Stable Harmony

Nils Kurbis

1. Gentzen's Thesis and Two Notions of Harmony

Gentzen observed that there is a 'remarkable systematic' in the 'inference patterns' for symbols of the calculus of natural deduction. This lead him to put forward what might be called *Gentzen's Thesis*: 'The introductions constitute, so to speak, the "definitions" of the symbols concerned, and the eliminations are in the end only consequences thereof, which could be expressed thus: In the elimination of a symbol, the formula in question, whose outer symbol it concerns, may only "be used as that which it means on the basis of the introduction of this symbol".'[1]

Gentzen's Thesis invites fleshing out in a comprehensive theory, which is what Michael Dummett and Dag Prawitz aim to provide with their *proof-theoretic justification of deduction*.[2] They employ the following a version of Gentzen's Thesis: The meaning of a logical constant may be defined by its introduction and elimination rules if they are *in harmony*.

What is 'harmony'? *Two* explanations of this notion can be extracted from the literature which lead to two different definitions:

1.) Harmony is a feature detectable in the rules governing a logical constant: Introduction and elimination rules for a constant are *in harmony* if the latter can somehow be *read off* the former.

2.) Harmony holds if the rules for a constant yield a suitable induction clause for a *normalisation theorem*: Introduction and elimination rules for a constant are *in harmony* if *maximal formulas* may be removed from deductions.

[1] Gentzen (1935), p.189.
[2] See Dummett (1993) and Prawitz (1965).

Notice the difference between these two notions of harmony: Harmony$_1$ is a feature of rules *independently* of a logic they are part of. It is a common feature all rules of a certain kind are supposed to exhibit. It is to do with a special *uniformity in the form* of these rules. Contrary to that, harmony$_2$ is a feature rules can only have *relative to a logic* they are part of. There is *no* suggestion that it is a *common* feature of all rules which occur in logics which normalise.

An important relation seems to hold between the two notions of harmony: It has not been spelled out what harmony$_1$ consists in, what it means to *read off* introduction from elemination rules and *vice versa*. But it is perfectly clear what harmony$_2$ is. Thus harmony$_2$ suggests itself as a formally precise replacement of harmony$_1$.

2. The Problem with Stability

I have oversimplified Dummett's and Prawitz' version of Gentzen's Thesis. It should read: The meaning of a logical constant may be defined by introduction and elimination rules if they are *stable*. Stability is explained in terms of harmony$_1$: Stability holds if harmony$_1$ holds *and* the introduction rules for a constant can somehow be *read off* its elimination rules.[3]

This creates a problem: It is stability, hence harmony$_1$, which is crucial to Dummett's and Prawitz' programme. But the notion of 'reading off' introduction from elimination rules and *vice versa* has not been made precise. Rather, as noted at the very end of the last section, if anything, harmony$_1$ has been replaced by harmony$_2$. Thus we have no formally precise explanation of stability, and hence no formally precise foundation for Dummett's and Prawitz' proof-theoretic justification of deduction!

3. Making Harmony$_1$ Precise

The aim of the rest of this paper is to meet the challenge the last section poses and to define a formally precise notion of harmony$_1$. Gentzen provides the clue to the answer I shall propose: 'By making these thoughts [about the 'remarkable systematic' in the inference patterns] more precise it should be possible to establish on the basis of certain requirements that the elimination rules are functions of the corresponding introduction rules.'[4] My strategy is

[3]Cf. Dummett (1993), p.287.
[4]Gentzen (1935), p.189.

to specify the function Gentzen speaks of.

Gentzen does not give the function which maps introduction to elimination rules and to my knowledge neither does anyone else. This is remarkable, given there seems to exist a certain consensus amongst workers in the field of how to 'read off' elimination rules from introduction rules.[5] Dummett tries to capture some of Gentzen's idea with his notions of harmony and stability, but what he has to say on that issue is a long way from the envisaged mathematical precision of Gentzen's remark.

Previous discussions have, to my knowledge, always assumed that there is *one* kind of rule of inference—and then the meanings of the connectives are given either uniformly by introduction rules *or* by elimination rules, the others being determined relative to them by some principle of harmony. Maybe this is why the function Gentzen speaks of has not been found. In contrast, I shall give *two* kinds of general forms of rules of inference, one where an introduction rule for a connective is given and one where an elimination rule is given, and I shall specify for each kind a general method of reading off the elimination and introduction rules for the connective in such a way that the rules may be said to be 'in harmony'. Moreover, the process can be reversed, which corresponds to Dummett's notion of stability. I shall, however, define a notion of stability which is different from Dummett's, although arguably it captures what Dummett intended to capture with his notion.

Although I won't go into any further details, let me note here that there are also general forms of reduction procedures for removing maximal formulas and any logic with only connectives governed by stable rules of either of the two types fulfils the requirements of the proof-theoretic justification of deduction, in particular, deductions normalise. Logics containing only connectives governed by rules of either of the two types may be called *quasi-intuitionist*. The results can be extended to cover also *quasi-classical* logics by allowing rules of another type, namely versions of the negation rule *consequentia mirabilis*. It is interesting that proving normalisation theorems for quasi-classical logics, which I can demonstrate for classical and relevance logic, cannot proceed in as general a fashion as in the case of quasi-intuitionist logics.[6]

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[5]As demonstrated by, for instance, Dummett (1993), p.294ff. Zucker and Tragesser have to admit that their method for determining elimination rules from introduction rules allows there to be cases where a set of introduction rules is given but 'there does not seem to be a suitable set of [elimination] rules' ((1978), p.506).

[6]All this is covered in my Ph.D. thesis.

4. Formal Preliminaries

I shall presuppose familiarity with the framework of substructural logics and restrict myself here to explaining what is idiosyncratic to my formalism.

Lists are collections of formulas structured by punctuation marks. For instance, $((A, B); C), (A; B)$ is a list. Workers in the field in general restrict consideration to two kinds of punctuation marks, as e.g. in some substructural formulations of the relevant logic **R**. I allow logics to have any number of punctuation marks.

$\Theta(X_1 \dots X_n)$ is used to designate a list with immediate sublists $X_1 \dots X_n$ which are combined in a specific way by punctuation marks. Θ designates the structure of the list, which may be represented by using variables ranging over lists. For example, let Θ be $(\xi_1; \xi_2), \xi_3$. Then $\Theta((A, B)\ C\ (D; B))$ is the result of replacing ξ_1 by (A, B), ξ_2 by C and ξ_3 by $(D; B)$, i.e. $(((A, B); C), (D; B))$.

The following makes no mentioning of *structural rules* that allow the replacement of lists with other lists in the antecedents of consecutions $X \vdash A$. Only the general structure of lists is taken into account.

5. The General Forms of Rules of Inference

I shall now give the general forms of two kinds of rules of inference and the function that maps introduction/elimination rules to elimination/introduction rules.

5.a The General Form of Rules of Type One

If a connective Ξ is governed by rules of type one, it has *one* introduction rule of the form:

$$\frac{\Phi(X\ A_1 \dots A_h) \vdash B_1 \qquad \dots \qquad \Psi(X\ A_k \dots A_l) \vdash B_p}{X \vdash \Xi\, \overline{x}\, A_1 \dots A_l B_1 \dots B_p}$$

where there are no formulas on X containing the variables on the sequence $\overline{x} = x_1 \dots x_v$ free. To be governed by *harmonious rules* of inference, Ξ is required to have p elimination rules, i.e. one for each premises and each of the form:

$$\frac{Z \vdash \Xi\,\overline{x}\,A_1 \ldots A_l B_1 \ldots B_p \qquad Y_i \vdash A_i[\overline{x}/\overline{t}] \qquad \ldots \qquad Y_j \vdash A_j[\overline{x}/\overline{t}]}{\Theta(Z\ Y_i \ldots Y_j) \vdash B_o[\overline{x}/\overline{t}]}$$

where $\overline{t}$ is free for $\overline{x}$.

I leave it to the reader to figure out how the process of 'reading off' the elimination rules for Ξ from its introduction rule can be reversed so that the introduction rule is 'read off' the collection of elimination rules.

Examples of Rules of Type One

5.a.i ***Verum***

$\top$ is governed by an introduction rule with no premises:

$$X \vdash \top$$

Hence $\top$ has no elimination rule.

5.a.ii Conjunction

$\wedge$ is governed by an introduction rule with two premises and no discharged hypotheses:

$$\frac{X \vdash A \qquad X \vdash B}{X \vdash A \wedge B}$$

Hence $\wedge$ has two elimination rules without minor premises:

$$\frac{X \vdash A \wedge B}{X \vdash A} \qquad\qquad \frac{X \vdash A \wedge B}{X \vdash B}$$

5.a.iii Universal Quantification

$\forall x$ is governed by an introduction rule with one main premises and no discharged hypotheses, where x does not occur free in any formula on X:

$$\frac{X \vdash Fx}{X \vdash \forall x Fx}$$

Hence $\forall x$ has one elimination rule, where t is free for x in Fx:

$$\frac{Y \vdash \forall x Fx}{Y \vdash Ft}$$

5.a.iv Implication

$\to$ is governed by an introduction rule with one premise and one discharged hypothesis:

$$\frac{X, A \vdash B}{X \vdash A \to B}$$

Hence it has one elimination rule with one minor premise:

$$\frac{Y \vdash A \to B \qquad Z \vdash A}{Y, Z \vdash B}$$

Whether $\to$ is material or relevant depends on the structural rules that hold for the comma in the logic.

5.a.v Biconditional

$\leftrightarrow$ is governed by an introduction rule with two premises and one discharged hypothesis for each premise:

$$\frac{X, A \vdash B \qquad X, B \vdash A}{X \vdash A \leftrightarrow B}$$

Hence it has two elimination rules, each with one minor premise:

$$\frac{X \vdash A \leftrightarrow B \qquad Y \vdash A}{X, Y \vdash B} \qquad\qquad \frac{X \vdash A \leftrightarrow B \qquad Y \vdash B}{X, Y \vdash A}$$

Whether $\leftrightarrow$ is material or relevant depends on the structural rules that hold for the comma in the logic.

5.b The General Form of Rules of Type Two

If a connective Ξ is governed by rules of type two, it has *one* elimination rule of the form:

$$\frac{Z \vdash \Xi\,\overline{x}\,D_1 \ldots D_n \quad Y(\Phi(D_1 \ldots D_i)) \vdash E \quad \ldots \quad Y(\Psi(D_l \ldots D_n)) \vdash E}{Y(Z) \vdash E}$$

where none of the variables on the sequence $\overline{x} = x_1 \ldots x_v$ is free in E and any formula on Y.

To be governed by *harmonious rules* of inference, Ξ is required to have p introduction rules, where p is the number of minor premises, one for each, and each of the form:

$$\frac{X_j \vdash D_j[\overline{x}\,/\,\overline{t}\,] \qquad \ldots \qquad X_k \vdash D_k[\overline{x}\,/\,\overline{t}\,]}{\Theta(X_j \ldots X_k) \vdash \Xi\,\overline{x}\,D_1 \ldots D_n}$$

where $\overline{t}$ is free for $\overline{x}$

I leave it to the reader to figure out how the process of 'reading off' the introduction rules for Ξ from its elimination rule can be reversed so that the elimination rule is 'read off' the collection of introduction rules.

Examples of Rules of Type Two

5.b.i ***Falsum***

$\perp$ is governed by an elimination rule without minor premises:

$$\frac{X \vdash \perp}{Y(X) \vdash A}$$

Hence $\perp$ has no introduction rule.

5.b.ii Existential Quantification

$\exists x$ is governed by an elimination rule with one minor premise and one discharged hypothesis, where y is not free in C and any formula on Y:

$$\frac{X \vdash \exists x Fx \qquad Y(Fx) \vdash C}{Y(X) \vdash C}$$

Hence it has one introduction rule, where t is free for x in Fx:

$$\frac{Z \vdash Ft}{Z \vdash \exists x Fx}$$

5.b.iii Truth Constant t for the Empty List

t is governed by an elimination rule with one minor premise where the empty list 0 occurs in the place of a discharged assumption:

$$\frac{X \vdash \mathbf{t} \qquad Y(0) \vdash C}{Y(X) \vdash C}$$

Hence **t** has one introduction rule with no premises:

$$0 \vdash \mathbf{t}$$

5.b.iv Fusion

The constant $\times$ is governed by an elimination rule with one minor premise and two discharged hypotheses:

$$\frac{X \vdash A \times B \qquad Y(A;B) \vdash C}{Y(X) \vdash C}$$

Hence it has one introduction rule with two premises:

$$\frac{X \vdash A \qquad Y \vdash B}{X;Y \vdash A \times B}$$

5.b.v Disjunction

The constant $\vee$ is governed by an elimination rule with two minor premises, each with one discharged assumption:

$$\frac{X \;\vdash\; A \vee B \qquad Y(A) \;\vdash\; C \qquad Y(B) \;\vdash\; C}{Y(X) \;\vdash\; C}$$

Hence $\vee$ has two introduction rules with one premise each:

$$\frac{X \;\vdash\; A}{X \;\vdash\; A \vee B} \qquad\qquad \frac{X \;\vdash\; B}{X \;\vdash\; A \vee B}$$

6. Formally Precise Definitions of Harmony and Stability

We can now give a formally precise definition of harmony: introduction and elimination rules for a constant Ξ are *in harmony* or a logical constant Ξ is governed by *harmonious* rules iff either (i) Ξ is governed by an introduction rule of type one and elimination rules determined from it by the method of section 5.a, or (ii) Ξ is governed by an elimination rule of type two and introduction rules determined from it by the method of section 5.b.

This definition allows restrictions on the lists occurring in the rules such as the restrictions on the elimination rule for quantum disjunction or the introduction rule for **S4**-necessity. Let's call such restrictions *list restrictions*. List restrictions do not affect whether the rules are of a specific form, hence do not affect harmoniousness. However, we can exclude rules with list restrictions from being stable by adopting the following definition: introduction and elimination rules for a constant Ξ are *stable* or a logical constant Ξ is governed by *stable* rules iff the rules for Ξ are harmonious and without lists restrictions.

Notice that these definitions sever the ties between harmony and normalisation that is prevalent in the literature. It is not the case that deductions in any logic containing only constants governed by harmonious rules normalise. For instance, this is not the case for deductions in a logic containing only implication and **S4**-necessity. However, if all rules are stable, then normalisation is guaranteed.

Department of Philosophy
University College London
Gower Street
London
WC1E 6BT
email: n.kurbis@ucl.ac.uk

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