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Proofs, Necessity and Causality

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Abstract

There is a long tradition in logic, from Aristotle to Gödel, of understanding a proof from the concepts of necessity and causality. Gödel's attempts to define provability in terms of necessity led him to the distinction of formal and absolute (abstract) provability. Turing's definition of mechanical procedure by means of a Turing machine (TM) and Gödel's definition of a formal system as a mechanical procedure for producing provable formulas prompt us to understand formal provability as a mechanical causality. We propose a formalism which makes explicit the mechanical causal nature of a TM's work. We claim that Gödel's axiomatized ontotheology and his ontological proof give a clue for the understanding of the concept of absolute provability and the pattern of the corresponding absolute completeness proof, respectively.

Keywords: proof, necessity, causality, Turing machine, justification logic, Gödel

Introduction

When in 1933 Gödel published his modal translation of intuitionistic logic [15], he joined, under modern presuppositions, the long tradition of the understanding of proof and inference from the concept of necessity. For Aristotle, the "necessary following" of a conclusion from its premises is the essence of a syllogism [1, *An. Pr.* A 24b 18–20]. For Kant, who was highly respected by the founders of modern logic like Frege, Hilbert and Gödel,[1] an inference is the "cognition of the *necessity* of a proposition by means of the subsumption of its condition under a given universal rule" [29, refl. 3201, cf. 3196, 3198] (our emph.). By using $\mathsf{S4}$ propositional modal logic and prefixing each subformula (possibly except conjuncts) with the necessity operator B, Gödel wanted to make explicit the provability-related meaning of sentences in accordance with the informal Brouwer-Heyting-Kolmogorov semantics for intuitionistic logic. As is well known, one result was that B was too general, because, as shown in [15], $B\neg B\, 0=1$ turned out to be provable, thus violating Gödel's second incompleteness theorem. Gödel obtained a similar result again several years later,

[1] See, for example, [11, §89], [25, p. 376] and [20, pp. 384–387].

when, in his lecture at Zilsel's (1938) [17], he changed the non-constructive Bp ('there is a proof of p') for the constructive zBp, q and zBp ('z proves q from p', 'z proves p'): according to Gödel, $aB\,\forall u\,\neg uB\,0 = 1$ follows from any aBq (cf. a proof, for instance, in [30]). Obviously, these notions of provability (B) do not refer to formalized provability, but relate to a more general notion – "absolute" (or "abstract") provability ("provable in the absolute sense" [17, p. 101], "absolute notion" of "demonstrability" [18, 23]), i.e., provability independent of any given formalism [40, pp. 187–188]. Gödel remarks that the idea of "absolute proof" is not consistent with Brouwerian intuitionism because of Brouwer's exclusion of the reference to "all" proofs [40, 6.1.13, 6.1.15 on p. 188].[2]

Two problems remained open in Gödel's [15] and [17]: (1) the problem of the concept of formal provability, and (2) the problem of the concept of a provable evidence independent of a given formalized system ("absolute provability").

(1) The first problem was resolved on the basis of Turing's "absolute" concept of mechanical procedure by means of a Turing machine [39] (see [18]). On this ground, Gödel defined a "formal system" ("formalism") as a "mechanical procedure for producing formulas, called provable formulas", which includes a mechanical procedure for the application of each rule of inference [16, 'Postscriptum' 1964 pp. 369–370, p. 346].[3] Accordingly, formal provability is not simply $\mathsf{S4}$ necessity B, but is constructively defined as equivalent with an ideal mechanical device – a Turing machine for writing down axioms and their consequences [19, p. 308], possessing its own mechanical necessity of work.[4] Furthermore, along the lines of Wittgenstein's reflections on a machine (or a "picture of it") in general, a Turing machine can be conceived as a universally understood "symbol" which "in itself" contains and shows (without the help of any formalism) the way of its work (the possibilities of its motions) [41, pp. 78–79].[5]

[2] But see, for instance, [17, pp. 97–101] or [40, 8.6.27 p. 280] on Heyting's presupposing of a general concept of proof.

[3] Cf. [19, p. 308]. For Turing's idea of a machine dealing with axiomatic systems, see [39, pp. 118, 135, 138].

[4] We remark that, for Gödel, a formal system is a deductive calculus since he defines "provable formula" in a formal system as the last formula in a sequence of axioms and immediate consequences (by rules of inference) [16, p. 346]. Manzano and Alonso remark that a general computational account of logic might be too wide. If the completeness of logic is defined by the existence of "an algorithm which recursively enumerates the truths (validities) of that logic", then "[i]n principle, it would not be necessary to have a deductive calculus for the logic; any recursive procedure able to generate logical truths will do", contrary to the usual view on a logic as additionally comprising the deductive calculus for its provability relation [33, p. 51]. Cf. Gödel's disjunction: "…it [= computability] is merely a special kind of demonstrability or decidability" [18, p. 150].

[5] Drawings of machines contained in works by Leonardo da Vinci (1881–1991), Faust Vrančić (1616/17) and Georg Andreas Böckler (1661), possessed by Wittgenstein, might have prompted him to come to his abstract, "symbolic" notion of

(2) The problem of "absolute" provability is open. The concept of "absolute proof", if self-applied, leads to unsolved "intensional paradoxes", independent of the language used and its semantics (cf. "I am not provable", "not applying to itself", "not meaningfully applicable to itself" [40, pp. 187–188, 271, 279–280]). In addition, as Gödel remarked in [18], "absolute" provability is non-constructive because any attempt to formalize it leads to an extension of a given formalism by new axioms (in distinction to a specific formal provability concept as defined in [16, p. 346]). In particular, the notion of absolute provability does not satisfy Gödel's constructivity requirements from [17], especially because it is neither defined in a formal definition nor is its behaviour derived by formal rules of inference, and, in addition, because "all" proofs are not "surveyable" (enumerable) [17, pp. 103, 91]. However, it is not excluded that a "system" of absolute provability is complete in the sense of decidability, say, of set theory from the present axioms extended by some stronger new axiom of infinity [18, p. 151]. This is so because the general notion of provability should lead to a description of the way to generate higher and higher formal systems by supplying new axioms that have some general characterization, e.g., for set theory: how to extend the theory by generating new axioms.

First, we will focus on the reduction of the formal provability concept to Turing machines (TMs) and represent the mechanical causality of a TM by a formalism implemented on Gödel's justification logic [17]. Thereafter, we will briefly comment on the problem of absolute provability within a wider context of Gödel's formal modal (implicitly causal) ontology.

Mechanical nature of formal provability

In accordance with Gödel, the essence of formal provability is *mechanical* necessity, which is reducible to mechanical *causality* as a specific kind of necessity and can be adequately represented by a Turing machine.[6] Like necessity in general, the mechanical nature, too, of formal reasoning, is clearly recognizable already in Aristotle's definition and treatment of syllogism, according to which the conclusion is computed (*syllogismós*) from the premises (cf., for Kant, the determination of the syllogistic cognition in accordance with a rule).[7]

a machine (see [37] and `http://digitalcollections.mcmaster.ca/russell-lib/media/machinae-novae-fausti-verantii-siceni`).

[6] "More exactly, a formal system is nothing but a many-valued Turing machine which permits a predetermined range of choices at certain steps. The one who works the Turing machine can, by his choice, set a lever at certain stages. This is precisely what one does in proving theorems within a formal system." Of course, "[s]ingle-valued Turing machines yield an exactly equivalent concept of formal system" [40, 6.4.5 p. 204].

[7] Aristotle conceived inference as computation (*syllogismós*) of the relation of two terms by means of their relation to a middle term, where computation happens "automatically": the conclusion follows from "something else", the premises, necessarily and without any external help ("just by the fact that the premises obtain" [1, A1, 24b 18-22]).– For Kant, the major

Gödel's 1938 version of the logic of proofs [17] axiomatically describes the behaviour of proofs by means of "proof terms" (cf. Gödel's a, $f(z,u)$ and z' introduced instead of general B), whose inner functional structure (application, sum, confirmation) reveals the way the evidence of the proven proposition is generated. In this style of logic, the problem of the *formal* presentation of the concept of logical-arithmetical (formal) provability is solved by Artemov's "logic of proofs", LP [2, 3].[8] We claim that, with some adjustments, proof terms of [17] can be reinterpreted as mechanical causes of a Turing machine's work. Accordingly, not only can each Turing machine be equivalently presented by a formal first-order inference (for a standard way, see in [5]), but rather the causes of its work and their structure can be made explicit by causal terms constructed in an analogous, but not identical way as the proof terms of the logic of proofs. An essential reason for the difference in the behaviour between cause terms and proof terms is that TM obviously cannot behave in a non-consistent way (performing contradictory actions at the same time) and this impossibility should somehow ensue from TM's own causal structure (a causal counterpart of the above-mentioned $aB\,\forall u\,\neg uB\,0=1$[9]), despite a formal proof system not being able to generate a proof of its own consistency (see *Introduction* here).

Turing justification logic TJL

We will describe a TM in causal terms by using appropriately modified justification logic so that TM halts iff the associated causal logic inference is valid (as is, analogously, the case with the standard association of classical first-order inference with halting TM machines [5]). We choose a Gödelian justification logic approach in order to obtain a close relationship of causality with the provability and proof terms of the logic of proofs, as well as in order to preserve the closest connection with the concept of necessity in standard modal logics ($\mathbf{CK}, \mathbf{C4}, \mathbf{C5}$ and the rule **ACau** below as causal variants of standard $\mathbf{K}, \mathbf{4}, \mathbf{5}$ and the necessitation rule, respectively).

A TM's mechanical behaviour, often presented by quadruples $\langle q, M_j, E, q'\rangle$, can be expressed in a formal logical system with explicit causal terms: q is a given "internal" state of a TM (Turing's "m-configuration"), M_j the scanned symbol (e.g., '1' or a blank, '0'), E is the newly written symbol M_k or a move to the left or right on the TM's tape, and q' the resulting "internal" state of the TM. To this end, we modify and causally re-interpret first-order justification

premise of a syllogism is a *general rule*: if the condition (e.g., subject term) of the rule is satisfied, then the determination of the cognition by the "assertion" (a predicate) of the rule is necessarily (since "a priori") brought about [*bewirkt*] [28, B 360–361].

[8] [2] and [17] were published simultaneously and independently of one another in 1995. For semantics of LP, see Mkrtychev in [35] and Fitting in [8] and later.

[9] Such a theorem could be easily proved in an appropriately enriched causation (justification) logic [30].

logic FOLP (see [9, 10, 4], for a causal second-order variant, cf. [30]) and obtain the system TJL.

The *vocabulary* of the language $\mathcal{L}\mathsf{TJL}$: individual variables x, y, z, x_1, ... (t informally as time variable), individual constants 0 and a finite number of constants q_i; relation symbols $@^2, M_0^2, M_1^2, Q^2, =, <$; causal constants $c, c_1, c_2, \ldots$; function symbols $s, +, \cdot, !, ?, \mathsf{gen}_x, \mathsf{gen}_y, \ldots$; parentheses. Operator symbols are $\neg$, $\rightarrow$, $\forall$ (other propositional and quantification operators classically defined) and the simbol $:$.

Individual terms (w, w_t a time term) are individual variables, individual constants and terms $s(w)$. *Causal terms* (u, v) are constants q_i, causal constants and causal compound terms $(u + v), (u \cdot v), !u, ?u, \mathsf{gen}_x(u)$.

Definition 1 (Formula) $\phi ::= @(w_t, w) \mid M_0(w_t, w) \mid M_1(w_t, w) \mid Q(w_t, w) \mid w_1 = w_2 \mid w_1 < w_2 \mid u\colon \phi \mid \neg\phi \mid (\phi_1 \rightarrow \phi_2) \mid \forall x\phi$

Informally, $@(w_t, w)$ means 'TM at time t scans square w'; $M_i(w_t, w)$: 'at time t, i is written in square w', where $i \in \{0, 1\}$; $Q(w_t, w)$: 'TM is at time t in state w'; $u\colon \phi$ means 'u causes ϕ'. In addition, s is the successor function and, finally, $+, \cdot, !, ?, \mathsf{gen}$ are sum, application, affirmation, limitation and generalization of causes, respectively. Inversion s^{-1} is defined in the familiar way. We will write $Q(t, q_i)$ instead of $Q(w_t, w)$ if $w = i$. We will use $1, -1, 2$, etc. as abbreviations for $s(0), s^{-1}(0), s(s(0))$, etc.

If $x \in \mathsf{free}(\phi)$ and $x \notin \mathsf{free}(u)$, then x is bound in $u\colon \phi$, where $\mathsf{free}(\phi)$ is the set of free variables in ϕ.

We now define a *system* TJL, comprising logical and arithmetical axioms, causal axioms, rules of inference, as well as special TM axioms (different for different TMs) that make TJL a family of systems.

Logical axioms:

CPC classical propositional tautologies,
$\forall$a $\forall x\phi \rightarrow \phi(w/x)$, w is free for x in ϕ,
$\forall$b $\forall x(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \forall x\psi)$, $x \notin \mathsf{free}(\phi)$,
=1 $w = w$,
Subs $w_1 = w_2 \rightarrow (\phi(w_2/x) \rightarrow \phi(w_1/x))$, w_1 and w_2 are free for x in ϕ.

We add the *arithmetical* axioms for s and $<$ (see [5]): $\forall x\forall y(s(x) = s(y) \rightarrow x = y)$, $\forall x\forall y(s(x) = y \rightarrow x < y)$, $\forall x\forall y\forall z((x < y \wedge y < z) \rightarrow x < z)$, $\forall x\forall y(x < y \rightarrow x \neq y)$.

Causal axioms:

CMon $u\colon \phi \rightarrow (u + v)\colon \phi$, $\quad v\colon \phi \rightarrow (u + v)\colon \phi$,
CK $u\colon (\phi \rightarrow \psi) \rightarrow (v\colon \phi \rightarrow (u \cdot v)\colon \psi)$,
C4 $u\colon \phi \rightarrow\ !u\colon u\colon \phi$,
C5 $\neg u\colon \phi \rightarrow\ ?u\colon \neg u\colon \phi$,
C$\forall$ $u\colon \phi \rightarrow \mathsf{gen}_x(u)\colon \forall x\phi$, $\quad x \notin \mathsf{free}(u)$,
T $\forall t\forall x((@(t, x) \vee M_k(t, x) \vee Q(t, x)) \rightarrow 0 \leq t)$
1@ $\forall t\forall x(@(t, x) \rightarrow \forall y(y \neq x \rightarrow \neg @(t, y))$,

1S $\forall t\forall x(M_i(t,x) \to \neg M_j(t,x))$ $(i \neq j)$.

Inference rules:

MP $\vdash \phi \to \psi \,\&\, \vdash \phi \implies \vdash \psi$,

U $\vdash \phi \implies \vdash \forall x\phi$,

ACau (axiom causation): $\vdash \phi \implies \vdash c\colon \phi$, where $\langle c, \phi\rangle \in \mathcal{CS}$ and
$\mathcal{CS} \subseteq \mathit{Causal\,Constants} \times \mathit{Axioms}$ (surjective),

e.g., informally: $\langle a, \forall x\phi \to \phi\rangle$, $\langle b, \bigwedge_{1\leq i\leq n} \phi_i \to \phi_{1\leq k\leq n}\rangle$ and $\langle c, (\phi_1 \to (\phi_2 \to \ldots(\phi_n \to (\phi_1 \wedge \ldots \wedge \phi_n)))))\rangle$.

We will use e for a (possibly complex) cause of a behaviour according to arithmetic laws. Monotonicity (**CMon**) shows that a cause u is conceived as being sufficient, i.e., no new, adjoining factors can prevent its effect in the presence of u (e.g., we are considering only a given TM, ideally, without any external disturbing factors). According to the axiom **CK**, a causal nexus (u) as applied to the distal cause (v) gives a compound proximal cause ($u \cdot v$). Axiom **C4** expresses that cause u has $!u$ as a cause that affirms and activates u. Similarly, **C5** states that cause u has its causal limitation $?u$. **C**$\forall$ introduces cause $\mathsf{gen}_x(u)$ of a family of effects instantiating the same property ϕ.

Each TM has its initial configuration and instructions (quadruples) for its work (change of a given configuration), which we causally describe by special axioms of TJL. Prefix '$q_i\colon$ ' indicates the "internal state" of a TM, which causes the TM to behave in a specific way in dependence of an outer configuration of the machine (location of the head on the tape, scanned symbol) at a moment t.[10] For simplicity, we consider TMs with only one argument.

Special TM axioms.

IC $q_0\colon (@(0,0) \wedge Q(0,q_n) \wedge M_1(0,0) \wedge \ldots \wedge M_1(0,w_m) \wedge \forall x(x \neq w_{0\leq k\leq m} \to M_0(0,x)))$, or $q_0\colon (@(0,0) \wedge Q(0,q_n) \wedge \forall x M_0(0,x))$

IC′ $@(0,0)\wedge Q(0,q_n)\wedge M_1(0,0)\wedge\ldots\wedge M_1(0,w_m)\wedge\forall x(x \neq w_{0\leq k\leq m} \to M_0(0,x))$ or $@(0,0) \wedge Q(0,q_n) \wedge \forall x M_0(0,x)$

M1 $q_m\colon \forall t\forall x((@(t,x) \wedge M_0(t,x)) \to (@(s(t),x) \wedge M_1(s(t),x) \wedge Q(s(t),q_n) \wedge \forall y((y \neq x \wedge M_k(t,y)) \to M_k(s(t),y))))$

M0 $q_m\colon \forall t\forall x((@(t,x) \wedge M_1(t,x)) \to (@(s(t),x) \wedge M_0(s(t),x) \wedge Q(s(t),q_n) \wedge \forall y((y \neq x \wedge M_k(t,y)) \to M_k(s(t),y))))$

L $q_m\colon \forall t\forall x((@(t,x) \wedge M_j(t,x)) \to (@(s(t),s^{-1}(x)) \wedge Q(s(t),q_n) \wedge \forall y(M_k(t,y) \to M_k(s(t),y))))$

R $q_m\colon \forall t\forall x((@(t,x) \wedge M_j(t,x)) \to (@(s(t),s(x)) \wedge Q(s(t),q_n) \wedge \forall y(M_k(t,y) \to M_k(s(t),y))))$

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[10] In traditional, Aristotelian terms, state u might be understood as an "efficient" cause (in the presence of symbol k, u determines TM to do ϕ), symbol k as the formal cause (inscripted shape, "species"), and the tape as the "material" cause. There is no final cause except in the sense of a possible result of the TM's work, which may be thought of as a goal corresponding to the TM's designer's intention and as embodied by her/him in the design of the TM.

CTE $q_m\colon \forall t\forall x(\phi \to \psi) \to \forall t\forall x((\phi \wedge Q(t, q_m)) \to \psi)$, where ϕ and ψ have the form of the corresponding main antecedent and consequent, respectively, in the axioms **M1, M0, L** or **R**.

If a TM halts, we assume that a *conclusion* which is a disjunction of the sentences of the form $f(q_1, \ldots, q_n)\colon \exists t\exists x(@(t,x) \wedge M_k(t,x) \wedge Q(t,q_n))$, for some f and $q_1, \ldots, q_n$ occurring in specific TM axioms, is provable from the axioms. A sentence $\exists t\exists x(@(t,x) \wedge M_k(t,x) \wedge Q(t,q_n))$, without the presence of a special TM axiom of the form $q_n\colon \forall t\forall x((@(t,x) \wedge M_k(t,x)) \to \psi$ (ψ as in **CTE**), is an instruction to *halt*.

We call a special set of TM axioms together with the conclusion a *TM descriptive inference*.

Example 2 (A simple TM) *Let us take a very simple example of a TM that writes down the string '11' on the squares* 0 *and* -1 *on the TM's initially blank tape [5, pp. 26–27, shortened].*

$$q_1 M_0 M_1 q_1,\ q_1 M_1 L q_2,\ q_2 M_0 M_1 q_2.$$

The work of this TM is described by means of an inference where the premises (axioms) 1–4 describe the initial configuration of the TM and the instructions for its work, while the conclusion contains the instruction to halt in state q_2 *with* M_1 *at the scanned square.*

1	$q_0\colon (@(0,0) \wedge Q(0,q_1) \wedge \forall x M_0(0,x))$	
$1'$	$@(0,0) \wedge Q_0(0,q_1) \wedge \forall x M_0(0,x)$	
2	$q_1\colon \forall t\forall x((@(t,x) \wedge M_0(t,x)) \to (@(s(t),x) \wedge M_1(s(t),x)$ $\wedge\, Q(s(t),q_1) \wedge \forall y((y \neq x \wedge M_k(t,y)) \to M_k(s(t),y))))$	$q_1 M_0 M_1 q_1$
3	$q_1\colon \forall t\forall x((@(t,x) \wedge M_1(t,x)) \to (@(s(t),s^{-1}(x))$ $\wedge\, Q(s(t),q_2) \wedge \forall y(M_k(t,y) \to M_k(s(t),y))))$	$q_1 M_1 L q_2$
4	$q_2\colon \forall t\forall x((@(t,x) \wedge M_0(t,x)) \to (@(s(t),x) \wedge M_1(s(t),x)$ $\wedge\, Q(s(t),q_2) \wedge \forall y((y \neq x \wedge M_k(t,y)) \to M_k(s(t),y))))$	$q_2 M_0 M_1 q_2$
$\vdash$	$f(q_0, \ldots, q_2)\colon \exists t\exists x(@(t,x) \wedge M_1(t,x) \wedge Q(t,q_2))$, *for some* f	*halting*

Schematic f *in the last line indicates a compound causal term to be constructed from* $q_0, \ldots, q_2$.

We now construct the prefix (mechanical cause) of the conclusion (halting) for the above example.

Example 3 (Causal justification) *In the following proof, 'Pr' stands for a premise from the inference above. In line 8,* e *is used for the cause of* $-1 \neq 0$. PREF*(n) in lines 11, 13 and 14 is short for the whole prefix in line n. Symbol* a' *in line 14 is the justification for the existential generalization on the basis of term* a.

1	$((c \cdot (b \cdot q_0)) \cdot (a \cdot (b \cdot q_0)))\colon (@(0,0) \wedge M_0(0,0))$	*Pr1*
2	$(a \cdot (a \cdot q_1))\colon ((@(0,0) \wedge M_0(0,0)) \to (@(1,0) \wedge M_1(1,0)$ $\wedge Q(1,q_1) \wedge \forall y((y \neq 0 \wedge M_k(0,y)) \to M_k(1,y))))$	*Pr2*
3	$((a \cdot (a \cdot q_1)) \cdot ((c \cdot (b \cdot q_0)) \cdot (a \cdot (b \cdot q_0))))\colon (@(1,0) \wedge M_1(1,0)$ $\wedge Q(1,q_1) \wedge \forall y((y \neq 0 \wedge M_k(0,y)) \to M_k(1,y)))$	*1, 2*
4	$(a \cdot (a \cdot q_1))\colon ((@(1,0) \wedge M_1(1,0)) \to$ $(@(2,-1) \wedge Q(2,q_2) \wedge \forall y(M_k(1,y) \to M_k(2,y))))$	*Pr3*
5	$((a \cdot (a \cdot q_1)) \cdot (b \cdot ((a \cdot (a \cdot q_1)) \cdot ((c \cdot (b \cdot q_0)) \cdot (a \cdot (b \cdot q_0))))))\colon$ $(@(2,-1) \wedge Q(2,q_2) \wedge \forall y(M_k(1,y) \to M_k(2,y)))$	*4, 3*
6	$(a \cdot (b \cdot q_0))\colon M_0(0,-1)$	*Pr1*
7	$(a \cdot (b \cdot ((a \cdot (a \cdot q_1)) \cdot ((c \cdot (b \cdot q_0)) \cdot (a \cdot (b \cdot q_0))))))\colon$ $((-1 \neq 0 \wedge M_0(0,-1)) \to M_0(1,-1))$	*3*
8	$((a \cdot (b \cdot ((a \cdot (a \cdot q_1)) \cdot ((c \cdot (b \cdot q_0)) \cdot (a \cdot (b \cdot q_0)))))) \cdot$ $((c \cdot e) \cdot (a \cdot (b \cdot q_0))))\colon M_0(1,-1)$	*7, 6*
9	$(a \cdot (b \cdot$ $((a \cdot (a \cdot q_1)) \cdot (b \cdot ((a \cdot (a \cdot q_1)) \cdot ((c \cdot (b \cdot q_0)) \cdot (a \cdot (b \cdot q_0))))))))\colon$ $(M_0(1,-1) \to M_0(2,-1))$	*5*
10	$((a \cdot (b \cdot$ $((a \cdot (a \cdot q_1)) \cdot (b \cdot ((a \cdot (a \cdot q_1)) \cdot ((c \cdot (b \cdot q_0)) \cdot (a \cdot (b \cdot q_0)))))))) \cdot$ $((a \cdot (b \cdot ((a \cdot (a \cdot q_1)) \cdot ((c \cdot (b \cdot q_0)) \cdot (a \cdot (b \cdot q_0)))))) \cdot$ $((c \cdot e) \cdot (a \cdot (b \cdot q_0)))))\colon M_0(2,-1)$	*9, 8*
11	$((c \cdot (b \cdot$ $((a \cdot (a \cdot q_1)) \cdot (b \cdot ((a \cdot (a \cdot q_1)) \cdot ((c \cdot (b \cdot q_0)) \cdot (a \cdot (b \cdot q_0)))))))) \cdot$ $\text{PREF}(10))\colon (@(2,-1) \wedge M_0(2,-1))$	*5, 10*
12	$(a \cdot (a \cdot q_2))\colon ((@(2,-1) \wedge M_0(2,-1)) \to$ $(@(3,-1) \wedge M_1(3,-1) \wedge Q(3,q_2) \wedge \forall y((y \neq -1 \wedge M_k(2,y))$ $\to M_k(3,y)))$	*Pr4*
13	$((a \cdot (a \cdot q_2)) \cdot \text{PREF}(11))\colon$ $(@(3,-1) \wedge M_1(3,-1) \wedge Q(3,q_2) \wedge \forall y((y \neq -1 \wedge M_k(2,y))$ $\to M_k(3,y)))$	*12, 11*
14	$(a' \cdot (b \cdot ((a \cdot (a \cdot q_2)) \cdot \text{PREF}(11))))\colon$ $\exists t \exists x(@(t,x) \wedge M_1(t,x) \wedge Q(t,q_2)$	*13*

In line 1, $c \cdot (b \cdot q_0)$ causes $M_0(0,0) \to (@(0,0) \wedge M_0(0,0))$ (where $b \cdot q_0$ causes $@(0,0)$), and $a \cdot (b \cdot q_0)$ causes $M_0(0,0)$ (for cause terms a and b, see **ACau** *and $\mathcal{CS}$). In lines 7-10, the justification is calculated of the required instantiation of the formula $M_k(2,y)$ (see universal conjuncts within lines 3 and 5) for the symbol $k = 0$ written on the square $y = -1$, unchanged from the beginning of the work of the machine.*

The causal structure behind the work of a TM (causal prefixes) is increasingly more complicated and does not reduce just to the current "internal state" q_i that the TM is in at a time moment t. This structure includes, besides TM's internal states, some logical and arithmetical laws that "ontologically" (objectively) govern the TM's behaviour.

Example 4 (Factivity) *Factivity, $u\colon \phi \to \phi$, should be separately proved by means of* **CTE**, *on the supposition of the initial configuration (arguments). In our Example 2 of a TM, this amounts to several simple deductive steps.*

1	$@(0,0) \wedge Q(0,q_1) \wedge \forall x M_0(0,x)$	*Pr1′*
2	$M_0(0,0)$	*1*
3	$(@(0,0) \wedge M_0(0,0) \wedge Q(0,q_1)) \to$ $(@(1,0) \wedge M_1(1,0) \wedge Q(1,q_1) \wedge$ $\forall y((y \neq 0 \wedge M_k(0,y)) \to M_k(1,y)))$	*Pr2,* **CTE**
4	$@(1,0) \wedge M_1(1,0) \wedge Q(1,q_1) \wedge$ $\forall y((y \neq 0 \wedge M_k(0,y)) \to M_k(1,y))$	*1, 2, 3* **MP**
5	$(@(1,0) \wedge M_1(1,0) \wedge Q(1,q_1)) \to$ $(@(2,-1) \wedge Q(2,q_2) \wedge \forall y(M_k(1,y) \to M_k(2,y)))$	*Pr3,* **CTE**
6	$@(2,-1) \wedge Q(2,q_2) \wedge \forall y(M_k(1,y) \to M_k(2,y))$	*4, 5* **MP**
7	$M_0(0,-1)$	*1*
8	$M_0(1,-1)$	*4, 7*
9	$M_0(2,-1)$	*6, 8*
10	$(@(2,-1) \wedge M_0(2,-1) \wedge Q(2,q_2)) \to$ $(@(3,-1) \wedge M_1(3,-1) \wedge Q(3,q_2) \wedge$ $\forall y((y \neq -1 \wedge M_k(2,y)) \to M_k(3,y)))$	*Pr4,* **CTE**
11	$@(3,-1) \wedge M_1(3,-1) \wedge Q(3,q_2) \wedge$ $\forall y((y \neq -1 \wedge M_k(2,y)) \to M_k(3,y))$	*6, 9, 10 MP*

Descriptive inferences and halting

Proposition 5 *A Turing machine TM halts iff the TM descriptive inference is valid.*

Proof. In the usual way (see [5] for a standard case). For the left to right direction, we prove by induction: if TM does not halt before time t, PREM $\vdash$ Des(t) (where PREM is the set of special TM axioms for an inference in TJL, and Des(t) is a TJL description of the whole TM configuration at time t). Suppose the claim holds for time t, and suppose that TM does not halt before $t+1$. We prove that PREM $\vdash$ Des($t+1$). The TM's step from t to $t+1$ is accounted by a premise of the TM descriptive inference. If the TM halts at t, Des(t) is accounted by the conclusion of the TM descriptive inference.

(1) Let us give as an example a premise of the form **M1**:

$$q\colon \forall t \forall x((@(t,x) \wedge M_0(t,x)) \to$$
$$(@(s(t),x) \wedge M_1(s(t),x) \wedge Q(s(t),q') \wedge \forall y((y \neq x \wedge M_k(t,y)) \to M_k(s(t),y))))$$

By $\forall$**a** and **CK**, the premise is instantiated for a particular time moment t (as in Example 3, e.g., line 12):

$$(a\cdot(a\cdot q))\colon ((@(t,x)\wedge M_0(t,x))\rightarrow \\ (@(s(t),x)\wedge M_1(s(t),x)\wedge Q(s(t),q')\wedge\forall y((y\neq x\wedge M_k(t,y))\rightarrow M_k(s(t),y))))$$

By inductive hypothesis, the antecedent under $a\cdot(a\cdot q)$ is already accounted for by PREM for some series of internal states $q_0,\ldots,q_n$ embodied in a cause $f(q_0,\ldots,q_n)$ (cf. line 11 of Example 3), and thus:

$$((a\cdot(a\cdot q))\cdot f(q_0,\ldots,q_n))\colon (@(s(t),x)\wedge M_1(s(t),x)\wedge \\ Q(s(t),q')\wedge\forall y((y\neq x\wedge M_k(t,y))\rightarrow M_k(s(t),y)))$$

(cf. line 13 of Example 3), which is the description, in TJL, of time $t+1$.

(2) If TM stops at t, the derived Des(t) immediately gives the conclusion (by existential generalization on t and the scanned square at t), without any instruction for a further change. ■

Models and adequacy

Models

We now give a possible model-theoretic semantics for TJL, by generalizing the model definition informally given in [5] and extending it by the causal influence relation In.

Definition 6 (Model, $\mathfrak{M}$) *Model is an ordered quintuple $\langle\mathbb{Z},\mathbf{Q},R,In,V\rangle$, where*

1. *the domain is the set of integers, $\mathbb{Z}$,*
2. *$\mathbf{Q}$ is a finite subset of $\mathbb{Z}$,*
3. *R is a set of pairs $\langle\langle m,n\rangle,\langle X,n'\rangle\rangle$, where $m\in\{1,0\}$, $n,n'\in\mathbf{Q}$, and $X\in\{M_{k\neq m},s,s^{-1}\}$,*
4. *if $(\phi\rightarrow\psi)\in In(u)$ and $\phi\in In(v)$, then $\psi\in In(u\cdot v)$; if $\phi\in In(u)$ then $\phi\in In(u+v)$, and if $\phi\in In(v)$ then $\phi\in In(u+v)$; if $\phi\in In(u)$ then $u\colon\phi\in In(!u)$; if $\neg\phi\in In(u)$ then $u\colon\neg\phi\in In(?u)$; if $\phi\in In(u)$ then $\forall x\phi\in In(\mathsf{gen}_x(u))$, where $x\notin\mathsf{free}(u)$.*[11]
5. (a) *$V(0)=0$, $V(q_i)=i\in\mathbf{Q}$, $V(s)$ is a successor function from $\mathbb{Z}$ to $\mathbb{Z}$,*
   (b) *$\langle 0,0\rangle\in V(@)$; $\langle 0,n\rangle\in V(Q)$ for $n\in\mathbf{Q}$; there is $\Sigma=\{0,\ldots,n\}$ or empty, such that $\langle 0,i\rangle\in V(M_1)$ for each $i\in\Sigma$, and $\langle 0,j\rangle\in V(M_0)$ for each $j\in\mathbb{Z}\setminus\Sigma$,*

[11] In is analogous to $*$ of [35] and $\mathcal{E}$ of [10].

(c) *if* $\langle \mathbf{t}, o\rangle \in V(@)$, $\langle \mathbf{t}, o\rangle \in V(M_m)$ *and* $\langle \mathbf{t}, n\rangle \in V(Q)$, *then*

$$\begin{cases} \langle \mathbf{t}+1, o\rangle \in V(@), \langle \mathbf{t}+1, o\rangle \in V(M_k), & \textit{if } \mathbf{M1/0}^* \in \\ \quad \textit{and for each } o' \neq o, & In(q_n) \\ \quad \langle \mathbf{t}+1, o'\rangle \in V(M_i) \textit{ if } \langle \mathbf{t}, o'\rangle \in V(M_i) & \\ \langle \mathbf{t}+1, s(o)\rangle \in V(@) \textit{ or } \langle \mathbf{t}+1, s^{-1}(o)\rangle & \textit{if } \mathbf{R}^* \in In(q_n) \\ \quad \in V(@), \textit{and for any } o', & \textit{or } \mathbf{L}^* \in In(q_n), \\ \quad \langle \mathbf{t}+1, o'\rangle \in V(M_i) \textit{ if } \langle \mathbf{t}, o'\rangle \in V(M_i)) & \end{cases}$$

while $\langle \mathbf{t}+1, n'\rangle \in V(Q)$,
where $\langle m, n\rangle R\langle X \in \{M_k, s, s^{-1}\}, n'\rangle$, $\mathbf{t}, o \in \mathbb{Z}$, *and* $0 \leq \mathbf{t}$
($\mathbf{M1/0}^*, \mathbf{R}^*$ *and* $\mathbf{L}^*$ *are the matrices*

$$\forall t\forall x((@(t,x) \wedge M_m(t,x)) \to (@(s(t),x) \wedge M_{k\neq m}(s(t),x) \wedge Q(s(t),q_{n'}) \\ \wedge \forall y((y \neq x \wedge M_i(t,y)) \to M_i(s(t),y)))),$$
$$\forall t\forall x((@(t,x) \wedge M_m(t,x)) \to (@(s(t),s^{-1}(x)) \wedge Q(s(t),q_{n'}) \\ \wedge \forall y(M_i(t,y) \to M_i(s(t),y))))$$
$$\forall t\forall x((@(t,x) \wedge M_m(t,x)) \to (@(s(t),s(x)) \wedge Q(s(t),q_{n'}) \\ \wedge \forall y(M_i(t,y) \to M_i(s(t),y)))),$$

respectively),

(d) $V(=)$ *and* $V(<)$ *are evaluated as usual.*

R semantically describes TM quadruples (m referring to the subscript of M_m). We note that the only causal terms referring by definition to domain objects are internal states. Otherwise, causes are defined only implicitly, by the influence function In, which maps a causal term to a subset of formulas, and the meaning of such causes is left to be purely intensional (without any extension associated by a model).

Definition 7 (Variable assignment, a**)** *For variable assignment* a, $a(x) \in \mathbb{Z}$.

The denotation of individual term w in $\mathfrak{M}$ for a will be expressed by $[\![w]\!]^{\mathfrak{M}}_a$.

Definition 8 (Satisfaction)

1. $\mathfrak{M} \models_a \Phi(w_1, w_2) \Leftrightarrow \langle [\![w_1]\!]^{\mathfrak{M}}_a, [\![w_2]\!]^{\mathfrak{M}}_a\rangle \in V(\Phi)$, *for* $\Phi \in \{@, Q, M_k\}$,
2. $\mathfrak{M} \models_a w_1 = w_2 \Leftrightarrow [\![w_1]\!]^{\mathfrak{M}}_a = [\![w_2]\!]^{\mathfrak{M}}_a$,
3. $\mathfrak{M} \models_a w_1 < w_2 \Leftrightarrow [\![w_1]\!]^{\mathfrak{M}}_a < [\![w_2]\!]^{\mathfrak{M}}_a$,
4. $\mathfrak{M} \models_a \neg\phi \Longleftrightarrow \mathfrak{M} \not\models_a \phi$,
5. $\mathfrak{M} \models_a \phi \to \psi \Longleftrightarrow \mathfrak{M} \not\models_a \phi$ *or* $\mathfrak{M} \models_a \psi$,
6. $\mathfrak{M} \models_a \forall x\phi \Longleftrightarrow$ *for each* $n \in \mathbb{Z}, \mathfrak{M} \models_{a[n/x]} \phi$,
7. $\mathfrak{M} \models u\colon \phi \Longleftrightarrow \phi \in In(u)$.

The work of a TM can be described in an obvious ("self-evident") way by using TM terms, i.e., in terms of reading and writing of symbols or moving left or right on the TM's tape, depending on the TM's internal states. Thus, the above definition of satisfaction can be replaced by a description in TM terms in a familiar way. For example,

1. TM is at the time m in the state n iff $\mathfrak{M} \models_a Q(m, q_n)$,
2. TM is at the time m at the square n iff $\mathfrak{M} \models_a @(m, n)$
3. For TM, at the time m, 1 is written in the square n or n is blank iff $\mathfrak{M} \models_a M_1(m, n)$ or $\mathfrak{M} \models_a M_0(m, n)$, respectively.

In general, the truth of each sentence ϕ of the formal language of TJL can be expressed by a corresponding non-formalized sentence F in TM terms by using, in addition, some usual paraphrasing for logical terms and syntax. TM terms are clearly understandable without being formalized and can exactly depict the form of a TM's work.[12] Hence, they are not "informal" eventhough they are non-formalized and independent of formal systems.[13]

Soundness and completeness

Theorem 9 (TM-soundness) *If* $\vdash \phi$ *then* $\models \phi$.

Proof. We give some examples.

(a) Axiom **CTE** (for **M1**). Let $\mathfrak{M} \models_a q_m \colon \forall t \forall x((@(t, x) \wedge M_0(t, x)) \rightarrow (@(s(t), x) \wedge M_1(s(t), x) \wedge Q(s(t), q_n) \wedge \forall y((y \neq x \wedge M_k(t, y)) \rightarrow M_k(s(t), y))))$, and let $\mathfrak{M} \models_{a[\mathbf{t}/t,\, o/x]} @(t, x) \wedge M_0(t, x) \wedge Q(t, q_m)$. Thus, $\langle \mathbf{t}, o \rangle \in V(@)$, $\langle \mathbf{t}, o \rangle \in V(M_0)$ and $\langle \mathbf{t}, m \rangle \rangle \in V(Q)$, and hence (according to the first assumption, Definition 6 for R and V, and Definition 8), $\langle 0, m \rangle R \langle 1, n \rangle$ holds. Accordingly, $\mathfrak{M}$ and $a[\mathbf{t}/t,\, o/x]$ satisfy $@(s(t), x), M_1(s(t), x)$ and $Q(s(t), q_n)$, and for all other squares $o' \neq o$ nothing changes in $\mathbf{t} + 1$. This satisfies the consequent in **M1**, i.e., $@(s(t), x) \wedge M_1(s(t), x) \wedge Q(s(t), q_n) \wedge \forall y((y \neq x \wedge M_k(t, y)) \rightarrow M_k(s(t), y))$ which is ψ of **CTE** (see special TM axioms above).

(b) The proof of the soundness for causal axioms is similar as in [35] (with In for $*$). E.g., for **CK**, let $\mathfrak{M} \models_a u \colon \phi \rightarrow \psi$ and $\mathfrak{M} \models_a v \colon \phi$. Thus $\phi \rightarrow \psi \in In(u)$ and $\phi \in In(v)$, implying $\psi \in In(u \cdot v)$, and hence, $\mathfrak{M} \models_a (u \cdot v) \colon \psi$. The additional case of $\mathsf{gen}_x(u)$ is proved in an analogous way.

We now give the semantic account of TM descriptive inferences as a characteristic example. Let $\mathfrak{M}$ and a satisfy an axiom of the form **IC**$'$. Let also, for instance, $\mathfrak{M} \models_a q_m \colon \forall t \forall x((@(t, x) \wedge M_k(t, x)) \rightarrow (@(s(t), s(x)) \wedge Q(s(t), q_n) \wedge \forall y(M_k(t, y) \rightarrow M_k(s(t), y))))$ (axiom scheme **R**). In addition, let $\mathfrak{M}$ and $a[\mathbf{t}/t,\, o/x]$ satisfy $@(t, x), M_k(t, x)$ and $Q(t, q_m)$. Hence, by Definition 8,

[12] Cf. [18] on the "absolute definition" of Turing computability.

[13] See [6] on Gödel's "absolute" concepts as "formal" in the sense of "universal applicability" ("without any restriction of type") and as related to Platonic "forms".

$\mathfrak{M} \models_{a[\mathbf{t}/t,\,o/x]} (@(t,x) \wedge M_k(t,x)) \rightarrow (@(s(t),s(x)) \wedge Q(s(t),q_n) \wedge \forall y(M_k(t,y) \rightarrow M_k(s(t),y)))$. Since **CTE** is semantically valid, it follows that $\mathfrak{M} \models_{a[\mathbf{t}/t,\,o/x]} @(s(t),s(x)) \wedge Q(s(t), q_n) \wedge \forall y(M_k(t,y) \rightarrow M_k(s(t),y))$. Similarly for other TM-premises. By induction, with some logical semantics, for any $\mathbf{t}$ and o in the work of the TM, $\mathfrak{M} \models_{a[\mathbf{t}/t,\,o/x]} @(t,x) \wedge M_k(t,x) \wedge Q(t,q_j)$ for some j. This can be existentially generalized on t and x (a semantically valid step), in which way we obtain a true halting conclusion if no axiom is valid containing the instruction on how to continue further from the internal state j and scanned 'k'. ■

Definition 10 (Saturated set Γ_ω^{max} of closed sentences) *A saturated set of closed sentences is maximal consistent (consistent, includes ϕ or $\neg\phi$ for each sentence ϕ) and ω-complete (includes an instantiation of each $\exists$-sentence).*

Let $\mathcal{L}\mathsf{TJL}^k$ be as $\mathcal{L}\mathsf{TJL}$, extended by infinitely many witnesses (individual constants not in the vocabulary of $\mathcal{L}\mathsf{TJL}$).

Lemma 11 *Each consistent set of closed sentences of $\mathcal{L}\mathsf{TJL}$ can be extended to a saturated set Γ_ω^{max} of sentences of $\mathcal{L}\mathsf{TJL}^k$.*

Proof. A usual method of proof, extending a given consistent set to its saturated superset by using witnesses. ■

Definition 12 (TM canonical model, $\mathfrak{M}^c$) *In the following, w is a closed term, and $[\![w]\!]^c$ is its meaning in the canonical model $\mathfrak{M}^c$.*

1–2. as in Definition 6,

3. (a) $\langle m,n\rangle R\langle M_{k\neq m}, n'\rangle$ iff $q_n\colon \forall t\forall x((@(t,x) \wedge M_m(t,x)) \rightarrow (@(s(t), x) \wedge M_{k\neq m}(s(t),x) \wedge Q(s(t),q_{n'}) \wedge \forall y((y \neq x \wedge M_k(t,y)) \rightarrow M_k(s(t), y)))) \in \Gamma_\omega^{max}$,

(b) $\langle m,n\rangle R\langle s, n'\rangle$ iff $q_n\colon \forall t\forall x((@(t,x) \wedge M_m(t,x)) \rightarrow (@(s(t),s(x)) \wedge Q(s(t),q_{n'}) \wedge \forall y(M_k(t,y) \rightarrow M_k(s(t),y)))) \in \Gamma_\omega^{max}$ (similarly for $s^{-1}(x)$ instead of $s(x)$),

4. $In(u) = \{\phi \mid u\colon \phi \in \Gamma_\omega^{\max}\}$,

5. (a) $V(0), V(q_i), V(s)$ as in Definition 6,

(b) $\langle 0,0\rangle \in V(@)$, $\langle 0, [\![w]\!]^c\rangle \in V(M_k)$ iff $M_k(0,w) \in \Gamma_\omega^{max}$, $\langle 0, [\![q_i]\!]^c\rangle \in V(Q)$ iff $Q(0,q_i) \in \Gamma_\omega^{max}$,

(c) conditions of Definition 6,

(d) $V(=) = \{\langle [\![w_1]\!]^c, [\![w_2]\!]^c\rangle \mid w_1 = w_2 \in Q_\omega^{\max}\}$, $V(<) = \{\langle [\![w_1]\!]^c, [\![w_2]\!]^c\rangle\rangle \mid w_1 < w_2 \in Q_\omega^{\max}\}$.

Lemma 13 (TM Canonical satisfaction) $\mathfrak{M}^c \models_c \phi$ *iff* $\phi \in \Gamma_\omega^{max}$ *(ϕ is a closed formula of $\mathcal{L}\mathsf{TJL}^k$).*

Proof. We elaborate specific cases of atomic and justification sentences.

1. Atomic case. The thesis holds for time 0 (Definition 12, cases 4a–b) and timeless atomic sentences ($=$, $<$). By induction on time, we prove the thesis for atomic sentences in general. By inductive hypothesis, $\mathfrak{M}^{\mathfrak{c}} \models @(w_t, w)$ iff $@(w_t, w) \in \Gamma_\omega^{max}$, $\mathfrak{M}^{\mathfrak{c}} \models M_k(w_t, w)$ iff $M_k(w_t, w) \in \Gamma_\omega^{max}$, and $\mathfrak{M}^{\mathfrak{c}} \models Q(w, q_i)$ iff $Q(w, q_i) \in \Gamma_\omega^{max}$. (a) Suppose that R of $\mathfrak{M}^{\mathfrak{c}}$ associates with k (of M_k) and i (of q_i) a new pair, e.g., $\langle 0, q_1\rangle R\langle 1, q_1\rangle$ ($k = 0, i = 1$), that is, TM continues to work after $[\![w_t]\!]^c = \mathbf{t}$ in $[\![w_{t+1}]\!]^c = \mathbf{t}+1$. According to definitions 6 and 8, from $@(w_t, w)$, $M_k(w_t, w)$ and $Q(w_t, q_1)$ satisfied by $\mathfrak{M}^{\mathfrak{c}}$, it follows that $\mathfrak{M}^{\mathfrak{c}}$ satisfies $@(w_{t+1}, w)$, $M_1(w_{t+1}, w)$ and $Q(w_{t+1}, q_1)$ and for each $w' \neq w$, $M_k(w_{t+1}, w')$ is satisfied if $M_k(w_t, w')$ is. However, according to Definition 12, $\langle 0, q_1\rangle R\langle 1, q_1\rangle \in R$ iff the corresponding axiom $q_1 : \forall t \forall x((@(t, x) \wedge M_0(t, x)) \to (@(s(t), x) \wedge M_1(s(t), x) \wedge Q(s(t), q_1) \wedge \forall y((y \neq x \wedge M_k(t, y)) \to M_k(s(t), y)))) \in \Gamma_\omega^{max}$. From this axiom, the same atomic sentences for w_{t+1} are provable (and are thus members of Γ_ω^{max}) which are satisfied by $\mathfrak{M}^c$. (b) If R does not associate any new $\langle X, q_j\rangle$ with $\langle k, q_i\rangle$, i.e., TM does not continue to work after $\mathbf{t}$, then there are no atomic sentences with $@$, M_k or Q that are true for $\mathbf{t}+1$. Equivalently, according to Definition 12, there is no corresponding axiom (TM premise) from which these atomic sentences could be provable.

2. The general case of $u\colon \phi$ is simple. Let $\mathfrak{M}^c \models u\colon \phi$. Thus, $\phi \in In(u)$ (Definition 8) from where, by Definition 12, it follows that $u\colon \phi \in \Gamma_\omega^{max}$. Also, $\mathfrak{M}^c \not\models u\colon \phi$ implies that $\phi \notin In(u)$, from where we obtain (Definition 12) that $u\colon \phi \notin \Gamma_\omega^{max}$.

   In addition, the In conditions 4 of Definition 6 hold for a canonical model. For $\mathsf{gen}_x(u)$, assume $\phi(x) \in In(u)$ ($x \notin \mathbf{free}(u)$). Thus $u\colon \phi(x) \in \Gamma_\omega^{max}$ (In in Definition 12), and therefore $\mathsf{gen}_x(u)\colon \forall x \phi(x) \in \Gamma_\omega^{max}$ (Axiom $\mathbf{C}\forall$). According to Definition 12 (In), it follows that $\forall x \phi(x) \in In(\mathsf{gen}_x(u))$. For other cases, cf. Lemma 10 of [35] (the case of $?u$ is proved similarly).

■

Theorem 14 (Completeness) TJL *is complete with respect to* TJL *models.*

Proof. In a familiar way: from the TJL consistency of a closed formula $\neg\phi$ the satisfiability of $\neg\phi$ follows by a canonical model. Thus, if ϕ is semantically valid, it is a theorem of TJL. ■

"Absolute provability", "absolute notions" and completeness

We, finally, add some remarks on the Gödelian notion of "absolute provability", independent of a given formal system.

Gödel anticipated the notion of "absolute provability" already in his completeness paper from 1929 [13, pp. 62–65], where he mentioned the unrestricted principle of excluded middle, for example, in the following sense: either the validity of ϕ is provable or ϕ should be refuted by a counterexample – not in the sense of the provability in a formal system, but in the sense of any provability means (not restricted to some specified formal system). This meaning of provability was "questionable" for Gödel in 1929. However, in 1933 [15], "absolute provability" was stated in a neutral way as belonging to the $\mathsf{S4}$-like "provability" operator B, and in 1938 [17], it was characterized as "curious" (*merkwürdig*, pp. 100-101). In 1946 [18], Gödel speaks with much conviction about an "absolute notion" of provability, adjoining it to the absolute notion of computability discovered by Turing in [39]. Finally, in his conversations with Wang (late 1960s/1970s), Gödel, on the one side, points to the "bankruptcy" (not just "misunderstanding") of our present theory with respect to abstract (absolute) notions (e.g., the general concepts "proof" and "concept") due to intensional paradoxes, and on the other to the possibility of a future solution of these paradoxes while, in the meantime, avoiding the self-application of abstract concepts [40, pp. 187–188, 270, 272–273].

It seems that a clue to the concept of absolute provability and the corresponding "absolute" completeness proof can be found in Gödel's ontological proof (in the system OB of his *ontologischer Beweis* [22], and in GO, Scott's slight emendation of OB [36]). There, a sort of (axiomatically described) paradigm for an "absolute" completeness proof can be recognized, deducing the existence ("realization") that exemplifies a "system" of properties of individual objects from the compatibility of the system. Instead of metatheoretical reasoning to bridge the gap between a formal system and its model theory, Gödel, in his ontological proof, introduces "abstract", higher-order reasoning, with "abstract" concepts of a property (second-order quantification), positive property (third-order property of positiveness, $\mathcal{P}$), possibility and necessity ($\Diamond, \Box$).[14] "Positiveness" can be understood as an abstract ("absolute") criterion for the building of a proof system, depending on what we want to have as a consistent system of values – in the "moral esthetic sense" [22, p. 404], or in a logical sense of "assertions (+ tautologies)" [22, p. 434][24, XIV 106] (cf. "pure 'attribution'" [22, p. 404]), in which case, according to Gödel, a simpler system would be possible [24, XIV 106] (cf. [22, p. 404]). Abstract provability is, again, necessity, but strengthened to $\mathsf{S5}$ propositional base (allowing "negative introspection") and having the abstract characterization of the "following from

[14] Cf. [23], where, in the context of the problem of the consistency of a formal system, Gödel (relying on Bernays' remarks) is emphasizing the need of the "abstract concepts" ("essentially of the second or higher order"). Abstract concepts "do not have as their content properties or relations of *concrete* objects (such as combinations of symbols), but rather of *thought structures* or *thought contents* (e.g., proofs, meaningful propositions, and so on)" for which "insights" are needed that are "derived" from a "reflection upon the *meanings* involved".

mere concepts" ("understandability", *Verständlichkeit*; cf. "following from the essence of ...", "following from the nature of ...", where "essence" and "nature" need not be explicitly definable).[15]

In Gödel's ontological proof, we encounter a similar pattern of reasoning as mentioned at the beginning of [13] from 1929: in 1929, from the consistency of a system of propositions (axioms), to the realization (model) of this system; in GO, from the *possibility* ($\Diamond$, "compatibility" of all "positive" properties[16] in one thing, $\Diamond\exists xGx$), to their "realization" in an *existing* thing (that possesses all positive properties), $\Box\exists xGx$ (where $Gx =_{def} \forall X(\mathcal{P}X \to Gx)$).[17] Thus, if we apply the decidability formulation of 1929, the completeness can, in GO, be formulated in the following way:

> Either the system of all positive properties is refutable, or there is x that has all positive properties.

Let us note that Gödel's "abstract" approach to provability and its completeness differs from the reductionist completeness proof of [13, 14], which reduces the completeness (in the form of the principle of excluded middle between formal refutation and the possession of a model), step by step, to the case of skolemized first-order formulas and to the propositional logic case.

Whereas Henkin's approach consists in building a canonical model defined by means of expressive features of a given formal language (see, for instance, [32]),[18] Gödel exceeds a given language and its predicates by abstract concepts and new primitive terms in order to come, by explicit abstract reasoning, to an adequate realization of a given consistent "system" of predicates together with the abstract concepts introduced. The realization is achieved in the instantiating entity x for the system of positive properties, which is completely determined by the possession of positive properties (i.e., this possession makes its "essence") – Gödel's counterpart of a Henkin-style canonical model as being entirely determined by a saturated set of formulas for which it is built.

Eventually, if absolute provability is understood from "positiveness" taken as the criterion of the choice of a proof system, absolute provability and ontological necessity coincide:

> The positive and the true sentences are the same, for different reasons". [22, pp. 432–433][24, XIV 104]

In addition, in accordance with Gödel's basic philosophical views, we can assume that the abstract structure comprising all positive properties should

[15] See [24, XIV 118–119], [22, pp. 403, 435] and [19, p. 313].

[16] Including propositions, as closed properties, if the full comprehension scheme for GO is accepted.

[17] On Gödel's characteristic pattern of reasoning from possibility to existence, also in cosmology and philosophy of time, see [42, 43].

[18] "... maximal consistent set as an oracle and as building blocks for the model" [32, p. 158].

have an objective, and moreover, a causal, but non-mechanical, meaning [31]. For Gödel, concepts, and thus necessity as "following from mere concepts" ("following from the nature of ...") are not just our constructs ("creations"), but have an objective character and causal sense.[19] Besides general statements on "axioms causing theorems" and a "fundamental theorem causing its consequences" [40, pp. 120, 320], Gödel also seems to have in mind some "absolute" counterpart of a universal Turing machine, which he sometimes describes in Aristotelian terms as an "active intellect" working on the "passive intellect" (cf. head and tape of a Turing machine),[20] and as the influence of concepts on our mind [40, 4.4.7 on p. 149].[21] The development of our mind[22] and of our perception of concepts lies behind and inspires the development of the formalization work, which, in turn, corrects our uncertain conceptual perception and leads to its further precision and enrichment.

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[19] See [22, p. 432] and [24, p. 104] on the Kantian categories, including necessity, which should be understood, in a Gödelian view, from the concept of cause.

[20] "The active intellect works on the passive intellect which somehow shadows what the former is doing and helps us as a medium" [40, pp. 235, 189].

[21] See also Gödel's reflections on Kant's synthetic unity of consciousness, with category as "generating unities out of manifolds" [21, p. 268]. On a causal interpretation, see in [31].

[22] Cf. [23] on Turing's philosophical error.

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