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Early Russell on Types and Plurals

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In 1903, in *The Principles of Mathematics* (*PoM*), Russell endorsed an account of classes whereupon a class fundamentally is to be considered many things, and not one, and used this thesis to explicate his first version of a theory of types, adding that it formed the logical justification for the grammatical distinction between singular and plural. The view, however, was short-lived—rejected before *PoM* even appeared in print. However, aside from mentions of a few misgivings, there is little evidence about why he abandoned this view. In this paper, I attempt to clarify Russell's early views about plurality, arguing that they did not involve countenancing special kinds of plural things distinct from individuals. I also clarify what his misgivings about these views were, making it clear that while the plural understanding of classes helped solve certain forms of Russell's paradox, certain other Cantorian paradoxes remained. Finally, I aim to show that Russell's abandonment of something like plural logic is understandable given his own conception of logic and philosophical aims when compared to the views and approaches taken by contemporary advocates of plural logic.

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1 Introduction

It would be an understatement to say that in the early part of his career, Bertrand Russell was interested in the nature of plurality. This interest represents the common core between his work in the foundations of mathematics and his metaphysical stance taken against the idealist monists of his day. Perhaps unlike some contemporary theorists, Russell consistently and throughout his career[1] connected issues regarding the philosophical logic of plural constructions with those regarding the nature of classes. Despite the later appropriation of the word "class" in certain iterative set theories, it must be remembered that Russell never understood a class as anything like an iterative mathematical structure. Instead, Russell took giving an account of classes to be part of the core of logical theory, and originally conceived of classes as involved in all categorical judgments. The topic in Russell's mind is essentially the same as that of specifying the logical form of propositions about a group of entities where the claim made is extensional, i.e., depends only on the make-up of the collection. The exact attitude Russell took here changed often and rapidly in the first decade of the 20th century, culminating in the so-called "no-classes" theory of classes given in *Principia Mathematica* (*PM*). Prior to this, however, Russell explored several very interesting alternatives, one of which being the first version of the theory of types outlined in Appendix B of his 1903 classic *The Principles of Mathematics* (*PoM*). Although it bears many structural similarities to the theory of types of *PM*, the philosophical understanding of the theory was quite a bit different in 1903, and there is evidence that Russell had abandoned the early view already by the time *PoM* appeared in print. It is interesting to consider why this may have been. Unfortunately, we have little to go on except the misgivings Russell stated about the theory even while advancing it.

In this paper, I undertake three things. (1) Firstly, in secs. 2 and 3, I clarify how Russell understood plurality in his early work, and how this formed the basis for his first theory of types. In so doing, I argue against a common reading of early Russell according to which he acknowledged a special kind of "plural thing" distinct from individuals. (2) Secondly, in sec. 4, I discuss the difficulties Russell himself found with this approach, and speculate as to why he might have given the view up. In so doing, I present at least two Cantorian paradoxes which are not solved merely by thinking of classes as pluralities rather than as individuals. These contradictions have been discussed before, but, to my knowledge, not in their specifically pluralist formulations. (3) Finally, and in conclusion, I differentiate Russell's conception of logic from another conception which I think is more widespread today, and suggest that, keeping this conception in mind, it is easier to understand why Russell moved away from logical views endorsing a kind of plural predication in his later works. In particular, I suggest that if one is interested in analyzing states of affairs or facts, rather than reasoning or language, it is less clear that countenancing a form of plural predication (beyond that involved in interpreting higher-order logic) is necessary.

2 "Whatever are, are many"

Russell regarded the problem of how to understand the nature of plurality as, in a nutshell, the difficulty of reconciling the obvious fact or truism that each thing is just one thing, or an individual, with the possibility of plurality, that there are many things, i.e., that there are some thing*s* of which it is true to say that *they* are not one but many. On the one side of this dilemma we find Russell's conviction at the time of *PoM*[2] that there was an all-embracing category of *entities*, *terms*, *individuals* or *logical subjects* embracing any kind of being whatever, and that each member of this category can be counted as one. Here is a characteristic passage:

> Whatever may be an object of thought, or may occur in any true or false proposition, or can be counted as *one*, I call a *term*. This, then, is the widest word in the philosophical vocabulary. I shall use as synonymous with it the words unit, individual, and entity. The first two emphasize the fact that every term is *one*, while the third is derived from the fact that every term has being, *i.e. is* in some sense. A man, a moment, a number, a class, a relation, a chimaera, or anything else that can be mentioned, is sure to be a term; and to deny that such and such a thing is a term must always be false. (Russell 1931, §47)

Russell here makes it clear that he regards every entity as an individual, and regards every individual as one thing rather than many. The reason he believes it must always be false to deny that something is a term or individual is that he thinks to do so would involve a self-contradiction; in any proposition of the form *A is not a term*, *A* is the logical subject, and is thus made into an object of thought or subject of predication, and this shows that it must be a term, and hence the proposition asserting that it is not reveals its own absurdity (Russell 1931, §47; cf. Frege 1980, pp. 134, 138).

On the other side of the dilemma is the apparently obvious fact that sometimes predications of numbers other than one are appropriate; some things are two in number, or three, or four, etc. But if each thing is an individual, including as we have seen, each class that we can mention, and each individual is to be counted as one, then how could such predications ever be true? What are the subjects of such predications?

Russell's solution to this dilemma in *PoM* is to distinguish between a class as one and a class as many. When the members of a class do form a complete whole, and can be treated as a single thing, the single thing involved is the class as one. The class as one is an individual, and, as the label implies, counts as one. A class as many, however, is not a thing at all, or at least not one thing; it is as many things as there are members of the corresponding class as one. The most direct way to speak of a class as many is extensionally, by directly naming each of the members, and adjoining the names with "and". Numbers other than one may be truly predicated of the grammatical subjects so formed, thus, e.g., it is true that "John and Paul and Ringo and George are four". In the proposition expressed by this sentence, however, Russell insists, there is no one logical subject. John and Paul and George and Ringo—*they* (plural)—are the logical subjects. Russell sees in this a way out of the dilemma:

> But can we now avoid the contradiction always to be feared, where there is something that cannot be made a logical subject? I do not myself see any way of eliciting a precise contradiction in this case. . . . In such a proposition as "*A* and *B* are two," there is no logical subject: the assertion is not about *A*, nor about *B*, nor about the whole composed of both, but strictly and only about *A* and *B*. Thus it would seem that assertions are not necessarily *about* single subjects, but may be about many subjects; and this removes the contradiction . . . (Russell 1931, §74)

The suggestion seems to be something like this: although every thing which is a logical subject is an individual, and thus one, some propositions have more than one logical subject, and while each of these logical subjects is itself one thing, it is possible to predicate something of them collectively, so that no one of them is *the* logical subject. So while every proposition of the form "*A* is one" or "*A* is an individual" is true, the proposition expressed by "*A* and *B* are one" or "*A* and *B* are an individual" is not of this form, and so it may be false. Similarly, while every proposition of the form "*A* is many" is false, the proposition "*A* and *B* are many" is not of that form. As Russell puts the point succinctly later, "For although whatever is, is one, yet it is equally true that whatever are, are many" (§127).

Russell's terminology on these issues can cause confusion and sometimes misrepresents his position—especially the talk of "*a* class as many" as if it were *a* thing, and his description of a class as many as "*a* collection" (§130) or "*an* object" (§497) distinct from an individual. Such locutions have naturally led certain interpreters to the conclusion that Russell endorsed a bifurcated metaphysics where some things are singular and some are plural, and acknowledged different kinds of logical subject positions in propositions differentiated by what sort of thing occupies them. For example, Proops (2007, pp. 3–6) has argued that the individual variable in Russell's early philosophical logic was not fully unrestricted, at least partly[3] on the grounds that it excludes such "objects" as *William and Mary*. But I think this way of stating the conclusion is misleading. William and Mary are not a thing, they are two things. To say that the individual variable does not have *them* as a value is not for that variable to be restricted, it is only for it to be incapable of taking on *two values* at once. Russell's view is that there are, in the end, only individuals, and that a class as many is only a way of speaking about the members of the class as one in a way that does not reduce to singular predication of each of the members separately, *or* to a predication made about the class as one. Thus, instead of propositions which have a special, non-singular, logical subject, we have instead propositions which simply have no one logical subject (cf. §70). Indeed, this is demanded by the method of "avoiding the contradiction" of things which cannot be made subjects; the assertion is made "strictly and only about *A* and *B*" and not about one thing at all, and *a fortiori* not about a special "plural thing". That Russell does not recognize a special category of plural thing during this period is also attested in his correspondence. Thus in July 1902 he wrote to Frege that “a class consisting of more than one object is in the first place not *one* object but many” (Frege 1980, p. 137), and in 1906, while summarizing the development of his views in a letter to Jourdain, he wrote:

> You will see that in my book (p. 104, art. 104) I suggest that certain functions do not determine a *class as one*. This is practically the same doctrine as that they do not determine a class, for a *class as many* is not an entity. ... My book gives you all my ideas down to the end of 1902: the doctrine of types (which in *practice* is almost exactly like my present view) was the latest of them. (Grattan-Guinness 1977, p. 78)

This was during the period of Russell’s “substitutional theory”, when he explicitly took the view that “there is really nothing that is not an individual” (Russell 1973b, p. 206) and “adhere[d] with drastic pedantry to the old maxim that, ‘whatever is, is one’.” (Russell 1973c, p. 189)

3 Types in Appendix B

It seems clear that had he not discovered the class paradox bearing his name, Russell would have held that for each class as many, there was a corresponding class as one, and vice versa. The class paradox itself Russell blamed on a violation of Cantor’s powerclass theorem (§100), i.e., the result that every class must contain more subclasses than members. This means in particular that the collection of all individuals must contain more subclasses than members, i.e., that there must be more classes of individuals than individuals. However, if each class of individuals itself forms an individual, i.e., forms a class as one, Cantor’s principle is obviously violated. For Russell at the time, the lesson of the paradox seemed to be, as clear also in the letter to Jourdain quoted above, that certain propositional functions, while they determine a class as many, do not determine a class as one:

> In terms of classes the contradiction appears even more extraordinary. A class as one may be a term of itself as many. Thus the class of all classes is a class; the class of all terms that are not men is not a man, and so on. Do all the classes that have this property form a class? If so, is it as one a member of itself as many or not? If it is, then it is one of the classes, which, as ones, are not members of themselves as many, and *vice versâ*. Thus we must conclude again that the class which as ones are not members of themselves as many do not form a class, or rather, that they do not form a class as one, for the argument cannot show that they do not form a class as many. (Russell 1931, §101)

More generally, Russell held that an expression for a class did not name a class as one (but only a class as many) when the defining propositional function involved what he called a “quadratic form”, understood as involving an assertion about something, where what the assertion is and what the assertion is about are somehow linked and depend on each other, and so cannot be varied independently (§103). Russell did not give in *PoM* (or elsewhere) a completely formal, precise or spelled out definition of exactly under what conditions a propositional function was to be regarded as quadratic. The strategy taken in Appendix B of *PoM*, where Russell first gave a version of the theory of types, skirted the need to do so by recommending that classes as many rather than classes as one be taken as the basis for a symbolic logic of classes. The response to the Cantorian worry mentioned above then is that while it is true that there must be more classes *as many* than individuals, classes as many are just that—many entities, not one—and so we are not forced to countenance as many individuals as classes as many after all.

In particular, the suggestion was that in any proposition of the form:

$$x \epsilon \alpha$$

The x and the α must be taken as having different logical types, so that, for example, if x is a variable for an individual, α would be a variable for a class as many of individuals, which we have seen, is not one thing at all, but rather, (potentially) *many* individuals. A caveat is that under the views Russell adopted here, a late addition to the text, Russell believed that a singleton class as many, or even the empty class as many, could also be countenanced, and in the former case, distinguished from the member which it has. Singleton classes as many, rather than individuals, then were taken as the subjects of predications of the number one, so that all numbers were taken as predications made about classes as many. These views went against positions taken in the body of the text (§§72–73), and which, one might think, go better naturally with the overall view that a class as many just is its members taken plurally. Yet, Russell continues to believe, explicitly despite this, that, "[t]his constitutes, in a kind of way, a justification for the grammatical distinction of singular and plural" (§497). The theory of types adopted here in effect distinguishes between individuals, pluralities of individuals, and pluralities of pluralities. Yet, it is primarily flanking the two sides of "ϵ" that Russell believes that type-restrictions must be obeyed. Unlike the kinds of type theories found in his later work, Russell here acknowledges there can be combined types, and that some other predicates and relations apart from "ϵ" may be type agnostic. One example he gives (§497), is identity. Hence, Russell believes it is meaningful (though certainly false) to claim of a single entity that it is identical to many entities. Numbers too, Russell regards as falling into their own type, since one can predicate *two* both of a class (plurality) of individuals with two members, and of a class of classes (plurality of pluralities) with two members. It seems that Russell at this point hoped to avoid the result which one finds in later versions of type theory that there are distinct numbers for distinct types. To capture mathematics fully, it will also be necessary for there to be classes of numbers, which would presumably be distinct either from classes of individuals, or even from classes of classes of individuals, or anything in that hierarchy. Unfortunately, Russell tells us very little about higher types, other than to mention that they exist:

> The next type after classes of individuals consists of classes of classes of individuals. Such are, for example, associations of clubs; the members of such associations, the clubs, are themselves classes of individuals. . . . There is a progression of such types . . . (§497)

From a philosophical standpoint, Russell's reticence about higher types is unfortunate, since it is not altogether clear how to understand a class of classes from an entirely pluralist standpoint. For example, the class of classes named by "the French and the English" would seem, at least in some sense, to have two members, the French, and the English, but if the French and the English are not thought of as classes as one here, it is difficult to understand how together they can be two. There is no textual evidence, however, as far as I can find to hint that Russell saw the difficulty here.

Another complication in his views at the time is that in addition to classes and individuals, Russell speaks of something he calls "ranges", and with these too, it is rather obscure how the basic philosophical foundation of distinguishing singular entities from pluralities of entities applies to them. To understand what Russell means by a "range", one must turn to Russell's engagement with Frege's philosophy, something which Russell also took up only while finishing *PoM*, resulting in his writing Appendix A (on Frege). Russell's term "range" here derives from Frege's term "*Werthverlauf*", nowadays usually translated as "value-range" or "course-of-values". In the logical system of Frege's *Grundgesetze der Arithmetik*, all first-level functions were taken to have a value-range. In the case of what Frege calls concepts, the referents of grammatical predicates, Frege identified their value-ranges with the extensions of concepts, or classes. But Frege acknowledges value-ranges for other functions as well, which was important for Frege's method for capturing the extension of a relation. Frege treated these with what he called a "double value-range" (Frege 2013, §36). For example, consider what Frege would have written as $\acute{\alpha}\acute{\varepsilon}(\varepsilon > \alpha)$. Strictly speaking, this is the value-range of the function $\acute{\varepsilon}(\varepsilon > (\ ))$, i.e., the function whose value for 4 as argument is $\acute{\varepsilon}(\varepsilon > 4)$, the class of all numbers greater than 4, and whose value for 7 as argument is $\acute{\varepsilon}(\varepsilon > 7)$, the class of all numbers greater than 7, and so on. This value-range does the logical work one would expect of the extension of the *greater-than* relation in Frege's logic.

It is not entirely clear how much of Frege's precise understanding of value-ranges Russell meant to co-opt here,[4] though certainly not all of it, since in the case of extensions of concepts, Frege was crystal clear that he did not regard the extension of a concept as in any sense a plurality or aggregate of entities, and regarded all value-ranges as falling in the same type as objects. Russell here, however, wants to apply the overall distinction between singular entities and so-called "objects", which are in some sense not singular, so as to place "ranges" into the second category, and into a distinct logical type. He describes them as if they were collections of "couples with sense" (§497), or ordered pairs in contemporary vocabulary. Then, a claim to the effect, e.g., that a certain couple (x, y) fell into the "range" of some relation would, like the claim that one individual was a member of a class, be a claim involving a distinction of logical type among the relata, so that one could not claim that a relation made up one of the parts of a couple falling in its own extension. This Russell hoped would solve a paradox similar to the class paradox but involving relations.[5]

It is interesting the extent to which the changes Russell made to his conception of classes as many, and the apparent clash between some of them and the underlying philosophy of chap. VI, compare to concerns that contemporary theorists have over the nature of plural logic or plural predication generally.

Interest in this subject was revived by George Boolos's 1984 paper "To Be is to Be a Value of a Variable (or to be Some Values of Some Variables)", in which Boolos suggests that higher-order logic might be interpreted pluralistically. On this reading, a second-order variable would be taken as having as its values groups of individuals taken plurally, rather than taken as ranging over possibly ontologically dubious entities such as sets, type-stratified attributes in intension, or "propositional functions". Others such as Thomas McKay (2006) and David Lewis (1991) have argued for acknowledging plural quantification and plural predication on more general philosophical grounds. Such thinkers have struggled exactly with issues concerning whether or not a plural interpretation of higher-order logic can be successfully applied to a logic of relations or their extensions (see, e.g., Lewis 1991, chap. 3, Hazen 1997), whether or not a single individual can count as a value of a plural variable (see, e.g., Boolos 1984, p. 67), and so on.

4 Russell's Misgivings About His Own Theory

For Russell's own part, his exploration of views of this stripe was short-lived. Already by the time *PoM* appeared in print, Russell seems to have given up on it. The immediate cause seems to have been the publication of the second volume of Frege's *Grundgesetze*, to which Frege had, spurred by letters from Russell, hastily added an Appendix on Russell's paradox. Reading this prompted Russell to add the following note to Appendix A (on Frege) of *PoM* at the very last minute before *PoM* was published:

> The second volume of *Gg.*, which appeared too late to be noticed in the Appendix, contains an interesting discussion of the contradiction (pp. 253–265), suggesting that the solution is to be found by denying that two propositional functions which determine equal classes must be equivalent. As it seems likely that this is the true solution, the reader is strongly recommended to examine Frege's argument on the point.

The first manuscripts of Russell's from the post-*PoM* period see him exploring views of the sort espoused by Frege in the appendix, although it didn't take long for Russell to abandon that approach either. By mid-1903 he had moved on to a theory according to which classes were regarded as superfluous in the context of mathematical logic, and a logic of functions substituted instead,[6] and by later in that year, he returned to the theory according to which only some propositional functions determine classes, those that are not "quadratic", but with the logic focused on classes as one rather than on classes as many. Indeed, the distinction between classes as one and classes as many never reappears in Russell's manuscripts, somewhat surprisingly given the sheer number of alternatives he explores in the years that follow.[7]

Russell does not tell us anywhere exactly why he abandoned this particular understanding of classes or types, and we are left to speculate. There are, however, a number of hints in *PoM* itself about why he would have been less than fully happy with the theory and eager to consider alternatives. The first, and I think, least important, comes in a footnote to §58 of *PoM*, where Russell writes:

> I shall use the word *object* in a wider sense than *term*, to cover both

singular and plural, and also cases of ambiguity, such as "a man". The fact that a word can be framed with a wider meaning than *term* raises grave logical problems. Cf. §47.

In §47, as we have already seen, Russell had argued that "term", "individual", "unit" or "entity" were synonyms and that any one of them represented "the widest word in the philosophical vocabulary". He similarly argued that to deny of anything that it is a term must always be false, since in order to deny that the thing in question is a term, one must make it into a logical subject and thus into a term. But the vocabulary of "objects" here seems to suggest that there are some "objects", i.e., plural things, which are not individuals or units or entities, i.e., a vocabulary wider than that of terms or entities. But I think it would be a mistake to read too much into this as an explanation for why Russell was unhappy with his account of classes as many, given that he explicitly argued in §74 that the "contradiction always to be feared" of something that cannot be made into a logical subject was avoided so long as a class as many is regarded always and strictly as many entities and not one, as I have argued we must understand Russell's classes as many. It certainly may be that Russell retained certain misgivings about his *terminology*, i.e., that by describing many things as "an object" or "a collection" or "a class as many", the grammar invites us to think of many objects as if they made up one thing or one logical subject, which is of course precisely what we must not do. Yet it is hard to imagine Russell abandoning an otherwise sound philosophical position merely on the basis that its formulation used misleading language, especially so long as the misapprehensions it caused could be cleared up.

A more telling, and not unrelated, hint about Russell's direction of thought occurs in §76, where he writes:

> It is plain that, since a class, except when it has one term, is essentially many, it cannot be *as such* represented by a single letter: hence in any possible Symbolic Logic the letters which do duty for classes cannot represent the classes *as many*, but must represent either class-concepts, or the wholes composed of classes, or some other allied single entities. And thus ϵ cannot represent the relation of a term to its class as many; for this would be a relation of one term to many terms, not a two-term relation such as we want. This relation might be expressed by "Socrates is one among men"; but this, in any case, cannot be taken to be the meaning of ϵ.

Here Russell is explaining why it is preferable for the development of symbolic logic to understand ϵ as representing a relation between an individual and a class as one, rather than between an individual and a class as many. However, in the logic of Appendix B, one is in effect taking ϵ to be the second relation Russell mentions here, that of "being among", i.e., the relation which many contemporary plural logicians represent with the sign $\prec$ instead. The reasons Russell is uncomfortable with this are perhaps not entirely clear from this passage alone, and it is not hard to imagine a contemporary advocate of plural quantification insisting that the attitude that a single sign cannot have many semantic values at once, or that a single variable cannot have many assignments at once, is just a singularist prejudice without merit.

Russell's worries, however, are, I think, rather more subtle and insightful, and connect centrally with his attitude about how these views fare vis-à-vis the logical paradoxes. There are two ways, according to Russell, for a proposition to come to be about a class as many. The first is for the class as many to be specified extensionally by listing all of the entities in question. Thus for example, if one writes:

George is one among John and George and Paul and Ringo. (1)

Here one has said something that seems almost trivially true, and there seems to be no direct violation of Russell's contention that to speak of a class as many requires using multiple symbols to represent its multiple members. Here, the right side of the "is one among" relation uses many signs to speak of many people. Suppose, however, that I write:

George is one among guitarists. (2)

Here, rather than specifying a class as many directly, I make use of a concept. I am in no position to name every guitarist who has ever (and will ever!) perform, and so it is impossible for me to use the method of extension to get at the class of guitarists; I can only do so via intension. Russell's worry I think, is that if we use something of the form "$x \in \alpha$" to represent an individual's being one among a certain plurality, then insofar as it is appropriate to use a single letter α for the plurality, the α must be taken as not *directly* representing the plurality, but rather as directly giving us either a class-concept, or concept of a class (denoting concept),[8] which, in turn, represents the class as many at another level of removal. So even though there are many guitarists, and *they* might be taken to be what (2) is about, the concept is itself still one thing, and so the number of things *directly* indicated and the number of symbols, still must match.

The intensional aspects of plural logic, from what I've seen, have received relatively little attention from contemporary theorists and logicians, and have the potential for raising serious concerns. From Russell's perspective, within this lay the threat of completely undermining the very *raison d'être* of the pluralist understanding of class logic to begin with: the response to Cantor. So long as for every plural collection of individuals there is some one thing, some one individual, that they all have in common, even if that thing is an intension rather than something extensional, we will end up positing as many individuals as classes, and Cantor's theorem will be violated. Indeed, already in chap. X of *PoM* Russell had stated the paradox in terms of class-concepts as follows:

> Let us now state the same contradiction in terms of class-concepts. A class-concept may or may not be a term of its own extension. "Class-concept which is not a term of its own extension" appears to be a class-concept. But if it is a term of its own extension, it is a class-concept which is not a term of its own extension, and *vice versâ*. (§101)

Things are no better with a plural understanding of classes if the symbolism presupposes that pluralities are in general, and always can be, represented by means of concepts. On that approach it might be better to state the problem this way: some concepts which denote collections of things are parts of those collections. The concept indicated by "concepts" is itself one of the many things it collects together, but the concept indicated by "cats" is not a cat. Let us call those concepts which, like *cats*, and most other examples which come readily to mind, are not among the collections they pick out, "ordinary concepts". Now let us ask whether or not the concept *ordinary concepts* is one among ordinary concepts, and we find that it is just in case it is not. Contradiction. Russell was, I think, worried that if we regard ϵ as suggested above, this version of the paradox will be hard to resist.[9]

This segues nicely into the last bits of textual evidence indicating the ways in which Russell may have been less than fully happy with the doctrine of types advocated in Appendix B, which involve doubts he expresses in Appendix B itself. The appendix begins:

> The doctrine of types is here put forward tentatively, as affording a possible solution of the contradiction; but it requires, in all probability, to be transformed into some subtler shape before it can answer all difficulties. In case, however, it should be found to be a first step towards the truth, I shall endeavor in this Appendix to set forth its main outlines, as well as some problems which it fails to solve. (§497)

What problems does it fail to solve? Russell summarizes them at the end of §498 thusly:

> ... the number of propositions is as great as that of all objects absolutely, since every object is identical with itself, and "x is identical with x" has a one-one relation to x. In this there are, however, two difficulties. First, what we called the propositional concept appears to be always an individual; consequently there should be no more propositions than individuals. Secondly, if it is possible, as it seems to be, to form ranges of propositions, there must be more such ranges than there are propositions, although such ranges are only some among objects (cf. §343). These two difficulties are very serious, and demand a full discussion.

Note that in order for the plural approach to afford an adequate answer to Cantor, it must not only be impossible to generate a class (as one) or class-concept for every plurality, but it must be *completely impossible* to generate a distinct entity of a given type for each plurality of things of that type. We have already seen a potential problem with each plurality being denoted by a concept. Another point of concern is whether or not it is possible to generate a distinct proposition for each plurality.[10] Recall that at this time, Russell regarded a proposition as a mind-independent complex entity, and not as something linguistic or even as something semantic or intentional-with-a-t, and thus, they too can be among the members of a class, or other range.

In the appendix, Russell lists two ways of generating a distinct proposition for each class of propositions. One, evident in the quotation above, is to consider the proposition that that class is self-identical. Another, is to consider, for each class of propositions what Russell calls its "logical product", i.e., the proposition that all propositions in the class are true (§500). For present purposes, however, I think it is best to give a slightly different example, to make it unequivocal that the problem threatens even for contemporary plural logics, whether or not they regard pluralities as in any sense a kind of class or directly related to class theory. If there are propositions, and they can be included among pluralities, then so long as there is any plural predicate applicable to pluralities of propositions, then it seems that, contrary to Cantor, there are as many propositions as collections or pluralities thereof. We may take as our example plural predicate the predicate "are many"—if you don't like this example, then choose any other plural predicate which you prefer. Applying Cantor's diagonal method, we get the following contradiction. Some propositions of the form "*p*s are many" are such as to be among the propositions they are about. Thus, for example, the proposition *truths are many* is itself a truth, and so it is one among the propositions it is about. However, the proposition *falsities are many* is true, not false, and so it is not one among the propositions it is about. Let us call propositions like the second example, which are of the form "*p*s are many", but are not included in what they are about, "groovyprops". Now let us consider the proposition *groovyprops are many*. This proposition is true, but never mind that. The more interesting question is: is it among the propositions it is about? I.e., is it a groovyprop? If it is, then it must not be, and if it is not, then it would be. Contradiction.[11]

Russell was explicit that he believed that there to be such a "close analogy" between paradoxes of this ilk and the more simple class version of Russell's paradox that they "must have the same solution, or at least very similar solutions" (§500). But yet there is no obvious way to extend the general solution to the class paradox which takes classes to be many entities rather than one so that it solves these paradoxes as well. In search of a more unified solution, Russell seems to have given up on views of this sort altogether. Of course, there may have been other reasons Russell was unhappy with his early theory of types; for example, he may have begun to appreciate the sorts of difficulties with a plural interpretation of higher types, such as classes of classes like "the French and the English" as discussed in sec. 3. If so, however, it does not seem to be reflected in his letters or manuscripts.

Of course, it is one thing to notice that a plural understanding of classes leaves certain paradoxes unsolved; it is another thing to discover another tack which might do better. In between abandoning plurals and finally settling on the ramified theory of types of *PM*, Russell considered in turn a number of other approaches he also discarded. Unfortunately, we cannot survey all this work here. Immediately after abandoning plurals he seems to have been inspired by the appendix Frege added to volume II of his *Grundgesetze* in which he discussed Russell's paradox, as evinced by a last minute note Russell added to the end appendix A of *PoM* endorsing Frege's solution. In Frege's appendix, he blamed the contradiction on his Basic Law V, which asserts that the extension of concept F is identical to the extension of concept G just in case F and G are coextensive. Frege goes on to generalize the result to show not only that Law V is false, but that there is no way to correlate objects with concepts such that different objects are always correlated with two concepts that are not coextensive. Frege concluded that it must be possible for two non-coextensive concepts to have the same "extension". Russell seems to have held hope for awhile that this insight would provide a way not only for blocking the usual class-form of Russell's paradox, but also other forms, including the propositional form, as he mentions in a 1903 letter to Frege (Frege 1980, p. 160). This would be to adopt a different identity criterion for propositions, so that a proposition *there are many things with property F* might be the *same proposition* as the proposition *there are many things with property G* even when *F* and *G* are not coextensive; indeed, that very proposition might fall under one but not the other. Eventually Russell seems to have grown dissatisfied with this approach as well (see Urquhart 1994, pp. 3–4), but we must leave discussion of later developments in Russell's thought for another occasion.

5 Conclusion: Are Plurals Superfluous?

Russell gave up his early plural understanding of classes, but was he right to do so? This is a large question, and we shall not be able to give it an answer here, but perhaps we can gesture at possible routes for further exploration. The first thing to note is that the kinds of Cantorian contradictions mentioned in the last section, although important and, I think, unduly neglected in current discussions, are not obviously decisive. While these paradoxes are not semantic in the sense of involving language, or intentional-with-a-t or mental entities, they are semantic in the sense of being intensional-with-an-s, or in the sense of involving entities which might often be described as *meanings*: concepts and propositions. Since on Russell's understanding, and most others, concepts and propositions are not linguistic entities, it is not clear that the usual ways of addressing the semantic paradoxes, with a Tarskian hierarchy of languages or similar, would work. Nevertheless, it would not be unreasonable still to hold that their genesis lies not with the plural conception of classes with which they were formulated above, but rather with the assumptions they rely upon regarding the nature and existence of these intensional entities.

These are difficult issues, but it is perhaps worth briefly mentioning why Russell or someone with broadly similar commitments would not have taken one approach. Contemporaries who combine set theory with plural logic often employ a "limitation of size" approach to sets: certain *things* (plural) form a set when they are not too many. In the case of the paradoxes discussed in the previous section, the analogous move would be to claim that there is no concept picking out a certain plurality if the plurality is too large, or that there can be no proposition about a plurality if the plurality is too large. In these cases, however, the limitation of size approach does not seem very intuitively plausible. Is it really reasonable to conclude that there is not even such a *concept* as *entities* or *self-identical things*, merely because there are too many entities or too many self-identical things? After all, we seem to be making use of just such concepts (or similar concepts) in stating our philosophical views. Similarly, it would seem almost unintelligible to claim that there are certain collections so large that there can be no propositions about them. For one, that very statement seems to assert the sort of proposition it is meant to rule out. One must also remember that for early Russell, propositions were not first and foremost meanings; they could be meant, of course (i.e., they could be semantic *values*), but a proposition generally was taken to consist of the actual entities it was about and the actual relations between them. An early Russellian proposition can basically be understood as a state of affairs, and Russell himself explicitly identified a proposition, when true, with a fact (see Russell 1994a, p. 75 and Russell 1994b, p. 492). For a Russellian, to claim that there are collections too large for there to be propositions about them is tantamount to claiming that there are collections about which there are no facts, or no truths. This conclusion too seems too harsh.

Putting the paradoxes aside for the moment, are there specific ways in which Russell's later philosophy suffers in virtue of having given up this plural conception of classes? This too is too large an issue to tackle at once. To do it justice, we would have to formulate Russell's mature theory of types, and evaluate it in comparison to this earlier theory. While I think strides have been taken in recent years towards getting a better handle on exactly how Russell's mature theory of types should be understood, what its philosophical underpinnings are, and what solution it offered for these and other paradoxes, it would again take us too far afield to discuss these issues fully here.[12] Instead, I would like only to *begin* to address the question of how I think Russell would answer those advocates of plural logic who would regard Russell as being more or less on the right track at the time of *PoM*, and off-track in advocating what at least appear to be strongly "singularist" sentiments in *PM* and elsewhere in his later career. It has been argued, for example, that Russell would have had a better time justifying his axiom of reducibility had he adopted a plural quantification understanding of quantification over predicative propositional functions in *PM* (see, e.g., Yi (2013)).

One could sort advocates (or at least arguments in favor) of plural quantification and plural predication in logic into two categories: those who urge it as a way of interpreting or explaining the nature of higher-order constants and variables and a way of understanding the logical subjects of statements of number on the one hand, and those who would advocate it as something we must acknowledge as something in addition to, or over and above, traditional higher-order logics, type theories or set or class theories, on the other.

Russell's own interests in a kind of plural logic would put him squarely in the first camp here. Of course, by the time of *PM*, Russell thought he had found an alternative way of understanding higher-order quantification and the nature of numbers with which he was, obviously, at least as happy. Without delving fully into the nature of the views he held then, it is worth in this context, however, noting why it was that Russell himself never put much emphasis on the differences between his mature theory of types and the early theory of types of Appendix B of *PoM*, even though in some ways they seem radically different. As Russell notes in the letter to Jourdain quoted above, there are some ways in which his later views were in many ways similar to these early views. In particular, Russell held that in predications of number, the predication is in some ameliorated sense "about" a class, but both early and late, Russell thought the appearance of a singular logical subject here to be an illusion. On his early view, a class as many was not one thing but many, and hence, to make a predication of a number above zero or one is not to make a claim about a special "object" which happens to be more than one. On his later, "no class" view of classes, again, to make a claim about a class is not to make a claim about any one particular "thing"; the claim needs fuller analysis, and the claim is really one *about* all or some of the *members*, as defined by some specifiable propositional function. Nonetheless, it is not the case, as is often held about Russell, that he understood the true logical subjects of such predications of number to be propositional functions. Like classes, Russell at the time of *PM* considered a propositional function to be an "incomplete symbol", something which contributes to the meaning of a complete formula in which it appears but without having a single entity or thing (even an abstract thing) as its semantic value, as I have argued elsewhere. (See especially Klement 2010a and Klement 2013.) So Russell's views both early and late adhere to the "old maxim" that every genuine thing is an individual, and that, when we predicate many or more than one, what we predicate many *of* is not a genuine thing, but just the appearance of one. We cannot pursue further here the topic of which of the two precise ways of making good on this conclusion—the 1903 method or the 1910 method (or the many other similar methods Russell explored between and after these accounts)—is preferable here. Arguably at least, Russell's later higher-order logic suffices for the analysis of those propositions many have argued that one needs *either* higher-order logic *or* plural quantification to capture. Any statement of a plural logic which makes use only of plural *quantification* and the plural "is among" relation will have a translation in Russell's higher-order logic using instead higher-order quantification and higher-order predication instead. This would include, for example, the Kaplan-Geach sentence, "some critics only admire one another". If Russell's analysis of number ascriptions is unobjectionable, then statements of cardinality, finite or infinite, can be captured as well, e.g., "the planets circling the sun are eight in number."

Assuming for the moment that the higher-order logic of *Principia Mathematica*, and Russell's understanding of it, is adequate for these purposes, what of arguments from the other category of plural logic advocates? Some would no doubt insist that even with an adequate higher-order logic, Russell is missing something in his later views by not acknowledging a distinct logical form of plural predication. If we bracket for the moment the best understanding of type theory or class or set logic, is there a solid reason for acknowledging a kind of logical form in which a certain predicate or relation is predicated of a certain number of things, plurally, where the number of things involved need not be specified in advance? Personally, this question reminds me of Wittgenstein's claim in the *Tractatus* (§5.5541) that it is impossible to foresee *a priori* whether or not logic will need to make room for a 27-termed relation. We are left only to look for examples of propositions which would be impossible to analyze except by taking them to involve additional kinds of plural predication (and not simply plural quantification and the "is among" relation).

With regard to the examples that contemporary advocates tend to give, I find myself wishing to make a distinction that

I think Russell would also make, between two conceptions of logic. On the one hand, one might take logic to consist in the study of forms and patterns of reasoning, and thus to focus centrally on the forms which are used in the languages we actually use when reasoning, and perhaps also, the language of thought (if there is such). In nearly all natural languages,[13] there is of course a grammatical distinction between plural and singular, and it becomes very awkward to try to force certain plural constructions into a form of expression using only singular constructions without getting the feeling that something is lost.[14] On another conception of logic, the one Russell wrote about, logic involves studying something like the metaphysical forms of structures in the world itself, so that, as Russell put it later on, "logic is concerned with the real world just as truly as zoology, though with its more abstract and general features" (Russell 1919, p. 169). Again, the "propositions" the logical form of which Russell meant to analyze were understood by him as states of affairs or facts, not as intentional entities. Neither of these conceptions of logic is the "right" conception: we may define words however we want, and both are worthy areas of intellectual study. But when we remind ourselves which of them Russell was interested in, we may look differently at, and be more forgiving for, his singularism.

It is on this second conception of logic that it is hard to see in advance what will be necessary, and it is only *after* we have an analysis of the "correct" logical forms of the facts involved that it is possible to decide what forms are required. Russell and Whitehead's *PM* was dedicated to giving a full analysis of the propositions of mathematics, and they found, that, when analyzed, nothing approximating plural predication was necessary. If they are right—which of course is controversial—that plurals are not necessary in mathematics, where plurality one might think, is essential, it becomes more and more difficult to believe it will be necessary somewhere else.

I shall confine further discussion to a single example of a proposition which involves allegedly unanalyzable plural predication from McKay (2006, p. 20), slightly modified:

The students surround the building. (3)

The statement (3), McKay would tell us, is about the students, *them*, plurally. No one student by him or herself surrounds the building, and it is awkward at least to describe things as if what is surrounding the building is a class of students, or a mereological fusion of students, or any other singular entity somehow made up of the students. Hence, McKay concludes that in the subject position of the surrounding relation the relatum is plural: the students, they, occupy this position. McKay is certainly right that it is awkward to reword this sentence in any way that avoids a plural construction. This, I think, is sufficient to establish that plural constructions will be an ineliminable part of the project of logic if logic is taken in the first sense mentioned above. We reason differently and think differently regarding plural collections than we do about singular things.

But if our interest is not on how we reason, or how ordinary language works, but on the metaphysical structure of the kind of facts involved with (3), then the issue does not seem so clear to me. The first thing to note is that (3) is vague. It is easy to imagine configurations of students that clearly count as their surrounding the building, and it is easy to imagine configurations of students in which they are clearly not surrounding the building. But it is also easy to imagine configurations in which it is neither clearly true nor clearly false, e.g., if there is a larger gap at one place in the circle around the building than at others. (The notions of *student* and *building* are probably also vague, but never mind that for now.) Unless we are prone to swallow ontological vagueness in addition to linguistic vagueness,[15] it is already clear that (3) would need analyzing into something more precise. To make it more precise, we could, for example, set a limit of m meters as the largest gap that may exist between the students, and a distance of n meters as the largest possible from the center of the building. (Students likely do not count as surrounding a building, even if they form a perfect unbroken circle, if that circle is many many miles in radius.) Because "surround" is vague, it would be arbitrary to pick any values of m or n as the "right" values to take, but we cannot make progress without presupposing certain values for them. Now, let us pick a certain direction away from the center of the building and dub that direction 0°, and then we can speak of the various directions away from the center of the building using real numbers from 0° to 360° modularly. Then, we might suggest, as a first pass, an analysis somewhat like the following for (3):

For each direction d from the center c of the building, there exists a point p not more than n meters from c along the radius extending from c at d degrees, such that there is a student s who is equidistant from c as is p, and who is not more than m meters from p, *and* for each student x, if x is standing along the radius from c defined by e, there is a another student y standing at a degree greater than e who is not more than m meters away from x, and yet another student z standing a lesser degree than e who is also not more than m meters away from x. (4)

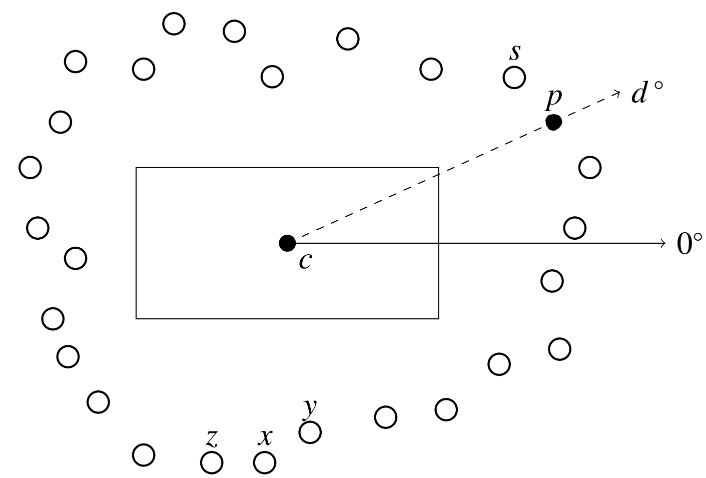

Figure 1: The students surround the building.

This analysis, no doubt, could be improved upon by analyzing what a direction is, what a student is, taking into account the shape of the building, making the times involved explicit, and so on.[16] The analysis as given is also limited to two-dimensional forms of surrounding. To cover statements such as "the spaceships surrounded the satellite", the third spatial dimension would have to be brought in. I shall not attempt to make the necessary modifications, but for the most part, allowing for a third (or greater) spatial dimension would mainly consist in modifying what is meant by "each direction" away from the center point to involve more than only a single real number degree coordinate. Together with, and assuming the legitimacy of Russell's replacement for plural quantification and ascriptions of cardinality, a three dimensional analysis could also handle such a sentence as "the asteroid is surrounded by billions of particles of sand". Similar and most likely simpler modifications could be made to produce a one-dimensional notion of surrounding which might be useful in analyzing such statements as "this bout of depression in Susan's life was surrounded by periods of happiness" or even "six is surrounded by prime numbers".

I think this takes us far enough to realize that there is little reason to think that the final analysis of (3) will involve plural predication. Indeed, plural predication has already completely disappeared from (4). Now, if our interest were logic in the first sense, this may not be relevant. Clearly, no one prior to my writing this has anything like (4) clearly in mind when asserting or reasoning with (3). But Russell for one is clear that he does not expect that analysis will preserve "what we meant all along", and that is not its goal (Russell 1956b, p. 180). Russell is interested in forms of facts and states of affairs, not the forms of reasoning. It seems to me at any rate that (4) brings us far closer to the form of the *facts* involved than (3) does pre-analysis. (4), unlike (3), provides insights into the basic analytic facts about surrounding, i.e., that if something is surrounded, it is difficult to reach it or leave it without passing close by to what is surrounding it, and so on. These are the benefits of analysis.

Obviously, nothing is settled by examining one specific example. To fully make the point, additional examples would have to be given, and even then, it seems that by this method one can at best establish that as one analyzes a proposition, the need for plural predication becomes less and less likely. Whether or not it will be possible to eliminate fully plural predication may come down to what we take to be the basic or unanalyzable sorts of facts to be. This is itself something Russell changed his mind about. In the later 1910s and 1920s, Russell (e.g., 1956a and 1956b) would not even have taken the fundamental laws of physics to be unanalyzable facts, and would instead analyze matter in terms of sense-data. At other times, Russell may have located "the fundamental level" (if indeed there is such) elsewhere, and contemporary Russellians may do so as well. Unfortunately, I am not well-versed enough in contemporary physics to know whether or not plural properties are naturally involved in stating any fundamental physical laws. When it comes to the kinds of examples proferred by the contemporary proponents of plural logic, my reaction is often similar to the students surrounding the building example: it seems clear in most such cases that further analysis is possible. The popularity of such arguments brings me to worry whether anything is left of the old school analytic philosophy championed by Russell. At any rate, I believe it would be very premature to declare the issues discussed here, the "hoary problem" of the one and the many, as anything close to settled in Russell's disfavor.

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Notes

[1]See, e.g., Russell (1919, p. 181), Russell (1994a, p. 437), Whitehead and Russell (1925–1927, pp. 35–36) and for discussion how this connection plays out in Russell's (and Whitehead's) later work, see Oliver and Smiley (2005, pp. 1039–48).

[2]There is some disagreement among commentators about how long Russell held on to this conviction; I think quite long, see Klement (2004).

[3]Proops's argument is more general than this; my comment here is meant only to respond to the exclusion of "plural terms" as grounds for thinking that the individual variable is not unrestricted.

[4]It is, I think, clear from Russell's notes for Appendix A of *PoM*, and elsewhere, that Russell at least had a good understanding of most of these aspects of Frege's understanding of value-ranges (Linsky 2004, 2005), and I have discussed their impact on Russell's philosophy in a slightly later period elsewhere (Klement 2003).

[5]Russell discusses this paradox in §102 of *PoM* as well as in letters to Frege (Frege 1980, pp. 144, 147–48).

[6]For discussion, see Landini (1992) and Klement (2005).

[7]On all these points, see Urquhart (1994, pts. I–III).

[8]Russell discusses the difference in his mind between class-

concepts and concepts of a class in §67 of *PoM*. The difference is important for understanding Russell's views at the time, but I think not so very important for understanding the argument being made here, so I am glossing over the difference intentionally.

[9]Immediately after the passage quoted from §101 describing the paradox in terms of class-concepts, Russell writes that "[t]hus we must conclude, against appearances, that "class-concept which is not a term of its own extension" is not a class-concept". Russell's suggestion here is that one can evade the paradox for class-concepts by adopting a sparse ontology for them, i.e., that while simple count nouns might indicate class-concepts, complex count noun expressions need not. But a sparse view of class-concepts could not serve as the basis for a symbolic logic of classes understood as pluralities, since the latter are abundant. Presumably in claiming that "class-concept which is not a term of its own extension" was not a class-concept, Russell did not mean to deny that there are such class-concepts and that they form a plurality, but only that there is some class-concept which they fall under. Hence in a symbolic logic in which the "α" in "$x \epsilon \alpha$" is always understood as a class-concept, certain pluralities or collections would be left out of consideration. More likely, however, Russell would regard the single concepts indicated by single expressions used when a class as many is spoken of by intension rather than by extension as what he calls "concepts of a class" (denoting concepts) rather than class-concepts (see note 8); however, the paradox can just as easily be stated for one as opposed to other, so long as it is possible to speak of a denoting concept itself as opposed to its denotation. It is interesting to note, here, how closely Russell's work on paradox-solving is interwoven with his theory of denoting, which helps explain why he was so interested in the latter in the years immediately following, resulting of course in his famous theory of descriptions in 1905.

[10]In being aware of such problems, Russell anticipated by a full century problems which have only recently been discussed in the contemporary plurals literature; see, e.g., Spencer (2012).

[11]For further discussion of these paradoxes and similar ones in the offing, see my 2010b.

[12]For my own reading of Russell, the most influential recent works are Landini (1998); Stevens (2005) and my own contributions 2004; 2010a; 2013.

[13]I have been told that in Japanese, nouns are not marked for gender or number, but I am not familiar with Japanese personally.

[14]Oliver and Smiley (2005, pp. 1047-48), for example, complain about Russell's later understanding of plural descriptions but do entirely on the basis of a mismatch between it and certain data from ordinary language.

[15]Which, of course, Russell himself was not, when he later broached the issue—see Russell (1923).

[16]It should be noted that even as it is, it does not entail that the students must surround the building in a perfectly or even nearly perfectly circular structure.

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