6 Logical Form and the Development of Russell's Logicism

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1. Introduction

While there are no doubt more nuanced varieties worth considering, one straightforward version of logicism is the thesis that mathematical truths simply *are* logical truths. Today, a typical characterization of a logical truth is one that remains true under all (re)interpretations of its non-logical vocabulary. Roughly, this means that something can be a logical truth only if all other statements of the same form are also true. "*Fa* ⊃ (*Rab* ⊃ *Fa*)" can be a logical truth because not only it, but *all* propositions of the form "*p* ⊃ (*q* ⊃ *p*)" are true. It does not matter what "*F*", "*R*", "*a*" and "*b*" mean, or what specific features the objects meant have. Applying this conception of a logical truth in the context of our crude form of logicism seems to present an obstacle. "Five is prime", at least on the surface, is a simple subject-predicate assertion, and obviously, not all subject-predicate assertions are true. How, then, could this be a logical truth? Similarly, "7 > 5" asserts a binary relation, but obviously not all binary relations hold. In what follows, I shall call this the *logical form problem* for logicism.

A proponent of logicism might respond in many ways. On the more radical side, one might reject the previous characterization of a logical truth, and propose a different one. Alternatively, one might accept it as a general characterization of logical truths in a strict sense, but instead argue that mathematical truths are "logical" in a different, extended, or amended sense. Perhaps mathematical truths are only *analytic*, and share the universality and privileged epistemological status of logical truths without, strictly speaking, being such. However, I wish to focus my concern in what follows on what happens if one accepts the usual assumption that formal generality is at least a necessary condition for being a logical truth, and also holds to a strict form of logicism that insists that mathematical truths are logical truths in this very sense. There seem to remain two options worth considering. The first, which I'll call *option A*, would admit that "five is prime" is of subject-predicate form, but insist that it also has a more specific form that is universally true. After all, "*Fa* ⊃ (*Rab* ⊃ *Fa*)", is an instance not just of the form "*p* ⊃ (*q* ⊃ *p*)", but

also of the more generic form "$p \supset q$", not every instance of which is true. Taking this option would mean holding that such classifications as subject-predicate, binary-relational, and so on are too coarse-grained to capture fully the precise logical form of some of the truths that fall under such broad classifications. The second, *option B*, instead denies that truths such as "five is prime" and "7 > 5" really have the logical forms their surface syntax seems to suggest. While no true subject-predicate statement is logically necessary, there are mathematical truths that are worded in ordinary language in a way that seems subject-predicate, but these truths, when properly understood or analyzed, have different forms, and indeed, fully general and logically necessary ones.

It might be thought that I am here omitting a third option, which would be to consider "five", "seven", "prime", ">", and so on themselves to be logical constants. But depending on the details, I believe this suggestion collapses to one or the other of option A and option B. If "prime" is considered, syntactically, a predicate, and "five" a normal subject, then we have a version of A. By taking these words to be logical constants, one is insisting that "five is prime" is a *maximally specific* subform of the generic subject-predicate form. If one instead takes "five" and "prime" to be logical constants but *not* to fall under the usual syntactic categories of subject and predicate, one is in effect adopting option B, as this truth is not taken to have the logical form it seems to have.

In either approach, it should be noted that it is not enough simply to hold that statements about numbers, say, have a different logical form from statements about other things. "7 > 5" is a mathematical truth, but "4 > 5" is not, and so, for a logicist pursuing one of these options, the former must have a different (specific) logical form than the latter has, despite the apparent identity in their surface forms and despite the only difference between them being which number is involved. The logical form of a statement about a number or other mathematical entity must not just be affected by the *kind* of (abstract) entity apparently referenced but by the particularities of the specific member of the kind.

I believe the distinction between options A and B can shed light on the development of Bertrand Russell's logicism, particularly during the period of transition between his two major logicist works, *The Principles of Mathematics* from 1903 and *Principia Mathematica*, whose first volume was published in 1910. I do not mean to suggest that Russell himself explicitly considered the puzzle in such a simplistic form. If nothing else, the characterization of logical truths as those that remain true under all reinterpretations of their non-logical vocabulary is one that became standard only later,[1] perhaps in part because of the work of Russell and those

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[1]There were, of course, earlier anticipations of this approach, for example, Bolzano (1837), but there is no indication Russell was aware of them.

influenced by him, such as Wittgenstein. Nonetheless, I think Russell at least implicitly struggled with the choice between these two options, and that it is useful to think of the development of his views as a migration away from option A towards option B. This migration was driven largely by his responses to the paradoxes hampering his logicist work, which forced a reorientation on how to think about "abstract objects" generally. Russell is explicit in many places that the development of his views in metaphysics, his endorsement of "logical constructions over inferred entities", and his taking apparent names of entities having "smooth logical properties" as "incomplete symbols" rather than genuine names, all of which are central to my account, were elements of his philosophy that developed while working through the problems facing his logicist philosophy of mathematics in the first decade of the 20th century (e.g., *PoM*[2], 2nd ed., x–xi; *PLA*, 160, 234–35; *LA*, 161–69; *MPD*, 83–85). While Russell may not have explicitly formulated what I have called the logical form problem for logicism, there is evidence that his philosophy can be seen as having evolved to solve it.

I also think that framing Russell's views in this way helps to explain what is unique about his form of logicism, and how it stacks up in comparison to other thinkers both within and outside of the logicist program. Along the way, I also hope to say a bit about Russell's attitude about abstraction principles and "defining" by abstraction, as these have become so important in contemporary discussions of forms of logicism, and were already considered important in Russell's day.

2.   The Birth of Russell's Logicism and Peano's Logic

It would be a natural assumption to think that Russell's logicism was initially inspired by late 19th-century logicists such as Dedekind and Frege, but it would be a mistaken one. Although Russell knew Dedekind's technical work, he does not seem to have been much influenced by its underlying philosophy. While Russell was eventually influenced by Frege, he only read Frege carefully when *The Principles of Mathematics* was near completion, after Russell had already become convinced of logicism. Russell arrived at the position in a somewhat different fashion.

Russell's intellectual interests were always extraordinarily broad. His education at Cambridge was focused equally on mathematics and on "moral science" (philosophy), finishing his Tripos in the former subject in 1893 and in the latter in 1894. As early as 1895, he had a plan to produce two series of books, one on the philosophies of the various sciences, and one on social questions, which he hoped would eventually "meet in a

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[2]Abbreviations are typically used for references to Russell's own works; the abbreviations are given after their titles in the References section.

synthesis" (*Auto* I: 185).   The first book intended to be a part of the former series was his 1897 *Essay on the Foundations of Geometry*. Although it is often critical of Kant, this book is still recognizably situated within the Kantian tradition on geometry, and is dedicated largely to making room for non-Euclidean projective geometries within that approach. The influence of the neo-Hegelian British Idealist tradition that Russell was trained in is also very much in evidence. That tradition holds that one cannot without falsification separate any intellectual subject matter from another and treat it in isolation; attempts to do so result in antinomies or contradictions. Russell's work during this period largely deals with such antinomies. On the one hand, geometrical objects such as continuous quantities and points are thought of as intrinsically identical and therefore identifiable only by their relationships to other objects of similar kinds. On the other hand, it was held by those in the neo-Hegelian tradition that all relations to other things must be grounded in the intrinsic natures of the relata—this is this so-called "doctrine of internal relations". Early Russell was prone to accepting such contradictions as unavoidable, at least prior to the unification of the logic of all sciences.

Russell's thinking on these matters changed in 1898 due to a number of factors. Discussions with G. E. Moore made him more sympathetic to a more realist philosophy rejecting the doctrine of internal relations. Whitehead, Russell's mathematical mentor, published his *Universal Algebra*, which brought to the fore aspects of mathematics that were hard to accommodate within the mostly quantity-focused conception of mathematics found in the Kantian tradition, including the algebra of logic itself. Indeed, Whitehead there defines mathematics as "the development of all types of formal, necessary, deductive reasoning" (Whitehead 1898, vi), which could itself be read as an endorsement of a form of logicism. The influence reveals itself in his changing the title of the work on arithmetic Russell was planning from "On Quantity and Allied Conceptions" to "An Analysis of Mathematical Reasoning" (*Papers* 2, 155–242). The title changed again, to "The Fundamental Ideas and Axioms of Mathematics" (*Papers* 2, 261–305), and finally to *The Principles of Mathematics*. However, even the first draft of the *Principles* (1899–1900; *Papers* 3, 9–180) did not endorse logicism. Russell had been convinced that not only mathematics generally, but even the algebra of logic in particular, made use of not-specifically-logical notions. Most significantly, Russell then understood Boolean class-logic as involving whole/part relationships. This likely changed only when Russell was introduced to Peano's work at the International Congress of Philosophy in Paris in August 1900, which he later described as one of the most important events in his intellectual development (*MMD* 12). Unlike earlier thinkers in the Boolean tradition, Peano distinguished between the logical form of claims that an individual is a member of a class and the logical form of claims that one class is a subset of another. Peano understood the latter relationship as involving an

implication for all values of a variable. Russell quickly mastered Peano's logical techniques in the final months of 1900. Around the same time, he was putting the final touches on his book on Leibniz, and in that process had diagnosed many of the problems as he saw them with Leibniz's philosophy as involving overly strong assumptions about the reducibility of relational propositions to subject-predicate form.

Between dropping the assumption that logical relationships are to be analyzed as part/whole relationships, and concluding that relations in general need not be understood as reducible to other forms, Russell came to the view that the most important concept for mathematics is that of a variable, and that formal implications, or quantified conditionals, were of crucial importance for mathematics. Indeed, by the time he finished *Principles*, he had come to the conclusion that all truths of mathematics could be seen as formal implications and that the constants used in such implications were all logical. It is at this point that Russell adopted logicism.

It is worth considering in more detail what it was about Peano's logic that made such a difference for Russell. For an example, let us consider his views on (cardinal) numbers. In the 1899 (pre-Peano) draft of *Principles*, Russell held that numbers formed a system of concepts related to each other. While they *applied* to collections, they were not the same as the collections they applied to, and arithmetical addition could not be reduced to the adding of entities to form collections. An entity "added" to itself in the latter sense (*A* and *A*) did not make up two things, whereas for arithmetical addition, it was true that $1+1 = 2$. Russell seems to conclude from this alone that numbers could not be defined (*Papers* 3, 15–16). It is not exactly clear how one would fill in the argument to this conclusion, but looking back, one cannot help thinking that Russell's pre-Peano logic did not allow for a coherent conception of what a *defined* concept applicable to collections would look like. It would seem one would be defining a collection of collections. Traditionally understood, the part/whole relationship, however, is transitive, so it would seem that the defining features of the collections in the collection of collections would also have to apply to the elements in the individual collections, as they too, are "parts" of it. Exactly how this makes Russell's remarks about the incompatible notions of "addition" decisive remains somewhat obscure, but it is not difficult to see why an alternative approach was attractive to him once it became available.

Peano's distinction between a class being a member of another and being a subset of it removed most of these difficulties. Peano used an epsilon (ϵ) for the membership relation and a horseshoe ($\supset$)[3] for the subset relation.

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[3]Yes, $\supset$ was first used for the subset relation, not the superset relation. It was Russell himself who reversed this to the now-standard $\subset$.

Since Peano read "ϵ" as "is" (shorthand for the Latin/French "est" or Greek ἐστί ; see Peano 1895–1908, II: 6), Russell interpreted Peano as understanding this relation as both that between a member and a class, and between an object and a predicate/concept, accusing Peano as having "quite consciously" identified classes with their defining class-concepts or predicates (*PoM*, §69). This reading of Peano is probably dubious, but what is important is that since Russell himself did not go along with the identification, he thought there were two possible uses of ϵ: as a membership sign, and also basically as a copula, or relation between an object and a concept applicable to it. Moreover, Peano introduced a symbol, $\ni$, pronounced "such that", which could be used to define new notions in terms of existing ones by binding a variable x (Peano 1895–1908, II: 7). In particular, "$x \ni ...x...$" means the class of all xs such that ...x... is true of them, roughly akin to the modern notation $\{x|...x...\}$. Peano at least implicitly accepted the naïve class theory schema:

$$y \; \epsilon \; (x \ni ...x...) = ...y...$$

That is, y is a member of the class of all xs such that ...x... iff ...y.... In Russell's interpretation of Peano, however, $\ni$ could also be read as a device for defining complex predicates, and the previous as asserting that y falls under the concept of being an x such that ...x... iff ...y.... Either way, it provides a mechanism for forming complex mathematical definitions in terms of more basic logical vocabulary, a mechanism that did not suffer from the difficulties brought on by the simplistic logic Russell had previously been working in.

In *Principles*, Russell holds that numbers can be defined either as properties of classes, or as classes of classes, settling on the latter as most consistent with mathematical practice and most convenient for a symbolic treatment of the subject. Along with this, he chose to make use of ϵ as the membership relation in his own symbolic work most of the time. It is fairly clear how one would go about making use of Peano's $\ni$ to give an explicit definition of the class of all empty classes (Russell's zero), or the class of all unit classes (Russell's one), and so on. Moreover, it is clear how one would define the class of all those classes which can be obtained by adding one member to some class in another class of classes, thereby allowing a definition of the "successor" relation, in a way that does not conflate the addition of individuals with mathematical addition.

By giving explicit definitions of the cardinal numbers, identifying them with classes of classes alike in cardinality, Russell was deviating from Peano's own views. Peano had explicitly considered and rejected defining the number of a class as the class of classes alike in cardinality (Peano 1895–1908, III: 70), arguing that the class of all unit classes had properties specific to it considered as a class that the number one seemed to lack. Peano instead introduced cardinal numbers by an abstraction principle,

roughly identical to the one now commonly called Hume's Principle, taking numbers to be objects to which all and only classes that can be put in 1–1 correspondence had a unique relation. He and his school in general defended definitions by abstraction in mathematical practice. As we now know from contemporary proponents of "abstractionism" (e.g., Hale and Wright 2001), one can get quite far using this method in capturing the objects and results of arithmetic. And of course this difference between Russell and Peano was not limited to cardinal numbers, but also ordinals and other mathematical entities that might be similarly introduced or defined. Russell defined what he called "the principle of abstraction" thus:

> Every transitive symmetric [equivalence] relation, of which there is at least one instance, is analyzable into joint possession of a new relation to a new term, the new relation being such that no term can have this relation to more than one term, but that its converse does not have this property.
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> (*PoM*, §220)

Russell thought that this principle did not need to be justified by a new "creative" form of mathematical definition but instead could be *proven* simply by taking the relation and new term in question to be membership and the equivalence class formed by the equivalence relation. He writes:

> Wherever Mathematics derives a common property from a reflexive, symmetric and transitive relation, all mathematical purposes of the supposed common property are completely served when it is replaced by the class of terms having the given relation to a given term …
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> (*PoM*, §111)

Along with this, Russell rejected Peano's supposition that abstraction principles gave us access to a common predicate shared by relata of the equivalence relation, noting that it is unclear that there is a unique such predicate. In the case of cardinal numbers, for example, *being similar (equinumerous) to a* and *being similar to b* do not seem to be the same property or concept, even if *a* and *b* are similar, so neither would seem appropriate to be *the* unique number they share. No doubt Russell thought this was another instance in which Peano conflated the class concept or predicate with the class itself; only the classes themselves have extensional identity conditions.

Let us return to the logical form problem and our options A and B. There is no question that the adoption of Peano's symbolic logic was pivotal for Russell's philosophical development, and those familiar with Russell's later philosophy are no doubt aware that he took (at least his own) symbolic logic to present a better picture of the real "logical form" of the world than ordinary language does. This attitude, of course, is necessary for something like option B, in which even the language of ordinary mathematics

is considered misleading about its "true" logical form. But at least at first, Russell continued to think that surface grammar, even in ordinary language, is generally a reliable guide to logical form (as made explicit in *PoM*, §46). And in his logicist work, his solution is probably best considered a version of option A. Note that Peano's ɔ allows one to form syntactically complex terms, and, under Russell's other interpretation of it, syntactically complex predicates. Taking this syntactic complexity to be indicative of more complex logical forms, this allows for forms which remain subject-predicate, but are not *merely* subject-predicate. This may even make room for logically general or necessary subforms. Consider, for example:

$$a \; \epsilon \; (x \ni x = a)$$

In one interpretation, this asserts that *a* is in the class of things identical with *a*; on the other, it claims that *a* has the quality of being identical with *a*. The only non-logical constant here is "*a*", but its interpretation seems not to matter: all instances of this specific form seem to be true, even though this form is an instance of the more generic form, *a ɛ b*, which does have false instances. For early Russell, of course, logical forms attached first and foremost to propositions, understood as mind- and language-independent complexes, and only derivatively to sentences. Syntactically simple expressions could be used as shorthands for complex notions. If we understand "five", "seven", ">", and "prime" as such shorthands, one could similarly argue that the form of the *proposition* expressed by "five is prime" is similarly of a more specific, but still subject-predicate, form, and indeed, one without false instances. Going into the precise details would require delving into the exact logicist definitions of "five" and "prime", which is rather involved. Hopefully it is clear enough how the basic approach might work.

3.   The Paradoxes and the Theory of Incomplete Symbols

From 1901 until the publication of *Principia Mathematica* in the early 1910s, Russell's views changed often and rapidly, deviating further and further from the rather simple approach described previously. The principal driving catalyst of these changes was the desire to solve logical and other paradoxes threatening the logical basis of his logicism. Some of these changes are evident even by the time *Principles* was published in 1903. Russell's working notes for the various solutions he tried over the years that follow have mostly been preserved and are now published in volumes 4 and 5 of his *Collected Papers*. Even a fairly rudimentary summary of the twists and turns of his thought is not possible here.[4]

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[4]Russell himself provided a summary of at least the early years of this development in a letter to Philip Jourdain, which makes for a useful comparison. See Grattan-Guinness (1977), 78–80.

Instead, I shall focus on general themes, in particular, how it is that Russell was pushed more and more away from views compatible with option A responses to the logical form problem and more and more towards option B.

It is worthwhile first to note certain commitments that option A seems to require, or at least steer one towards. The approach makes use of complex terms and/or complex predicates, and requires this complexity to have significance at the level of logical form. For Russell, or someone with similar commitments, this means this complexity exists at the level of the proposition, or the objective content of the truths in question. In option A, the terms and predicates are still terms or predicates: their role would seem to be to represent objects and qualities/relations, just like any other terms and predicates. How does the complexity in the term transfer to a corresponding complexity in the content?

Two possible answers suggest themselves. Perhaps there are special objects possessing an "inner" logical form so that any proposition about them, merely by virtue of being about *them*, has a different logical form than it would have if it were about an object without such an inner form, or having a different inner form. The number five itself has a kind of special logical nature so that to assert that *it* is prime is to assert something of a different specific logical form than found in the false proposition that four is prime. Another possible answer would be that the more nuanced complexity in the logical form comes not from the objects meant, but from a complexity in the *representation of them* in the proposition. This second kind of answer collapses to the first if the way in which the proposition represents an object must always be simply having the object itself as a constituent; in that case, the complexity in the representation would be a complexity in the represented. Russell's *early* views, however, allowed that a proposition could be about an object by virtue of having a different, represent*ing* constituent. Russell called represent*ing* constituents "denoting concepts", or later, "denoting complexes". Others might prefer to think of Fregean senses or suchlike. In this approach, one needn't insist that the numbers four and five *themselves* have a different "inner" logical form, but one would insist rather that the propositions expressed by "four is prime" and "five is prime" nonetheless have different specific logical forms because of a difference in the constituents of the propositions having these forms that do the work of *representing* four and five. These representing entities have different forms, and this is reflected in the overall forms of the propositions. In an option A response, however, both "four is prime" and "five is prime" are still subject-predicate propositions, and so share a more generic logical form. It should be noted both kinds of answers commit one to *things* having a kind of complex inner logical nature that affects the logical form of the propositions into which they enter. The only difference lies in whether these things are the meanings or the things meant.

It is now well understood that being too liberal about what abstract or logical entities one postulates to exist can lead to trouble. The simplistic option A approach sketched in the previous section is no exception. In one interpretation, "$x \ni (x \sim\epsilon\ x)$" means the class of non-self-membered classes; in another, it means the quality of non-self-predicability. With the naïve assumptions Russell took from Peano, either leads to a contradiction when we ask whether it bears the appropriate interpretation of ϵ to itself. Interestingly, Russell's initial reaction to the two forms was different. Already by the time *Principles* was published, he concluded that it is a mistake to think that a complex predicate exists for every open sentence, or for what Russell called a "propositional function", by which he meant the objective content of an open sentence. While there may be propositions of the form "u is not u", there is no such predicate as being an x such that x is not an x; non-self-predicability is not a predicate (*PoM*, §101; cf. §84). A device for forming complex predicates allowing any arbitrary open sentence would therefore not be allowed in his logical language. Presumably Russell would not have regarded the ordinary language sentence "Blueness is non-self-predicable" as nonsense, or as not expressing any kind of proposition, but if it does express one, it could not express one of subject-predicate form; perhaps it expresses the *negation* of the subject-predicate proposition "Blueness is blue", but a negation of a subject-predicate proposition is not, or at least not always, also a subject-predicate proposition. Note that this already requires acknowledging that the grammatical form of a sentence is not always indicative of the logical form of the proposition expressed.

It took Russell longer to abandon complex terms for classes formed from arbitrary open sentences, though his notation changed from Peano's "$y \ni (...y...)$" to something akin to Frege's value-range notation "$\acute\epsilon(...\varepsilon...)$" and eventually to the circumflex notation "$\hat{x}(...x...)$" found in *Principia Mathematica*. All other complex terms in his formal work were derivative of this notation; for example, early Russell made use of Peano's functor ı, mapping a singleton to its sole member, to form definite descriptions. However, his exact understanding of these notations varied widely over the years. It is of course inconsistent to think that a class, itself understood as an individual, exists for every open sentence using a variable for individuals ...x..., where the class is itself a possible value for that variable x, and the membership of the class consists of all and only the values of x for which ...x... holds. Since Russell regarded mathematical objects such as numbers, and any others which might be introduced by "abstraction principles", as classes, the logical form of class-notation was central to his logicism. Are these terms to be understood as terms in anything like the usual sense, so propositions expressed by formulas using them have at least the same generic form, even if a different specific form, from other singular predications?

As Russell notes in various places, he discovered "Russell's paradox" when considering Cantor's powerclass theorem that every class has more subclasses than members (*PoM* §100; *IMP* 136; *MPD* 75). This would seem to mean that at least some, if not all, classes of individuals cannot themselves be considered individuals, or else the class of individuals would have all its subclasses as members, which is impossible if it has more subclasses than members. At the time of finishing *Principles* (see chapter vi; appendix B), Russell distinguished classes-as-one from classes-as-many, with the latter being considered irreducibly plural. But this means that a proposition of the form "$x \, \epsilon \, \alpha$", where α is a class-as-many, is not a binary relation after all, as it has more than two relata. Here we again see slippage towards option B, as the apparent logical form of "$x \, \epsilon \, \alpha$" is not taken at face value. But at least at first he continued to hold that at least some membership propositions, those where the class was a class-as-one, did have their apparent logical form. Typically, as we have seen, a class was referenced by use of a class abstract of the form "$\hat{x}(\varphi(x))$". At first Russell took whether the class denoted could be considered a class-as-one, or a single thing, as determined by features of the defining function $\varphi(x)$, calling those that could not be considered to define classes "quadratic forms". However, he found it difficult to isolate a specific category of forms which should be considered quadratic.

In mid-1903, Russell made his first attempt to develop a "no classes" theory in which apparent discourse about classes was to be replaced with discourse about their defining functions [*Papers* 4, 49–73; *LtF*]. He re-purposed Frege's notation "$\dot{\epsilon}(...\epsilon...)$" for value-ranges of functions as a function abstract notation for the functions themselves. Yet, if these functions are considered discrete and individual "things" which can enter into logical forms as "units", the improvement over a realism about classes is unclear. One still needs to consider the function $\sim\varphi(\varphi)$, true for a given argument which is a function that is not satisfied by itself. Later, Russell thought to replace taking functions as separable entities with the notion of a substitution of one entity for another in a constant proposition, so that the work done by, for example, "x is human" could be done instead by the proposition "Socrates is human" and various substitutions for Socrates within it (*Papers* 5, 90–296). However, he found that this approach succumbed to another problem he was also already aware of at the time of publishing *Principles* (§500): Cantor's theorem also seems to pose a problem for the totality of propositions. If a different proposition exists for every class of things, and propositions themselves are things, Cantor's theorem suggests there must be more propositions than propositions. Russell-style paradoxes can then be generated. Consider, for example, propositions of the form "for all propositions p, if $\varphi(p)$ then p", where $\varphi(x)$ is a condition on propositions the proposition in question does not satisfy. Being such a proposition is itself a condition on propositions,

$\psi(x)$; now consider the proposition "for all propositions p, if $\psi(p)$ then p"; it itself satisfies $\psi(x)$ just in case it does not.[5]

A crude summary of the difficulty Russell was facing during these years is that Cantor's theorem will create a problem for any theory that tries to make a single thing "out of" many things in a generic fashion. Classes-as-one are a very simplistic way of considering many things as one, but the problem also arises for complex entities such as Russellian propositions and entities derivative from them like "propositional functions". Considered as objective complexes, propositions can be made up of any combination of entities in any form. They can have any number of constituents. So long as there is at least one such complex for any collection of individuals, there must be more such propositions than there are individuals. The same goes for defining conditions or "propositional functions", which, if taken objectively, could be of arbitrary complexity. Note, moreover, that it does not necessarily help merely to adopt a less objective account of propositions or propositional functions, that is, one that denies that they can be of arbitrary complexity or have any number of constituents. To get around the problems raised by the diagonal paradoxes Cantor's argumentation leads us to, one would have to find a principled reason to deny the existence of the "paradoxical" propositions, classes or functions in particular. This seems to pose a definite obstacle for thinking of propositions, functions or classes as single things, or at least as single things in the same sense as their basic constituents or urelements, and if so, then statements "about" such entities, if indeed it remains appropriate to think of them as entities, would seem to have a different form than statements about these more basic elements.

Notice moreover that propositions (and related entities like "propositional functions") are *precisely* the things we naturally think to *have* logical form. We noted earlier that option A responses to the logical form problem for logicism seem to have a need to posit entities that have a special logical form as part of their nature, either as special varieties of subjects and predicates, or as representatives thereof. But assuming the identity conditions of these special entities are determined by their defining logical forms, and such entities exist for any arbitrary chosen proposition or propositional form, we are in a position in which a violation of Cantor's theorem or other paradoxical consequence is a definite threat. And note moreover that the switch from the represented entity to the representative of that entity does little good, so long as the representatives themselves are entities and it is possible to disambiguate between propositions about the representative and propositions about the represented things. Consider *any* kind of complex term that makes use of

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[5]See Landini (1998) for a discussion of various forms of such "propositional" paradoxes and how they fared and developed within the context of Russell's "substitutional theory".

a propositional form as a constituent, such as Peano's "$x \ni \varphi(x)$", later description notations such as "$(\imath x)\varphi(x)$", or even something like "$\#x : \varphi(x)$" read as "the number of x such that $\varphi(x)$". Whether or not the entities *denoted* by such complex terms are so numerous as to generate a violation of Cantor's theorem, if the contribution made by such terms to the logical forms in which they appear as meanings or representatives are single things, and are distinct for distinct $\varphi(x)$'s, there is a potential violation of Cantor's theorem and a related diagonal contradiction. Early Russell would have considered a complex term "$x \ni \varphi(x)$" as contributing a denoting complex which is not itself the class but a representative that denotes the class. The complex itself would have its own individual identity determined by the defining condition $\varphi(x)$, and a coextensional $\psi(x)$ would give rise to a distinct denoting complex "$x \ni \psi(x)$", even if the two denoting complexes denote the same collection of things. But if we can talk about denoting complexes themselves in addition to what they denote,[6] then we can consider properties of them, including the property W a denoting complex of the form "$x \ni \varphi(x)$" has when it itself does not satisfy $\varphi(x)$. Does the denoting concept "$x \ni W(x)$" satisfy W? It does just in case it does not. This contradiction does not make use of the denotations of such representative entities. Even if classes are not single entities, if their representatives in logical forms *are* single entities, trouble brews. And the problem is just as bad for other kinds of complex terms such as those of the forms "$(\imath x)\varphi(x)$" and "$\#x : \varphi(x)$".[7]

In later reminiscences about his work on attempting to solve the paradoxes, Russell makes note of his suspicion that the theory of denoting was important for their solution, a suspicion he claimed turned out to be correct [*Auto* I:229; Grattan-Guinness (1977), 78]. Hopefully, the connection is beginning to become clear. Prior to developing the theory of descriptions in 1905, Russell did not have a clear sense of how a complex term could be used without the logical form represented being one containing single entities of a problematic sort. The details of Russell's new theory of definite descriptions are now well known. A term of the form "$(\imath x)\varphi(x)$" is not to be taken as a self-standing constituent of the logical form of the proposition represented by a sentence in which it appears, but instead must

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[6]In his well-known article "On Denoting" (*Papers* 4, 414–427), in which Russell argues against Frege-style meaning/denotation or sense/reference distinctions, Russell gave what is now considered a very obscure argument against thinking that it is possible to disambiguate discourse about a denoting complex from discourse about what it means: the so-called "Grey's Elegy Argument". But it should be noted that even if this argument fails, we are simply impaled on a different horn of the dilemma, as it then becomes possible to formulate paradoxes such as this.

[7]See Klement (2014) for a fuller discussion of the possibility of paradoxes such as these in a Russellian context.

be unpacked contextually. In particular, a formula of the form $\psi[(\imath x)\varphi(x)]$ is taken as abbreviating:

$$(\exists x)[\varphi(x) \mathbin{.} (y)(\varphi(y) \supset y = x) \mathbin{.} \psi(x)]$$

Part and parcel of this theory is the rejection of taking "$F[(\imath x)\varphi(x)]$" to have subject-predicate form, even when F is a simple predicate, despite the *apparent* similarity in form with a simple subject-predicate statement "*Fa*". There quite simply is no *single* constituent of the previous more complex formula corresponding to the description, no one entity with an inner connection to the form $\varphi(x)$. Of course, there needs to be an object in the domain of quantification which uniquely satisfies $\varphi(x)$, but there needn't be distinct such objects for every propositional form.

By the time of *Principia Mathematica* (see, e.g., *PM*, introd. chapter 3), Russell saw the theory of descriptions as only one component of a broader theory of "Incomplete Symbols", the other key component of which was his oxymoronically named "no classes theory of classes". According to that theory, a class abstract "$\hat{x}\varphi(x)$" is similarly not taken to represent a single constituent of the resulting logical form. Instead $\psi[\hat{x}\varphi(x)]$ is taken as shorthand for:

$$(\exists f)((x)(f!x \equiv \varphi(x)) \mathbin{.} \psi(f!\hat{x}))$$

To give an example, to say that a is a member of the singleton $\hat{x}(x = a)$ is to say:

$$(\exists f)((x)(f!x \equiv x = a) \mathbin{.} f!(a))$$

There is some f true of all and only those things identical to a, and a satisfies f. Despite appearances, $a \,\epsilon\, \hat{x}(x = a)$ has a much more complicated form than subject-predicate. There is no one single individual entity, either a meaning or a denotation, corresponding to the class abstract "$\hat{x}(x = a)$". There is some disagreement about the nature of the higher-order quantifiers and the "propositional functions" they allegedly quantify over among interpreters of Russell, but it is at any rate clear that Russell did not think of functions as additional "individuals", nor statements about functions of one type as having the same logical form as statements about those of another type. A claim about a mathematical "object" such as the number 1 is really a claim making use of a specific logical form: in the case of the number 1, this form involves uniqueness. To say that the class of φs is a member of 1 is really to make a complex statement involving this form, one that can be shown to be logically equivalent with the claim that there is a unique φ making use only of standard logical rules governing quantifiers and truth-functional connectives. Similar treatment was given to higher-order statements about classes of classes, as well as an approach

to "relations in extension". Put all together, and applied to suitably contextually defined mathematical notions such as "5" and "7", one arrives at a theory in which something such as "7 > 5" is interpreted as really having a much more complicated form than that of a simple binary relation, without a unique single constituent corresponding to any of the signs "5", "7" or ">". When unpacked, this resolves into a form that makes use of nothing but variables and logical constants such as quantifiers and truth-functional propositional operators. And moreover, this form will be without false instances.

It should be clear that according to our earlier classification, Russell's theory of incomplete symbols is squarely an option B approach. The apparent logical form of a statement need not track the apparent surface form of a sentence we use to express it. Of course, it is possible to invent a notation, such as the logical language of *PM*, which, when written without abbreviations or convenient shorthands, does reflect the actual logical form of what is expressed with its syntax. But it would be cumbersome to make use of such a language without introducing notations that *mimic* basic logical forms such as subject-predicate, binary relational, and so on without actually being such. Most inferential patterns applicable to a true binary relation "aRb" will have an analogue for "7 > 5", for example, generalizing to "$(\exists x)(aRx)$" from the former is akin to generalizing to "$(\exists \alpha)(7 > \alpha)$". This makes it seem as if we are dealing with the "same" form, but on further scrutiny, the latter has a more complicated form that this abbreviated method of representation lacks. To think about a specific number, one is thinking about a specific kind of logical form. While the rules for "unpacking" two different numerals, say "7" and "4", at least at the first step, are analogous so in that sense they "share" a (generic) form, what they are unpacked into has different (specific) resulting forms, explaining how it is that "7 > 5" could unpack into a universally valid logical form whereas "4 > 5" does not so unpack, where these results presuppose nothing beyond the usual assumptions of classical logic.

Russell's theory of incomplete symbols remained a staple of his philosophy for the remainder of his career. While the 1925–1927 second edition of *Principia Mathematica* saw him experiment with a more austere propositional system and more stringent assumptions about extensionality and the existence of "predicative" functions, Russell never returned to anything like an option A approach according to which complex terms for mathematical entities were taken at anything like face value. In terms of his ontology of mathematical or abstract entities, Russell's views were mostly settled by 1910.

4.   Abstract "Objects" as Fragments of Form

Any form of logicism that holds on to the view that logic has a special relationship with form must also hold that the "objects" of interest to

mathematicians, numbers and suchlike, also have a special relationship to logical form. According to Russell's mature views, nearly all the apparent "things" of interest to mathematicians, including not just numbers but all classes, were considered what he called "logical fictions". The expressions that apparently stand for them are considered incomplete symbols instead: rather than referring to things these expressions are meaningful in a different way. They contribute fragments of logical form to the overall statements they are used within, and these fragments are not themselves additional things. If there is something in common between the generic forms various categories of such fragments take—as with, different (cardinal) natural numbers—and a variable of an appropriate type can be used whose values range over their differences, then it will be possible to quantify over such apparent "objects" as well. For example, *PM* *20.07 introduces a proxy for quantification over classes (including the Frege-Russell numbers) by means of higher-order quantification. Statements apparently "about" numbers can come out as true, and even quantified existence claims postulating, for example, the existence of numbers with such-and-such characteristics, become possible. But at the level of ultimate, fundamental objects, there are no classes or numbers in addition to the more basic elements of more basic logical forms. We are able to invent notations that mimic what goes on at more basic forms where genuine reference to "real" objects takes place, but the actual logical forms are quite different.

It is interesting to contrast this with other popular approaches to abstract objects. At least one major strand in the "abstractionist" school[8] takes inspiration from Frege's notion of "recarving content" (Frege 1884, sec. 64). The logically complex statement to the effect that there exists a 1–1 mapping between the objects falling under concepts F and G can be "recarved" or "recast" or "reframed" as an identity statement between the number of Fs and the number of Gs. Thus the two sides of an instance of Hume's Principle, for example, can have the same content "carved" in different ways. But carved as an identity statement, "$\#x : Fx = \#x : Gx$" really does have the same logical form as any other identity statement between objects, and indeed, through such recarving we can become aware of genuine objects not referred to in the other recarving. The same content can have multiple logical forms, and only on one "carving" of that content is it about objects, but somehow by virtue of this possibility, the same content seen as having a simple logical form can also be seen as having a different, logically richer, logical form.

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[8]This school is obviously made up by many thinkers whose precise views differ; here I must make do with a crude summary that may oversimplify the precise attitudes of individuals within it. For some discussion, see, for example, Hale and Wright (2001), 91-116, Potter and Smiley (2004).

In Russell's view too, it is possible to take what is *apparently* a simple identity statement in a certain notation with incomplete symbols and recognize it as also having a more complicated, logically richer, form as well, one such that the equivalence of the two sides of Hume's Principle, for example, can be demonstrated using only the standard logical rules for the quantifiers and propositional connectives. With Russell's general strategy of using the corresponding equivalence classes to form explicit definitions of terms introduced by "abstraction principles", the abstraction principle itself becomes a demonstrable theorem, requiring no additional assumptions beyond what is necessary for establishing that the relation in question is an equivalence-relation. What is different in the two approaches is that for Russell, the apparently simple logical form on the one side of the principle is only an apparent simplicity. The terms in the identity statement are not actual terms, and this method of reframing content to *appear* to have a different form does not make it *actually* have that form. Therefore, any claim that this reframing provides knowledge of any "new" identities is at best misleading.[9] One advantage of the Russellian approach is that abstraction principles need not be taken as in any sense basic, definitional, or logically analytic in an extended or different sense. Another advantage of the Russellian approach is that it obviates the seemingly difficult task of explaining how the same content could have different but incompatible logical forms at once.[10]

But this isn't to say that the Russellian approach does not have its disadvantages as well. Because the terms apparently standing for abstract entities are not truly referential, they cannot be used to establish new existence theorems not already demonstrable. In particular, Russell could not use any version of the Fregean bootstrapping argument to establish an infinity of individuals. As numbers are identified with classes, having extensional identity conditions, only those numbers applicable to appropriately sized collections can be proven distinct: inapplicable numbers are all "empty" and the same. This is the source of the issues surrounding the so-called "axiom of infinity" in Russell's logicism (see *IMP*, chapter xiii). In order to preserve certain truths commonly held to be mathematical ones as fully logically general, such as the Peano-Dedekind axiom that no two natural numbers share a successor, Russell had to add an assumption of an infinity of individuals as an antecedent to his own statement of these principles.[11]

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[9]Hale and Wright (2009), 190, contrast their approach instead with a "Tractarian" ontology of structured facts, but of course the metaphysics of Wittgenstein's *Tractatus* is largely inspired by Russell.

[10]See Klement (2012) for a fuller comparison between these approaches.

[11]For a fuller discussion of this issue and the extent to which it poses a problem for Russell's logicist project, see Klement (2019), 168–173.

Russell's attitude also was not limited to the kinds of abstract objects considered in mathematics. Mature Russell also denied the existence of semantic entities, such as Fregean senses or his own earlier "denoting concepts", although he held that something like Frege's distinction between sense and reference was preserved in his theory of descriptions, so long as it was not taken at face value or interpreted too literally (*KAKD*, 157). We have seen some of his motivation for holding this, and some of the dangers of adopting too-liberal assumptions about the existence of such things. Unfortunately, other traditions have done relatively little by way of addressing such concerns, and so it is hard to form a comparison. It is often noted, for example, that Hume's Principle, unlike some other abstraction principles like Frege's Law V, is consistent, even when the terms "$\#x : Fx$" are taken as representing objects in the domain of the individual variables. Since distinct concepts often have the same number, there needn't be more numbers than concepts applicable to them, and so this by itself doesn't violate Cantor's theorem. But note that this does not address the status of the *meanings* or *senses* of such complex terms as "$\#x : Fx$"; if these are entities that enter into logical forms as *units*, it would appear that there must be many such entities, and the changes to the paradox described in the previous section about denoting complexes of the form "$x \mathrel{\backepsilon} \varphi(x)$" to form one about those of the form "$\#x : Fx$" are trivial and easily made. That the Russellian approach sidesteps such worries is at least arguably a large point in its favor.

One aspect of Russell's views however that I do find worrying is his eventual insistence that propositions too be considered "logical fictions", and not as entities that enter, as units, into facts. In 1910, Russell adopted his "multiple relations" theory of judgment, whereupon such "propositional attitudes" as belief, desire, and so on are not to be thought of as a binary relation between a subject and a "proposition", as a whole, but as having a more complicated logical form. This theory spurred a lot of discussion and a lot of criticism, including by Wittgenstein, and Russell himself eventually became dissatisfied with it. For someone mainly interested in the relationship between logic and mathematics, this issue might seem not to be of central importance, and may be thought better left to epistemologists or philosophers of mind. But in addition to the need for a good theory of judgment, it leaves a bit unclear exactly *what* it is that has logical form, an issue that had been clear while Russell still believed in propositions as mind-independent complexes. The nature of logical form and its relation to linguistic form is clearly an issue central to logicism. A natural answer is that logical form deals with forms of *facts*. But what are they? Is the real logical form of discourse about them just as complicated as discourse about numbers, and is positing facts as dangerous as positing propositions? The official position of the 1910 Introduction to *PM* (44–46) seems to have been that there are "complexes" or "facts" corresponding only to *elementary* truths, and that no single fact or complex

corresponds to a quantified truth. But notice that we are always dealing with quantifiers when unpacking incomplete symbols such as descriptions and class terms. If one claims that the "true" logical form of "$\psi(\hat{x}\varphi(x))$" is not subject-predicate, but a complicated form involving higher-order quantification, what is it that *has* this "true" logical form? Apparently, not a proposition, and not a fact. It cannot be something linguistic either, because it's precisely the linguistic form that is supposed to be misleading as to the actual form. Perhaps a suitably Russellian answer can be given, but any such answer is likely to be more complicated than our discussion up until now would seem to suggest, possibly involving a fairly philosophically committal semantic theory.

In conclusion, Russell gives a compelling, if not fully complete or fully satisfying account of the relationship between thinking about apparent mathematical objects such as numbers, and thinking about logical forms, which is often taken to be the heart of logic. Personally I find the overall approach to be at least as compelling as the often obscure and less developed suggestions to the effect that discourse about abstract objects can be taken to "recarve" content of more complicated forms. It also provides a relatively attractive solution to what I have called the problem of logical form for logicism, even if it leaves us with the impression that almost no discourse in mathematics (or possibly in other scientific endeavors) has the actual logical form we are tempted to take it as having. This impression can even lead us to doubt whether any actual discourse has the simple logical forms we often assume it to have, and even to begin to lose sight of why we believe in such forms in the first place.

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