Arthur Prior's Proofs of the Necessities of Identity and Difference

Nils Kürbis

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Abstract

This paper draws attention to a proof of the necessity of identity given by Arthur Prior. In its simplicity it is comparable to a proof of Quine's, popularised by Kripke, but it is slightly different. Prior's Polish notation is transcribed into more familiar idiom. Prior's proof is followed by a proof of the necessity of difference, possibly the first such proof in the literature, which is also repeated here and transcribed. The paper concludes with a brief discussion of Prior's views on identity and difference over time.

There has recently been some interest in the origin of proofs of the necessity of identity.[1] The first to have established such a result was Ruth Barcan (Barcan, 1947). The formal systems of modal logic she uses are second order quantified versions of the now slightly arcane Lewis system S2 and of the much more widely known S4.[2] Barcan proves the material equivalence of $x = y$ and $\Box\, x = y$ in S2 and their strict equivalence in S4. As Barcan's proofs are carried out in the original versions of S2 and S4, axiomatised on the basis of a primitive connective for strict implication, they inevitably carry with them a certain ballast. Her derivations are, in the words of Leech, austere. There is a much simpler proof, known to many through Kripke's work. Reasoning informally, everything is necessarily self-identical; if $x = y$, then by Leibniz' Law, x and y share all properties, so in particular y must share x's property of being necessarily identical to x. In symbols, the following is an instance of Leibniz' Law: $\forall x, y\ \ (x = y \supset (\Box\, x = x \supset \Box\, x = y))$, hence $\forall x, y\ (x = y \supset \Box\, x = y)$ by classical logic from $\forall x\, \Box\, x = x$. Kripke attributes this simple proof to Wiggins. Burgess has traced its origin to Quine, who records it in two papers published in 1953.

This note draws attention to the fact that Arthur Prior also found a simple proof of the necessity of identity no later than the publication of Quine's proof.

[1] See (Burgess, 2014) and (Leech, 2024). This paper was prepared in response to discussions with Jessica Leech.

[2] Hughes and Cresswell inform us that S2 was Lewis's preferred system, which suffices to explain Barcan's choice. It results by adding the K and T axioms and the rules 'If A is a tautology of the propositional calculus or an axiom, then $\vdash \Box A$' and 'If $\vdash A \supset B$, then $\vdash \Box(\Box A \supset \Box B)$' to the classical propositional calculus. See (Hughes and Cresswell, 1996, 200).

Prior's proof is on p. 205f of the second edition of *Formal Logic*. This edition, says the preface to it, left the substance of the text of the first edition untouched. In a footnote to p. 205, Prior attributes his proof, 'in substance', to Barcan.[3] He does not mention Quine. This indicates that Prior was not aware of Quine's proof at the time of writing *Formal Logic*. Prior was a prolific referencer, so had he known Quine's proof, we can be fairly confident that he would have said so.[4] Another reason is that he probably could not have seen Quine's proof in print at the time of the completion of *Formal Logic*. Although the first edition of *Formal Logic* was published only in 1955, the preface is dated to May 1953. Assuming the preface was written last and any changes to the body of the text afterwards restricted to minor corrections at proof stage, we can conclude with some confidence that Quine's publications were not available to Prior at the time and may not yet have been available in print at all.

Prior's proof is not the same as Quine's. It is slightly longer, because spelled out in all axiomatic details, and Prior derives the necessity of self-identity from more fundamental principles. Despite the acknowledgement of Barcan, it is fair to say that Prior's proof is rather more straightforward than hers.

Prior's proof is in Polish notation. Unfortunately, this way of writing logical formulas is no longer as *en vogue* as it was in the 1950s, and so another aim of this paper is to make this proof more accessible to the contemporary reader. I shall transcribe Prior's proof in more familiar notation and accompany the transcription with an analysis of how each step in the proof is derived.

Prior's proof of the necessity of identity is followed by a proof of the necessity of difference. As far as I know, this is the first such proof in the literature. I will transcribe it below, too. Prior deduces the necessity of difference from the necessity of identity in S5.

Prior's proof of the necessity of identity looks like this:

Df. I: $Ixy = \Pi\phi C\phi x\phi y$

1. $C(\Pi\phi f\phi)(f\phi)$

2. $CCpCqrCqCpr$

3. Ixx

   $1\ f/C'x'y \times$ Df. $I = 4$

4. $CIxyC\phi x\phi y$

   $2\ p/Ixy, q/\phi x, r/\phi y = C4{-}5$

5. $C\phi xCIxy\phi y$

   $3 \times \mathrm{RB2} = 6$

6. $LIxx$

   $5\ \phi/LIx' = C6{-}7$

7. $CIxyLIxy$

I is the symbol for identity, Π for the universal quantifier, C for material implication, L for necessity. I will use the more common $=$, $\forall$, $\supset$ and $\Box$. I'll

[3] Prior discusses Barcan's paper also in (Prior, 1956), where he shows that the Barcan Formula $CM\Sigma x\varphi x\Sigma xM\varphi x$, or in more common notation, $\Diamond\exists x\varphi x \supset \exists x\Diamond\varphi x$, which Barcan adds as an axiom to S2 and S4, is derivable in (a version of) quantified S5.

[4] A referee pointed out that Prior also refers to Quine at various points in the book.

use $\mathbf{X}$ as a schematic letter for second order predicates or a third order variable. This corresponds to Prior's f. Instead of ϕ I'll use F for first order predicates. The expressions $f/C'x'y$, ϕ/LIx' and p/Ixy etc. indicate substitution: in the first case of second order, in the second case of first order, in the third case of propositional order. The apostrophe marks 'gaps' in the predicate, to be filled to form sentences, in the first case to be replaced by first order predicates, in the second case by singular terms. Below I'll use '...' for this purpose.

The proof appeals to the following definition and rules:

> Definition. $x = y$ is short for $\forall F(Fx \supset Fy)$.
> Detachment: If A and $A \supset B$ are theorems, so is B.
> Rule RB2 (Necessitation): If A is a theorem, so is $\Box A$.
> Substitution: If A is a theorem, so are substitution instances of A.

In more familiar notion, the proof proceeds as follows:

1.	$\forall F\mathbf{X}(F) \supset \mathbf{X}(F)$	The axiom of second order universal instantiation.
2.	$(p \supset (q \supset r)) \supset (q \supset (p \supset r))$	A theorem of the propositional calculus.
3.	$x = x$	$\forall F(Fx \supset Fx)$ is a theorem of second order logic, apply the definition. Prior does not tell us this, but it is evident.

Substituting $\mathbf{X}$ by $\ldots x \supset \ldots y$ in line 1 gives $\forall F(Fx \supset Fy) \supset (Fx \supset Fy)$, which by the definition is:

4. $x = y \supset (Fx \supset Fy)$

Substituting p by $x = y$, q by Fx and r by Fy in line 2 gives:

$$(x = y \supset (Fx \supset Fy)) \supset (Fx \supset (x = y \supset Fy))$$

Use line 4 and detach:

5. $Fx \supset (x = y \supset Fy)$

Apply RB2 (Necessitation) to line 3:

6. $\Box\, x = x$

Substituting F in line 5 by $\Box\, x = \ldots$ gives $\Box\, x = x \supset (x = y \supset \Box\, x = y)$. Use line 6 and detach:

7. $x = y \supset \Box\, x = y$

In the footnote to p. 205, Prior observes that the proof is acceptable also to those who reject the Leibnizian definition of identity, if line 1 is dropped and lines 3 and 4 are adduced as axioms (with ϕ/F schematic in the latter). Examples of such philosophers are those who follow Quine in objecting to one half of the definition on the grounds that it requires second order logic, or Black, who argued that different things can have all properties in common (Black, 1952). Quine himself would of course also object to the use of modal logic.

Prior's proof is noteworthy: it shows that the necessity of identity follows from quite uncontroversial principles. The only modal principle appealed to is Necessitation, which is very plausible indeed: every logical truth is necessary. Speaking in the syntactic mode, the necessity of what is provable from no premises is provable from no premises.

The material equivalence of $x = y$ and $\Box\, x = y$ follows by an appeal to the T axiom. Their strict equivalence follows by another application of Necessitation.

Prior's proof of the necessity of difference (p. 206) looks like this:

1. $CIxyLIxy$
2. $CCpqCNqNp$
3. $CCpqCCqrCpr$
4. $CLpp$

   $2\, p/Ixy, q/LIxy = C1{-}5$
5. $CNLIxyNIxy$

   $5 \times \mathrm{L2} = 6$
6. $CNLIxyLNIxy$

   $2\, p/LIxy, q/Ixy = C4\, p/Ixy{-}7$
7. $CNIxyNLIxy$

   $3\, p/NIxy, q/NLIxy, r/LNIxy = C7{-}C6{-}8$
8. $CNIxyLNIxy$

N stands for negation, for which I'll use $\neg$. The proof appeals to rule L2 of Prior's formalisation of S5, which results by adjoining the following to the classical propositional calculus:

> L1: If $A \supset B$ is a theorem, so is $\Box A \supset B$.
> L2: If $A \supset B$ is a theorem, so is $A \supset \Box B$, provided all atomic formulas in A are modalised.

A modalised formula is one that is in the scope of a modal operator.

In more familiar notation, the proof is as follows:

1.	$x = y \supset \Box\, x = y$	The conclusion of the previous proof.
2.	$(p \supset q) \supset (\neg q \supset \neg p)$	A theorem of the propositional calculus.
3.	$(p \supset q) \supset ((q \supset r) \supset (p \supset r))$	A theorem of the propositional calculus.
4.	$\Box p \supset p$	The T axiom, provable in S5 by L1 from $p \supset p$.

Substitute p by $x = y$ and q by $\Box\, x = y$ in line 2, use line 1 and detach:

5. $\neg\Box\, x = y \supset \neg\, x = y$

$\neg\Box\, x = y$ is modalised, so apply L2 to line 5:

6. $\neg\Box\, x = y \supset \Box\neg\, x = y$

Substituting p by $\Box\, x = y$ and q by $x = y$ in line 2 gives:

$$(\Box\, x = y \supset x = y) \supset (\neg\, x = y \supset \neg\Box\, x = y)$$

The antecedent of this is line 4 with p substituted by $x = y$, so detach:

7. $\neg\, x = y \supset \neg\Box\, x = y$

Substituting p by $\neg\, x = y$, q by $\neg\Box\, x = y$, r by $\Box\neg\, x = y$ in line 3 gives:

$$(\neg\, x = y \supset \neg\Box\, x = y) \supset ((\neg\Box\, x = y \supset \Box\neg\, x = y) \supset (\neg\, x = y \supset \Box\neg\, x = y))$$

Use lines 7 and 6 and detach:

8. $\neg\, x = y \supset \Box\neg\, x = y$

The material equivalence of $\neg\, x = y$ and $\Box\neg\, x = y$ follows by the T axiom, their strict equivalence by Necessitation, which is derivable by L2.

Prior comments on the necessity of identity that it is 'odd, but not incredible when one reflects both on its meaning and on its proof', and that its consequence in S5, the necessity of difference, 'is harder to accept'.[5] Prior explains the former in the way one would expect: 'Any object is necessarily identical with itself, and in every case the only object that is in fact identical with a given object will be itself, i.e. the object that is necessarily identical with it' (p. 206). His doubts about the latter turn on the possibility of giving sense to

(∗) 'This thing, which is not in fact that thing, yet might have been that thing'.

in such a way that it may be true. Prior holds that the necessity of difference could be defended if it is argued that the system 'assumes a fixed range of objects, each with its own fixed identity' over which variables range, and that it is arguable that the necessity of identity and even of self-identity 'reflect the same assumption'. But then, writes Prior, if (∗) may indeed be understood so that it may be true, this sense cannot be expressed in the system. Prior leaves these issues there and does not discuss further what he has in mind.[6]

Two later papers shed some light on the issue. In 'Opposite Number', Prior considers the question of one thing becoming two. His example is one in which persons reproduce like amoebae through fission. These cases are not quite as stated in (∗), but they do show how the necessity of difference might fail. If x

[5] These and the following citations are from p. 206f.

[6] A referee enquired about Prior's reference to vol. i, ch. xiv of C. D. Broad's *Examination of McTaggart's Philosophy* in a footnote to p. 207. Prior's proofs of the necessities of identity and difference are not to be found there, although the chapter contains considerations that imply that a thing could not be another thing. According to Broad, (∗) has sense only if 'this' and 'that' are not proper names of objects with which one is acquainted, i.e. their referents are in fact picked out via definite descriptions rather than ostension. In some of his discussion, Broad commits himself to 'a fixed range of objects, each with its own fixed identity', but his discussion is of relevance rather to another major concern of Prior's than one thing's possibly being another, namely the possibility of things coming into and going out of existence. And this, when it comes to simple substances, is either not possible at all or not intelligible to us according to Broad (cf. p. 271f).

splits into y and z, Prior argues, we can make sense of 'y and z are now different, but were once the same' so that it is true. After fission, y and z are different, but before they were identical, as both were identical to x. Prior acknowledges that there are two other options of looking at such situations, namely that x ceases to exists and y and z come into being, or that x was the two objects y and z all along, but argues that his way is preferable. Prior also briefly mentions the complementary case of fusion of objects.

The topic is picked up again in 'Time, Existence and Identity'. While in the earlier paper Prior has no qualms about regarding Leibniz' Law of the indiscernibility of identicals as a fundamental logical principle, in the later paper, Prior now expresses doubts about its validity which are motivated by cases of fission or fusion.

This is not the place to go into the further conclusions for tense logic that Prior drew from fission and fusion. These cases are well known now from later discussions in the philosophy of mind and the metaphysics of personal identity. Prior's work is often unjustly forgotten, and I should hope that the present paper entices its readers to engage with his work and study Prior's discussion of the issues raised above themselves.

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