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ON MINIMAL MODELS FOR PURE CALCULI OF NAMES

Abstract. By pure calculus of names we mean a quantifier-free theory, based on the classical propositional calculus, which defines predicates known from Aristotle's syllogistic and Leśniewski's Ontology. For a large fragment of the theory decision procedures, defined by a combination of simple syntactic operations and models in two-membered domains, can be used. We compare the system which employs 'ε' as the only specific term with the system enriched with functors of Syllogistic. In the former, we do not need an empty name in the model, so we are able to construct a 3-valued matrix, while for the latter, for which an empty name is necessary, the respective matrices are 4-valued.

Keywords: calculus of names; Leśniewski's Ontology; cardinality of models; Horn theories; axiomatic rejection

Introduction

Despite the long history of development of the logical theory of names, starting from Aristotle, the calculus still attracts the attention of philosophers and logicians. One of the approaches, introduced by Jan Łukasiewicz, is to build it in the form of a quantifier-free theory, based on the classical propositional calculus (CPC). We will refer to such a theory, which includes notions taken from Aristotle's syllogistic and Stanisław Leśniewski Ontology, as a pure calculus of names (PCN). As semantic counterparts of such axiomatic systems set theoretic models are built. One of the problems investigated in this paradigm is computational complexity of various fragments of the theory of names, which is often ex-

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pressed in terms of a minimal size of models that can be used for decision procedures (see [3, 12, 4]).

In [10] Andrzej Pietruszczak presents a quantifier-free Horn[1] axiomatisation of PCN using Leśniewski's ε and functors a and i from syllogistic as primitive terms. The Łukasiewicz's style axiomatic refutation, with technical details elaborated by Jerzy Słupecki (see [8, 13]), is applied to a variant of the system in [7]. In the same paper, the results concerning a minimal size of a model, established for Łukasiewicz's syllogistic in [4], is extended to the considered fragment of Leśniewski's Ontology.

In the present paper we use the same methodology for the quantifier-free fragment of Ontology with ε as the only primitive term, axiomatised by Arata Ishimoto in [2]. The domain of the models that have to be considered can be, as in the system from [7], reduced to two members, but an empty name is not necessary for the refutation of any formula. Thus, we are able to construct a 3-valued matrix for the logic, while the respective matrices for the system with functors ε, a and i are 4-valued.

The results gathered in the present paper were first publish in Polish (in a different arrangement) in [6][2].

In Section 1 the language of PCN is presented, in Section 2 the axiomatic system and its rejected counterpart for the theory with Leśniewski's ε as the only operator on names is given, in Section 3 the same is presented for the system enriched with operators of syllogistic and in Section 4 a decision procedure based on minimal models is defined and discussed.

1. Language of PCN

We shall consider two systems that slightly differ in their language — in one of them there is only one specific operator — Leśniewski's 'est', in the other operators constructing positive sentences of syllogistic are also present. We shall refer to those systems as ε-system and F-system respectively. Let us start with presenting the alphabet of PCN on the basis of which we shall define our two languages. It consists of:

- name variables: A, B, C, ...;

[1] We employ the notion of a Horn formula from the theory of logic programming, see e.g. [1]. It will be formally defined in the following section.

[2] Actually the present paper was first submitted to Logic and Logical Philosophy journal before the book [6] was finished.

- binary operators of the calculus of names:
  ε (for Leśniewski's 'est'),
  $\mathfrak{a}$ (for a functor building universal affirmative sentences of syllogistic understood in a "strong" sense),
  $\mathfrak{i}$ (for a functor building particular affirmative sentences of syllogistic),
- operators of classical propositional calculus: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\to$ (implication), $\equiv$ (equivalence).

Moreover, we use the following metalanguage symbols to refer to:

- name variables: $\mathcal{X}$, $\mathcal{Y}$, ...;
- propositional formulae of the system: α, β, γ, ..., α_1, α_2, ...;
- substitutions: e, e_1, e_2, ...;
- assertion: $\vdash$ ('$\vdash \alpha$' means that α is a thesis and '$\nvdash \alpha$' means that α is not a thesis);
- rejection: $\dashv$ ($\dashv \alpha$ means that α is rejected)

Now we can define (in Backus-Naur notation) the language L_ε of ε-system as follows:

$$\alpha ::= \mathcal{X}\ \varepsilon\ \mathcal{X} \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \to \alpha \mid \alpha \equiv \alpha$$

and the language L_F of F-system as follows:

$$\alpha ::= \mathcal{X}\ \varepsilon\ \mathcal{X} \mid \mathcal{X}\ \mathfrak{a}\ \mathcal{X} \mid \mathcal{X}\ \mathfrak{i}\ \mathcal{X} \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \to \alpha \mid \alpha \equiv \alpha,$$

Obviously the latter language is an extension of the former one.

An *atomic formula* (or *atom*) is any formula of the following forms: $\mathcal{X}\varepsilon\mathcal{Y}$, $\mathcal{X}\mathfrak{a}\mathcal{Y}$, $\mathcal{X}\mathfrak{i}\mathcal{Y}$. An arbitrary finite conjunction of atomic formulae will be called *elementary formula*. Any atomic formula and any implication $\alpha \to \beta$, where α is an elementary formula and β is an atomic formula, will be called a *Horn formula*.

2. ε-system and its axiomatic refutation counterpart

We shall use the axiomatisation of ε-system presented by Ishimoto in [2]. This system is defined in the language L_ε by the rules of Modus Ponens (MP) and substitution for name variables (Sub) of the usual schemata. The axioms of this system are: all the substitutions of CPC theses in the language L_ε and the following specific axioms:

(a1) $$A\ \varepsilon\ B \to A\ \varepsilon\ A$$

(a2) $$A \,\varepsilon\, B \wedge B \,\varepsilon\, C \to A \,\varepsilon\, C$$

(a3) $$A \,\varepsilon\, B \wedge B \,\varepsilon\, C \to B \,\varepsilon\, A$$

A formula of L_ε is a *thesis* of ε-system iff it is an axiom of this system or can be obtained from axioms of this system by means of the rules (MP) and (Sub).

The system is complete with respect to a set theoretic model[3] and a separation theorem with respect to Ontology holds for it, i.e. if a quantifier-free formula, built only with the use of name variables, predicate ε and propositional functors, is a thesis of Ontology, then it is a thesis of ε-system (see [2]).

In order to define axiomatic rejection we introduce the rejected axioms ($a1^{-1}$) and the following rejection rules: rejection by Modus Ponens (MP^{-1}), rejection by substitution (Sub^{-1}) and rejection by composition[4] ($Comp^{-1}$). The rejected axiom is the following formula:

($a1^{-1}$) $$A \,\varepsilon\, C \wedge B \,\varepsilon\, C \to A \,\varepsilon\, B.$$

The rules rejection by Modus Ponens and substitution take respectively the following forms:

(MP^{-1}) $$\frac{\vdash \alpha \to \beta \qquad \dashv \beta}{\dashv \alpha}$$

(Sub^{-1}) $$\frac{\dashv e(\alpha)}{\dashv \alpha}$$

where α, β are formulae of L_ε and e is a substitution. Moreover, the rule rejection by composition takes the following form:

($Comp^{-1}$) $$\frac{\dashv \alpha \to \beta_1 \quad \ldots \quad \dashv \alpha \to \beta_n}{\dashv \alpha \to \beta_1 \vee \cdots \vee \beta_n} \quad n \geqslant 1,$$

where $\alpha \to \beta_i$ is a Horn formula of L_ε $(1 \leqslant i \leqslant n)$.

DEFINITION 1. A formula is rejected in ε-system iff it is a rejected axiom ($a1^{-1}$) or can be obtained from other rejected formulae and theses of the system by means of one of the rules: (MP^{-1}), (Sub^{-1}) or ($Comp^{-1}$).

We will show that the system of refutation is adequate, i.e. that any formula is either a thesis of ε-system or is rejected and no formula is both a thesis and a rejected formula.

[3] We shall deal with models for the calculus of names in Section 4.

[4] This rule is equivalent to the one introduced by Słupecki for rejection in Łukasiewicz's system of syllogistic.

By CPC and, respectively, by (MP) or (MP^{-1}) we obtain:

Lemma 1. *If* $\vdash \alpha \equiv \beta$ *then*: $\vdash \alpha$ (*resp.* $\dashv \alpha$) *iff* $\vdash \beta$ (*resp.* $\dashv \beta$).

It is easy to check that rejected axiom ($a1^{-1}$) is false in the standard interpretation of Ontology. That gives the justification for the following observation.

Observation 1. *Rejected axiom* ($a1^{-1}$) *is not a theorem of* ε*-system.*

We also obtain the following observation:

Observation 2. *No rejected formula is a thesis of* ε*-system.*

In the proof of Observation 2 we use the following McKinsey's lemma:

Lemma 2 ([9]). *If all axioms of a quantifier-free theory based on CPC are Horn formulae, then for any elementary formula* α *and any atoms* $\beta_1, \ldots, \beta_n$ *of the language of this theory*: $\vdash \alpha \to \beta_1 \vee \cdots \vee \beta_n$ *if and only if ether* $\vdash \alpha \to \beta_1$ *or* $\ldots$ *or* $\vdash \alpha \to \beta_n$.

Proof of Observarion 2. By induction on the length of (rejection) proof (see Definition 1).

Firstly, by Observation 1, the rejected axiom ($a1^{-1}$) is not a thesis.

Let $\dashv \gamma$ and for some β we have that $\vdash \gamma \to \beta$ and $\dashv \beta$. Then, by the inductive hypothesis, $\nvdash \beta$. So also $\nvdash \gamma$, by (MP).

Let $\dashv \gamma$ and for some e we have that $\dashv e(\gamma)$. Then, by the inductive hypothesis, $\nvdash e(\gamma)$. So also $\nvdash \gamma$, by (Sub).

Let $\dashv \gamma$ and for some $\alpha, \beta_1, \ldots, \beta_n$ we have that $\gamma = \alpha \to \beta_1 \vee \cdots \vee \beta_n$, $\dashv \alpha \to \beta_1, \ldots, \dashv \alpha \to \beta_n$. Then, by the inductive hypothesis, $\nvdash \alpha \to \beta_1, \ldots, \nvdash \alpha \to \beta_n$. So $\nvdash \alpha \to \beta_1 \vee \cdots \vee \beta_n$, by Lemma 2. □

Lemma 3. *The following formulae are rejected*:

(1) $A \,\varepsilon\, B \to B \,\varepsilon\, A$,

(2) $A \,\varepsilon\, B \to B \,\varepsilon\, B$

(3) $B \,\varepsilon\, B$

(4) $\neg A \,\varepsilon\, A$

Proof. For (1):

1. $\vdash_{\mathrm{CPC}} (A\varepsilon C \wedge C\varepsilon B \to A\varepsilon B) \to ((B\varepsilon C \to C\varepsilon B) \to (A\varepsilon C \wedge B\varepsilon C \to A\varepsilon B))$
2. $\vdash A \,\varepsilon\, C \wedge C \,\varepsilon\, B \to A \,\varepsilon\, B$ (a2)
3. $\vdash (B \,\varepsilon\, C \to C \,\varepsilon\, B) \to (A \,\varepsilon\, C \wedge B \,\varepsilon\, C \to A \,\varepsilon\, B)$ (MP): 1, 2
4. $\dashv A \,\varepsilon\, C \wedge B \,\varepsilon\, C \to A \,\varepsilon\, B$ ($a1^{-1}$)

5. $\dashv B \varepsilon C \to C \varepsilon B$ (MP^{-1}): 3, 4
6. $\dashv A \varepsilon B \to B \varepsilon A$ (Sub^{-1}): 5

For (2):

1. $\vdash_{\mathrm{CPC}} (A\varepsilon B \wedge B\varepsilon B \to B\varepsilon A) \to ((A\varepsilon B \to B\varepsilon B) \to (A\varepsilon B \to B\varepsilon A))$
2. $\vdash A \varepsilon B \wedge B \varepsilon B \to B \varepsilon A$ (a3)
3. $\vdash (A \varepsilon B \to B \varepsilon B) \to (A \varepsilon B \to B \varepsilon A)$ (MP): 1, 2
4. $\dashv A \varepsilon B \to B \varepsilon A$ (1)
5. $\dashv A \varepsilon B \to B \varepsilon B$ (MP^{-1}): 3, 4

For (3):

1. $\dashv A \varepsilon B \to B \varepsilon B$ (2)
2. $\vdash B \varepsilon B \to (A \varepsilon B \to B \varepsilon B)$ CPC
3. $\dashv B \varepsilon B$ (MP^{-1}): 2, 1

For (4):

1. $\vdash_{\mathrm{CPC}} (A \varepsilon B \to A \varepsilon A) \to ((A \varepsilon A \to B \varepsilon A) \to (A \varepsilon B \to B \varepsilon A))$
2. $\vdash A \varepsilon B \to A \varepsilon A$ (a1)
3. $\vdash (A \varepsilon A \to B \varepsilon A) \to (A \varepsilon B \to B \varepsilon A)$ (MP): 1, 2
4. $\dashv A \varepsilon B \to B \varepsilon A$ (1)
5. $\dashv A \varepsilon A \to B \varepsilon A$ (MP^{-1}): 3, 4
6. $\vdash \neg A \varepsilon A \to (A \varepsilon A \to B \varepsilon A)$ CPC
7. $\dashv \neg A \varepsilon A$ (MP^{-1}): 6, 5 □

LEMMA 4. *If α is a elementary formula or disjunction of atoms of L_ε, then $\dashv \alpha$ and $\dashv \neg \alpha$.*

PROOF. We substitute 'B' (resp. 'A') for all variables appearing in α (resp. $\neg \alpha$) and as a result we obtain a formula equivalent in CPC to $B \varepsilon B$ (resp. $\neg A \varepsilon A$), which is rejected by Lemma 3. So we use Lemma 1 and (Sub^{-1}). □

LEMMA 5. *Any Horn formula of L_ε is either a thesis or a rejected formula of ε-system.*

PROOF. Let us first notice that, by Lemma 4, any atomic formula is rejected.

There are two possible consequents of a nonatomic Horn formula of the language L_ε: (i) $\mathcal{X} \varepsilon \mathcal{X}$ and (ii) $\mathcal{X} \varepsilon \mathcal{Y}$, where $\mathcal{X} \neq \mathcal{Y}$. We will consider the two cases separately.

(i) Let α be a formula of the form $\beta \to \mathcal{X}\varepsilon\mathcal{X}$, where β is an elementary formula. If β has $\mathcal{X}\ \varepsilon\ Y$ as one of its conjuncts, than α is a thesis, by (a1) and CPC. Otherwise, we can apply a substitution e such that we substitute 'A' for $\mathcal{X}$ and 'B' for any other variable. Then, $e(\beta)$ is a conjunction of atoms from $\{B\ \varepsilon\ B, B\ \varepsilon\ A\}$. Thus, since $B\ \varepsilon\ A \to B\ \varepsilon\ B$ is a substitution of axiom (a1), $\vdash e(\alpha) \to (B\ \varepsilon\ A \to A\ \varepsilon\ A)$ holds in ε-system. Since $B\,\varepsilon\,A \to A\,\varepsilon\,A$ is a rejected formula, by the rule (MP^{-1}) we have $\dashv e(\alpha)$ and by the rule (Sub^{-1}) also $\dashv \alpha$.

(ii) Let α be a formula of the form $\beta \to \mathcal{X}\,\varepsilon\,\mathcal{Y}$, where β is an elementary formula and $\mathcal{X}, \mathcal{Y}$ are different variables. It follows from (a2) and CPC that for any variables $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$, $n \geqslant 2$:

$$(\star) \qquad \vdash \mathcal{Z}_1\ \varepsilon\ \mathcal{Z}_2 \wedge \mathcal{Z}_2\ \varepsilon\ \mathcal{Z}_3 \wedge \ldots \wedge \mathcal{Z}_{n-1}\ \varepsilon\ \mathcal{Z}_n \to \mathcal{Z}_1\ \varepsilon\ \mathcal{Z}_n\,.$$

We will state that $\mathcal{X}$ *is connected to* $\mathcal{Y}$ in β, if β contains $\mathcal{X}\ \varepsilon\ \mathcal{Y}$ as a conjunct or there exist variables $\mathcal{V}_1, \ldots, \mathcal{V}_n$ ($n \geqslant 1$) such that β contains all atoms from $\{\mathcal{X}\ \varepsilon\ \mathcal{V}_1, \mathcal{V}_1\ \varepsilon\ \mathcal{V}_2, \ldots, \mathcal{V}_n\ \varepsilon\ \mathcal{Y}\}$ as conjuncts. Thus, by ($\star$), if $\mathcal{X}$ is connected to $\mathcal{Y}$ in β, then α is a thesis. Moreover, by axiom (a3), if $\mathcal{Y}$ is connected to $\mathcal{X}$ in β and β contains $\mathcal{X}\ \varepsilon\ \mathcal{Z}$ as a conjunct, then α is also a thesis. Furthermore, from that fact and (a2) we can deduce that if there exist variables $\mathcal{Z}$ and $\mathcal{V}$ such that $\mathcal{X}$ and $\mathcal{Y}$ are connected to $\mathcal{Z}$ and β contains atom $\mathcal{Z}\ \varepsilon\ \mathcal{V}$, then α is a thesis as well.[5]

If, on the other hand, β does not contain any of the above mentioned sets of atoms, then α is a rejected formula. We have to consider two subcases: (a) $\mathcal{Y}$ is connected to $\mathcal{X}$ in β and (b) $\mathcal{Y}$ is not connected to $\mathcal{X}$ in β.

(a) β contains no atom of the form $\mathcal{X}\ \varepsilon\ \mathcal{Z}$. We use a substitution e defined for the case (i), and obtain $e(\alpha)$ which is a conjunction of atoms from $\{B\varepsilon B, B\varepsilon A\}$. Thus $\vdash e(\alpha) \to (B\varepsilon A \to A\varepsilon B)$. Since $B\varepsilon A \to A\varepsilon B$ is a rejected formula (see (1) and (Sub^{-1})) we have $\dashv e(\alpha)$, by (MP^{-1}). So $\dashv \alpha$, by (Sub^{-1}).

(b) We use a substitution e_1 such that we substitute 'C' for any variable to which both $\mathcal{X}$ and $\mathcal{Y}$ are connected in β (β does not contain any atom of a form $\mathcal{Z}\ \varepsilon\ \mathcal{V}$ because in such a situation α would be a thesis), 'A' for any other variables to which $\mathcal{X}$ is connected in β and any variables connected to $\mathcal{X}$ in β and, finally, 'B' for any other variable. Since $\mathcal{X}$ and $\mathcal{Y}$ are not connected to each other in β, the $e_1(\beta)$ is a conjunction of atoms from $\{A\ \varepsilon\ C, B\ \varepsilon\ C, A\ \varepsilon\ A, B\ \varepsilon\ B\}$. Thus

[5] The simplest example of such a formula is: $A\ \varepsilon\ C \wedge B\ \varepsilon\ C \wedge C\ \varepsilon\ D \to A\ \varepsilon\ B$.

$\vdash e_1(\alpha) \to (A\ \varepsilon\ C \wedge B\ \varepsilon\ C \to A\ \varepsilon\ B)$. Hence, by (MP^{-1}) and (a1^{-1}), we obtain that $\dashv e_1(\alpha)$. So $\dashv \alpha$, by (Sub^{-1}). □

We are now ready to prove the refutation adequacy.

THEOREM 1 (Refutation adequacy of ε-system). *Any formula of* L_ε *is either a thesis or a rejected formula of* ε*-system and no formula of* L_ε *is both a thesis and a rejected formula.*

PROOF. Since ε-system is built on CPC, for which any formula can be transformed into its conjunctive normal form. So for any formula γ of L_ε there is a formula γ^* such that γ^* has conjunctive normal form and $\gamma \equiv \gamma^* \in$ CPC. Thus, for some formulae $\gamma_1^*, \ldots, \gamma_n^*$ $(n \geqslant 1)$ we have

$$\vdash \gamma^* \equiv \gamma_1^* \wedge \ldots \wedge \gamma_n^*,$$

where for any $i \leqslant n$ there are some atoms $\alpha_1^i, \ldots \alpha_{k_i}^i, \beta_1^i, \ldots, \beta_{m_i}^i$ of L_ε $(k_i + m_i \geqslant 1)$ such that:

$$\vdash \gamma_i^* \equiv \neg\alpha_1^i \vee \cdots \vee \neg\alpha_{k_i}^i \vee \beta_1^i \vee \cdots \vee \beta_{m_i}^i$$

So, there are the following three cases. If $k_i > 0$ and $m_i > 0$, then

$$\vdash \gamma_i^* \equiv \alpha_1^i \wedge \cdots \wedge \alpha_{k_i}^i \to \beta_1^i \vee \cdots \vee \beta_{m_i}^i$$

If $m_i = 0$, then $k_i > 0$ and

$$\vdash \gamma_i^* \equiv \neg(\alpha_1^i \wedge \cdots \wedge \alpha_{k_i}^i)$$

If $k_i = 0$, then $m_i > 0$ and

$$\vdash \gamma_i^* \equiv \beta_1^i \vee \cdots \vee \beta_{m_i}^i$$

Notice that $\dashv \gamma$, if there is $i \leqslant n$ such that:

(a) $k_i = 0$ or
(b) $m_i = 0$ or
(c) $k_i > 0$, $m_i > 0$ and for any $j \leqslant m_i$: $\dashv \alpha_1^i \wedge \cdots \wedge \alpha_{k_i}^i \to \beta_j^i$.

Indeed, in the cases (a) and (b) $\dashv \gamma_i^*$, by lemmas 1 and 4. In the case (c) $\dashv \gamma_i^*$, by Lemma 1 and (Comp^{-1}). Since $\gamma \to \gamma_i^* \in$ CPC, so we use (MP^{-1}).

Thus, we suppose that for any $i \leqslant n$ we have that: $k_i > 0$, $m_i > 0$ and for some $j_i \leqslant m_i$ it is not the case that $\dashv \alpha_1^i \wedge \cdots \wedge \alpha_{k_i}^i \to \beta_{j_i}^i$. Then $\vdash \alpha_1^i \wedge \cdots \wedge \alpha_{k_i}^i \to \beta_{j_i}^i$, by Lemma 5. Hence, by some laws of CPC, for any $i \leqslant n$, $\vdash \gamma_i^*$; so $\vdash \gamma^*$.

To complete the proof we use Observation 2. □

3. F-system

F-system can be defined in L_F as an extension[6] of ε-system obtained by adding the following axioms:

(a4) $$A \,\varepsilon\, B \to A \,\mathfrak{a}\, B$$

(a5) $$A \,\mathfrak{a}\, B \wedge B \,\varepsilon\, C \to A \,\varepsilon\, C$$

(a6) $$A \,\mathfrak{i}\, B \wedge B \,\varepsilon\, B \to B \,\varepsilon\, A$$

(a7) $$A \,\mathfrak{i}\, B \to B \,\mathfrak{a}\, B$$

(a8) $$A \,\mathfrak{a}\, B \to A \,\mathfrak{i}\, B$$

(a9) $$A \,\mathfrak{a}\, B \wedge B \,\mathfrak{a}\, C \to A \,\mathfrak{a}\, C$$

(a10) $$A \,\mathfrak{i}\, B \wedge B \,\mathfrak{a}\, C \to C \,\mathfrak{i}\, A$$

This system is presented in details in [7]. Let us just make a few remarks about it. The axiomatisation of F-system presented here is not independent because axioms (a2) and (a3) can be derived from the remaining ones. Further notions of Leśniewski's Ontology can be introduced to F-system by quantifier-free definitions. The axiomatisation is equivalent to the one presented earlier in [10] — the difference is that axioms (a5) and (a6) are used instead of the formula

(5) $$A \,\mathfrak{a}\, B \wedge B \,\varepsilon\, B \wedge A \,\mathfrak{i}\, C \to A \,\varepsilon\, C$$

from [10], as they are shorter. Pietruszczak presents also other axiomatisations of name calculus in [10, 11].

The system has its refutation counterpart defined by the rejection rules (MP^{-1}), (Sub^{-1}), ($Comp^{-1}$) and the following rejected axioms:

($a2^{-1}$) $$A \,\varepsilon\, C \wedge B \,\varepsilon\, C \to A \,\mathfrak{i}\, B$$

($a3^{-1}$) $$B \,\varepsilon\, B \to A \,\mathfrak{i}\, A$$

Definition 2. A formula is rejected in F-system iff it is a rejected axiom ($a2^{-1}$) or ($a3^{-1}$) or can be obtained from other rejected formulae and theses by means of one of the rules: (MP^{-1}), (Sub^{-1}) or ($Comp^{-1}$).

The following theorem is presented and proved in [7].

Theorem 2 (Refutation adequacy of F-system). *Any formula of the language of F-system is either a thesis or a rejected formula of F-system and no formula of the language is both a thesis and a rejected formula.*

[6] By an extension of a system we understand any system in which one can prove more formulae.

4. Model based decision procedures

4.1. Models for PCN

We shall base our decision procedures on standard for the calculus of names model structures investigated in depth in [10]. Let $M = \langle D, V\rangle$, where D is a nonempty set and V is an interpretation function (valuation) from the set of name variables to the powerset of D, be a model. Satisfaction conditions for atomic formulae are as follows:

$$\begin{aligned} M \models \mathcal{X}\ \varepsilon\ \mathcal{Y} \quad &\text{iff} \quad V(\mathcal{X}) \subseteq V(\mathcal{Y}) \text{ and } |V(\mathcal{X})| = 1\,,\\ M \models \mathcal{X}\ \mathfrak{a}\ \mathcal{Y} \quad &\text{iff} \quad V(\mathcal{X}) \subseteq V(\mathcal{Y}) \text{ and } V(\mathcal{X}) \neq \emptyset\,,\\ M \models \mathcal{X}\ \mathfrak{i}\ \mathcal{Y} \quad &\text{iff} \quad V(\mathcal{X}) \cap V(\mathcal{Y}) \neq \emptyset\,. \end{aligned}$$

Satisfaction for propositional operators is classical. We say that a formula is *valid* (resp. *valid in a class of models*) iff it is satisfied in all models (resp. all models from this class).

LEMMA 6. *Every thesis of* ε*-system* (*F-system*) *is valid.*

PROOF. All axioms are valid and (MP) and (Sub) lead from valid formulae to other valid formulae. □

Both ε-system and F-system are sound and complete with respect to the model. Moreover, Pietruszczak shows in [10, 11] that for ε-system it is enough to consider models with nonempty subsets of a domain only.

In the following sections we will show a decision procedure for both considered systems in which models with domains reduced to two elements are sufficient.

4.2. Models for Horn formulae of ε-system

Let ♡ and ♢ denote arbitrary objects constituting a two-membered domain. Now, let $x := \emptyset$, $y := \{\heartsuit\}$, $z := \{\diamondsuit\}$ and $v := \{\heartsuit, \diamondsuit\}$. To define a class of models adequate for Horn formulae of ε-system[7] we use nonempty subsets of the domain, i.e. sets y, z and v as possible model counterparts of variables. The interpretation of atoms can be presented in the form of the following ε-matrix, in which the value 1 stands for truth and the value 0 for falsehood:

[7] I.e. a class of models such that a Horn formula is a thesis of ε-system iff it is valid in this class. For example, that is the class of models of the form $\langle\{\heartsuit, \diamondsuit\}, V\rangle$, where $V(\mathcal{X}) \neq \emptyset$ for any variable $\mathcal{X}$.

ε	y	z	v
y	1	0	1
z	0	1	1
v	0	0	0

We shall write that a formula is *valid in* ε*-matrix* iff for all valuations of name variables in the set $\{y, z, v\}$, the formula takes the value 1. Thus, to check whether a Horn formula is a thesis of ε-system one should check its validity in the ε-matrix. The following theorem shows that the procedure is accurate.

Theorem 3. *Every Horn formula of* ε*-system is a thesis iff it is valid in* ε*-matrix.*

Proof. "$\Rightarrow$" By Lemma 6, all theses are valid, so they are also valid in ε-matrix.

"$\Leftarrow$" We receive an equivalent statement:

(†) If a Horn formula is not a thesis, then it is not valid in ε-matrix.

Since, by Lemma 5, every Horn formula is either a thesis or a rejected formula, the sentence (†) is further equivalent to:

(‡) If a Horn formula is rejected, then it is not valid in ε-matrix.

Let us now notice that, in the proof of Lemma 5, every rejected Horn formula is rejected using the rules (MP^{-1}) and (Sub^{-1}) only. Thus, to prove (‡) we use induction on the length of (rejection) proof.

Firstly, notice that the rejected axiom ($\mathrm{a1}^{-1}$) is not valid in ε-matrix. It is enough to put: y for 'A', z for 'B' and v for 'C'.

Let $\dashv \alpha$ and for some β we have that $\vdash \alpha \to \beta$ and $\dashv \beta$. Then, by Lemma 6, $\alpha \to \beta$ is valid in ε-matrix. Moreover, by the inductive hypothesis, β is not valid in ε-matrix. So also α is not valid in ε-matrix, since we employ the classical notion of implication.

Let $\dashv \alpha$ and for some substitution e we have that $\dashv e(\alpha)$. Then, by the inductive hypothesis, $e(\alpha)$ is not valid in ε-matrix. So also α is not valid in ε-matrix, since any substitution can only reduce the number of variables and for that reason cannot change a valid formula into a nonvalid one. □

4.3. Models for Horn formulae of F-system

To define a model for F-system we will employ the same domain as for ε-system and use arbitrary subsets of the domain as values for name variables.

The matrices for primitive constants are as follows.

ε	x	y	z	v
x	0	0	0	0
y	0	1	0	1
z	0	0	1	1
v	0	0	0	0

a	x	y	z	v
x	0	0	0	0
y	0	1	0	1
z	0	0	1	1
v	0	0	0	1

i	x	y	z	v
x	0	0	0	0
y	0	1	0	1
z	0	1	1	1
v	0	1	1	1

The following theorem is proved in [7]:

THEOREM 4. *Every Horn formula of F-system is a thesis iff it is valid in* εai*-matrices.*

To compare the results concerning ε-system with F-system let us notice that in both systems any Horn formula which is not a thesis is falsified in a model defined in a two membered domain. However, in the former system the empty set is not used. Thus the general result from [10, 11] can be transferred to minimal models for Horn formulae. Therefore, in ε-system the number of values in the matrix is reduced from 4 to 3.

4.4. Decision procedure

The procedure defined for Horn formulae in both presented systems can be, in a straightforward manner, extended for arbitrary formulae of the systems. Any formula should be syntactically transformed into its conjunctive normal form in CPC (see the proof of Theorem 1). Obviously, a formula is accepted if and only if all conjuncts are accepted. Each of these conjuncts is equivalent to a formula which has one of the following forms (where $k > 0$ and $m > 0$):

$$\alpha_1 \wedge \cdots \wedge \alpha_k \rightarrow \beta_1 \vee \cdots \vee \beta_m$$
$$\neg(\alpha_1 \wedge \cdots \wedge \alpha_k)$$
$$\beta_1 \vee \cdots \vee \beta_m$$

In the first case to check whether a conjunct is accepted or not, it is further transformed into a set of Horn formulae as in the rule (Comp^{-1})

inverted. One can check whether they are accepted or not using the procedure defined in the previous subsections. If some member of the set is accepted, then the formula is also accepted. Otherwise, i.e. if none of them is accepted, the formula is not accepted.

4.5. Comparison with other works

The comparison with the above mentioned results concerning the size of models from [3] and [12] is not straightforward because the systems and types of formulae considered there are different from the ones investigated in the present paper.

Johnson considers the system of syllogistic equivalent to the well known system of Łukasiewicz from [8] and the formulae of a special shape of *syllogistic chains*. He shows that for such formulae models defined in the domain of 3 members are sufficient. In the present paper we also use a special shape of formulae of our language, in this case Horn formulae of the PCN, and we limit the size of the domain to 2.

Pietruszczak, on the other hand, uses the language similar to the one we use. However, he applies the so called weak interpretation of sentences built with the functor '$\mathfrak{a}$', such that for every model $M = \langle D, V\rangle$:

$$M \models \mathcal{X}\,\mathfrak{a}\,\mathcal{Y} \quad \text{iff} \quad V(\mathcal{X}) \subseteq V(\mathcal{Y}).$$

Since the formula '$A\,\mathfrak{a}\,B \vee A\,\mathfrak{i}\,A$' is valid in that interpretation while the formulae '$A\,\mathfrak{a}\,B$' and '$A\,\mathfrak{i}\,A$' are not, the system does not allow for its Horn axiomatisation. This fact is important from the point of view of the present paper since we define the decision procedure on the basis of Horn formulae. Thus, the systems are not fully compatible. Still the functors of one can be defined in terms of the functors of the other (see [10, 11]).

Pietruszczak considers arbitrary formulae of the investigated fragments of the language of PCN. He establishes polynomial dependences of the size of the model required for the refutation of a nonvalid formula on the number of variables occurring in the formula. He determined this dependence for many different systems. For the fragment which is closest to the one from the present paper in which functors 'ε', '$\mathfrak{a}$' and '$\mathfrak{i}$' appear, the limit is $\frac{1}{2}n(n+3)$, where n is the number of variables occurring in a formula (see [12]).

The limit established in the present paper is constant, independent of the number of variables in a given formula, and thus is considerably lower. However, it applies only to Horn formulae. Thus, for the complete

comparison of the resulting procedure, the cost of transformation of an arbitrary formula into its conjunctive normal form has to be taken into account.

5. Conclusions

The main contributions of the paper are: axiomatic refutation counterpart of the system of a quantifier-free Ontology with 'ε' as the only primitive term from [2] and the matrix-based decision procedure for the system.

Adding a syntactic transformation to the model structure enables us to limit the size of models considerably. The size of a minimal model is constant (the domain of the model has two members) regardless of the number of variables in a formula.

The results concerning the system are compared with a similar system enriched with functors 'a' and 'i' of syllogistic. The difference between the systems is that in the simpler one the empty name is not necessary for the model, so we are able to construct a 3-valued matrix, while for the enriched one the empty name is necessary, and the respective matrices are 4-valued.

The paper is limited to logical matters. An attempt at their philosophical interpretation is presented in [5]. More detailed investigations into the computational aspect of the decision procedures are left for future works.

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