On the basic principle of number

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A history of the construction of number has been in line with the process of recognition about the properties of geometry. Natural number representing countability is exhibited on a straight line and the completeness of real number is also originated from the continuous property of the number line. Complex number on a plane off the number line is established and thereafter, the whole number system is completed. When the process of constructing a number with geometric features is investigated from different perspectives, it provides a new insight into the fundamental principle of number, which leads to a novel methodology in mathematics.

Development of concept

There are four foundational concepts in the construction of number. They are the origin(0, criterion point), unit magnitude(1), criterion direction(0→1), unit angle(or unit rotation, 1), respectively. The origin is a mathematical object without magnitude and it is a point geometrically. A unit magnitude is determined with natural number(N) 1, which is a point on the line. It requires two points(0, 1) for the determination and the other natural numbers can be constructed from it. The number 1 is a point on the line explicitly but it also refers a distance from the origin implicitly. Simultaneously, the criterion direction is arranged with an unit interval but its attribution is not conspicious due to the characteristic of criterion.
A change of direction is implemented from the criterion with an introduction of minus sign(-). An unit angle is determined with the two directions(-1←0→1), which are conversely related. The sequence of construction is the origin, a unit magnitude and the criterion direction simultaneously, and a unit angle last.

Once a unit interval is determined, the remaining natural numbers are constructed using mathematical induction which makes infinitely countable numbers. The numbers increase by a principle of summation, which originate from addictive identity(0, 0+1, 0+1+1, 0+1+1+1, ...). For a notation of these numbers, the modern decimal numeral system of which is easy to recognize is devised, which is based on the concept of positional rule. There were times when all natural numbers are denoted differently. Negative integers are expressed with minus sign. All numbers spaced evenly with unit interval are graduated and a skeleton on the infinitely long number line is completed.

For a detailed expression of the completeness of real number(R)[1], a principle of division of the unit interval into equal ten subsections is contrived and this procedure is repeated infinitely by mathematical induction. Figure 1 illustrates the mechanism and, however small the segments might be, the mechanisms are identical. It is possible to represent an infinitesimally small segment with end points and interval of the two, which correspond to rational numbers and the in-between irrational numbers, respectively. The former is a definite representation but the latter is an indefinite one. Therefore, the completeness of real number is associated with the property of straight line.

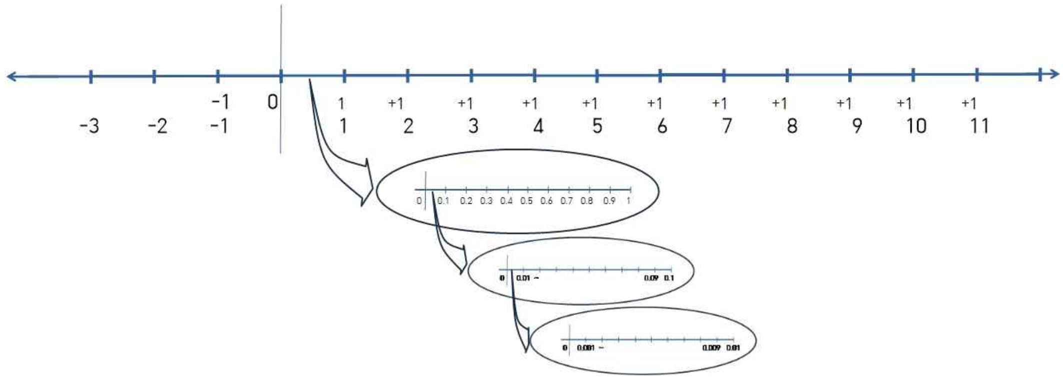

Figure 1. Construction of real number from natural number

The use of minus sign assigns a number an additional property, which functions for a change of direction and accompanies the introduction of complex number. The minus sign is a dynamic operator(rotation) and it also refers an object resulting from the operation with original object to which it is applied. It is analogous to the property of number on line, which is a position and also a distance from the origin.

The origin is a point but the criterion direction requires two points(0, 1). It is an arrow from the origin to the position one. A unit change of direction(unit rotation) from the criterion is actualized with the operator(-) and it corresponds to minus one(-1). Minus one is a result of one unit rotation from the criterion. Its direction(-1⟵0) is converse to the criterion. As a result, the location of the end of arrow is inversed and the direction is conversed. The opposite direction is a marked one, which is different from criterion. Normally, the criterion is unmarked in the organization of concept, which is similar to the tense and aspect of natural language. Following this procedure, an unit angle(1) is determined using the two directions, the criterion and the opposite(180˚) a result from an unit rotation.

A notation of unit rotation($(-1)^1$) is itself a number which refers magnitude and direction. An unit rotation counter clockwise from criterion accompanies a unit angle(1), which is a power of the notation. One application of the operator(-) composes an unit rotation which is a half rotation(180°) and two units rotation($(-1)^2$) correspond to one complete rotation(360°). Therefore, the period is 2 and it originates from an inherent property of number. The N time applications are notated as $(-1)^N$ and their positions correspond to $1(N=2n)$ and $-1(N=2n+1)$, respectively. Contrary to the mechanism of natural number, these numbers increase with a principle of variation by product, which originate from multiplicative identity(1, 1×(-1), 1×(-1)×(-1), 1×(-1)×(-1)×(-1), ...). The formal notation is only an abbreviated version of this. The multiplicative identity(1) and rotation operator (-) are essential and they constitute the base of the notation. Due to this feature, magnitude does not change but direction varies when the number N increases. A clockwise rotation can be expressed with negative integer in the power position. The mechanism of discrete variation through an unit rotation is completed and this is analogous to one of integer on the line.

A principle for the expression of continuous variation of angle is identical to that of real number from natural number. A unit angle is divided into ten equal subsections, which correspond to an interval $(-1)^{0.1}$. This process is repeated infinitely through an mathematical induction and it is possible to represent a continuity with infinitesimally small sections($(-1)^{0.1}$, $(-1)^{0.01}$, $(-1)^{0.001}$, ...). Figure 2 illustrates this with detailed subsections. Definite and indefinite representations of an angle are determined by the property of real number. It is possible to represent all angles from $(-1)^N$ to $(-1)^R$ with this method. It is a phase function with period 2 on unit circle of radius one. It is viable to express all directions using unit magnitude(1), rotation operator(-), principle of variation by product(an exponential function), and the property of real number(continuity), respectively. It is an introduction of a new notation for complex number and this is denominated as basic.

Notation

As the domain of function is defined on R, it is crucial to consider the phase of the function $(-1)^R$. Consideration of angle and location without phase leads to fallacy during mathematical manipulation. This is analogous to the established complex number system and this problem is solved by marking direction separately. For example, $\sqrt{-2}$ refers both magnitude and direction and therefore, the two properties should be manipulated separately during arithmetic operation. Misconception of these properties is subject to error to conclude that $\sqrt{-2} \bullet \sqrt{-3} \neq \sqrt{(-2)\times(-3)} = \sqrt{6}$. The solution for this problem is to mark i[2]

for the direction and then carry out the mathematical calculation such as $\sqrt{-2}\times\sqrt{-3}=\sqrt{2}\,i\times\sqrt{3}\,i=\sqrt{2}\times\sqrt{3}\times i\times i=-\sqrt{6}$. The expressions like $(-1)^{2n}=1$ and $(-1)^{2n+1}=-1$ are correct with regard to location but are not a rigorous ones when a phase is considered. Incorrect use of these equations cause a serious problem when the power of an exponential function is a main concern. It refers the phase of the number and is a principal factor during arithmetic operation. It is recommended to introduce a constant c= $(-1)^{1/\pi}$ in order to prevent the problem and adjust the period with radian. The notation c is related to i with the equation $c^{1\pi/2}=i$. A preliminary procedure for the notation of complex number is established and it is possible to represent all complex numbers with the equation $z=rc^{x}$, where r is a real number(R>0). In the following sections, the use of this notation is reserved for the sake of a familarity with the concept of $(-1)^{R}$. Instead, the notation $(-1)^{R}$ is substituted with $(-1)^{x}$, which is common in function.

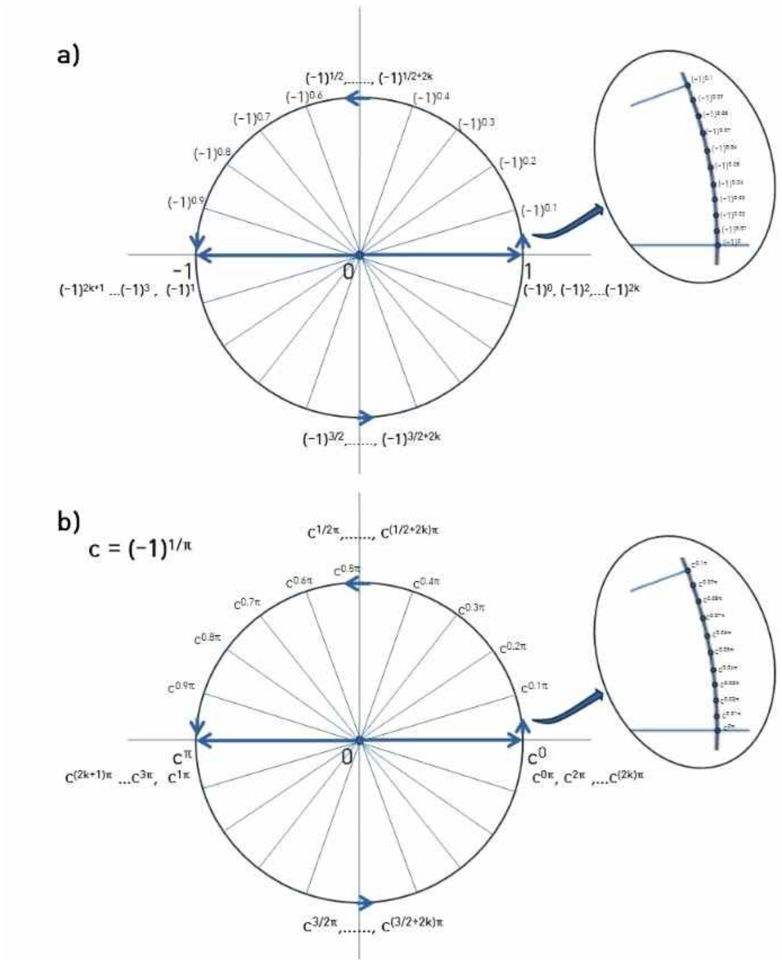

Figure 2. Schematic representation of the function $f(x)=(-1)^{x}$ (a) and $f(x)=c^{x}$ (b), respectively.

Exponential function

It is an exponential function $f(x) = (-1)^x$ with base(-1) and power(x), where the function also represents a number on complex plane. The base has a property of magnitude one and unit rotation, therefore the function represents a complex number on unit circle of radius one. It is a phase function with period 2, where x refers phase. An exponential function such as $g(x) = a^x\ (a>0)$ is a function only about magnitude. There are no indication of change of direction in the base. However, the base of the function $f(x) = (-1)^x$ is different. When x varies, the range of $f(x)$ also changes its direction because of minus sign but the magnitude remains the same due to the identity element one. Both $g(x) = a^x\ (a>0)$ and $f(x) = (-1)^x$ are exponential functions but the substantial meanings and actual results regarding exponential variation are different. The former means a rapid change but the latter indicate a periodic variation. Comprehensive meaning of exponential function is a variation by product through a base which has an unique property. It is possible to construct a complex exponential function $h(x) = (-a)^x\ (a>0)$. However, the magnitude and the direction what the function refers should be manipulated separately with particular care.

From a perspective of formalism, it is an exponential function with different explicit graph representation. Therefore, it is legitimate to apply the theorems of exponential function[3] to this function without any contradictions.

1) $(-1)^{x+y} = (-1)^x \bullet (-1)^y$
2) $(-1)^{x-y} = (-1)^x / (-1)^y$
3) $(-1)^0 = 1$
4) $(-1)^{nx} = (-1^x)^n$
5 $(-1)^x \bullet (-1)^x = (-1 \times -1)^x = (-1)^{2x}$
  ※ $(-1)^{2n} \neq 1, (-1)^{2n+1} \neq -1\ (n \in N)$
6) $f(x) = a^x = b^{k_a x} = b^{x \log_b a}$ $\qquad (a>0, b>0, a = b^{k_a}, k_a = \log_b a)$

$$= e^{x \ln a}$$

7) $f(x) = a^x,\ g(x) = b^x$ $\qquad (a > 0, b > 0)$

$$f(x) \bullet g(x) = a^x \bullet b^x = (ab)^x$$

$$= a^x \bullet a^{k_b x} = a^{x(k_b+1)} = a^{x log_a ab} \qquad (b = a^{k_b},\ k_b = \log_a b)$$

$$= b^{k_a x} \bullet b^x = b^{x(k_a+1)} = b^{x log_b ab} \qquad (a = b^{k_a},\ k_a = \log_b a)$$

$$= c^{k_{a1} x} \bullet c^{k_{a2} x} = c^{x(k_{a1}+k_{a2})} = c^{x \log_c ab} \qquad (c > 0, c^{k_{a1}} = a,\ c^{k_{a2}} = b)$$

$$= e^{x \ln ab}$$

8) $f(x) = (-a)^x,\ g(x) = (-b)^x$ $\qquad (a > 0,\ b > 0)$

$$f(x) \bullet g(x) = (-a)^x \bullet (-b)^x = (-1)^x a^x \bullet (-1)^x b^x = (-1)^{2x} \bullet (ab)^x$$

$$= (-1)^{2x} \bullet a^{x log_a ab}$$

$$= (-1)^{2x} \bullet b^{x log_b ab}$$

$$= (-1)^{2x} \bullet e^{x \ln ab}$$

The periodic characteristics have equivalent relation with trigonometric function as it is expressed on unit circle with radius one. There should be a particular care when the product of two numbers is interpreted. The product of numbers indicates a product of magnitude but it refers a summation with regard to direction.
Direction is an abstract concept contrived for a coordination of geometry from which magnitude is different. The lack of substantiality is distinctive from magnitude and its characteristics is a kinds of coordination. It is a virtual line on geometric structure, which reflects the cognition process of object. Consequentially, this polar coordination has an equivalent relation with cartesian coordinate on complex plane[4]. While the former is explicit on direction, the latter is evident on independent components. The two are related as follows.

1) $(-1)^x = \cos x + i \sin x$

2) $(-1)^{x+y} = (-1)^x \bullet (-1)^y$
$\cos(x+y) + i\sin(x+y) = (\cos x + i\sin x) \bullet (\cos y + i \sin y)$

3) $(-1)^{x-y} = (-1)^x / (-1)^y$
$\cos(x-y) + i\sin(x-y) = (\cos x + i\sin x)/(\cos y + i \sin y)$

4) $(-1)^0 = 1$
$\cos 0 + i \sin 0 = 1$

5) $(-1)^{nx} = (-1^{x})^{n}$
$\cos nx + i\sin nx = (\cos x + i\sin x)^{n}$

This function also obeys the differential rule of exponential function and has a corresponding relation with trigonometric function. As a result of first derivative, the magnitude does not change and the rate indicates only the variation of direction which is tangential on circle and with unit magnitude. Subsequent derivatives are obtained from within.

$$f(x) = (-1)^{x},$$

$$f'(x) = \lim_{h\to 0}\frac{(-1)^{x+h} - (-1)^{x}}{h}$$

$$= \lim_{h\to 0}\frac{(-1)^{x}((-1)^{h} - 1)}{h}$$

$$= (-1)^{x} \bullet f'(0)$$

$$f'(0) = \lim_{h\to 0}\frac{(-1)^{h} - 1}{h}$$

$$= \lim_{h\to 0}\frac{(\cos h + i\sin h) - (\cos 0 + i\sin 0)}{h}$$

$$= \lim_{h\to 0}\left[\frac{\cos h - \cos 0}{h} + i\frac{\sin h - \sin 0}{h}\right]$$

$$= -\pi\sin 0 + i\pi\cos 0$$

$$= i\pi$$

$$\therefore\ f'(x) = f'(0) \bullet f(x) = i\pi f(x)$$

$$f''(x) = -\pi^{2}f(x),\ f'''(x) = -i\pi^{3}f(x),\ f''''(x) = \pi^{4}f(x)$$

The characteristic of this function is identical to Euler's formula[4] and its Taylor series is as follows.

$$\begin{aligned}(-1)^{x} &= 1 + i\pi x - \frac{1}{2!}(\pi x)^{2} - i\frac{1}{3!}(\pi x)^{3} + \frac{1}{4!}(\pi x)^{4} + i\frac{1}{5!}(\pi x)^{5} \\ &\qquad - \frac{1}{6!}(\pi x)^{6} - i\frac{1}{7!}(\pi x)^{7} + \frac{1}{8!}(\pi x)^{8} + i\frac{1}{9!}(\pi x)^{9} - \mp + \ldots \\ &= \left(1 - \frac{1}{2!}(\pi x)^{2} + \frac{1}{4!}(\pi x)^{4} - \frac{1}{6!}(\pi x)^{6} + \frac{1}{8!}(\pi x)^{8} \mp \ldots\right) \\ &\qquad + i\left(\pi x - \frac{1}{3!}(\pi x)^{3} + \frac{1}{5!}(\pi x)^{5} - \frac{1}{7!}(\pi x)^{7} + \frac{1}{9!}(\pi x)^{9} \mp \ldots\right)\end{aligned}$$

The function $f(x) = (-1)^x$ is intuitive and is originated from the intrinsic property of number. However, the period of this function is different from radian which makes an equation simple in calculus. It can be adjusted by introducing a constant c=(-1)$^{1/\pi}$ and the function $f(x) = c^x$, where the period is 2π. Traditionally, radian is used to represent an angle for a couple of reasons. Distance and angle are principal factors in geometry, which are independent concept. The independence provokes an confusion when arithmetic operations are considered in calculus. This has been resolved through the relation between angle and its arc length, which are bijective and order-preserved. It is important to recognize the distinctions between what the number actually refers in geometry and what its sense is. All the previous equations are reformulated with radian as follows. The difference is the explicit and implicit representation of π, respectively.

Exponential function theorem

1) $c^{\theta_1 + \theta_2} = c^{\theta_1} \cdot c^{\theta_2}$

2) $c^{\theta_1 - \theta_2} = c^{\theta_1} / c^{\theta_2}$

3) $c^0 = 1$

4) $c^{n\theta} = (c^{\theta})^n$

5) $c^{\theta} \cdot c^{\theta} = (c \times c)^{\theta} = c^{2\theta}$

※ $(-1)^{2n} \neq 1, (-1)^{2n+1} \neq -1 \; (n \in N)$

6)$f(\theta) = a^{\theta} = b^{k_a \theta} = b^{\theta \log_b a}$ $\quad (a > 0, b > 0, a = b^{k_a}, k_a = \log_b a)$

$= e^{\theta \ln a}$

7) $f(\theta) = a^{\theta}, g(\theta) = b^{\theta}$ $\quad (a > 0, b > 0)$

$f(\theta) \cdot g(\theta) = a^{\theta} \cdot b^{\theta} = (ab)^{\theta}$

$= a^{\theta} \cdot a^{k_b \theta} = a^{\theta \log_a ab}$ $\quad (b = a^{k_b}, \; k_b = \log_a b)$

$= b^{k_a \theta} \cdot b^{\theta} = b^{\theta \log_b ab}$ $\quad (a = b^{k_a}, \; k_a = \log_b a)$

$= c^{k_{a1} \theta} \cdot c^{k_{a2} \theta} = c^{\theta \log_c ab}$ $\quad (c > 0, c^{k_{a1}} = a, \; c^{k_{a2}} = b)$

$= e^{\theta \ln ab}$

8) $f(\theta) = (-a)^{\theta},\ g(\theta) = (-b)^{\theta} \qquad (a > 0,\ b > 0)$

$$f(\theta) \cdot g(\theta) = (-a)^{\theta} \cdot (-b)^{\theta} = c^{\pi\theta}a^{\theta} \cdot c^{\pi\theta}b^{\theta} = c^{2\pi\theta} \cdot (ab)^{\theta}$$

$$= c^{2\pi\theta} \cdot a^{\theta \log_a ab}$$

$$= c^{2\pi\theta} \cdot b^{\theta \log_b ab}$$

$$= c^{2\pi\theta} \cdot e^{\theta \ln ab}$$

Trigonometric function

1) $c^{\theta} = \cos\theta + i\sin\theta$

2) $c^{\theta_1 + \theta_2} = c^{\theta_1} \cdot c^{\theta_2}$
$\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = (\cos\theta_1 + i\sin\theta_1) \cdot (\cos\theta_2 + i\sin\theta_2)$

3) $c^{\theta_1 - \theta_2} = c^{\theta_1}/c^{\theta_2}$
$\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2) = (\cos\theta_1 + i\sin\theta_1)/(\cos\theta_2 + i\sin\theta_2)$

4) $c^{0} = 1$
$\cos 0 + i\sin 0 = 1$

5) $c^{n\theta} = (c^{\theta})^{n}$
$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^{n}$

Differentiation

$f(\theta) = c^{\theta}$,

$$f'(\theta) = \lim_{h\to 0}\frac{c^{\theta + h} - c^{\theta}}{h}$$

$$= \lim_{h\to 0}\frac{c^{\theta}(c^{h} - 1)}{h}$$

$$= c^{\theta} \cdot f'(0)$$

$$f'(0) = \lim_{h\to 0}\frac{c^{h} - 1}{h} = \lim_{h\to 0}\frac{(-1)^{h} - 1}{h}$$

$$= \lim_{h\to 0}\frac{(\cos h + i\ \sin h) - (\cos 0 + i\sin 0)}{\theta}$$

$$= \lim_{h\to 0}\left[\frac{\cos h - \cos 0}{h} + i\frac{\sin h - \sin 0}{h}\right]$$

$$= -\sin 0 + i\cos 0$$

$$= i$$

$\therefore\ f'(\theta) = f'(0) \cdot f(\theta) = i f(\theta)$

$f''(\theta) = -f(\theta),\ f'''(\theta) = -i f(\theta),\ f''''(\theta) = f(\theta)$

Taylor series

$$c^{\theta} = 1 + i\theta - \frac{1}{2!}\theta^2 - i\frac{1}{3!}\theta^3 + \frac{1}{4!}\theta^4 + i\frac{1}{5!}\theta^5 - \frac{1}{6!}\theta^6 - i\frac{1}{7!}\theta^7 + \frac{1}{8!}\theta^8 + i\frac{1}{9!}\theta^9 - \mp + \ldots$$

$$= (1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \frac{1}{6!}\theta^6 + \frac{1}{8!}\theta^8 \mp \ldots)$$

$$+ i(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + \frac{1}{9!}\theta^9 \mp \ldots)$$

Conclusion

A new methodology on complex number is established. It originates from the intrinsic property of number when a direction is coordinated. This is an exponential function which satisfies the theorems of the function. However, the graphical representation is different from ordinary one and it exhibits a periodic characteristics. The period of this function is rectified to radian with a constant $c=(-1)^{1/\pi}$. A more thorough understanding of this function, especially with negative base, and comparison with Euler's formula are left for future works.

Acknowledgement

References

1) Completeness of the real numbers,
Retrived from https://en.wikipedia.org/wiki/Completeness_of_the_real_numbers

2) Erwin Kreyszig, Advanced Engineering Mathematics, Wiley; 7th edition (August 11, 1992), PP 706-750

3) Daniel Mathews et al, Exponential and logarithmic functions-A guide for teachers(Years 11-12), Education Services Australia(2013).

4) James Stwart, Calculus Early Transcendentals, Cengage Learning; 6th edition (June 7, 2007) PP A57-A63