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The paradoxes and Russell's theory of incomplete symbols

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Abstract Russell claims in his autobiography and elsewhere that he discovered his 1905 theory of descriptions while attempting to solve the logical and semantic paradoxes plaguing his work on the foundations of mathematics. In this paper, I hope to make the connection between his work on the paradoxes and the theory of descriptions and his theory of incomplete symbols generally clearer. In particular, I argue that the theory of descriptions arose from the realization that not only can a class not be thought of as a single thing, neither can the meaning/intension of any expression capable of singling out one collection (class) of things as opposed to another. If this is right, it shows that Russell's method of solving the logical paradoxes is wholly incompatible with anything like a Fregean dualism between sense and reference or meaning and denotation. I also discuss how this realization lead to modifications in his understanding of propositions and propositional functions, and suggest that Russell's confrontation with these issues may be instructive for ongoing research.

Keywords Bertrand Russell · Paradox · Meaning · Class · Denoting · Language · Symbolism · Descriptions · Logic

1 Introduction

Bertrand Russell claimed in his *Autobiography* and elsewhere (1998, p. 155; cf. Russell 1958, p. 79; Grattan-Guinness 1977, p. 80) that he discovered his theory of descriptions in 1905 while working to solve the logical paradoxes plaguing his work on the foundations of mathematics. It was, he writes, "the first step towards

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overcoming the difficulties which had baffled me for so long." Nonetheless, the close relationship in Russell's mind between the topics of 1905's "On Denoting"—the theory of descriptions and other quantifier phrases—and his work in solving the paradoxes is not fully understood. Connections have been noted, especially the similarity between Russell's contextual definitions of descriptions and of class terms in *Principia Mathematica* (explicitly drawn attention to in chap. 3 of *PM*'s introduction), whereupon both are "incomplete symbols". Still, it may be difficult to understand why Russell was even motivated to explore issues concerning the basic logical form of quantified and descriptive propositions while his chief philosophical occupation was finding a solution to (mostly) class-theoretic paradoxes. Isn't class or set-theory something significantly more mathematically advanced or specialized than the topic of the logical form of such basic propositions? Indeed, those accustomed to set theories like ZF or NBG can be surprised to learn that Frege and Russell intended to make use of classes in their logicist reductions of mathematics. Isn't set theory a part of mathematics, not logic proper, and so wouldn't this at best amount to reducing mathematics to one of its subdisciplines?

This attitude is a testament to the success of Fregean quantification theory. Prior to it, most logicians, especially those mathematically inclined, analyzed nearly all judgments as involving classes. In the Boolean tradition, a universal affirmative *all As are Bs* was in effect analyzed as $A \cap B = A$, where "A" and "B" represent classes. Even Peano, whose logical notation was significant steps closer to modern quantifier logic, analyzed a subject-predicate proposition *x is P* as "$x \epsilon P$," with "P" taken as standing, directly or indirectly, for a class (Peano 1967, p. 89). Russell (1931, §§69, 76) accused Peano of ambiguity, suggesting that "$x \epsilon P$" could be read as asserting a relationship either between x and a class, or between x and a class-concept, but these issues were closely tied in his mind as well. As Russell saw it, the paradoxes force a revision in our understanding of not just discourse about classes, but quantified and other propositions generally. Although its treatment of definite descriptions has made the biggest impression on the philosophical community, "On Denoting" deals with quantifiers generally, including *some*, *all* and *every*.

In this paper, I argue that Russell's theory of incomplete symbols (including his theory of descriptions) emerged in part from the realization, forced upon him by the paradoxes, that not only can a class not be thought of as a single thing, neither can the meaning or intension of any expression capable of singling out one collection (class) as opposed to another. If this is right, it shows that Fregean semantic "dualisms" (involving a sense/reference or meaning/denotation division) may be incompatible with the general Russellian tack for dealing with the paradoxes. I also explain how Russell's theory of "incomplete symbols" was later generalized to propositions and propositional functions in response to paradoxes almost identical to those that led him to rethink his earlier understanding of classes. I contend that the heart of Russell's solution to the paradoxes is not the theory of types, but rather the view that words or phrases that apparently stand for the kinds of entities that give rise to paradoxes are not to be taken at face-value. This is important not only for understanding the development and unity of Russell's views in philosophical logic, but may continue to be a source of inspiration and guidance for ongoing researches in these areas. These developments derived from Russell's early adoption of a (hyperintensional) theory of meaning focused on the notion of *propositions* and their make-up and constituents, and I begin there.

2 Propositions and their constituents

Although the opening passage of "On Denoting" speaks of "denoting phrases," we must remember that the "propositions" Russell meant to analyze were not understood by him as linguistic items. A proposition was understood roughly as a state of affairs, consisting of the actual entities it is "about". In the modern sense, these are intensional entities (indeed, hyperintensional), because their individuation conditions are finer-grained than either their truth-values or even the collections of possible worlds or situations in which they are true. Yet, Russellian propositions are not first and foremost semantic entities. They can be meanings, of course, or semantic values, but this is just to say that they can be meant, which would be true of any entity. A Russellian proposition is a mind- and language-independent complex unity, and when true, early Russell identified a proposition with a fact (see, e.g., Russell 1994c, p. 75 and Russell 1994d, p. 492).

Despite the objective nature of propositions, Russell posited a very close isomorphism between their structures and those of sentences used to express them. In §46 of his 1903 *The Principles of Mathematics*, Russell suggests that grammatical differences correspond to differences in the mode of occurrence of entities in propositions, and that almost every word in a sentence means a component of the corresponding proposition. There are exceptions, but these would be far fewer in a symbolic logical language where words not contributing to the proposition expressed are left out. Russell is usually read, rightly, as later moving away from the simplistic view of *Principles*, especially after "On Denoting". However, the differences would still mainly be evident for natural language. In his logical symbolism, Russell hoped to preserve an isomorphism between its formulæ and the propositions expressed as much as possible. If we assume we are working in a fully analyzed symbolic language, in which simple things are represented by simple expressions, defined signs are replaced by their definiens, and redundant primitives are eliminated, it would not be unfair to attribute to Russell a commitment to something like the following, which together I shall call LPI for *language-proposition isomorphism*:

1. A proposition has precisely as many constituents as a sentence in such a language expressing it has constituents, parts, or component expressions.
2. In such a language, two (closed) sentences express the same proposition if and only if they differ from each other by at most arbitrary choice of bound variable.

LPI would require further discussion to make it fully clear. A rough first interpretation of what is meant by a sentence's "component expressions" is that they are those involved in the recursive characterization of its syntactic structure, or those which would represent "nodes" in a tree-like depiction of its syntax. It follows that some complex component expressions would contribute parts to the proposition, even though these parts themselves have parts. Consider "$(p \supset q) \supset q$". Here the part "$(p \supset q)$" is a component expression under the characterization just given, whereas, e.g., "$\supset q) \supset$" would not be. This seems right, as Russell would understand the proposition that p implies q as a part of the more complex proposition that p's implying q implies q. This part itself has the propositions p and q and the relation of implication as parts.

Again considering only fully analyzed languages, LPI seems to imply something I call ECI for *expression-component isomorphism*:

1. Every closed component expression of a sentence makes a contribution to the proposition the sentence expresses;
2. This contribution is an object, though it may not be the semantic value of the expression;
3. Two component expressions make the same contribution to the proposition if and only if they differ from each other by at most arbitrary choice of bound variable.

The second clause of ECI here allows for what Russell called "denoting concepts": entities, which, when they occur in a proposition make the proposition not about them but rather about other objects to which they bear the relation of *denoting*. While the semantic value of the phrase "the Pope" may be Jorge Bergoglio, Bergoglio himself is not what this phrase contributes to the proposition; instead, it is a concept which denotes Bergoglio. Such phrases as "Bergoglio" and "the Pope" would likely not occur in a fully analyzed language, but if we suppose that they did, by the third clause above, they would make different contributions. However, when Russell still took class-abstracts to be complete symbols (prior to his "no-classes" theory), the third clause would allow that the difference between, e.g., "$\{x|x = x\}$" and "$\{y|y = y\}$" is unimportant so that these can contribute the same entity. However, this entity is likely not the universal class, but rather something denoting the universal class, as, according to ECI, these would contribute a distinct entity from "$\{x|x \notin \varnothing\}$", despite having the same semantic value.

3 Propositions about classes

Russell discovered the paradox involving the "class of all classes not members of themselves" in 1901 while considering Cantor's powerclass theorem.[1] By Cantor's result, every class, even an infinite one, has more subclasses than members. In other words, the powerclass of any class c is larger than c. Cantor's proof proceeds by a diagonal argument. Suppose, for reductio that c has as many members as its powerclass, i.e., suppose each member of the powerclass of c, i.e., each subclass of c, could be correlated with a distinct member of c. Some subclasses would be paired

[1] See, e.g., Russell (1931, §100), and his letter to Frege of 24 June 1902 in Frege (1980, pp. 133–34), and for historical context, see Griffin (2004).

up with members of c which they contain, some would not. Let s be the subclass of c of those members of c paired up with subclasses in which they are not included. Since s is a subclass of c it is a member of the powerclass of c, and thus must itself be paired up with some member m of c. However, is m a member of s? It is just in case it is not. This contradiction means that there must not be such a means of pairing up subclasses with members. Russell's paradox is just the contradiction that results if we consider classes to be individuals and attempt to pair up subclasses of the universal class of individuals with *themselves* as members: s and m then both become the class of classes not members of themselves, and a contradiction results from considering whether or not it is a member of itself.

Russell eventually concluded that there must be more classes of individuals than individuals and hence classes cannot themselves be individuals. But there is a bigger picture lesson to be drawn as well, which is that it is absolutely impossible to correlate each class with a distinct individual, however this might be attempted. I think this larger lesson is very important for understanding the direction of Russell's thought during these years. While he later gave a more generic diagnosis of the paradoxes (see Russell 1973c, pp. 141–42), early on, he seemed to see Cantor's result as lying behind most of the difficulties, despairing to Frege in September 1902 that "From Cantor's proposition that any class contains more subclasses than objects we can elicit constantly new contradictions" (Frege 1980, p. 149). It gives rise to a difficulty in understanding the make-up of propositions about classes. If an expression for a class, "$\{x| \ldots x \ldots\}$", counts as a component expression of a sentence, then by ECI, it contributes something to the proposition expressed, and this something would seem to be an individual. This goes against what the main lesson of Cantor's theorem appears to be.

In *Principles*, Russell was not yet prepared to claim that *no* class is an individual. There, he made a distinction between a "class as one" and a "class as many".[2] Some collections could be thought of as single things, and thus single constituents of a proposition—these are classes as one. Other collections, however, could only be considered many entities (Russell 1931, §§70, 101). The members of a class as many, then, could not be *one* constituent of a proposition, though they could be many constituents. To maintain the isomorphism between language and proposition, it would be necessary that a sentence represent a class as many with many expressions. Such is the case with, e.g., "John is one among John and Paul and George and Ringo," where the class as many *John and Paul and George and Ringo* is represented with many names, not one. However, this is almost never how we speak of classes. In a symbolic language, a class is typically represented with an abstract "$\{x| \ldots x \ldots\}$". Russell concluded that in symbolic logic, ϵ must not be understood as a relation to a class as many:

> It is plain that, since a class, except when it has one term, is essentially many, it cannot be *as such* represented by a single letter: hence in any possible Symbolic Logic the letters which do duty for classes cannot represent the classes *as many*, but must represent either class-concepts, or the wholes composed of classes, or

[2] For further discussion of his views at this time, see Klement (2014).

> some other allied single entities. And thus ϵ cannot represent the relation of a term to its class as many; for this would be a relation of one term to many terms, not a two-term relation such as we want. (Russell 1931, §76)

As we have already seen, when a class-abstract "$\{x| \ldots x \ldots\}$" is used in a sentence, the entity it contributes to the proposition expressed, while a single entity, is not the class itself. What is the entity it contributes, then, and are there as many such entities as there are classes thereof?

Besides this distinction, in *Principles*, Russell also discussed two other kinds of entities, class-concepts and concepts of a class, which were not the same (§67). Class-concepts are the entities contributed to propositions by count-noun phrases, e.g., "human" or "cat". Russell seems willing (§§57–58) to equate them with what he calls "predicates" (the contributions made by adjectives), understood basically as Platonic universals. For each class-concept α, there is a variety of denoting concepts: *all* α, *every* α, *any* α, *a(n)* α, *some* α, *the* α. A denoting concept of the form *all* α is what Russell calls "the concept of a class": it denotes the class as many of things bearing α. Early Russell took predicates or class-concepts to be individuals (§§49, 499), and so there must be fewer of them than there are classes. Otherwise, Cantor's theorem would be violated, and diagonalization would yield another version of Russell's paradox, as noted in §101:

> A class-concept may or may not be a term of its own extension. "Class-concept which is not a term of its own extension" appears to be a class-concept. But if it is a term of its own extension, it is a class-concept which is not a term of its own extension, and vice versâ. Thus we must conclude, against appearances, that "class-concept which is not a term of its own extension" is not a class-concept.

Russell's solution here is to deny that every class is defined by a class-concept or predicate. In effect, this is to adopt a *sparse* view of class-concepts/predicates. Notice that he is not denying that there is a class of all class-concepts not among their own extensions, or even that there is a defining propositional function of this class. Indeed, he claims that "every class can certainly be defined by a propositional function" (§103). If we use ϵ to mean not the relation of an individual to a class, but the relation of an individual to a class-concept, "$\sim(x \epsilon x)$" is meaningful and true for many values of x. However, there is no such class-concept which all and only such class-concepts share:

> It must be held, I think, that every propositional function which is not null defines a class, which is denoted by "x's such that ϕx." … But it may be doubted … whether there is always a defining predicate of such classes. Apart from the contradiction in question, this point might appear to be merely verbal: "being an x such that ϕx," it might be said, may always be taken to be a predicate. But in view of our contradiction, all remarks on this subject must be viewed with caution. (Russell 1931, §84)

A sparse view of predicates and concepts also helps with the version of the paradox involving predicates not predicable of themselves. Russell continues to believe that some predicates can be truly predicated of themselves (e.g., *Being*), and some can be falsely predicated of themselves (e.g., *Existence*), but there is no predicate all and only those predicates in the latter group share (§78).

So for Russell, not only are there classes whose members don't share a defining predicate or class-concept, but *it is possible to give instances of such classes*, and not just extensionally. The class of all class-concepts not members of their own extension is an example. If we are not giving this class by extension, how are we able to talk about it? If denoting concepts of classes are always derived from class-concepts or predicates, as Russell seems to suggest in *Principles* (§58), then we cannot speak of them by means of denoting *concepts* either. He allows them to be denoted by class-abstracts defined using open-sentences ("propositional functions"). I have been using "$\{x|\ldots x\ldots\}$", but Russell's own notation and terminology varied during these years. Early on, Russell (following Peano) wrote instead "$x \ni \ldots x \ldots$", pronouncing $\ni$ as "such that", but claims that "*such that* contains a primitive idea, but one which it is not easy to disengage from other ideas" (§33), intimating that it needs further examination. This notation falls away in his manuscripts, replaced by a succession of others. Charting the twists and turns of Russell's thinking on such matters, even only for 1902–05 would require a book-length treatment.[3] Here, I offer only a crude summary of what, he at *some* times believed. The development of his views were quite complicated, and I take the liberty of focusing on what I take to be the most important developments.

Beginning around mid-1903, Russell began using notation for function abstraction, akin to the modern $\ulcorner\lambda x(\phi(x))\urcorner$. He uses a variety of notations, finally settling on $\ulcorner\phi(\hat{x})\urcorner$ familiar to readers of *Principia Mathematica*. Because some notations he uses for function-abstracts are identical to those he at other times uses for class-abstracts, care must be taken to avoid misunderstanding. Therefore, I shall replace these notations with the contemporary $\ulcorner\lambda x(\phi(x))\urcorner$ except in direct quotations. Russell at times used a notation for classes that was derived from the notation for function abstraction. He might write something along the lines of $\ulcorner\mathrm{Kl}`\lambda x(\phi(x))\urcorner$ for the class of all $x$ which satisfy the function named by $\ulcorner\lambda x(\phi(x))\urcorner$. While the semantic value—the entity the resulting proposition is about—of the term $\ulcorner\mathrm{Kl}`\lambda x(\phi(x))\urcorner$ is a class, what it contributes to the proposition is not the class, but rather a "denoting complex". A denoting complex is like a denoting concept in that, when it occurs in a proposition, the proposition is not about it but about what it denotes. Unlike a denoting concept, it is derived from a complex propositional function rather than a (simple) class-concept. The sign "Kl" Russell regards as representing a "denoting function", which is a function whose values are not propositions but denoting complexes.[4] Russell still seems to hold ECI during this period. The denoting complex contributed by $\ulcorner\mathrm{Kl}`\lambda x(\phi(x))\urcorner$ is the same as that

[3] A summary of the changes to his *notation* can be found in Klement (2003).

[4] At times, Russell would have taken something like "Kl" as a primitive denoting function; at times instead he would have made use of a relation, where "u Kl $\lambda x\phi(x)$" would mean that u is the class determined by "$\lambda x\phi(x)$," and then define my "$\mathrm{Kl}`\lambda x(\phi(x))$" using the descriptive denoting function $\imath$, discussed below, i.e., as something like "$\imath`\lambda u(u \text{ Kl } \lambda x(\phi(x)))$". See, e.g., Russell (1994i, pp. 352–57), Russell (1994h, pp. 384, 397), and also note 5 below.

contributed by $\ulcorner \mathrm{Kl}`\lambda x(\psi(x))\urcorner$ just in case these expressions differ from each other by at most choice of bound variable. Hence, we might regard "$\mathrm{Kl}`\lambda x(x = x)$" and "$\mathrm{Kl}`\lambda y(y = y)$" as contributing the same denoting complex, but a different one from "$\mathrm{Kl}`\lambda x(x \notin \varnothing)$", despite denoting the same class. In addition to Kl, Russell makes use of only one other kind of denoting function, written "$\imath$".[5] Here, a denoting complex $\ulcorner \imath`\lambda x(\phi(x))\urcorner$ would denote the unique $x$ such that $\phi(x)$. As with Kl, the entity contributed by $\ulcorner \imath`\lambda x(\phi(x))\urcorner$ would differ from the entity contributed by $\ulcorner \imath`\lambda x(\psi(x))\urcorner$ unless these differed from each other only by choice of bound variable, even when the object denoted is the same.

In summary, during this period Russell regarded propositions involving class-abstracts as containing denoting complexes derived from propositional functions. He gave much the same analysis to propositions containing descriptions.

4 Paradoxes of denoting complexes and disambiguation

It is natural to worry that nothing is gained by denying a defining concept for every class, if there is still a defining function for every class, and every class is denoted by a denoting complex. Isn't a propositional function, or a denoting complex, a single thing (individual), and wouldn't there then be as many single things (individuals) as classes thereof, in violation of Cantor's theorem? Wouldn't then diagonal paradoxes threaten for them as well?

The worry is real, but there is an additional wrinkle not present in the cases considered earlier. Because denoting entities make propositions in which they occur not about them, but about the entities they denote, in order to formulate a paradox "about" them one needs a mechanism for speaking about denoting complexes themselves as opposed to their denotations. Likewise, it is also difficult to formulate a paradox involving propositional functions. Russell's exact understanding of the nature of propositional functions changed often during this period, but through much of it he regarded them also as a kind of denoting entity. Russell understood variables as denoting their values, and understood a propositional function as a proposition-like complex containing one or more variables—a "dependent variable" denoting those propositions resulting from what its constituent variable denotes. Speaking of a propositional function also requires some means of disambiguating between meaning and denotation. The notation for function abstraction, Russell concluded, does just that:

> The circumflex has the same sort of effect as inverted commas have. E.g., we say
>
> Any man is a biped;
> "Any man" is a denoting concept.

[5] See Russell (1994a, p. 298); Russell (1994i, p. 355). In other manuscripts, Russell writes instead "$(\imath\hat{x})\phi`\hat{x}$", with the circumflex notation indicating that the descriptive functor is applied to a function-abstract similar to the contemporary lambda notation, and similarly "$\hat{x}(\phi`\hat{x})$" for the class-abstract (Russell 1994j, p. 81; Russell 1994g, p. 89; Russell 1994f, p. 105; Russell 1994h, p. 384).

> The difference between $p \supset q \ .\!\supset\!. \ q$ and $\hat{p} \supset \hat{q} \ .\!\supset\!. \ \hat{q}$ corresponds to the difference between any man and "any man". (Russell 1994b, pp. 128–29)

I have argued elsewhere that it is only when Russell began employing function abstraction notation that he was able to formulate and thus recognize the paradox involving propositional functions that do not satisfy themselves.[6] It is, I think, the connection between disambiguating between meaning and denotation, or denoting concepts themselves and what they denote, and the possibility of paradox that led Russell to concentrate on such issues during these years. (We shall return to propositional functions in sec. 6 below.)

In the Gray's Elegy passage of "On Denoting," Russell argues that any attempt to disambiguate between meaning and denotation leads to "inextricable tangles" and must ultimately fail. His reasoning is notoriously obscure. Let us, however, put it aside and suppose that there were some formal means for speaking of a denoting complex itself as opposed to its denotation. In "On Denoting" and in manuscripts, Russell uses double inverted commas. However, I have herein been following the American English convention of using " and " for quotations and mentioned expressions. If we use hard brackets instead, we may contrast:

$$x = \mathrm{Kl}`F \tag{1}$$

$$x = [\mathrm{Kl}`F] \tag{2}$$

In (1), x is identified with the class of all Fs, but in (2), x is identified with a denoting complex which denotes this class.

Assuming then, that denoting complexes can be considered individuals, and there are as many such denoting complexes as there are classes, we can generate a Cantorian diagonal paradox. Firstly, let W be a propositional function satisfied by all and only denoting complexes that denote a class in which they are not included:

$$W =_{\mathrm{df}} \lambda x((\exists F)(x = [\mathrm{Kl}`F] \ \& \sim\! F(x)))$$

This propositional function defines a class of denoting complexes, Kl‘W, which would be denoted by the denoting complex [Kl‘W]. Is *this* denoting complex in that class, i.e., does [Kl‘W] satisfy W? Firstly, suppose that it does:

$$W([\mathrm{Kl}`W]) \tag{3}$$

By the definition of W and conversion, we get:

$$(\exists F)([\mathrm{Kl}`W] = [\mathrm{Kl}`F] \ \& \sim\! F([\mathrm{Kl}`W])) \tag{4}$$

Call the function existentially posited here G. We then have both:

$$[\mathrm{Kl}`W] = [\mathrm{Kl}`G] \tag{5}$$

$$\sim\! G([\mathrm{Kl}`W]) \tag{6}$$

However, (5) seems to imply that $W = G$ if we accept ECI. The reason for this is that $\ulcorner[\mathrm{Kl}`\phi]\urcorner$ would seem to stand for the propositional constituent contributed by

[6] This goes against the widespread view that the propositional functions version of the paradox can be found in *Principles*; I argue otherwise in Klement (2005).

an expression of the form ⌜Kl‘ϕ⌝. If two such expressions contribute the same entity just in case they differ from each other by at most choice of bound variable, then ⌜[Kl‘ϕ]⌝ and ⌜[Kl‘ψ]⌝ will stand for the same denoting complex just in case ϕ and ψ differ from each other by at most that much. In that case, they represent the same function. Hence, Russell is committed to:

$$(\forall F)(\forall G)([\text{Kl`}F] = [\text{Kl`}G] \supset F = G) \tag{7}$$

From (5) and (7) then, we get:

$$W = G \tag{8}$$

From (6) and (8) we get:

$$\sim W([\text{Kl`}W]) \tag{9}$$

This contradicts (3). Because (3) implies a contradiction, it is false, and (9) true. But then, by the definition of W, and ordinary logical manipulations, we have:

$$(\forall F)([\text{Kl`}W] = [\text{Kl`}F] \supset F([\text{Kl`}W])) \tag{10}$$

This yields:

$$[\text{Kl`}W] = [\text{Kl`}W] \supset W([\text{Kl`}W]) \tag{11}$$

As the antecedent here is an evident truth, the consequent follows, which contradicts (9). Hence we have a genuine antinomy.

Some remarks are in order. First, nothing in the paradox requires thinking of a class as one thing rather than many. We never make use of the denotations of denoting complexes of the form [Kl‘F], only the complexes themselves. If such complexes denote “classes as many,” this does nothing to block the contradiction. Second, nothing requires that a class be denoted by only one denoting complex. If each class is denoted by more than one denoting complex, this makes the violation of Cantor’s theorem worse, not better. Third, nothing is solved by denying that every class is presented by a denoting complex. It is natural to think, for example, that infinitely large random collections of entities with no easily specified common characteristic would not be presented by a denoting complex. However, assuming there to be such collections would not solve the contradiction unless one were to insist that the class of Ws is such a collection. However, W itself gives the membership conditions for its class, and hence it is not so easy to deny such a denoting complex as [Kl‘W] which denotes that class.

Moreover, the contradiction goes through just as well with other sorts of denoting complexes. In particular, one can replace “Kl” with “$\imath$” throughout in the deduction above, and still arrive at an antinomy. The resulting contradiction can be put into more ordinary language as follows: some denoting complexes of the form [the x such that Fx] satisfy Fx, whereas others do not. For example, [the self-identical thing] is self-identical, whereas the denoting complex [the cat] is not a cat. Let x satisfy the propositional function W' just in case it is a denoting complex of this form that does not satisfy its defining function. Now consider the denoting complex [the x such that $W'x$]: does it satisfy W'? It does just in case it does not.[7] This shows that it is not enough to deny both the reality of classes and even denoting complexes for classes; any way of generating an *object* that would be distinct for distinct collections of things (or distinct definable propositional functions) will lead to a diagonal paradox.

Admittedly, I know of no text or manuscript in which Russell explicitly considered denoting complex paradoxes. He was certainly aware, however, of the impossibility of generating a distinct individual for every class or defining function, and his explicit consideration of the class-concept paradox in *Principles* lends support for thinking that he would have known of their possibility. The most likely explanation for the lack of explicit formulation is that, as noted earlier, he thought it impossible to disambiguate between propositions about denoting complexes and those about their denotations. The interpretation of his argument for this—the "Gray's Elegy Argument" of "On Denoting"—is very controversial.[8] I cannot hope to develop my own here. Engagement with this issue, however, would have been unavoidable given his need to address the logical paradoxes. Interestingly, Russell might instead have made use of a dilemma in "On Denoting" against the meaning/denotation distinction. Either it is coherent to disambiguate between meaning and denotation, denoting complex and what it denotes, or it is not. If it is possible to disambiguate, then paradoxes of denoting complexes can be formulated, resulting in contradictions. If it is not possible to disambiguate, the entire theory differentiating them is incoherent. Either way, the meaning/denotation distinction must be rejected.

5 Incomplete symbols

In "On Denoting," Russell introduced not only his famous theory of definite descriptions, but a theory of "denoting phrases" or quantified terms generally, including "all humans" and "some cats". In *Principles*, Russell took such phrases to mean denoting concepts derived from *class*-concepts; indeed, the denotation of "all humans" was supposedly a class as many. It is no wonder that 1905 represented a sea change year for Russell, not only in his understanding of descriptions, but also in his understanding of class talk. His first public endorsement of a "no-classes" theory came in a paper delivered in November 1905 (Russell 1973c), where it was listed as one of three possible broad tacks for dealing with the logical paradoxes. The others were "the zigzag theory" and "the theory of limitation of size", though it is clear from a note added just before publication that the "no-classes" theory was Russell's favored solution. All three approaches begin with the recognition that one cannot countenance a class, understood as an individual, for every open sentence ("property" or "propositional function") definable as satisfied or not by individuals. On the "zigzag theory", whether or not a class is defined depends on the nature of

[7] For more on this paradox, see Klement (2008).

[8] For works discussing and debating its interpretation, see e.g., Kremer (1994), Demopoulos (1999), Makin (2000), Levine (2004), Salmon (2005), Simons (2005), and Brogaard (2006), among others; for a broader discussion of Russell's engagement with the topic of disambiguation, see Klement (2002b).

the defining property, whether or not it has a certain “zigzagginess”. Although Russell explored approaches of this stripe in some of his manuscripts, he never delineated formally the precise conditions under which a property defined a class in a suitable way. On the “limitation of size” approach, whether or not a property defines a class depends on how many individuals have the property, somewhat reminiscent of certain contemporary set theories that postulate a set (rather than a “proper class”) only when its membership can be built up iteratively. Russell was never very attracted to this approach, perhaps in part because of his original understanding of classes as involved in all categorical judgments. The forms of our judgments do not seem different depending on how many things are under discussion. Consider the forms of “2 is one among natural numbers” and “2 is one among self-identical things”. It seems odd to think the former can be analyzed as being about a class but not the latter because “too many” things are self-identical. Russell seems to have liked that the “no-classes” theory treated all kinds of apparent talk about classes uniformly, without making *ad hoc* distinctions between kinds. The other approaches had a certain “artificiality” “owing to absence of any broad principle” deciding which properties define classes and which do not.

Russell’s preference for the no-classes theory is even easier to appreciate given his general desire to give similar solutions to the various Cantorian paradoxes. What would a “zigzag” or “limitation of size” approach look like as an attempted solution to the paradoxes of denoting complexes considered in the previous section? Could one hold that if *F* is “zigzaggy”, then there is not even a denoting concept [Kl‘*F*]? This would seem to make it meaningless to even speak of the class of *F*s, whereas it is just under these conditions that it is tempting to think we would have a denoting complex that fails to denote. If there is such a denoting complex, however, nothing has been gained by denying the class. Similarly, it does not seem plausible to suppose whether or not a denoting complex [Kl‘*F*] or [℩‘*F*] exists depends on how many *F*s there are. Certainly, if there are suitably many, or even *more than one F*, [℩‘*F*] will fail to denote, but it would appear that the complex itself should remain either way. Surely one cannot settle the debate between Russell and the idealist monists of his day as to whether there is really only one ultimate truth, or infinitely many, by considering the make-up or constituents of the proposition “the truth is true”, so that “the truth” means a denoting complex just in case there really is just one truth. Of Russell’s three strategies, only a “no-denoting-complex” approach is even *prima facie* plausible for these paradoxes. While there may be some derivative sense in which one can speak of different meanings, this kind of discourse is not to be taken at face-value: there are no single things, no individuals, which are denoting complexes. As Russell put the point later on, “I believe that the duality of meaning and denotation, though capable of a true interpretation, is misleading if taken as fundamental” (Russell 1992b, p. 157).

Although this is not the only nor most explicit reason he gives for accepting the post-1905 theory of descriptions, it is clear that it in effect provides just such a “no-denoting-complex” solution to the kinds of paradox considered in the previous section. According to it, a description is an “incomplete symbol”. An incomplete symbol, generally, is an expression which is, or by a notational trick, appears to be, a syntactically unified component of a sentence but there is no one single entity which it contributes to the proposition (early on) or fact (later on) the sentence expresses. The notion of incomplete symbols may appear to fly in the face of ECI, but the notion can be made compatible with ECI if it is possible to eliminate incomplete symbols by means of contextual definitions in favor of a more primitive notation of which ECI still holds. This is just what the contextual definition of descriptive phrases in *Principia Mathematica* (*14) provides:

$$\alpha((\imath x)\beta x) =_{\mathrm{df}} (\exists x)((y)(\beta y \equiv y = x) \;\&\; \alpha(x))$$

Consequently, "$(\imath x)\beta x$" is not officially a component expression of sentences in which it appears, and hence, regarding it as not contributing a single entity to the proposition expressed is not incompatible with ECI. If we use the description "$(\imath x)(x$ is even & x is prime)", there is no *one* entity, no one thing, it contributes to the proposition, neither the number two nor some denoting complex. There are of course entiti*es*, plural, it contributes, such as whatever is involved in being even or prime, and (the ontological correlates of) various additional logical primitives, but there is no one thing, the *meaning*, of which we can even meaningfully ask whether or not *it* is an even prime. This dissolves the denoting complex paradox involving "$\imath$".

The form involving "Kl" is similarly dispatched by Russell's mature analysis of class-abstracts as "incomplete symbols" in, e.g., *Principia Mathematica*'s *20:

$$\alpha(\hat{x}\beta x) =_{\mathrm{df}} (\exists \phi)((x)(\beta x \equiv \phi ! x) \;\&\; \alpha(\phi \hat{x}))$$

This gives a contextual definition of those (apparent) terms Russell previously might have written as "Kl'F", now written as "$\hat{x}Fx$" instead. Because these terms are defined away, again we do not have a direct violation of ECI. The similarity between this contextual definition and the one given above for definite descriptions of course has not gone unnoticed, largely because Russell himself draws attention to it. There are dissimilarities, too, though I think sometimes the dissimilarities are exaggerated. For example, it is sometimes pointed out (e.g., in Kremer 2008), correctly, that this latter definition is ontologically eliminative in a way that the former contextual definition is not. Russell's analysis of "the author of Waverly is a poet" quantifies over individuals, and for it to be true, in the range of the individual quantifier there must be someone who wrote Waverly. This is not an "elimination" of authors. Russell's analysis of "the universal class is non-empty," on the other hand, involves higher-order quantification, and not quantification over (or at least not directly over) classes, and hence makes room for the supposition that there may be no classes at all. This difference, however, concerns the level of denotation. At the level of *meaning*, both contextual definitions are equally eliminative: the first allows us to deny the reality of entities of the form [$\imath$'F]; the second those of the form [Kl'F].

6 Propositional functions and the substitutional theory

A natural worry here, however, is that Russell's "no-classes" theory simply replaces classes or class-concepts with propositional functions. Whether classes or collections are genuine entities or mere *façons de parler*, Cantor's reasoning that there must be in some sense more collections than things still holds. Getting rid of classes or their denoting concepts or complexes as single things is not an improvement if we have simply replaced them with propositional functions, regard propositional functions as "single things", and posit as many such things as we previously would have posited classes. This is certainly an issue of which Russell was aware, but once again, the attitude he took changed many times during these years, and here a crude summary must suffice.

As we saw in sec. 4 above, early on, Russell took a propositional function itself to be a kind of denoting entity, and thus, referring to a propositional function itself as opposed to one of its values required a device for speaking of a meaning as opposed to its denotation. The circumflex notation was taken to be such a device. Despite his misgivings about such disambiguating devices, it clearly would not have been adequate for Russell simply to conclude that this meant that a paradox involving a function satisfied by all and only those $\phi\hat{x}$'s such that $\sim\phi(\phi\hat{x})$ was itself therefore unintelligible. The circumflex notation is used in the contextual definition of class-abstracts, and hence some interpretation of it must be possible.

In Russell's early metaphysics, every singular entity was taken to be of the same logical type, which he variously called the type of "terms", "entities" or "individuals" (Russell 1931, §47). During the period in which he regarded propositional functions realistically, he claimed that a function "like everything else, is an entity" (Russell 1994f, p. 100), by which he meant that one could appear in a proposition as a logical subject or "term", i.e., as something it was about. Russell did not at this time distinguish different kinds of terms or logical subjects. If something was meaningful for one entity, it was meaningful for any other. After 1905, Russell's inclination therefore was not to regard a propositional function as a different *kind* of entity, but rather as a non-entity, a way of speaking, much like a class or a denoting complex. In an early draft of *Principia Mathematica*, he wrote, "A function must be an incomplete symbol. This seems to follow from the fact that $\phi!(\phi!\hat{x})$ is nonsense" (Russell 1907b). If a symbol of the form "$\phi!\hat{x}$" referred to some single thing, one could not avoid the question as to whether or not it satisfied itself. Hence, Russell was tempted to believe that such *apparent* terms must be meaningful in a different way; they must be "incomplete symbols". The difficulty of course, is that Russell made use of quantifiers over "propositional functions" as well as the circumflex notation in his no-classes theory.

From late 1905 through much of 1906 or 1907, Russell had hoped to avoid commitment not only to classes but also to "propositional functions" by means of what has come to be called his "substitutional theory" (see especially Russell 1973d, Russell 1973b; for discussion see Landini 1998). On this approach, rather than speaking of a propositional function, e.g., "$\hat{x}$ is human," as a single entity, Russell hoped to make use of a pair of entities, a proposition, e.g., *Socrates is human*, and an entity, e.g., Socrates, to be substituted-for in that proposition. The approach focused around a four-place relation, written:

$$p\frac{x}{a}!q$$

This means that q results by substituting x for a wherever it occurs as term or logical subject in p. The notion of different values of a "function," $\phi\hat{x}$, was replaced by speaking of the different results of substituting things for a in proposition p. This allowed Russell to avoid commitment to propositional functions as genuine entities by claiming that discourse apparently about such entities is eliminable in favor of "substitutional matrices", p/a. Since a matrix is not one entity, but two, it is nonsense to speak of a matrix taking "itself" as argument, writing, e.g., $p\frac{p/a}{a}!q$, as this would involve treating a four-place relation as if it were a five-place relation, generating nonsense. The theory thus emulates a theory of types without postulating types of entities. Indeed, this was Russell's official explanation of why type-distinctions are necessary when speaking of "propositional functions" as late as his well-known "Mathematical Logic as Based on the Theory of Types" (Russell 1956b, p. 77), written in 1907. Indeed, what most pleased Russell about this theory is it allowed him to maintain that "there is really nothing that is not an individual" (Russell 1973b, p. 206) and "while allowing that there are many entities, it adheres with drastic pedantry to the old maxim that, 'whatever is, is one' " (Russell 1973d, p. 189).

This method of treating propositional function expressions as "incomplete symbols" or treating propositional functions as mere *façons de parler* had a cost, however, which was that it required thinking of propositions themselves as genuine entities, and values of the individual variable, and there were problems in this view he did not know how to solve.

7 Paradoxes of propositions

The problems plaguing his theory of propositions were in many ways simple corollaries of the Cantorian worries he had about classes, or their representatives, generally. If there is *any* way of generating a distinct *genuine* entity for each class or collection of things then Cantor's theorem will be violated, and it will be possible to derive a diagonal contradiction. Getting rid even of denoting complexes as single entities, and instead taking class-abstracts to contribute in a more complicated way to the propositions they express does not help if the propositions themselves are single entities, and a different one can be generated for each different "class" thereof.

Russell had already been aware of problems of this sort while composing the *Principles*, and in Appendix B (§500) he discussed just such a diagonal paradox. If propositions are entities, then they too can be members of classes. By Cantor's theorem, there ought to be more classes of propositions than propositions. However, it seems possible, for each class of propositions, to generate a distinct proposition, such as the proposition that every proposition in that class is true. Some propositions that assert that every member of a class of propositions is true are in the class they are about; some are not. For example, the proposition *every member of the class of true propositions is true* is a true proposition, whereas the proposition *every member of the class of subject-predicate propositions is true* is not itself a subject-predicate proposition. Call the class of propositions of this form that are not in the class they are about, w. Call the proposition *every member of w is true*, r. Is r in w? It is if and only if it is not. Contradiction.

Russell formulated this contradiction using a variant of Peano's logical notation in a letter to Frege in 1902 (Frege 1980, p. 147; for discussion, see Klement 2001). Here, it is better to modernize the notation somewhat. For a formula A, I shall write $\ulcorner[A]\urcorner$ as a term for the proposition A expresses; this is meant to be consistent with our earlier use of "[...]", as of course, when A is a subformula of a larger formula, the proposition $[A]$ is what it contributes to the proposition expressed by the larger formula. We then define:

$$w =_{\mathrm{df}} \{q|(\exists m)(q = [(p)(p \in m \supset p)] \;\&\; \sim(q \in m))\}$$
$$r =_{\mathrm{df}} [(p)(p \in w \supset p)]$$

(In some places, Russell abbreviates "$(p)(p \in m \supset p)$" as "$\wedge$'m".) Russell then believes we can derive the contradiction:

$$r \in w \equiv \sim(r \in w) \tag{12}$$

Deriving this result requires that we assume:

$$[(p)(p \in w \supset p)] = [(p)(p \in m \supset p)] \supset w = m \tag{13}$$

At least while classes are treated as genuine entities, this seems well-motivated by LPI/ECI. Identical propositions must have identical constituents.

The formulation of the contradiction is complicated somewhat, however, when classes are eliminated as genuine entities in favor of their defining properties or propositional functions. Indeed, Russell had hoped this might provide a solution when he first began experimenting with the idea that "classes are entirely superfluous," as he put it in a follow-up letter to Frege of May 1903 (Frege 1980, pp. 158–60). Then, we might attempt to redefine w and r as follows:

$$w' =_{\mathrm{df}} \lambda q((\exists F)(q = [(p)(F(p) \supset p)] \;\&\; \sim F(q)))$$
$$r' =_{\mathrm{df}} [(p)(w'(p) \supset p)]$$

The contradiction analogous to (12) would be written:

$$w'(r') \equiv \sim w'(r') \tag{14}$$

Deriving this result requires assuming the analogue of (13), i.e.:

$$[(p)(w'(p) \supset p)] = [(p)(F(p) \supset p)] \supset w' = F \tag{15}$$

In his letter to Frege, Russell expressed hope that (15) could be denied, and cited a result from Frege's appendix to the second volume of *Grundgesetze* (Frege 2013, appendix) to the effect that, for every function from functions of objects to objects, there must be functions F and G for which F and G yield the same object as value when taken as argument to the higher-level function, but this value itself falls under F but not G (and hence F and G are not even coextensive much less identical). If we were to consider, e.g., $\lambda F([(p)F(p) \supset p)]$ as a function from functions to objects (propositions), Frege's result entails that there is an F and a

G (which might as well be w') such that $[(p)(F(p) \supset p)]$ and $[(p)(G(p) \supset p)]$ are the same, but F and G are not. Frege's reasoning was really just a *reductio* making use of the Cantorian diagonal contradiction that results otherwise. Russell seems to be pinning his hope then just on an argument to the effect that (15) *must* be false, because otherwise (14) would result. That is not a much of a "diagnosis" of the contradiction, nor does it explain how denying (15) is compatible with Russell's overall understanding of propositions and their constituents.

Is denying (15) consistent with, for example, LPI and ECI? This is not altogether clear. It would seem that $\ulcorner(p)(\phi(p) \supset p)\urcorner$ and $\ulcorner(p)(\psi(p) \supset p)\urcorner$ would differ by at most choice of bound variable just in case $\ulcorner\phi(p)\urcorner$ and $\ulcorner\psi(p)\urcorner$ differ by at most that much. Whether this is possible without $\phi(\ldots)$ and $\psi(\ldots)$ representing the same function depends exactly how function expressions are thought of as "entering in" to complex expressions. If function-abstracts are used, one may write three equivalent formulæ, one as "$\lambda q(q = q)(p)$," one as "$\lambda q(q = p)(p)$" and one as "$\lambda q(p = q)(p)$", all of which in effect (after conversion) assert $p = p$. Taking this notation at face-value, ECI would seem to require that these assert different propositions as well, as "$\lambda q(q = q)$" would seem to be a syntactically independent component expression of the first, and thus represent an object in the proposition expressed not represented by anything in the others. Recall that when Russell first began using function abstraction, he thought of the notation as representing a certain kind of meaning. While Russell still had a meaning/denotation distinction, identity conditions of propositions were sensitive to differences in meaning even when the denotation is the same. It is possible that one of Russell's reasons for abandoning his earlier, more-lambda-calculus-like abstraction notation $\ulcorner\dot{x}(\phi(x))\urcorner$ in favor of the simpler $\ulcorner\phi(\hat{x})\urcorner$ is that, with the latter notation, he never wrote an abstract along with its argument. Instead of "$(\hat{q} = \hat{q})(p)$", he would always simply write "$p = p$". This makes ECI more compatible with a coarser-grained understanding of propositions that are values of functions, and perhaps opens up room for thinking that $\ulcorner\phi(p)\urcorner$ and $\ulcorner\psi(p)\urcorner$ could represent the same proposition even when $\phi \neq \psi$, e.g., if ϕ is $\hat{q} = \hat{q}$ and ψ is $\hat{q} = p$. This *might* make it seem possible to deny (15) while not wholly abandoning ECI.

However, it seems that this general tack for attempting to solve the paradoxes of propositions is doomed to fail *in general* even if it does help with this particular formulation, as I think Russell himself came to realize. If instead of correlating each propositional function F with the proposition $[(p)(F(p) \supset p)]$ we simply correlate it with the proposition $[(x)F(x)]$, then no such response is possible. It is demonstrable that even if we allow for cases such as those considered above, $\ulcorner(x)\phi(x)\urcorner$ will differ at most by choice of bound variable from $\ulcorner(x)\psi(x)\urcorner$ only if "ϕ" and "ψ" differ from each other by at most this much, and hence, contribute the same component or components to the propositions expressed, and *a fortiori*, are coextensive. Indeed, in the lambda calculus, it is possible to use a primitive higher-type functor "Π" applied to a lambda abstract as a kind of quantifier; for example, Church (1940, p. 58) defines $\ulcorner(x)\phi(x)\urcorner$ as $\ulcorner\Pi(\lambda x(\phi(x)))\urcorner$. With the sign "$\Pi$", one can formulate a Cantorian paradox for propositions of the form $[\Pi(F)]$ using precisely the same deduction given for denoting complexes of the form [Kl'F] in sec. 4, simply by replacing "Kl" with "Π" throughout.

Some reflection shows that a Cantorian diagonal paradox of this sort will be forthcoming for any notation $\ulcorner C(F) \urcorner$, considered as a "complete expression" rather than an "incomplete symbol", which can contain an open sentence or propositional function-abstract, and which will be different for different (non-coextensive) function expressions/abstracts. Any such notation will, accepting ECI, require us to posit entities in such a way as to violate Cantor's theorem. This threatens the understanding of propositions as structured entities isomorphic with their linguistic representations. Moreover, so long as the entities postulated by expressions of the form $\ulcorner C(F) \urcorner$ are all taken as having the same logical type, even simple type-theory does not block the contradictions. The replacement for a logic of "propositional functions" in Russell's original substitutional theory only emulates a simple theory of types, and hence, a version of the Cantorian paradox of propositions from Appendix B of *Principles* could be formulated therein (for details, see Landini 1998, pp. 201–03).

Russell had discovered that such problems plagued the substitutional theory by mid-1906. In "On 'Insolubilia' and Their Solution by Symbolic Logic" (Russell 1973b), his reaction was to deny the reality of quantified propositions, and accept only quantified "statements". Every formula should be written in prenex normal form with all quantifiers "pulled out" to the beginning. Quantifiers would then not be allowed subordinate to truth-functional connectives or in a "term" for a proposition "[A]". Technically, this is compatible with ECI, because no closed subexpression contains a quantifier, and thus Russell need not postulate ontological correlates of such subexpressions. It poses an obstacle to LPI, however, as a complete quantified formula is not thought to express a proposition *at all*. Also, recall that on the substitutional theory's method of proxying discourse about propositional functions, a function is replaced by a "matrix" consisting of a proposition and a replaceable entity. Without quantified propositions, one cannot proxy a propositional function defined using quantification, which would be necessary for something like w' above. This provided a "solution" of sorts to the propositional paradox, but weakened the substitutional theory's higher-order logic to make it an inadequate basis for mathematics. Russell hoped to get around this with a "mitigating axiom" postulating (non-quantified) propositions equivalent to quantified formulæ even through substitutions (Russell 1973b, p. 201), but later discovered that this axiom led to a more complicated form of propositional paradox.[9]

Another strategy, taken up in "Mathematical Logic as Based on the Theory of Types," involved adopting a ramified hierarchy of different "orders" of propositions, so that a proposition quantifying over propositions of order n would be an order higher than n. If one were to replace function quantification with quantification over propositions per the substitutional theory, the definition of w' would involve not a quantifier $(\exists F)$ but a pair $(\exists p)(\exists a)$. However precisely we define r', it would be in terms of w', and thus use such a propositional quantifier. It would

[9] This more complicated form is described by Russell in a 1907 letter to Hawtrey, reprinted in, and discussed by, both Landini (1998, pp. ii, 231–33) and Linsky (2002b).

therefore be a higher-order than the potential arguments to w', and the question as to whether it satisfies w' would not arise. However, Russell still hoped to maintain the view of *Principles* that every thing is one thing, everything is an individual. When he first considered the notion of a hierarchy of propositions in Appendix B of *Principles*, he described the notion as "harsh and highly artificial" (§500). In his 1906 manuscripts, he called it "intolerable" (Russell 1906b). It is around this time that Russell began to doubt the reality of objective propositions, and although he did not make up his mind right away (—his indecision is apparent in, e.g., Russell 1907a—), he seems to have wholly abandoned propositions by the time of the 1910 introduction to *Principia Mathematica*.

Russell's motivations for abandoning his early view of propositions were no doubt complex,[10] but the proximate cause for his reconsideration was the difficulty posed by the propositional paradoxes. His explicit concern was that if propositions are taken realistically and not divided into a hierarchy, "we can establish a $1 \rightarrow 1$ [relation] of all classes of propositions to some propositions" (Russell 1906a), and "we can't escape such propositions as $\wedge$'$u \in u$" (Russell 1906b); clearly, the Cantorian paradoxes remained at the forefront of his mind.[11]

8 Propositions and propositional functions as incomplete symbols

In 1910, Russell adopted what is now widely known as the "Multiple Relations Theory of Judgment" (Whitehead and Russell 1925–1927, pp. 43–45; Russell 1992c), along with a correspondence theory of truth. On it, when I judge that *aRb*, rather than relating to the proposition [*aRb*], I am instead related separately to *a*, *R* and *b*. My judgment is true if there is a "complex," in which *a* is in fact related to *b* by *R*. Of this view, Russell writes:

> … a "proposition," in the sense in which a proposition is supposed to be *the* object of a judgment, is a false abstraction, because a judgment has several objects, not one …
> Owing to the plurality of objects in a single judgment , it follows that what we call a "proposition" … is not a single entity at all. That is to say, the phrase which expresses a proposition is what we call an "incomplete" symbol; it does not have a meaning in itself, but requires some supplementation in order to acquire a complete meaning. (Whitehead and Russell 1925–1927, p. 44)

Russell sought to extend his previous successes with treating symbols apparently standing for entities, which, if taken as complete symbols, lead to paradoxes, as incomplete symbols instead, to cover the propositional paradoxes as well.

There is, however, a difference. Consider our earlier definition of an "incomplete symbol" as "an expression which is, or by a notational trick, appears to be, a

[10] For works exploring *other* reasons for his change of mind, see, e.g., Linsky (1993) and Proops (2011).

[11] Recall that "$\wedge$'u" is his abbreviation for "$(p)(p \in u \supset p)$" (Russell 1931, §500). Such manuscripts show that Russell clearly has not forgotten about the Cantorian paradox of propositions from Appendix B of *Principles*, as alleged by Grattan-Guinness (2000, pp. 328, 364).

syntactically unified component of a sentence but there is no one single entity which it contributes to the proposition (early on) or fact (later on) the sentence expresses". We can divide incomplete symbols into two categories: *superficial* and *deep*. A superficial incomplete symbol is one which only *appears* to be a syntactically unified component expression due to a notational trick. The contextual definitions for descriptions and class-abstracts in *PM* make them appear to be syntactically unified components, but when the definitions are unpacked, they are revealed not to be. Deep incomplete symbols, however, would be syntactic units even in a fully articulated symbolism but nonetheless do not have single entities as their meanings. Because complete formulæ, which *appear* to stand for propositions, are obviously not always parts of larger expressions wherein they might be contextually defined away, they are not "incomplete symbols" in the syntactically eliminative way that descriptions and class-abstracts are. They are "deep" incomplete symbols. In a sense, however, Russell regards them as "parts" of a larger kind of symbolic act where the "supplementation" they need is provided by the context of assertion:

> This fact [that propositional formulæ are incomplete symbols] is somewhat concealed by the circumstance that judgment in itself supplies a sufficient supplement, and that judgment in itself makes no *verbal* addition to the proposition. Thus "the proposition 'Socrates is human' " uses "Socrates is human" in a way which requires a supplement of some kind before it acquires a complete meaning; but when I judge "Socrates is human", the meaning is completed by the act of judging, and we no longer have an incomplete symbol. (Whitehead and Russell 1925–1927, p. 44)

Although the sentence "Socrates is human" does not itself represent a single complex thing, there is a complex fact of my *judging* that Socrates is human which is pointed to not by the sentence alone but by the context of its being uttered by me at a given time. Nonetheless, at a purely syntactic level, it cannot be denied that a closed formula is by itself a syntactic unit, and such a formula may have other closed formulæ as parts which are also syntactic units. Because Russell no longer regards these as representing single entities, he has given up the letter if not the spirit of ECI and LPI.

As noted by Church (1976, p. 748n), with propositions dropped in *PM*, propositional functions become obscure. Certainly they are not mathematics-style functions with propositions as value. The substitutional theory's proxy for propositional functions in terms of substitutional matrices would no longer be available. The interpretation of Russell's account of propositional functions, and higher-order quantification generally, in *Principia Mathematica*, is controversial. I have defended my own views elsewhere (Klement 2004, 2010, 2013). I only note that in *PM*, a propositional function is said to be "not a definite object" (p. 48), and elsewhere "nothing but an expression" (Russell 1958, p. 53) and "merely a schema" (Russell 1956c, p. 234). In *PM*'s introduction (pp. 41–47), Russell describes a hierarchy of senses of truth, where the truth of higher-order formulæ—those involving higher-order quantification—is defined recursively in terms of their substitution instances, so that truth ultimately "bottoms out" in elementary propositions, whose truth is determined per the multiple-relations theory by the correspondence of the judgments they "incompletely" express with objective complexes. Russell later claimed that in "the language of the second order, variables denote symbols, not what is symbolized" (Russell 1940, p. 192). In *PM* itself, Russell describes elementary judgments alone as relating to reality; only they "point to" complexes (pp. 44, 46). While an elementary proposition can be the *value* of a propositional function, the function presupposes its values, not vice versa (p. 39). At this point, Russell's metaphysics contains only simple individuals, their properties and relations, and the complexes (facts) composed thereof (Cf. Whitehead and Russell 1925–1927, p. 43). Russell's engagement with the logical paradoxes influenced significantly not only the genesis of his theory of descriptions and incomplete symbols, but also his logical atomism, a label he began to use the year following the publication of the first volume of *PM* (Russell 1992a, p. 135).

It is sometimes maintained that Russell's claim that a description "the F" is an incomplete symbol which does not make a unified contribution to the propositions in which it appears is an exaggeration or simply untrue. Even while maintaining the core of Russell's analysis, "the F" could be regarded as a restricted quantifier or higher-type function, predicable of a lower-type function just in case it is satisfied by something that is uniquely F, i.e., something like $\lambda G((\exists x)((y)(Fy \equiv y = x) \;\&\; Gx))$.[12] As a criticism of Russell, this falls flat if it is true, as I think it is, that Russell would have regarded a function-abstract as itself an "incomplete symbol", not having even a meaning in isolation. By the time of his 1912 Cambridge lectures, as recorded in Moore's notes, Russell suggested that all the basic concepts of logic and mathematics, "or, not, true, 0, 1, 2, etc." are "incomplete symbols" (Moore forthcoming).

9 Conclusion

Russell once described his philosophical development as a "retreat from Pythagoras" (Russell 1958, chap. 17). Whereas when he first broke from the British Idealist tradition, he countenanced all sorts of abstract objects: numbers, classes, denoting complexes, functions, propositions, etc., eventually he slashed more and more away with Occam's razor. However, his was not a simple-minded attitude of "less is more" or "parsimony is preferable to bloat". Russell was acutely aware of the paradoxical complications that go along with significant ontological commitments to abstracta. He stressed as an advantage to Occam's razor that it "diminishes the risk of error" (Russell 1956c, p. 280)—the fewer posits you make, the less likely you are to run into problems such as the Cantorian paradoxes. As he put it, Occam's razor "swept away from the philosophy of mathematics the useless menagerie of metaphysical monsters with which it used to be infested" (Russell 1986, p. 11). It is not that the mature Russell simply refused to speak of "apparent" things like numbers, classes, propositions, meanings, etc.; the theory of incomplete symbols

[12] For discussion of this and related issues, see Neale (2002) and Linsky (2002a).

explains how discourse apparently about such entities is meaningful without reading our ontology off the surface grammar.

It seems then that the core of Russell's solution to the paradoxes was not the theory of types—which I think, even in 1910, was for Russell only a theory of types of expression, not of types of *entities*—but rather the doctrine of incomplete symbols, the standpoint that words or phrases that apparently stand for such problematic entities as classes or propositions must not be taken at face-value. If this is right, then Russell's solution to the paradoxes sits uncomfortably with any theory of meaning postulating a separate realm of semantic objects, whether they are denoting complexes, Fregean senses, structured propositions, or anything similar. Even contemporary versions of such views are probably not immune to the paradoxes which motivated Russell away from such positions (cf. Klement 2002a, chaps. 6–7, Deutsch 2008). To my knowledge, such worries have no widely accepted solution, and are scarcely discussed. Russell's own engagement with these issues may continue to be instructive for some time to come.

Problems may remain with Russell's position. One issue that deserves more careful scrutiny is Russell's doctrine of *facts* adopted after (the first edition of) *PM*. In *PM* itself, Russell seems only to regard elementary judgments or propositions as directly made true or false by mind and language independent facts (or complexes). In some later works (Russell 1984, pp. 114, 130, Russell 1956c, pp. 236–37), Russell postulated the presence of "general" and "existence" facts corresponding to quantified formulæ.[13] Despite his arguments for them, which we cannot consider here, it is natural to worry that Russell might then be committed to as many facts as collections thereof, contrary to Cantor's theorem. Couldn't one reinterpret "[*A*]" for any formula, quantified or elementary, as a term for the fact corresponding to *A*'s truth, or falsity, whichever the case may be, and formulate a paradox of facts similar to the paradox of propositions considered in sec. 7? By the time Russell acknowledged general facts, Wittgenstein had convinced him that facts cannot be named or treated as logical subjects (Russell 1956c, pp. 187–89). This would apparently rule out using "[*A*]" in this way, or speaking of classes or collections of facts in any straightforward manner. Unfortunately, Russell's own philosophical writings on the nature of facts are then somewhat mysterious, susceptible to something akin to Ramsey's (1990, p. 146) charge against Wittgenstein of trying to "whistle" what cannot be said. I do not know of a fully adequate response on Russell's part to this charge. Perhaps reverting to the position of *PM* according to which there are no general facts would be one option. In any case, there is much more fruitful philosophical discussion to be had about Russell's engagement with the paradoxes and the lessons it might teach us today.

[13] Landini (2011, p. 269) claims that even when Russell has general facts in his metaphysics, they ought not to be understood as truth-makers for general formulæ. This issue does not bear directly on the kind of Cantorian worry described here, however.

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