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New Logic and the Seeds of Analytic Philosophy
Boole, Frege

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Introduction

Simplistic accounts of its history sometimes portray logic as having stagnated in the West completely from its origins in the works of Aristotle all the way until the nineteenth century. This is of course nonsense. The Stoics and Megarians added propositional logic. Medievals brought greater unity and systematicity to Aristotle's system and improved our understanding of its underpinnings (see e.g., Henry 1972), and important writings on logic were composed by thinkers from Leibniz to Clarke to Arnauld and Nicole. However, it cannot be denied that an unprecedented sea change occurred in the nineteenth century, one that has completely transformed our understanding of logic and the methods used in studying it. This revolution can be seen as proceeding in two main stages. The first dates to the mid-nineteenth century and is owed most signally to the work of George Boole (1815–1864). The second dates to the late nineteenth century and the works of Gottlob Frege (1848–1925). Both were mathematicians primarily, and their work made it possible to bring mathematical and formal approaches to logical research, paving the way for the significant meta-logical results of the twentieth century. Boolean algebra, the heart of Boole's contributions to logic, has also come to represent a cornerstone of modern computing. Frege had broad philosophical interests, and his writings on the nature of logical form, meaning and truth remain the subject of intense theoretical discussion, especially in the analytic tradition. Frege's works, and the powerful new logical calculi developed at the end of the nineteenth century, influenced many of its most seminal figures, such as Bertrand Russell, Ludwig Wittgenstein and Rudolf Carnap. Indeed, Frege is sometimes heralded as the "father" of analytic philosophy, although he himself would not live to become aware of any such movement.

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Boole's Contributions to Logic

George Boole was a native of Lincolnshire, England, although he spent the later part of his life as a mathematics professor at Queen's University in Cork, Ireland. Although now best remembered as a logician, he published more in other areas of mathematics, especially on differential equations, analysis, and probability theory. These writings were highly regarded at the time, and his books *A Treatise on Differential Equations* (1859) and *A Treatise on the Calculus of Finite Differences* (1860) were widely used as textbooks. He wrote only three works on logic. *The Mathematical Analysis of Logic* was published in 1847, followed a year later by the shorter article, "The Calculus of Logic" (1848). The latter reached a larger audience, although the former was more sophisticated and more detailed. His techniques were refined and published in his best-remembered work, *An Investigation of the Laws of Thought*, in 1854. He corresponded with a number of highly respected mathematicians, and his friendship with Augustus De Morgan (1806–1871) is likely what sparked his interest in logic.

At the core of Boole's contributions to logic is the suggestion to treat it using algebraic means. He used variables such as x, y, z for what he at first called "election operators" (understood as operations that gathered things matching some feature into a class), and later simply for the classes themselves. Concatenation—usually used to represent multiplication in algebra—was used for "logical multiplication", or to form what is in common (now called "intersection") between two classes, xy. The sign "+" was similarly used for the sum (or "union") of two classes, or "logical addition", $x + y$. The numeral "1" was employed to represent "the Universe", the class of everything, and "0" to mean "Nothing" or the null class. With these suggestions, we can interpret such equations as the following:

$$x + y = 1 \tag{1}$$

$$xy = 0 \tag{2}$$

Here, (1) asserts that when x and y are put together, they make everything, so that everything must be in either x or y. Similarly, (2) asserts that the common area between x and y is empty, or that x and y are disjoint. The complement of a class x was represented as its "difference" from 1, i.e., $1 - x$. Boole stressed the similarities between the laws of this algebra and regular arithmetical principles, such as the commutativity of addition and multiplication, distributivity, and so on:

$$x + y = y + x \tag{3}$$

$$xy = yx \tag{4}$$

$$x(y + z) = xy + xz \tag{5}$$

$$(1 - x) + x = 1 \tag{6}$$

Interpreted logically, (5), for example, asserts that what is in common between class x and everything in either y or z is the class of things either in the commonality between x and y or the commonality between x and z. Boole, however, also made note of differences. In this system, the the commonality of a class and itself, xx or x^2, is identical with the class, so that unlike regular algebra we have:

$$x^2 = x \tag{7}$$

Among Boole's goals was to show that all the rules of Aristotelain syllogistic logic could be reduced to algebraic rules. This of course required capturing the four categories of judgment from the traditional square of opposition. Universal judgments can be captured very naturally:

A	:	All X are Y	:	$xy = x$ or $x(1-y) = 0$
E	:	No X are Y	:	$xy = 0$ or $x(1-y) = x$

To capture particular judgments, Boole introduced a special elective operator v, which selected an arbitrarily chosen non-empty class. He then had:

I	:	Some X are Y	:	$xy = v$ or $xv = yv$
O	:	Some X are not Y	:	$x(1-y) = v$ or $xv = (1-y)v$

Then, by introducing algebraic rules for reduction and the elimination of middle terms, he showed how the system captured the syllogisms valid in Aristotelian logic. He saw it as superior as it allowed for equations with more than two terms, and bypassed the usual focus on "moods" and "figures", which often required overly rigid principles for ordering premises.

Boole also explained that the system was not only suitable for capturing categorical logic, the logic of "primary propositions", but also, through a suitable reinterpretation, it could capture the logic of "secondary propositions", or what we would now call "propositional logic". Class terms could be interpreted as representing those circumstances (in his earlier work), or those times (in his later work) in which a given judgment is true. One could then understand 1 as the class of all times or circumstances, 0 as the null class again, so that these values could go proxy for truth and falsity, respectively. On this reinterpretation, logical multiplication comes to mean conjunction, and logical addition comes to mean disjunction, and $(1-x)$ can do for the negation of x. By and large the same rules carry over. In that case, (3) above asserts the commutativity of disjunction, or that *X or Y* is true if and only if *Y or X* is, and (6) the law of excluded middle, or that *either not-X or X* is true.

The Influence and Continuation of Boole's Work

Boole's mathematical approach and methodology for studying logic dominated the discipline in the second half of the twentieth century. Many of the most important writers on logic can be seen as elaborating upon and refining Boole's work, including Jevons (e.g., 1864), MacColl (e.g., 1880), Venn (1881), Schröder (1905), Peirce (e.g., 1885) and Whitehead (1898). Jevons in particular was responsible for doing away with certain idiosyncracies in Boole's writings and bringing things in to line with what today we consider standard "Boolean algebra". For example, Boole considered a term of the form "$x + y$" as "uninterpretable" if the classes x and y overlapped, which, in the secondary interpretation, also meant that he bypassed the decision between inclusive and exclusive disjunction, as "$x + y$" would also be considered uninterpretable when both are true. Jevons read "+" instead as the modern union operation, or as inclusive disjunction in the secondary interpretation, and adopted the law that $x + x = x$. Boole's curious use of the sign "v" was rendered unnecessary by representing particular statements instead as *inequalities*:

I	:	Some X are Y	:	$xy \neq 0$
O	:	Some X are not Y	:	$x(1 - y) \neq 0$

They similarly did away with the assumption that variables x, y, etc., must represent non-empty classes, thereby replacing the traditional square of opposition with the modern square instead. They also clarified the notion of "logical division" (inverse of multiplication), which Boole employed without adequately explaining. With these changes made, the result is more or less the form of Boolean algebra usually recognized and employed today, which was finally axiomatized in the early twentieth century (Huntington 1904). (However, something closer to Boole's original approach was eventually brought up to contemporary standards of rigor in Hailperin 1976.)

Working within the same tradition, Schröder, De Morgan and Peirce all made attempts to incorporate something like a logic of relations within the Boolean framework, and by means of the notions of bounded (logical) addition and multiplication, Peirce (1885) created something very similar to modern quantification theory, only slightly after Frege's independent advance. The logical notation of Giuseppe Peano (e.g., 1894), although it used different symbols, self-consciously emulated Boole's "dual interpretation" model, and many of his symbols are still used today in symbolic logic, primarily because of Peano's influence on Russell and Whitehead's *Principia Mathematica*.

Boole's influence has also been great outside the purely academic study of logic. The simplicity of the Boolean understanding of logical operations inspired the creation of mechanical devices capable of performing logical calculations. Jevons himself constructed such a machine in 1869 (see Jevons 1870), and interest in electrical implementations of the same idea soon grew. Although these were not successfully built until well into the twentieth century, such "logic circuits" are now a key concept in electrical engineering and an essential component of computers and similar devices. The theoretical understanding of such circuits has been important in the development of twentieth century information theory (see, e.g., Shannon 1948). In typed programming languages, variables representing truth or falsity are standardly said to be of the "Boolean" data type.

Frege's Career

Gottlob Frege was born in Wismar, Germany and spent most of his academic career teaching mathematics at the University of Jena. While a student at the University of Göttingen, Frege also took classes in philosophy, especially from the Neo-Kantian philosopher Hermann Lotze. The influence of Neo-Kantian philosophy can be seen in his anti-psychologism, his apriorism, his esteem for Leibniz, and his general distrust of philosophical naturalisms. He finished his doctoral dissertation in 1873 on planar geometry, under the guidance of Ernst Schering.

Although the Kantian influences on Frege's philosophy are unmistakeable, his principal lifelong intellectual project was to establish the un-Kantian thesis that the principles of arithmetic can be understood as analytically true. In particular, he held that they could be derived from a purely logical foundation, a position now known as *logicism* in the philosophy of mathematics. His first major work, *Begriffsschrift* (1879), laid out the core of his logical system. In that work, he also showed how the notion of *following in a series* could be analyzed purely logically, providing a logical basis for the principle of mathematical induction, which Kantians had long assumed required synthetic intuition to justify. His next major work was 1884's *Die Grundlagen der Arithmetik*. Prompted by the advice of his colleagues, therein Frege argued informally for his logical conception of numbers, and criticized rival views in the philosophy of mathematics. In the early 1890s, he published important articles outlining new developments in his views on meaning and philosophical logic, including his best known paper, "Über Sinn und Bedeutung" ("On Sense and Reference"), from 1892. These are perhaps best understood as prepatory studies for his magnum opus, *Grundgesetze der Arithmetik*, the first volume of which appeared in 1893. It was in this work Frege sought to provide complete demonstrations of the most important principles of arithmetic beginning only with logical axioms and inference rules. The first volume contained a part devoted to Frege's logical system, to which he now added a theory of extensions of concepts, or more broadly, value-ranges of functions. It also contained the beginning of his treatment of cardinal numbers, both finite and infinite. This treatment was completed in the second volume, published in 1902. The second volume also contained an unfinished treatment of the arithmetic of real numbers, built around the notion of magnitude, as well as polemical replies to alternative accounts, especially from the formalist school. Frege had planned to publish a third volume of *Grundgesetze*, which was to complete this treatment, and go on to more advanced arithmetic as well.

Unfortunately, while volume II was in the process of being printed in mid-1902, Frege was informed by Bertrand Russell that the theory of extensions added to the logic of volume I was inconsistent, leading to the antinomy now known as "Russell's paradox". Frege hastily prepared an appendix to the second volume discussing the problem and offering a tentative solution. Frege was in his own words, "thunderstruck" (Frege 1980, p. 132). He continued to do important work for a few more years, including his debate with David Hilbert (e.g., 1899) and followers over the proper philosophical understanding of axiomatic systems in mathematics, published between 1903 and 1906 as "Über die Grundlagen der Geometrie". However, he produced very little new work between 1906 and his retirement. After retiring, he published a series of important papers, including the influential paper on truth, "Der Gedanke" ("Thoughts"), in

1918. These were likely new drafts of material dating back to the late 1890s which he had once considered including in a planned textbook on logic (cf. Frege 1979, pp. 126–51). In the final year of his life, he seems to have come to the conclusion that the logicist thesis that arithmetic reduced to logic was untenable, and began to reconsider a more Kantian foundation for arithmetic. However, only a few scant writings from this period exist, and none were published during his lifetime.

It appears that Frege was a shy and reserved man by nature. He was right-wing politically, and diaries and correspondence found since his death reveal that he was rather anti-semitic and even expressed admiration for the young Adolf Hitler. These revelations have disappointed many of us among his philosophical progeny. Despite his shy personality, Frege was a harsh polemicist. He acknowledged his debts to past philosophical thinkers only rarely, and even more seldomly to other mathematicians. These factors are likely among the reasons Frege's works were underappreciated during his lifetime. His technical logical writings received mostly negative reviews, and his *Grundlagen* was largely ignored by German academics. Frege himself bemoaned the poor reception of his work in a number of places, including the foreword to *Grundgesetze* (p. xi). After Russell dedicated an appendix to Frege's work in his 1903 *The Principles of Mathematics*, Frege began to receive significant attention within the English-speaking world. This tendency grew after Frege's death, especially once his most important writings were translated into English in the mid-twentieth century. Frege now is almost universally heralded as a significant philosopher and groundbreaking logician. He seems to have had some foreknowledge of his eventual success. When bequeathing his *Nachlaß* to his son, Alfred, Frege wrote, "I believe there are things here which will one day be prized much more highly than they are now. Take care that nothing gets lost." Alfred gave the papers to Heinrich Scholz of the University of Münster for safe-keeping, where unfortunately most were destroyed in a World War II bombing raid.

Frege's Logical Symbolism and His Criticisms of Boole's

Although it is difficult to imagine Frege's own work having been possible without the immense new interest in symbolic and mathematical approaches to logic spurred largely by Boole's efforts a quarter century earlier, Frege himself offered nothing but criticisms of the Boolean approach. While Frege did not single out Boole himself for attack on this score, Frege would no doubt have objected to Boole's characterization of logic as studying "the laws of thought", at least if this is interpreted to mean the laws of the ways in which people in fact infer or reason. Frege insisted that the laws of logic would remain the same even if in fact no one's reasoning ever accorded with them. Frege would allow that they are laws of thinking only in a *normative* or *prescriptive* sense. They tell us which chains of reasoning are truth-preserving, and hence which inferences can be *justified*, not which inferences are in fact routinely made.

In a pair of essays published only posthumously (Frege 1979, pp. 7–52), Frege compared his own logical language to Boole's logical calculus with an eye towards assessing which fared better in realizing Leibniz's vision of a "logically ideal language". His evaluation was that Boole's system failed in realizing the goals of a "*lingua characterica*"—a logically perspicuous language in which the meaning of each symbol is made fully precise—and only succeeded, and there imperfectly, in realizing a "*calculus raticionator*", or a calculus in which precise rules can be set forth to determine what does, and what does not, count as a legitimate inference. Frege found Boole's approach wanting for a variety of reasons. Firstly, it reused mathematical notation inconsistently with its standard usage, which would greatly hinder its ability to represent the logic behind proofs *in mathematics* fully and without ambiguity. Secondly, despite Boole's interest in this problem, it cannot easily capture statements of multiple generality, such as "*every* person loves *some* city", nor disambiguate between different possible readings of such sentences. Thirdly, like all earlier approaches to logic, Boole's work bifurcated the logic of "primary propositions" (categorical logic) and the logic of "secondary propositions" (propositional logic) into distinct interpretations of the algebra, and thus was incapable of representing complex inferences involving steps of both kinds together at once. Frege claimed that his own logical language, which he called "concept-script" (*Begriffsschrift*), did better on all these counts. He also suggested that his language relied on fewer primitive symbols, and fewer basic inference rules.

The logical system Frege developed is in many ways similar to contemporary second-order predicate logic, so much so that he is often credited as having invented predicate logic. Strictly speaking, however, the language Frege developed is a "function calculus". He claimed the function/argument analysis of mathematical terms was richer and more exact than the subject/predicate analysis found in traditional logic. Consider the mathematical expression "$5 + 7$". This expression stands for the value of the addition function for the arguments five and seven. Frege thought of a function expression as being in a certain sense "incomplete" or "gappy" in that it affords a place or places for its arguments to be written. Hence we can think of the sign for addition not merely as "+" but as "$(\ ) + [\ ]$" or "$\xi + \zeta$", with the ξ and ζ here marking empty spots to be filled with arguments. In order to apply this kind of analysis to complex expressions more generally, Frege expanded the kinds of functional expressions he countenanced in his language to include those representing functions with arguments and values other than numbers. He understood *concepts* as a special kind of function. If the functor "$H(\xi)$" stands for the concept *being human*, and "s" stands for Socrates, then when the latter fills the argument spot of the former, we get the truth, "$H(s)$". Relations were understood as similar sorts of functions, only with multiple arguments; e.g., "$\xi \leq \zeta$" represents the relation of *being less-than-or-equal-to*.

In his early work (e.g., Frege 1879, §2), Frege leaves it a bit unclear what sorts of objects he takes the *values* of concepts and relations to be, speaking only of such things as "the circumstance that A" (e.g., the circumstance that Socrates is human). By the 1890s, however, once he had adopted the sense/reference distinction, he took their values to be one of the two special objects, the True, and the False, called "truth-values". Hence, we might have:

$$H(x) = \begin{cases} \text{the True,} & \text{if } x \text{ is human,} \\ \text{the False,} & \text{otherwise.} \end{cases}$$

Just as "$5 + 7$" by itself is just a complex name of a number, "$H(s)$" by itself is simply a name of one of the truth-values, the True or the False. To transform a name of a truth-value into something that represents an actual judgment, or to actually make a genuine assertion, Frege begins the expression with the sign "$\vdash\!\!-$". Thus:

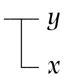
$$\vdash\!\!-\, H(s)$$

actually asserts that the concept $H(\xi)$ maps s to the True, or that Socrates is human. Frege calls the vertical line at the far left the "judgment stroke". If this is removed, leaving only the horizontal part, what remains, "$—\, H(s)$", is still only the name of a circumstance (earlier) or truth-value (later). This horizontal line was called the "content stroke", since it prefigures and is used to name the content which, once the judgment stroke is added, is asserted as true.

Frege's notation also made use of a "negation stroke" (somewhat analogous to the contemporary "$\neg$" or "$\sim$") and a "conditional stroke" (somewhat analogous to "$\to$" or "$\supset$"). These were used in the language by marking or branching the horizontal line to which they were always attached. The two-dimensional nature of the conditional stroke, with the antecedent term written below the consequent, makes Frege's notation somewhat unique, but he himself suggested that using all the dimensions available on a page was an advantage of his style. We might put their semantics in Frege's mature work as follows:

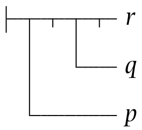
$$—\, x = \begin{cases} \text{the True,} & \text{if } x \text{ is the True,} \\ \text{the False,} & \text{otherwise.} \end{cases} \qquad \text{(horizontal/content stroke)}$$

$$-\!\!\top\!\!-\, x = \begin{cases} \text{the False,} & \text{if } x \text{ is the True,} \\ \text{the True,} & \text{otherwise.} \end{cases} \qquad \text{(negation stroke)}$$

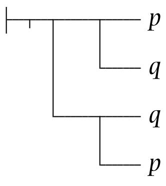


$$= \begin{cases} \text{the False,} & \text{if } x \text{ is the True and} \\ & y \text{ is other than the True,} \\ \text{the True,} & \text{otherwise.} \end{cases} \qquad \text{(conditional stroke)}$$

By combining these symbols, more complex propositional forms, including disjunctions, conjunctions and the like, could be represented:



(analogue of $p \to \neg(q \to \neg r)$ or $p \to (q \wedge r)$)

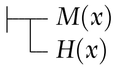


(analogue of $\neg((p \to q) \to (q \to p))$ or $(p \to q) \vee (q \to p)$)

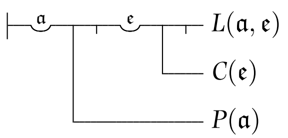
Frege understood variables (or as he called them, "Roman letters") as expressing generalities. Just as the mathematical formula

$$f(x) = x^2 + 1$$

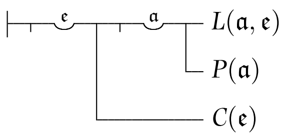
can be taken to assert that function $f(\xi)$'s value is one more than the square of its argument for *every* possible argument, Frege would understand



as asserting that it is true for *every* value of x that if it is human, then it is mortal (or that all humans are mortal). However, if the *scope* of the generality needed to be smaller than the entire proposition, Frege made use of Gothic (Fraktur) variables instead (always vowels), and marked the beginning of their scope with a concavity. This allowed him to distinguish between $\vdash\!\!\top\!\overset{\mathfrak{a}}{\smile}\!-\ H(\mathfrak{a})$ ("not everything is human"or "$\neg\forall x\, Hx$") and $\vdash\!\overset{\mathfrak{a}}{\smile}\!\top\ H(\mathfrak{a})$ ("everything is not human" or "$\forall x \neg Hx$"). Frege was thereby the first to understand quantifiers as scoped variable-binding operators, and the instigator of modern quantification theory. This represents perhaps the most important advance in his logical work. Existential quantification could be captured using universal quantification and negation. Among other things, by using different variables with different possible scopes, he was able to capture statements of multiple generality and correctly disambiguate their different readings, e.g.:



(every person loves some city *or other*)



(every person loves some *one* city)

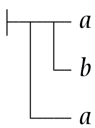
Frege understood quantifiers as *second-order* or *second-level* concepts. A first-order concept, such as $H(\xi)$ is understood as taking an object as argument and yielding a truth-value as value. A second-level concept takes a first-level concept (or other function) as argument instead, and also yields a truth-value as value. We have:

$$-\!\overset{\mathfrak{a}}{\smile}\!-\ \phi(\mathfrak{a}) = \begin{cases} \text{the True,} & \text{if function } \phi(\xi) \text{ yields the True as value for all arguments,} \\ \text{the False,} & \text{otherwise.} \end{cases}$$

Frege's logic was second-order, and so he also employed variables f, F, g, G, etc., for first-level functions (including concepts), as well as quantifiers, written, e.g., "$-\!\overset{\mathfrak{f}}{\smile}\!-\ \ldots \mathfrak{f} \ldots$". Both early and late, Frege's logical language also contained a sign for identity, at first written "$\xi \equiv \zeta$" and later changed to the usual mathematical equality sign "$\xi = \zeta$".

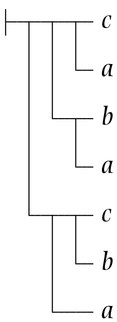
Frege's Logical Systems

While rival axiomatic systems for geometry (Euclidian, and non-Euclidian) were a subject of great intellectual curiosity in the nineteenth century, the logical basis of the demonstrations in them were typically taken for granted. They did not make what logical transformations they considered justified fully explicit. Frege hoped to remedy this in his logical language, and insist that proofs in it be "gapless", with every step made exact and precise so that no unnoticed assumption could sneak in. The result was the first rigorously axiomatized system of logic, where not only logical axioms were identified, but even the transformation and inference rules formulated exactly. The system of his 1879 *Begriffsschrift* contained nine axioms. Here I present each along with a contemporary near-equivalent and English gloss:



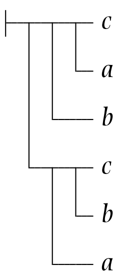
$p \rightarrow (q \rightarrow p)$ (A1)

If *A* then *B* (materially) implies *A*.



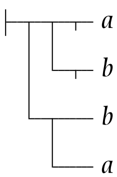
$[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$ (A2)

If *A* implies that *B* implies *C*, then if *A* implies *B*, then *A* implies *C*.



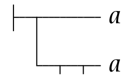
$[p \rightarrow (q \rightarrow r)] \rightarrow [q \rightarrow (p \rightarrow r)]$ (A3)

If *A* implies that *B* implies *C*, then *B* implies that *A* implies *C*.



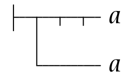
$(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$ (A4)

If *A* implies *B*, then not-*B* implies not-*A*.



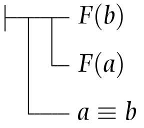
$$\neg\neg p \rightarrow p \tag{A5}$$

If not-not-*A* then *A*.



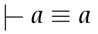
$$p \rightarrow \neg\neg p \tag{A6}$$

If *A* then not-not-*A*.

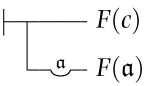


$$x = y \rightarrow (Fx \rightarrow Fy) \tag{A7}$$

If *a* is *b* then any concept holding of *a* also holds of *b*.



$$x = x \qquad / \qquad a \text{ is } a. \tag{A8}$$



$$(\forall x Fx) \rightarrow Fy \tag{A9}$$

A concept holding of all objects holds of any particular object.

Frege also included versions of (A9) for higher-order quantifiers. The inference rules of the system were *modus ponens* or detachment, universal generalization, and an implicit rule of substitution or replacement for free variables (which became explicit in *Grundgesetze*). Together (A1)–(A6) provide a complete axiomatization of propositional logic, in the sense that every truth-functionally tautologous form can be derived from them, although (A3) was later shown to be redundant. With the other axioms, this system is a consistent and Henkin-complete axiomatization of second-order logic with identity. Henkin-completeness is weaker than standard completeness. It is now known as a corollary of Gödel's incompleteness results that no finite or recursive axiomatization of second-order logic with standard semantics can be complete in the sense of capturing all logical truths, where these are understood as statements true in all models in which the range of the second-order quantifiers includes concepts for all subsets of the domain of the first-order quantifiers. (The notion of a finite or recursive axiom system is explained more below.) Henkin completeness requires only capturing those logical truths that are true in all domains where there are enough concepts to satisfy the comprehension principle (discussed below; see Henkin 1955). This is as complete as any second-order logic can be expected to be. That Frege managed to devise one without any knowledge of later logical meta-theory is a remarkable accomplishment, especially given that this is the first work even to employ quantifiers in the modern sense. The publication of *Begriffsschrift* alone makes 1879 one of the most significant years in the history of logic.

Frege believed that the axioms of logic were analytic, and that hence, so too were the theorems, including the basic principles of mathematics derivable from them. Nonetheless, unlike Kant and his followers, he denied that this meant that logic was purely formal and incapable of extending our knowledge in any substantive way. Frege understood the constants of his logical language as standing for functions, including concepts and relations, no less real than the meaning of "... is human". Logic has as its subject matter those functions and relations which bear a special relation to truth, which Frege thought was the guiding concept for logic, just as goodness was the guiding concept for ethics, or beauty the guiding concept for aesthetics (Frege 1984, p. 325). What's more is that Frege believed that we could recognize the existence of new concepts, relations and logical functions by taking a complex expression and removing one or more names from it. Frege's replacement rule for free function variables allows the substitution of any open expression, which makes it equivalent to the modern (impredicative) comprehension principle. In contemporary notation, this reads as follows, where "... x ..." is any open sentence containing "x" but not "F" free:

$$\exists F \forall x (Fx \leftrightarrow \ldots x \ldots)$$

Frege regarded this recognition of new concepts as the secret to the potential informativity of logic. Even when the axioms of logic are self-evident, and the inference rules obviously truth-preserving, the ability to recognize new concepts allows ampliative inference. As he put it, the true conclusions thereby reached are in one sense "contained within" the axioms, but only "as plants are contained in their seeds, not as beams are contained in a house" (Frege 1884, §88).

By the time of 1893's *Grundgesetze*, Frege had become convinced that logic provides us with knowledge not just of logical functions, concepts and relations, but through this means, also knowledge of certain logical objects with a special relationship to them. Mathematicians commonly speak of the "graph" of a function: for the *sine* function, it is a wave, for $x^2 + 2x + 1$, it is a parabola, etc. One of the words used by German mathematicians for a graph, "*Werthverlauf*" (translated as "value-range" or "course-of-values"), was adapted by Frege to speak of abstract objects which functions have in common when they have the same value for every argument, the same "graph", as it were, considered as an abstract object. In the case of concepts, i.e., functions from objects to truth-values, Frege identified their value-ranges with their *extensions*, as co-extensive concepts yield the same truth-values for the same arguments in every case.

The logical system of his *Grundgesetze* differed from that of *Begriffsschrift* by using fewer axioms but more inference rules. The logical cores of the systems were very similar. However, Frege added two new primitives. The first was a notation consisting of a Greek vowel written with a smooth-breathing accent followed by a function expression into the argument places of which the vowel was inserted. Frege understood this as a second-level function mapping a first-level function to its value-range. Another functor was introduced to be used along with it:

$\dot{\alpha}\phi(\alpha) =$ the value-range of function $\phi(\xi)$.

$$\backslash x = \begin{cases} \text{the sole object falling under the concept,} & \text{if } x \text{ is the value range of a concept under which only one object falls,} \\ x \text{ itself,} & \text{otherwise.} \end{cases}$$

Frege dubbed the second functor as "the substitute for the definite article", as under the appropriate circumstances, "$\backslash\dot{\alpha}F(\alpha)$" might be read "the F". Frege added the following axioms:

$$\vdash (\dot{\epsilon}f(\epsilon) = \dot{\alpha}g(\alpha)) = (\overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = g(\mathfrak{a})) \qquad \text{(Basic Law V)}$$

$$\vdash a = \backslash\dot{\epsilon}(a = \epsilon) \qquad \text{(Basic Law VI)}$$

The second, harmless, principle here asserts that a is *the* thing identical with a. Basic Law V, however, is not so innocuous, and has come to be a focal point in discussions of Frege's logic. It asserts that the truth-value of f and g having the same value-range is the same as the truth-value of f and g having the same value for every argument. Given what Frege understands value-ranges to be, this seems straightforward. In contemporary discussions, where the usual term/formula distinction is observed, using the set-theoretic notation $\{x|Fx\}$ for the extension of F, something similar to Frege's law might be put:

$$\forall F \forall G\big[\{x|Fx\} = \{x|Gx\} \leftrightarrow \forall x(Fx \leftrightarrow Gx)\big] \qquad \text{(BLV-modern)}$$

This asserts that F and G have the same extension just in case they are coextensive, which too seems self-evident.

Despite this, however, the presence of Basic Law V in a system in which the impredicative comprehension principle holds leads to inconsistency due to the contradiction from Russell's paradox. Extensions (and value-ranges generally) are objects, so some of them fall under the concepts of which they are the extensions, and some do not. The extension of the concept *extension* falls under its defining concept, whereas the extension of the concept *cat* is not a cat, and hence does not. Consider then the concept *extension that does not fall under its defining concept*. This concept has an extension, but does it fall under that very concept? It does just in case it does not. Let us use $W(\xi)$ to abbreviate:

$$\begin{array}{l} \top\!\overset{\mathfrak{f}}{\smile}\!\top f(\xi) \\ \qquad\;\, \llcorner\, \xi = \dot{\epsilon}\mathfrak{f}(\epsilon) \end{array} \qquad \text{(in contemporary notation } \exists F(\xi = \{x|Fx\} \wedge \neg F(\xi))\text{)}$$

An object falls under $W(\xi)$ just in case it does not fall under every concept of which it is the extension, or equivalently, just in case there is some concept of which it is the extension it does not fall under. This concept has an extension $\dot{\alpha}W(\alpha)$. Does it fall under W? Suppose it does. Then there is a concept F of which it is the extension such that it is not true that it falls under F. Since $\dot{\epsilon}F(\epsilon) = \dot{\alpha}W(\alpha)$, by Basic Law V, F and W are coextensive, and so $\dot{\alpha}W(\alpha)$ does not fall under W. Since assuming it does leads to the conclusion that it does not, $\dot{\alpha}W(\alpha)$ must not fall under W. But then it must in fact fall under every concept of which it is the extension, including W itself, and so we again arrive at the opposite result. Contradiction. In classical logic, everything can be proven from a contradiction, and so while it is possible to derive the basic principles of mathematics within Frege's *Grundgesetze* system, it is also possible to derive the negations of those very results: clearly a disaster.

After learning of the contradiction from Russell, Frege's own diagnosis was that Basic Law V was faulty. While many agree, it is also possible to blame the contradiction on impredicative comprehension. Notice that $W(\xi)$ above is defined using a second-order quantifier. If such quantifiers are not allowed within substituends for the f and g variables in Basic Law V, no contradiction results. Consistent fragments of Frege's system relying on weaker comprehension principles have been studied (Wehmeier 2004). Nonetheless, if Basic Law V is removed, and impredicative comprehension remains, one is again left with a consistent and Henkin-complete axiomatization of second-order logic. However, as such, the resulting system is too weak to derive the basic principles of arithmetic as Frege conceived them. Therefore, in the appendix dedicated to the problem in the 1902 second volume of *Grundgesetze*, Frege suggested using a weaker version of Basic Law V instead. The weaker version was later also shown to be flawed (Quine 1955; Landini 2006). It is not known whether or not Frege himself ever discovered the flaw in the revision, but in any case, the proposed third volume of *Grundgesetze* never appeared.

Frege's Philosophy of Mathematics

Although he is rightly best remembered for his positive arguments in favor of logicism, Frege also presented powerful and influential arguments against rival views. He soundly rejected psychologistic positions in both logic and mathematics. Numbers cannot be equated with mental images. Logical and mathematical truths are objective, and hence not dependent on the subjective states of any mind (Frege 1884, §26, Frege 1902, xviii–xxv). Both Mill's view that arithmetic consists of empirical truths based on observation as well as the Kantian view that arithmetic is grounded in the forms of experience (pure intuitions of space or time) were also dismissed as inadequate. Such positions cannot make sense, for example, of our knowledge of zero, as we have never had an experience of zero things, nor is there even the form of such an experience. These positions also fare poorly for large numbers, which are not sharply distinguished from small numbers, nor uniquely encountered in actual experience. He reserved his harshest criticisms for the formalist view that arithmetic can be understood as the study of uninterpreted formal systems, as advanced by figures such as Heine (1872), Frege's Jena colleague Thomae (1880), and, later, David Hilbert (1899). At its worst, he thought formalism was guilty of conflating symbols with what they mean, so that, for example, "$5 + 7 = 12$" in Arabic numerals would be considered a different truth from "V + VII = XII" in Roman numerals (Frege 1902, §100). At best, formalism involved a confusion of concepts and objects. By laying out a mathematical theory as a formal system, and suggesting that its subject matter is implicitly defined by the axioms of the system, one is merely defining a *concept* ("object satisfying the axioms" or similar). One, many, or even no, conglomerations of objects might fall under these concepts (Frege 1884, §97, Frege 1984, pp. 120–21). If the system of axioms is not given a specific interpretation, it has no specific content, and expresses no actual thoughts, true or false.

Frege put special emphasis on certain precepts as important when considering the nature of abstract objects such as numbers. One is his famous "context principle":

"never … ask for the meaning of a word in isolation, but only in the context of a proposition" (Frege 1884, p. x). Frege thought the best way to understand what number words mean, i.e., what numbers *are*, is to think about the truth-conditions of complete statements in which these words appear. He made special note that ascriptions of (cardinal) numbers in everyday use require the mentioning of a concept, or type of thing, being counted. Numbers cannot be ascribed to physical complexes outright: the same physical aggregate could be described either as *one* deck of playing cards, *fifty two* cards, *four* bridge hands, or as *goodness-knows-how-many* atoms. He accuses rival views about the nature of number as ignoring this fact. A sentence such as "Jupiter has 67 moons" seems to say how many times the concept *moon of Jupiter* is instantiated. In his *Grundlagen* he reaches the conclusion that *an ascription of number contains a assertion about a concept*, and insists strenuously that any adequate account of numbers must do justice to this.

Frege considered the view that numbers simply *are* second-order concepts, or concepts applicable to concepts. On this approach, zero would simply be the quantifier *nothing is …*, and one would be the concept applicable to concepts instantiated uniquely, and so on. Frege rejected this view on the grounds that it provides no interpretation of a numeral such as "3" by itself, but only of such phrases as "there are three ϕs". We would not be able to speak of numbers on their own, or make sense of such questions as to whether or not two numbers are the same, or whether or not the number 3 is identical with Julius Caesar. According to Frege's division of functions and objects, numerals as used in arithmetic have the syntax appropriate to name objects. Frege concludes that numbers must be objects. Nonetheless, we must consider the meaningfulness of number words in the context of propositions, which leads Frege to consider the truth-conditions of identity statements between numbers. Bearing in mind the connection between number ascriptions and concepts, and taking a cue from a passage in Hume, Frege makes note of the following principle—now commonly called *Hume's Principle*—governing numerical identity:

> The number of *F*s = The number of *G*s if and only if the *F*s and the *G*s can be put in 1–1 correspondence with each other. (HP)

The notion of 1–1 correspondence can be defined purely logically in second-order logic. It means that there is a relation that pairs up each *F* with exactly one *G*, and each *G* is paired up with exactly one *F*. In contemporary notation this can be put thus:

$$F(x) \cong_x G(x) =_{\mathrm{df}} \exists R\Big(\forall x\big(Fx \rightarrow \exists y(Gy \wedge Rxy)\big) \wedge \forall y\big(Gy \rightarrow \exists x(Fx \wedge Rxy)\big) \wedge \\ \forall x \forall y \forall z \forall w\big(Fxy \wedge Fzw \rightarrow (x = z \leftrightarrow y = w)\big)\Big)$$

Using "$\#x : Fx$" for "the number of *x* such that *Fx*", we might restate Hume's Principle as follows:

$$\#x : Fx = \#y : Gy \leftrightarrow F(x) \cong_x G(x) \tag{HP$'$}$$

Hume's principle takes the form of an *abstraction principle*. It introduces a functor, in this case the second-level functor "$\#x : \phi x$", by specifying under what conditions the values the function it represents yields are the same by citing a certain equivalence relation (symmetric, transitive and reflexive relation) holding between the arguments. Numbers applicable to concepts are the same when they are equinumerous: i.e., stand in 1–1 correspondence with each other. It should be noted that Frege's infamous Basic Law V also takes the form of an abstraction principle: value-ranges of functions are the same when those functions have the same value for every argument. Abstraction principles have been a subject of interest among mathematicians at least since Leibniz, and were also widely discussed by other nineteenth century figures including Grassmann and Peano, some of whom saw them as a kind of implicit definition. Frege considered the possibility that Hume's Principle might be considered as a kind of definition of cardinal numbers.

In the end, Frege rejected this suggestion. (HP′) does not provide an explicit definition of the notation "$\#x : Fx$", nor allow us to resolve it into primitive logical notation. It only fixes the truth conditions for *some* identity statements about numbers, not all. It does not tell us under what conditions "$\#x : Fx = q$" is true if "q" is not given in the form "$\#y : Gy$". So again, we cannot tell whether or not a number is identical with Julius Caesar. Yet, Frege did hold that Hume's Principle showed a way forward. Any correct definition of cardinal numbers must yield Hume's Principle as an analytic result. As we have seen, the mature Frege believed that logic could provide knowledge of certain logical objects: extensions of concepts most notably. In the *Grundlagen*, he therefore suggests defining "the number of Fs" as the extension of the concept *concept equinumerous with F*. Since equinumerosity is an equivalence relation, it would then follow that concepts F and G would have the same number just in case they were equinumerous, validating Hume's Principle. In the *Grundgesetze*, Frege allows the extensions of concepts to take over the role of concepts as the members of such extensions. (It is unclear how substantive this change is; see Klement 2012.) Frege then defined a function $\mathfrak{p}(\xi)$, which, when applied to the extension of a concept F, would yield the extension of the concept *extension of a concept equinumerous with F*, and hence, $\mathfrak{p}(\acute{\alpha}F(\alpha))$ became his version of "$\#x : Fx$". If we take Frege's talk of "extensions" as interchangeable with talk of "classes", then this definition makes numbers out to be classes of classes all of which have the same cardinality. After this definition was independently rediscovered by Russell, it has come to be known as the Frege-Russell definition of number. Using it, Frege was able to derive his version of Hume's Principle as a theorem from Basic Law V. Unfortunately, however, this wedded his theory of numbers to his (inconsistent) theory of extensions.

In his *Grundlagen*, Frege sketched informally how much of the core vocabulary of arithmetic could be defined logically, and how many of the basic principles—including a set of results equivalent to the Peano-Dedekind axioms for number theory—could be proven. Complete demonstrations of these principles were given in *Grundgesetze*. Roughly, 0 (zero) can be defined as the number of the null concept, *non-self-identity*, $\mathfrak{p}\acute{\alpha}{-\!\!\top\!\!-}(\alpha = \alpha)$. The successor function could be defined as yielding, for a given number n, the class of all extensions of concepts F such that for some x such that $F(x)$, the number belonging to the concept *being an F which is not x* is n. Speaking instead of classes, this will yield the class of all classes having one more member than the members of n. Using the logical notion of series introduced already in the *Begriffsschrift*, the concept of *natural number* could be defined as the class of all z in the series generated by the successor relationship starting with 0, or the class of all *z* falling under every concept held by 0 and always held by the successor of any *n* of which it holds. Importantly, Frege is able to prove that every natural number is non-empty by noting that for every natural number *n*, the subseries of natural numbers up to and including *n* has $n + 1$ members. There is one number up to and including 0, two numbers up to and including 1 (0 and 1), three numbers up to including 2, and so on. Frege thereby established that there are infinitely many natural numbers, a key step to obtaining results equivalent to the Peano-Dedekind axioms. Frege went beyond the arithmetic of finite numbers, pointing out that some concepts, including the concept *natural number* itself, apply to infinitely many things, and hence that there are infinite cardinals. As noted, he began to outline a treatment for real numbers as well before it was derailed by Russell's paradox.

In light of its inconsistency, Frege's logicist project cannot be considered an unqualified success, as he himself admitted. Yet, many key aspects of Frege's mathematical logic could perhaps be salvaged, and there have been many in the twentieth century who have attempted revisions, some of whom took themselves to be advancing their own form of logicism, others of whom took themselves to be offering a non-logicist but otherwise quasi-Fregean framework. For example, Whitehead and Russell offered a ramified theory of types (Whitehead and Russell 1914), and later, Ramsey offered a simpler theory of types (Ramsey 1925), which solved the logical paradoxes. Type theory, when combined with Russell's "no class" theory, applies something similar to Frege's hierarchy of levels (discussed below) to discourse about classes. It is then not possible to ask whether or not a class is a member of itself, only whether or not it is a member of a higher-type class. However, such systems require some non-obviously-logical principles such as an axiom of infinity to recover Peano arithmetic in its usual form. Quine (1937) offered us a Frege-inspired weaker, but still non-well-founded, theory of classes sufficient for arithmetic, though its consistency is still unproven. Boolos (1986) takes up Frege's suggestion from the Appendix to *Grundgesetze*, and offers a different, but consistent, modification of Basic Law V, called "New V". Recently a lot of attention has been paid to the fact that Frege's proofs of his equivalents to the Peano-Dedekind axioms only make essential use of Basic Law V in proving Hume's Principle, and that, once it is established, it plays the important role in the proofs. If notation such as "$\#x : Fx$" is taken as primitive, then (HP′) by itself as the sole addition to the usual axioms of second-order logic suffices for a consistent recapturing of all of Peano arithmetic. This result has been called "Frege's theorem", and many regard it as one of Frege's chief contributions to the foundations of mathematics. Indeed, Crispin Wright and Bob Hale (Wright 1983; Hale and Wright 2001) have advocated an "abstractionist" or "neo-Fregean" position that advocates postulating (HP′) and similar principles as analytic truths suitable for a kind of logicist treatment of mathematics, whether or not they are derivable from anything like set or class theory.

Another cause for concern for Frege's project came when Kurt Gödel (1931) proved that no finite or recursively axiomatizable system for mathematics that is expressively adequate to represent all recursive functions, can be both consistent and complete. Any such systems will be able to express mathematical truths not provable from the axioms. A recursively axiomatizable system is one in which there is a recursive procedure, or equivalently, a Turing-machine computable procedure, for determining what does and what does not count as a legitimate deduction in that system. One could add axioms to a system for mathematics, but if only a finite number are added, or an even a recursively decidable infinite number, there will still be unprovable truths. Only an inconsistent system, or one in which there is no recursive or Turing-machine computable algorithm for separating out good from bad deductions, can possibly provide a complete axiomatization of arithmetic. The extent to which this undermines Frege's logicist thesis depends greatly on how strong that thesis is taken to be. While it may be that no single recursive logical calculus suffices to derive all of mathematics, this does not obviously show that any portion of arithmetic is non-logical or non-analytic. Indeed, if Frege had been successful in providing a logical basis for all of standard Peano arithmetic—which encompasses nearly all the mathematics the average person learns in his or her lifetime—it would seem odd to suggest that mathematics as a whole has a non-logical metaphysical or epistemological foundation despite the fact that this portion does not. Nonetheless, Gödel's results surely complicate arguments in favor of logicism.

Frege on Objects and the Hierarchy of Concepts

We have seen that Frege modeled the syntax of his logical notation on the function/argument analysis of complex mathematical expressions. He believed that, despite certain complications, the same kind of segmentation could be applied to ordinary language. The phrase "the capital of Sweden" could be divided into a function expression "the capital of ξ," and a phrase for its argument, "Sweden". The proposition "Jupiter is larger than Mars" can be divided into function and argument in multiple ways, e.g., into the name "Jupiter" and the concept expression "ξ is larger than Mars", or into the name "Mars" and the distinct concept expression, "Jupiter is larger than ζ", or into the two names "Jupiter", "Mars" and the relational expression, "ξ is larger than ζ". The hallmark of a function expression is its having a spot (or spots) ready to receive the expression for its argument(s). The hallmark of the name of an object—including complete sentences which Frege understood as names of truth-values, only with added assertoric force (the natural language substitute for his judgment stroke)—lies in its lack of any such gap or incompleteness. Frege believed that this syntactic distinction reflected a real difference between the things the different expressions stood for. Not only are concept expressions gappy, but concepts themselves are *unsaturated* or have a *predicative nature*. In contrast, objects are self-subsistent or complete. It is the unsaturated nature of concepts that makes them suitable to be the references of concept expressions, and the saturated nature of objects that makes them suitable to be the references of names. In any complex expression, at least one phrase must represent a function: a series of names by itself cannot form a coherent whole. Complete names can only refer to objects, and incomplete expressions can only refer to concepts or other functions.

Frege also believed that functions differed depending on the nature of their arguments, forming a hierarchy. A monadic first-level function, such as $H(\xi)$, has a single argument spot which must be completed by an object. A polyadic or relational first-level function would have multiple arguments, e.g., $R(\xi, \zeta)$, but it would still take objects as argument. A monadic second-level function would be a function whose argument is a first-level function. A second-level concept is a second-level function whose value is always a truth-value, such as the quantifier $-\!\overset{\mathfrak{a}}{\smile}\!-\ \phi(\mathfrak{a})$. For Frege, the difference between first- and second-level functions is just as thoroughgoing as the difference between first-level functions and objects. A second-level concept cannot be predicated of an object or another second-level function. This is shown by the fact that the gap in the expression "$-\!\overset{\mathfrak{a}}{\smile}\!-\ \phi(\mathfrak{a})$" must be replaced by something itself gappy, so that it has a place to receive the bound object variable "$\mathfrak{a}$", but only a sign for a first-level function has the right form. When they come together, the first- and second-level functions "mutually saturate". A third-level function would be a function taking a second-level function as argument. Consider the higher-order quantifier $-\!\overset{\mathfrak{f}}{\smile}\!-\ \mu_\beta(\mathfrak{f}(\beta))$. Its argument spot is represented by the portion $\mu_\beta(\ldots\beta\ldots)$. Only a symbol such as "$-\!\overset{\mathfrak{a}}{\smile}\!-\ \ldots\mathfrak{a}\ldots$" can fit here, because by means of its bound variable, it could complete the argument spot of the variable bound by the higher-order quantifier, in this case resulting in $-\!\overset{\mathfrak{f}}{\smile}\!-\!\overset{\mathfrak{a}}{\smile}\!-\ \mathfrak{f}(\mathfrak{a})$.

Frege maintained that conflating concepts and objects, or functions of various levels, was a frequent source of confusion. We have already seen that he criticized certain formalist positions on such grounds. For Frege, the fundamental meaning of existence is given by the existential quantifier $\top\!\overset{\mathfrak{a}}{\smile}\!\top\ \phi(\mathfrak{a})$, which is a second-level concept, not a first-level one, giving a concrete interpretation to the Kantian dictum that "existence is not a predicate". Frege diagnosed the problem with the ontological argument for the existence of God as wrongly treating existence as a first-level concept. This quantificational account of existence was taken up in the twentieth century by Russell (e.g., 1905) and Quine (1948), and represents the main alternative to the position found in the works of Alexius Meinong (e.g., 1899), which admits both objects which have and which lack existence.

This view is not without its difficulties or puzzles. Pressed by Benno Kerry (1887), for example, Frege was compelled to admit that the phrase "the concept *horse*", since it has no gap for an argument, cannot refer to a concept. Frege concluded that there is a special object for every concept (or other function) which goes proxy for it when such locutions are used. "The concept *horse*" refers to an object, not a concept. But then can Frege even coherently state the distinction without violating his own principles? Consider the predicate "... is an object". If it stands for a first-level concept, it is true of all arguments; if something else, then it is true of none. Is it meaningful, by Frege's own lights, then, to speak of a distinction between concepts and objects, or postulate entities which are not objects? Frege admits the difficulty, claiming that the very nature of the distinction forces him to speak imprecisely, adding that he hoped readers "would meet me halfway" and "not bedrudge of pinch of salt" (Frege 1984, p. 193). This puzzle is sometimes thought to prefigure the discussion of "ineffable truths" in Wittgenstein's *Tractatus*, i.e., those that can only be "shown" not "said".

Frege on Meaning and Truth

A perhaps even more influential distinction made by Frege is that between the sense (*Sinn*) and reference (*Bedeutung*) of expressions. Frege introduced this distinction in his mature work to resolve what he considered a puzzle about identity. Consider:

$$\text{the morning star} = \text{the morning star} \tag{8}$$

$$\text{the morning star} = \text{the evening star} \tag{9}$$

These are both true, but (8) seems *trivial* in a way that (9) does not. However, the expressions flanking "=" in (9) have the same content, both with each other and with the corresponding parts of (8). So what accounts for the difference in cognitive significance? In his early work, when he employed the sign "≡" for "identity of content", he assumed it must be understood not as a genuine relation between objects, but rather as a relation between expressions. It must always be trivial to be told that a thing is itself, but it could be informative to be told two expressions have the same content ((Frege, 1879, §8)). However, in his mature work, he realizes that (9) must be understood as informing us something *about* the morning star itself, not merely about language. He therefore bifurcates his earlier notion of content into the notions of sense and reference. The reference is the actual object a name stands for. "The morning star" and "the evening star" have the same reference: the planet Venus. The sense of an expression is its "mode of presentation", or the way in which the reference is picked out. The expressions "the morning star" and "the evening star" differ in sense because they pick out Venus in virtue of different properties it has. (8) and (9) differ in sense, which explains their difference in cognitive significance.

Frege puts this distinction to a variety of uses. In earlier work, he had worried that in a mathematical equation such as "$5 + 7 = 4 \times 3$" the two sides cannot in all ways be considered "the same": one is a sum, the other a product, etc. Once adopting the sense-reference distinction, however, he saw no reason to differentiate between mathematical equality and identity. The distinction also made it possible for him to regard concepts and relations as functions with truth-values as values. In his mature logic, all true propositions consist of the judgment stroke preceding a name of the True. Obviously, it would be absurd to hold that all true propositions have the same meaning, full stop. Frege now held that although the references of all true (and of all false) propositions are alike, they differ in the senses they express, which he called *thoughts*. Although Frege considered it as a defect of ordinary language to be avoided in a logically rigorous language, Frege thought some expressions expressed a sense but nonetheless lacked a reference, such as in fiction, e.g., "Odysseus" or "Sherlock Holmes", or even in mathematics and science, e.g., "the least rapidly converging series". Because these expressions lack reference, so do any propositions in which they appear. Since Frege regards the reference of a complete proposition to be its truth-value, this would mean that such sentences as "Sherlock Holmes lived at 221B Baker Street in London" are neither true nor false.

Lastly, Frege used the distinction to account for why it is not always possible to substitute apparently co-referring expressions in a sentence while preserving the truth-value. Consider:

$$\text{Ptolemy believes } \underline{\text{the morning star} = \text{the morning star}} \tag{10}$$

$$\text{Ptolemy believes } \underline{\text{the morning star} = \text{the evening star}} \tag{11}$$

These differ from each other only by the substitution of one term referring to Venus for another. Since Frege thinks that the reference of an expression is a function of the references of its parts, it is difficult to see how (10) could differ in truth-value from (11). But clearly this is at least possible. Frege's solution is to insist that in certain contexts, such as in what we now call "propositional attitude reports" (beliefs, desires), words shift from having the usual or "direct" references to having their "indirect" references, which are their customary senses. Since (8) and (9) have different senses, when they are embedded in the underlined contexts in (10) and (11), they refer to different thoughts. Belief is a relation between a believer and a thought, not a believer and a truth-value. It is possible for Ptolemy to believe one thought and not the other. To be substitutable in such "indirect" contexts, words would have to share not only their customary references, but their customary senses as well. Frege's writing on this subject marks the starting point for most contemporary discussion in analytic philosophy about identity puzzles and the logic of propositional attitude reports (e.g., Salmon 1986).

Frege does not tell us much about the precise nature of senses. Some passages suggest that he thinks of a sense as containing a property or set of descriptive information, and the reference is the object which uniquely satisfies that property, if there is one. (This reading is presupposed, e.g., in Kripke 1980.) It is unclear, however, if this account covers all senses. Frege seems to think that each person can think about him or herself in a unique way, using a unique sense not fully graspable by others. This prefigures the contemporary discussion of the unique nature of *de se*, or self-directed, mental attitudes (Perry 1977). Frege applies the distinction also to other kinds of expressions, such as those for functions and concepts of different levels. Frege considers the identity relation to be a first-level relation that can only hold between objects. However, in posthumously published remarks (Frege 1979, pp. 118–25), he suggests that there is an analogous relation of "coinciding" which holds between functions that have the same value for every argument. Coextensive concepts such as ξ *has a heart* and ξ *has a kidney* would coincide, since they would map all the same objects to the same truth-values. Clearly, however, the phrases "ξ has a heart" and "ξ has a kidney" are not synonymous in *every* respect. Frege concludes that these too differ in sense, but not reference, noting that his sense/reference distinction plays a role similar to the more traditional distinction between the intension and the extension of a predicate. Frege regards the sense of a complete proposition as in some metaphorical way a complex or whole containing the senses of the parts. Thoughts have a kind of *unity* to them provided by the senses of the function expressions, which Frege too regards as unsaturated or incomplete. Exactly how the "unsaturatedness" of these senses is to be understood, and whether or not it differs from the kind of unsaturatedness that the functions that are their references have is a thorny issue (see e.g., Klement 2002, chap. 3).

Frege's contention that the reference of a proposition is its truth-value strikes many as idiosyncratic. Frege gave two sorts of reasons for it. The first involves a kind of parity of interests. We seem to be interested in or concerned with whether or not the subexpressions of a proposition have reference in precisely those situations in which we are concerned with their truth. If a story is told for fun, we are indifferent about whether or not the character names actually refer. However, when investigating the truth of a historical legend, it will matter to us whether or not the names refer to actual historical figures. The other reason, less emphasized by Frege but subsequently more influential, is based on his assumption that the reference of the whole must be determined by the references of the parts. He asks, rhetorically, "what feature except the truth-value can be found that ... remains unchanged by substitutions" in which "a part of the proposition is replaced by an expression with the same reference" (Frege 1984, p. 164)? A more fleshed-out argument was later given by Alonzo Church (1956, p. 25). If we assume that the reference of the whole is unchanged both by such substitutions and by relatively innocuous-seeming modifications to grammatical form, then all the propositions in the following series would seem to have the same reference:

> Sir Walter Scott is the author of Waverly.
> Sir Walter Scott is the writer of twenty nine Waverly novels.
> The number of Waverly novels written by Sir Walter Scott is twenty nine.
> The number of counties in Utah is twenty nine.

However, the last seems to have nothing interesting in common with the first apart from its truth-value, confirming Frege's suggestion that only the truth-value could be the reference. This argument has come to be called "the Frege-Church slingshot".

Thoughts, the senses of propositions, are the bearers of truth. However, Frege argues that the relationship of a thought to its truth-value must be understood as one of sense and reference, not of a thing and its attribute. Frege points out that "it is true that ..." adds nothing to the sense of a proposition it is applied to, and so truth cannot be a genuine property. Frege argues against the possibility of defining truth in a non-circular way. Truth cannot be correspondence in a strong sense, as nothing completely corresponds with anything except itself. If truth relies on a weaker, or partial, notion of correspondence in a certain respect, we would be led back to the question as to whether or not it was *true* that there was such a correspondence, creating a vicious regress. Frege thinks the same reasoning undermines any other way of defining truth. Rather, Frege thinks that the activity of forming judgments and making logical inferences indirectly reveals what truth is: "the meaning of the word *true* is spelled out in the laws of truth", i.e., the laws of logic (Frege 1984, p. 352). Frege claims that thoughts are timeless, objective abstract entities existing in a "third realm" distinct from both the physical, concrete world, and the subjective realm of ideas. Thoughts must be finer grained than the entities they are about, and so cannot be made up of these entities. He remarks, "Mont Blanc with its masses of snow and ice is no part of the thought that Mont Blanc is over 4000m high" (Frege 1980, p. 187). Moreover, unlike ideas, thoughts are not private to the individual. They can be shared. Thoughts and their relationship to truth and falsity are not created by our acts of thinking nor destroyed by their cessation. Although thoughts affect the physical world only through acts of cognition, one must not conflate the act of thinking with its content, the thought.

Analytic Philosophy and the Impact of the New Logic

Analytic philosophy has been perhaps the most successful philosophical movement of the twentieth century. While there is no one doctrine that defines it, one of the most salient features of analytic philosophy is its reliance on contemporary logic, the logic that had its origin in the works of Boole and Frege and others in the mid-to-late nineteenth century. The works of many key figures in the movement are only fully intelligible when taken together with their works on formal logic. Kripke's and Marcus's views on names and essence must be considered in relation to their work on modal logic, David Lewis's modal realism in relation to his counterpart theory, David Kaplan's theories of language in relation to his logic of demonstratives, and the list goes on.

Analytic philosophy is sometimes also called "Anglo-American philosophy". The label is misleading given both the many contributions from Continental Europeans, and the spread of the movement to many other parts of the globe. However, no one did more to promote the "new logic" in the philosophical communities of the English-speaking world than Bertrand Russell. Along with his Cambridge colleague, G. E. Moore, Russell had broken from the idealistic monism prevalent in late-nineteenth century Britain such as found in the writings of F. H. Bradley (see e.g., Hylton 1990). Although he already respected Boole, Russell first appreciated the philosophical significance of the "new logic" when he met Peano at the International Congress of Philosophy in 1900. Peano's symbolism was based largely on Boole's, but with certain changes more in line with Frege's approach. Russell found that Peano's logic made possible a new mode of philosophical analysis (see Levine 2016), and put the final pieces in place for him to complete his ongoing research on the foundations of mathematics. Russell independently came to hold an even stronger form of logicism than Frege's. He only read Frege closely when *The Principles of Mathematics* was near completion in 1902, though he immediately recognized the greater importance and profundity of Frege's approach to logic. Despite discovering the inconsistency in the *Grundgesetze* system, he expressed nothing but effusive praise and respect for Frege from then on. Russell once wrote, "when I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth" (quoted in van Heijenoort 1967, p. 127). Russell's championship of modern logic in something like the Fregean form has had an effect on the history of philosophy the full extent of which we cannot yet fully gauge.

Austrian thinker Ludwig Wittgenstein was led away from engineering towards philosophy when he discovered Russell's *The Principles of Mathematics* while an engineering student at the University of Manchester. He was particularly interested in the appendix on Frege, and the discussion of Russell's paradox. In 1911, he went to speak with Frege in Jena about a possible solution. Wittgenstein later recalled that Frege "wiped the floor" with him in their discussion (see Drury 1984, p. 110), but nonetheless Frege recommended that Wittgenstein go to Cambridge to study with Russell, which he did. Frege and Wittgenstein corresponded and had a few additional meetings. In the preface to his enormously influential *Tractatus Logico-Philosophicus*, Wittgenstein singles out only Russell and "the great works of Frege" as stimulation for his thoughts.

The *Tractatus* and its account of logical truth was influential on another group of thinkers formed around Moritz Schlick in the 1920s, the so-called "Vienna Circle", proponents of logical positivism. Perhaps the most philosophically sophisticated among them, Rudolf Carnap, had been a student of Frege's while an undergraduate at Jena, and his notes from Frege's 1911–13 lectures have survived. Carnap describes Frege's influence as "the most fruitful inspiration I received from university lectures" (Reck and Awodey 2004, p. 18). The strictly formal and syntactic understanding Carnap had of not only logic but, for parts of his career, all of philosophy, could only have been possible due to the mathematization and formalization of logic brought on by the works of Boole, Frege and others in the late nineteenth century. Both Boole and Frege are mentioned in the Vienna Circle's unofficial manifesto (Hahn, Neurath, and Carnap 1929).

Frege was praised by similar groups in Germany and Poland. Polish logician Stanisław Leśniewski is the first known to have discovered the formal flaw in the revised version of Law V published in the appendix to volume II of *Grundgesetze* (Sobociński 1949–1950). While the logical system of *Grundgesetze* could not be accepted as is, a rich variety of ways of "fixing Frege" have been researched, and will continue to be.

Despite the initial underappreciation Frege complained of, his work is now extremely widely read. His influence on contemporary philosophies of mathematics, on debates in the theory of meaning, the nature of truth, the logic of propositional attitudes and *de se* attitudes, have already been mentioned. More than 125 years after their publication, Frege's *Grundlagen* is now probably the most widely read book on the philosophy of mathematics, and his "Über Sinn und Bedeutung" one of the most widely read articles in the philosophy of language. The logical systems made possible by the work of Boole, Frege, and others of the late nineteenth century have changed philosophy and computing forever. Seeds even of twenty-first century analytic philosophy were planted in the rich logical soil of the nineteenth century. For example, the earlier nineteenth century works of Bernard Bolzano (especially Bolzano 1837) include anticipations of Tarski's work on the concept of logical consequence (Tarski 1936), and the recent hot topic of metaphysical grounding (e.g., Fine 2012). Bolzano's writings have only recently gained the attention of analytic philosophers, but now that they have, they will no doubt become more influential. The seeds of nineteenth century logic will continue to be reaped for years to come.

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