thus, (q) $\rightarrow$ (2_i) $\wedge$ ¬(2_i). Thus – by (C) – (1_i) and (3_i) imply that the agent knows ¬(q):

(2_{i+1}) the agent knows that the object is not i+1 inches tall.

Thus, from (KK), (C), and (2_i), we can infer (2_{i+1}).

Repeating the argument for values of i ranging from 0 to 664, we have

(2_0) An agent knows that the object is not 0 inches tall.

(2_{664}) An agent knows that the object is not 664 inches tall.

However, suppose that the object is in fact 664 inches tall and grant the factivity of knowledge (modal axiom T: $\Box\phi \rightarrow \phi$). Then ($2_{664}$) is false. So, from (1), (2_0), (C), and (KK), we can derive a false conclusion, (2_{664}).

(C) is a principle of deductive closure, and thus arguably ought to be preserved. Williamson takes (2_i) to be a truism, and (1) to be defensible. He thus argues that we ought to reject the KK principle.

In this essay, I endeavor to provide a novel account which permits the retention of both classical logic as well as a modal approach to the phenomenon of vagueness, while salvaging the ability of subjects to satisfy necessary conditions on there being iterated epistemic states. I will argue that – despite the invalidity of modal axiom 4 – a distinct means of securing an iterated state of knowledge concerning one's first-order knowledge that a particular state obtains is by availing of fixed point, non-deterministic automata in the setting of coalgebraic modal logic.

The modal μ-calculus is equivalent to the bisimulation-invariant fragment of monadic second-order logic.[3] μ(x). is an operator recording a least fixed point. Despite the non-transitivity of sorites phenomena – such that, on its epistemic interpretation, the subsequent invalidation of modal axiom 4 entails structural,

[3]Cf. Janin and Walukiewicz (1996).

higher-order epistemic indeterminacy – the modal μ-calculus provides a natural setting in which a least fixed point can be defined with regard to the states instantiated by non-deterministic modal automata. In virtue of recording iterations of particular states, the least fixed points witnessed by non-deterministic modal automata provide, then, an escape route from the conclusion that the invalidation of the KK principle provides an exhaustive and insuperable obstruction to self-knowledge. Rather, the least fixed points countenanced in the modal μ-calculus provide another conduit into subjects' knowledge to the effect that they know that a state has a determinate value. Thus, because of the fixed points definable in the modal μ-calculus, the non-transitivity of the similarity relation is yet consistent with necessary conditions on epistemic determinacy and self-knowledge, and the states at issue can be luminous to the subjects who instantiate them.

In the remainder of the essay, we introduce labeled transition systems, the modal μ-calculus, and non-deterministic Kripke (i.e., μ-) automata. We recount then the sorites paradox in the setting of the modal μ-calculus, and demonstrate how the existence of fixed points enables there to be iterative phenomena which ensure that – despite the invalidation of modal axiom 4 – iterations of mental states can be secured, and can thereby be luminous.

A labeled transition system is a tuple comprised of a set of worlds, M; a valuation, V, from M to its powerset, $\wp$(M); and a family of accessibility relations, R. So LTS = $\langle$M,V,R$\rangle$ (cf. Venema, 2012: 7). A Kripke coalgebra combines V and R into a Kripke functor, σ; i.e. the set of binary morphisms from M to $\wp$(M) (op. cit.: 7-8). Thus for an s$\in$M, σ(s) := [σ_V(s), σ_R(s)] (op. cit.). Satisfaction for the system is defined inductively as follows: For a formula ϕ defined at a state, s, in M,

$[\![\phi]\!]^M$ = V(s) [4]
$[\![\neg\phi]\!]^M$ = S – V(s)
$[\![\bot]\!]^M = \emptyset$
$[\![\mathrm{T}]\!]^M$ = M
$[\![\phi \vee \psi]\!]^M = [\![\phi]\!]^M \cup [\![\psi]\!]^M$
$[\![\phi \wedge \psi]\!]^M = [\![\phi]\!]^M \cap [\![\psi]\!]^M$
$[\![\diamond_d\phi]\!]^M = \langle \mathrm{R}_d\rangle [\![\phi]\!]^M$
$[\![\Box_d\phi]\!]^M = [\mathrm{R}_d][\![\phi]\!]^M$, with
$\langle \mathrm{R}_d\rangle(\phi)$:= {s$\in$S | R_d[s] $\cap\ \phi \neq \emptyset$} and
$[\mathrm{R}_d](\phi)$:= {s$\in$S | R_d[s] $\subseteq \phi$} (9)
$[\![\mu\mathrm{x}.\phi]\!] = \bigcap\{\mathrm{U} \subseteq \mathrm{M} \mid [\![\phi]\!] \subseteq \mathrm{U}\}$ (Fontaine, 2010: 18)
$[\![\upsilon\mathrm{x}.\phi]\!] = \bigcup\{\mathrm{U} \subseteq \mathrm{M} \mid \mathrm{U} \subseteq [\![\phi]\!]\}$ (op. cit.; Fontaine and Place, 2010),

A Kripke coalgebra can be represented as the pair (M, σ: S $\rightarrow$ **K**A) (Venema, 2020: 8.1)

In our Kripke colagebra, we have M,s $\Vdash \langle\pi^*\rangle\phi \iff (\phi \vee \diamond_s\langle\pi^*\rangle\phi)$ (Venema, 2012: 25). $\langle\pi^*\rangle\phi$ is thus said to be the *fixed point* for the equation, x $\iff \phi \vee \diamond$x, where the value of the formula is a function of the value of x conditional on the constancy in value of ϕ (38). The smallest solution of the formula, x $\iff$

[4] Alternatively, M,s $\Vdash \phi$ if s$\in$V(ϕ) (9).

$\phi \vee \diamond$x, is written μx.$\phi \vee \diamond$x (25). The value of the least fixed point is, finally, defined more specifically thus:

$[\![\mu x.\phi \vee \diamond x]\!] = V(\phi) \cup \langle R\rangle([\![\mu x.\phi \vee \diamond x]\!])$ (38).

A non-deterministic automaton is a tuple $\mathbb{A} = \langle A, \Xi, Acc, a_I\rangle$, with A a finite set of states, a_I being the initial state of A; Ξ is a transition function s.t. Ξ: A $\rightarrow \wp$(A); and Acc $\subseteq$ A is an acceptance condition which specifies admissible conditions on Ξ (60, 66).

Let two Kripke models $\mathbb{A} = \langle A, a\rangle$ and $\mathbb{S} = \langle S, s\rangle$, be bisimilar if and only if there is is a non-empty binary relation, Z $\subseteq \mathbb{A}$ x $\mathbb{S}$, which is satisfied, if:

(i) For all a$\in\mathbb{A}$and s$\in\mathbb{S}$, if aZs, then a and s satisfy the same proposition letters;
(ii) *The forth condition.* If aZs and $R_\triangle a, v_1 \ldots v_n$, then there are $v'_1 \ldots v'_n$ in $\mathbb{S}$, s.t.
• for all i $(1 \leq i \leq n)$ $v_i Z v'_i$, and
• $R'_\triangle s, v'_1 \ldots v'_n$;
(iii) *The back condition.* If aZs and $R'_\triangle s, v'_1 \ldots v'_n$, then there are $v_1 \ldots v_n$ in $\mathbb{A}$, s.t.
• for all i $(1 \leq i \leq n)$ $v_i Z v'_i$ and
• $R_\triangle a, v_1 \ldots v_n$ (cf. Blackburn et al, 2001: 64-65).

Bisimulations may be redefined as *relation liftings*. We let, e.g., a Kripke functor, **K**, be such that there is a relation $\overline{\mathbf{K}} \subseteq \mathbf{K}$(A) x $\mathbf{K}$(A') (Venema, 2020: 81). Let Z be a binary relation s.t. Z $\subseteq$ A x A' and $\wp\overline{Z} \subseteq \wp$(A) x $\wp$(A'), with

$\wp\overline{Z}$:= {(X,X') | $\forall$x$\in$X$\exists$x'$\in$X' with (x,x')$\in$Z $\wedge$ $\forall$x'$\in$X'$\exists$x$\in$X with (x,x')$\in$Z} (op. cit.). Then, we can define the relation lifting, $\overline{\mathbf{K}}$, as follows:

$\overline{\mathbf{K}}$:= {[(π,X), (π',X')] | $\pi = \pi$' and (X,X')$\in\wp\overline{Z}$} (op. cit.), with π a projection mapping of $\overline{\mathbf{K}}$.[5]

The relation lifting, $\overline{\mathbf{K}}$, associated with the functor, **K**, satisfies the following properties (Enqvist et al, 2019: 586):

- $\overline{\mathbf{K}}$ extends **K**. Thus $\overline{\mathbf{K}}f = \mathbf{K}f$ for all functions f: $X_1 \rightarrow X_2$;
- $\overline{\mathbf{K}}$ preserves the diagonal. Thus $\overline{\mathbf{K}}\mathrm{Id}_X = \mathrm{Id}_{KX}$ for any set X and functor, Id, where Id_C maps a set S to the product S x C (583, 586);
- $\overline{\mathbf{K}}$ is monotone. R $\subseteq$ Q implies $\overline{\mathbf{K}}$R $\subseteq \overline{\mathbf{K}}$Q for all relations R,Q $\subseteq$ X_1 x X_2;
- $\overline{\mathbf{K}}$ commutes with taking converse. $\overline{\mathbf{K}}R^\circ = (\overline{\mathbf{K}}R)^\circ$ for all relations R $\subseteq$ X_1 x X_2;
- $\overline{\mathbf{K}}$ distributes over relation composition. $\overline{\mathbf{K}}$(R ; Q) = $\overline{\mathbf{K}}$R ; $\overline{\mathbf{K}}$Q, for all relations R $\subseteq$ X_1 x X_2 and Q $\subseteq$ X_2 x X_3, provided that the functor **K** preserves weak pullbacks (op. cit.). Venema and Vosmaer (2014: §4.2.2)

[5] The projections of a relation R, with R a relation between two sets X and Y such that R $\subseteq$ X x Y, are
X $\longleftarrow(\pi_1)$ R $(\pi_2)\longrightarrow$ Y such that π_1((x,y)) = x, and π_2((x,y)) = y. See Rutten (2019: 240).

define a weak pullback as follows: 'A weak pullback of two morphisms f : X → Z and g : Y → Z with a shared codomain Z is a pair of morphisms $p_X : P \to X$ and $p_Y : P \to Y$ with a shared domain P, such that (1) $f \circ p_X = g \circ p_Y$, and (2) for any other pair of morphisms $q_X : Q \to X$ and $q_Y : Q \to Y$ with $f \circ q_X = g \circ q_Y$, there is a morphism $q : Q \to P$ such that $p_X \circ q = q_X$ and $p_Y \circ q = q_Y$. This pullback is "weak" because we are not requiring q to be unique. Saying that [a set functor] T : **Set** → **Set** preserves weak pullbacks means that if $p_X : P \to X$ and $p_Y : P \to Y$ form a weak pullback of $f : X \to Z$ and $g : Y \to Z$, then $Tp_X : TP \to TX$ and $Tp_Y : TP \to TY$ form a weak pullback of $Tf : TX \to TZ$ and $Tg : TY \to TZ$'.

The philosophical significance of the foregoing can now be witnessed by defining the μ-automata on an alphabet; in particular, a non-transitive set comprising a bounded real-valued, ordered sequence of chromatic properties. Although the non-transitivity of the ordered sequence of color hues belies modal axiom 4, such that one can know that a particular point in the sequence has a particular value although not know that one knows that the point satisfies that value, the chromatic values, ϕ, in the non-transitive set of colors nevertheless permits every sequential input state in the μ-automaton to define a fixed point. In order for there to be least and greatest fixed points, there must be monotone operators defined on complete lattices. As Venema (2020: A-2) writes: "A *partial order* is a structure $\mathbb{P} = \langle P, \leq \rangle$ such that $\leq$ is a reflexive, transitive and antisymmetric relation on P. Given a partial order $\mathbb{P}$, an element $p \in P$ is an *upper bound* (*lower bound, respectively*) of a set $X \subseteq P$ if $p \geq x$ for all $x \in X$ ($p \leq x$ for all $x \in X$). If the set of upper bounds of X has a minimum, this element is called the *least upper bound*, *supremum*, or *join* of X, notation: $\bigvee X$. Dually, the *greatest lower bound*, *infimum*, or *meet* of X, if existing, is denoted as $\bigwedge X$ ... A partial order $\mathbb{P}$ is called a *lattice* if every two-element subset of P has both an infimum and a supremum; in this case, the notation is as follows: $p \wedge q := \bigwedge\{p,q\}$, $p \vee q := \bigvee\{p,q\}$ ... A partial order $\mathbb{P}$ is called a *complete lattice* if every subset of P has both an infimum and a supremum ... A complete lattice will usually be denoted as a structure $\mathbb{C} = \langle C, \bigvee, \bigwedge \rangle$." "Let $\mathbb{P}$ and $\mathbb{P}'$ be two partial orders and let f: P → P' be some map. Then f is called *monotone* or *order preserving* if $f(x) \leq' f(y)$ whenever $x \leq y$ ..." (3.1). "Let $\mathbb{P} = \langle P, \leq \rangle$ be a partial order, and let f: P → P be some map. Then an element $p \in P$ is called a *prefixpoint* of f if $f(p) \leq p$, a *postfixpoint* of f if $p \leq f(p)$, and a *fixpoint* if $f(p) = p$. The sets of prefixpoints, postfixpoints, and fixpoints of f are denoted respectively as PRE(f), POS(f) and FIX(f). / In case the set of fixpoints of f has a least (respectively greatest) member, this element is denoted LFP.f (GFP.f, respectively)" (3-2). The Knaster-Tarski Theorem says, then, that, for a complete lattice, $\mathbb{C} = \langle C, \bigvee, \bigwedge \rangle$, with f: C → C being monotone, f has both a least and greatest fixpoint, LFP.$f = \bigwedge$PRE(f), and GFP.$f = \bigvee$POS(f) (op. cit.).

The epistemicist approach to vagueness relies, as noted, on the epistemic interpretation of the modal operator, such that the invalidation of transitivity and modal axiom 4 ($\Box\phi \to \Box\Box\phi$) can be interpreted as providing a barrier to iterated epistemic states, i.e. a necessary condition on self-knowledge. Crucially, μ-automata can receive a similar epistemic interpretation.[6] An epistemic interpretation of a μ-automaton is just such that the automaton operates over epistemically possible worlds. The automaton can thus be considered a model for an epistemic agent. The transition function accounts for the transition from one epistemic state to another, e.g. as one proceeds along the stages of a continuum. A fixed point operator on a given epistemic state, e.g. $\Box(\phi)$ where $\Box$ is interpreted so as to mean knowledge-that, amounts to one way to iterate the state. If one knows a proposition ϕ, the least fixed point operation, μx.$(\Box(\phi))$, records an iteration of the epistemic state, knowledge of knowledge, and similarly for belief. Thus, interpreting the μ-automaton epistemically permits the fixed points relative to the arbitrary points in the ordered continuum to provide a principled means – distinct from the satisfaction of the KK principle – by which to account for the pertinent iterations of epistemic states unique to an agent's self-knowledge.

The fixed point operators in the modal μ-calculus can be rendered hyperintensional, by defining the elements in the sets in the semantics for the operators above, such that they are hyperintensional parts of epistemically possible worlds, rather than whole epistemically possible worlds. [See Fine, 2017a,b,c, for a presentation of truthmaker semantics, and Elohim (2024) for detailed discussion of multi-hyperintensional semantics, which incorporates topics (i.e. subject matters), truthmakers, and two-dimensional indexing.] The semantics for each operator can then remain as presented in the foregoing, while changing the sets and their subsets to hyperintensional epistemic states or verifiers instead of worlds.

The fixed point approach to iterated epistemic states will provide a compelling alternative to the KK principle, if Williamson's argument against the KK principle does not hold for all ancestral relations of knowledge but rather only for specific applications of luminosity and modal axiom 4. If Williamson's argument does not generalize to all ancestral relations of knowledge, then one can avoid the objection that the fact that μx.$(\Box(\phi))$ entails that one knows that one knows that ϕ is such that the state collapses just to KK such that the state would rarely be satisfied in light of the argument against the KK principle. An iteration procedure via a fixed point operation on a knowledge state is distinct from an application of the KK principle, i.e. an application of modal axiom 4, and provides a novel formal method for accounting for the iteration of epistemic states.

[6] For more on the epistemic μ-calculus, see Bulling and Jamroga (2011); Bozianu et al (2013); and Dima et al (2014). For an examination of the modal μ-calculus and common knowledge, see Alberucci (2002).

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