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A State-of-Affairs-Semantic Solution to the Problem of Extensionality in Free Logic

If one takes seriously the idea that a scientific language must be extensional, and accepts Quine’s notion of truth-value-related extensionality, and also recognizes that a scientific language must allow for singular terms that do not refer to existing objects, then there is a problem, since this combination of assumptions must be inconsistent. I will argue for a particular solution to the problem, namely, changing what is meant by the word ‘extensionality’, so that it would not be the truth-value that had to be preserved under the substitution of co-extensional expressions, but the state of affairs that the sentence described. The question is whether or not elementary sentences containing empty singular terms, such as ‘Vulcan rotates’, are extensional in the substitutivity sense. Five conditions are specified under which extensionality in the substitutivity sense of such sentences can be secured. It is demonstrated that such sentences are *state-of-affairs-as-extension-related extensional*. This implies (in accordance with the basic idea of state-of-affairs semantics) that such sentences are also truth-value-related extensional in Quine’s sense, but not truth-value-as-extension-related extensional.

states of affairs, non-existence, extension, extensionality, free logic

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1. The problem of extensionality in free logic

A language must meet at least two requirements to be adequate for the purposes of science: it must be extensional in the substitutivity sense, and it must contain empty expressions.

Extensionality is considered by many (e.g., Quine) as a criterion of adequacy for a scientific language. A language is extensional in the substitutivity sense if, and only if, (iff) all of its sentences are extensional in this sense. Furthermore, a sentence is extensional iff it is always possible to substitute for each other within that sentence, expressions which have the same extension, without changing that sentence's extension. Hence, a sentence is non-extensional in the substitutivity sense iff it is not always possible to substitute for each other within that sentence, expressions which have the same extension, without changing that sentence's extension.

The intuitive basis of extensionality is as follows: whatever is true of an individual is independent of the expressions which have the individual as extension; this truth does not suddenly become untrue merely because other expressions can also have the individual as extension. For example, if it is true of an individual that it has a certain property, then it does not "lose" this property due to other expressions also having that individual and its property as extension. All that matters here is that the replacing expressions and the replaced expressions have the same individual and the same property as extensions. Assume that a sentence such as 'The earth rotates' has the extension that a certain individual (the earth) has a certain property (to rotate). Assume furthermore, that the replacing expressions and the replaced expressions have the same individual (the earth) and the same property (to rotate) as extensions. As a result, the extension of our sentence is not changed at all; for in our sentence, expressions are replaced by expressions which have the same extension as the replaced expressions. Hence, the sentences resulting from the substitutions under consideration have the extension that the earth has the property to rotate. Therefore, our sentence is extensional in the substitutivity sense. The first requirement thus demands that all sentences of a scientific language must be extensional in this sense.

Additionally, even in the scientific use of language, one can find examples of expressions, namely, singular terms which are *empty*; that is, having either nothing or something non-existent as extension; for instance, ‘the division of 1 by 0’, or ‘Vulcan’ – Vulcan[1] being a planet that causes, according to Leverrier, the perturbations in the orbit of Mercury. It turned out only later that Vulcan is non-existent. Science, as we know from its praxis, sometimes can only postulate the existence of certain objects. It must introduce singular terms that have these allegedly existent objects as extensions, thereby rendering one unsure as to whether or not these objects actually exist. It has even turned out that some of these allegedly existent objects do not exist. Hence, the singular terms introduced in order to speak of such objects are empty. For this reason, a scientific language must not only be extensional in the substitutivity sense, but it must also contain empty singular terms as a means to reconstructing adequately and analyzing logically the real scientific use of language in a fair way.

Lambert, one of the founders of free logic, puts forth a *metasemantic* argument, demonstrating that a language which contains such empty singular terms is non-extensional in the substitutivity sense. He argues that in single-domain-based free logic, it is not always possible to substitute co-extensional general terms (or predicates) for each other within elementary sentences containing empty singular terms, without changing the truth-value. And therefore, our two requirements for a scientific language seem to be faced with a dilemma: the admissibility of empty singular terms renders the affected language non-extensional. With regard to both requirements, the *problem of extensionality* arises in free logic with the question of whether and how extensionality in the substitutivity sense of a language containing empty singular terms can be secured.

In this paper, I will examine the problem of extensionality from a state-of-affairs semantic perspective, and I will propose a solution which proceeds from the assumption that sentences describe states of affairs (abbreviated: ‘states’). According to my proposed solution, the extensions of sentences are the states which they describe; that is, states are the *descripta* of sentences. The aim is to determine, from a state-of-affairs-semantic perspective, the conditions under which extensionality in the substitutivity sense of all elementary sentences containing empty singular terms (such as ‘Vulcan rotates’) can be secured in the framework of free logic. I will argue that there is no loss in going from *salva veritate* substitution of co-extensional general terms (or predicates)

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[1] For Vulcan see Baum [2], Lambert [7, p. 33], and Lambert [10, p. 35].

to *salvo statu rerum* substitution of co-extensional general terms (or predicates); the words ‘status rerum’ are the Latin translation[2] of ‘state of affairs’.

[2] I am indebted to Rodrigo for this translation.

2. Reformulating the problem of extensionality in terms of states

In philosophical logic, various semantic systems[3] are based on states. States might be understood as possible outcomes of actions and events, and as bearers of probabilities and metaphysical modalities. States are not propositions, neither in Frege's theory nor Russell's. While propositions are truth bearers (i.e., something that represents something else), states are never such truth bearers, but are only descripta or truth makers. Thus, propositions (or sentences) represent something else. For this reason, they can be true or false. However, states do not represent anything else. They are simply part of the world. To be true or false, something else must be represented. Since states do not represent anything else, they can be neither true nor false. However, they can obtain or not obtain, that is, hold or not hold.

What are states? A state is a way in which things in the world behave or do not behave (or might behave or might not behave). A state obtains (holds) iff things in the world behave in such a way as is demonstrated by that state. Two states are identical iff they involve the same objects and have the same necessary and sufficient condition of obtaining. An object (existent or non-existent) is everything that forms a unity and can thus be distinguished from other such unities. Assume that sentences can describe states. A sentence is true iff it describes a state that obtains.

Many ontological questions arise concerning the states that involve non-existent objects. One example would be the question of how their unity comes about. To answer this question, one can argue that two kinds of property[4] are predicable of non-existent objects. Some properties can serve to characterize how an object is (or what it is); thus, they can serve to determine the object's being-so, independently[5] of its being. Such properties are called 'nuclear properties'. The *being-so* can be understood as a set of subsets of the domain or as a set, or complex, or combination of nuclear properties,

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[3] For truth maker state-of-affairs semantics see van Fraassen [16], Fine [3] and for descripta state-of-affairs semantics see Taylor [15], Forbes [4].

[4] According to the common set-theoretical view, properties can be understood as subsets of the domain, and to exemplify a property can be understood as to be an element of the corresponding set.

[5] See Lambert [6] for the principle of the independence of being-so from being.

respectively. Thus, all properties that lie in a non-existent object's being-so are predicable of non-existent objects. Furthermore, non-nuclear "properties" are predicable of non-existent objects (such as: to exist, to be fictitious, to be self-identical, to be non-existent). For example, Vulcan and Sherlock Holmes form unities that can be distinguished from each other: while being a non-existent planet is truthfully predicable of Vulcan but not of Sherlock Holmes, being a fictitious detective is truthfully predicable of Sherlock Holmes but not of Vulcan. The unity of states involving non-existents then comes about either by the logical possibility of the predication of a property to a non-existent object, or by the logical possibility of the predication of a non-existent object's being-so to a property. There is a connection between these two types of predicability: A nuclear property is predicable of a non-existent object iff the non-existent object's being-so is predicable of the nuclear property. In other words, a non-existent object can exemplify a nuclear property iff the nuclear property can exemplify the non-existent object's being-so. For example, the non-existent object Vulcan can have the nuclear property of being a planet iff this property can lie in the combination of nuclear properties of Vulcan (in other words, can be among the nuclear properties of Vulcan).

Such states involving non-existents can obtain or not obtain in a way that is independent of whether they exist or do not exist. For example, the state that Sherlock Holmes is a fictitious detective obtains because certain things in the world behave in such a way as to create and maintain the fictitious detective Sherlock Holmes. For another example, the state that Vulcan is an existing planet does not obtain because certain things in the world do not behave in such a way as to render Vulcan an existing planet. Hence, whether such states obtain (or do not obtain) depends on certain things in the world behaving (or not) in accordance with what is demonstrated by these two states. The capability of either state to obtain or not obtain does not depend on its having existence (being). This is why the question of whether such states involving non-existents exist or not (or have being or not) can remain unanswered.

With regard to the question of extensionality, the original problem is that in single-domain-based free logic, it is not always possible to substitute co-extensive general terms (or predicates) for each other within elementary sentences containing empty singular terms, without changing the truth-value. This is established by Lambert's Non-

Extensionality-Argument[6]. I will reformulate this argument in terms of states, and I will propose a solution to the reformulated problem.

Assume that in single-domain-based free logic, the general terms (or predicates) (i) ‘(is) a thing such that it rotates’, (ii) ‘(is) a thing such that it rotates and it exists’, and (iii) ‘(is) a thing such that it rotates if it exists’ are co-extensive, that is, true (or false) of the same objects. One can now consider the states that are described by elementary sentences containing empty singular terms such as ‘Vulcan rotates’. Such states might obtain or not, or even have neither of these properties. Thus, three cases can be distinguished.

First case. Assume that the state described by the sentence

(1) Vulcan is a thing such that it rotates

obtains. Replacing the general term (or predicate) (i) in (1) with the co-extensive general term (or predicate) (ii) results in the sentence

(2) Vulcan is a thing such that it rotates and it exists.

To determine whether the state described by the sentence in (2) obtains, one can consider the state that is described by the following sentence

(3) Vulcan is a thing such that it rotates and Vulcan is a thing such that it exists.[7]

This state clearly does not obtain because this conjunctive state has another state as part, namely, the state that Vulcan exists. The latter state, however, clearly does not obtain because things in the world do not behave in such a way that Vulcan exists. Therefore, our conjunctive state does not obtain here because one of the two states combined in it does not obtain. The question now is whether the state described by (2) is identical to the state described by (3). To answer this question, identity criteria for states must be considered. If these two states are taken as being identical, then the state described by (2) does not obtain because it is identical to a state that does not obtain, namely, the state described by (3).

Second case. Assume that the state described by (1) does not obtain, and now replace the general term (or predicate) (i) in (1) with the co-extensive general term (or

[6] For this argument see Lambert [9, pp. 22–24], [8, pp. 95–97, pp. 109 f.], [11, pp. 278 f., p. 282], and [5, pp. 257–259].

[7] (3) results from applying the classical principle of term abstraction to (2).

predicate) (iii). Then, under the same procedure as in the first case, the state described by the resulting sentence turns out to be an obtaining one.

Third case. Assume that the state described by the sentence in (1) is neither an obtaining state nor a non-obtaining one. Then, as in the first case, the state described by (2) does not obtain.

The original problem is thus restated in terms of states: it is not always possible to substitute co-extensive general terms (or predicates) for each other within (1), without necessitating the changing of an obtaining state to a non-obtaining one, or vice versa; furthermore, in the third case, a state that is neither an obtaining state nor a non-obtaining one is changed to a non-obtaining state.

One can raise three objections to this argument. Firstly, one can argue that the general terms (or predicates) involved in the argument are not co-extensive. If the condition of being a non-existing planet is truthfully predicable of Vulcan, then all properties included in the property of being a planet are also truthfully predicable of Vulcan. If the property of being a rotating thing is included in the property of being a planet, then being a rotating thing must be truthfully predicable of Vulcan. Thus, the state described by (1) obtains if the condition of being a non-existing planet is truthfully predicable of Vulcan. However, the property of being a rotating thing is truthfully predicable of Vulcan only insofar as the condition of being a *non-existing* rotating thing is truthfully predicable of Vulcan, and not insofar as the condition of being an *existing* rotating thing is truthfully predicable of Vulcan because Vulcan does not exist. Hence, if being a rotating thing is truthfully predicable of Vulcan, then being a non-existing rotating thing is truthfully predicable of Vulcan. Furthermore, if being a non-existing rotating thing is truthfully predicable of Vulcan, then being a rotating thing and being an existing rotating thing are not truthfully predicable of the same objects. Therefore, the expressions '(is) a rotating thing' and '(is) an existing rotating thing' are no longer co-extensive if the property of being a rotating thing and the condition of being a non-existing rotating thing are both truthfully predicable of Vulcan (see (1) and (2)). This objection does not rely on the particular expression '(is) a rotating thing'; any such expression will do. For example, if the property of being a self-identical thing and the condition of being a non-existent self-identical thing are both truthfully predicable of

Vulcan, then the expressions ‘(is) a self-identical thing’ and ‘(is) an existing self-identical thing’ are not true of the same objects and thus are not co-extensive.

Secondly, if the states described by (2) and (3) turn out, however, to be non-identical, then the original problem (as restated in terms of states) can be resolved. An empty singular term either refers to nothing (that is, is irreferential) or refers to something non-existent (that is, is referential). If the singular term ‘Vulcan’ is irreferential, then it contributes no object to the state described by the sentence ‘Vulcan rotates’. Hence, it seems as though an object is missing in such states. My idea is to introduce non-existent objects and their condition of being-so, respectively, as markers of the “holes” in such states: certain set-theoretical objects can be considered, respectively, as formal-ontological representations of non-existent objects, and their (condition of) being-so. In this way, the necessary and sufficient condition of the obtaining of such states can be defined in the framework of a model[8] for free logic. The question now is: what are the appropriate hole-markers for the various single-domain-based systems of free logic?

For instance, in negative free logic such a hole-marker might be a non-existent object. This object can be represented by a non-element of the domain (e.g., the domain, its power set, etc.) if one assumes that no set has itself as member. The state described by the sentence ‘Vulcan rotates’ involves, then, a non-existent object (Vulcan), as represented by a non-element of the domain, and the set of rotating individuals of the domain. This state obtains iff the chosen non-element of the domain (that is, the formal-ontological representation of Vulcan) lies in the set of rotating individuals of the domain – a situation which is (as intended) never the case. As is required by negative free logic, this assumption guarantees that all elementary sentences containing empty singular terms are false if one assumes bivalence.

By contrast, in single-domain-based positive free logic, such a hole-marker might be a combination of properties of a non-existent object. This combination can be represented by a set of subsets of the domain (that is, the formal-ontological representation of the being-so of Vulcan). The state described by the sentence ‘Vulcan

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[8] For the models see Nolt [13, pp. 1032 f.], Morscher et al. [12, pp. 13 f.], and Antonelli [1, pp. 280–282].

rotates' involves, then, the being-so of the non-existent Vulcan, as represented by a set of subsets of the domain, and the set of rotating individuals of the domain. This state obtains iff the set of rotating individuals of the domain lies in the chosen set of subsets of the domain, which might or might not be the case. As is required by positive free logic, this assumption guarantees that some elementary sentences containing empty singular terms are true.

One can now easily verify that the predicative state (see (2)) and the conjunctive state (see (3)) do not involve the same objects: while the predicative state involves a hole-marker and the set of rotating individuals of the domain, the conjunctive state involves two states, namely, the two states that are its parts. According to the above identity criterion for states, the predicative state and the conjunctive state are therefore not identical.

Finally, one can prove that all elementary sentences containing empty singular terms, such as 'Vulcan rotates', are extensional in the substitutivity sense if one proceeds from the following five assumptions:

(A) Extensionality in the substitutivity sense is based on strong co-extensionality.

(B) Sentences describe states, and the states which they describe are their extensions.

(C) The two formal-ontological interpretations (explained above) introduce non-existent objects as hole-markers.

(D) Two states, described by two elementary sentences containing the same irreferential singular term, involve the same hole-marker (and if not, then non-extensionality is unavoidable).

(E) Two states are identical iff they involve the same objects and have the same necessary and sufficient condition of obtaining.

My proof can be found in section 4.

3. Various notions of extensionality in the substitutivity sense

In what follows, I define (and place against each other) two meanings of co-extensionality: weak versus strong co-extensionality. An *extension function* is a function *ext* from a set of expressions capable of having extensions to a set of objects admissible as extensions. An object *o* is the *extension* of an expression *A* at an extension function *ext* iff $ext(A) = o$. Furthermore, an expression *A has an extension* at an extension function *ext* iff there is an object *o* such that *ext*(*A*) = *o*.

(Df.WC) An expression A_1 is *weakly co-extensional* at an extension function *ext* to an expression A_2 :⇔

for all objects o_1, o_2 (if $ext(A_1) = o_1$ and $ext(A_2) = o_2$, then $o_1 = o_2$).[9]

Thus, two expressions in the same syntactical category are weakly co-extensional if they have no extension. By contrast,

(Df.SC) an expression A_1 is *strongly co-extensional* at an extension function *ext* to an expression A_2 :⇔

A_1 is weakly co-extensional at *ext* to A_2, and

there is an object o_1 such that ($ext(A_1) = o_1$), and

there is an object o_2 such that ($ext(A_2) = o_2$).

Strong co-extensionality thus implies weak co-extensionality, but not vice versa.[10]

Since the words 'to have the same extension' mean the same as 'to be co-extensional', the notion of extensionality in the substitutivity sense can be defined as follows:

(Df.SE) A sentence S_1 is *extensional* in the substitutivity sense :⇔

it is always possible to substitute co-extensional expressions for each other within S_1 salva extensione.[11]

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[9] Read the metalinguistic sign ':⇔' as 'is definitional equivalent to'.

[10] For ease of reading, I have suppressed the relativization of co-extensionality to an extension function in the other parts of the paper.

[11] That is, iff for all substitution-results S_2 and expressions A_1, A_2: if A_1 is co-extensional to A_2, then S_1 is co-extensional to S_2. Let the quantifier in 'for all substitution-results S_2' range over the results of replacing one or more occurrences of an expression A_1 in a sentence S_1 with an expression A_2 that is in the same syntactical category as A_1.

Hence, a sentence is *non-extensional* in the substitutivity sense iff it is not always possible to substitute co-extensional expressions for each other within that sentence salva extensione. The question of what kinds of objects (truth-values, states, etc.) the extensions of sentences are is left open. Thus, extensions are considered in a general and neutral way, and without regard for any specific choices of objects as particular extensions of sentences. When developing formal semantics one might well say that the set-theoretical objects, chosen as representatives for the extensions of sentences, are not yet philosophically interpreted as representatives for truth-values, states, or whatever. Since every property of extensionality in the substitutivity sense can be defined via one of the two relations of co-extensionality, one can accordingly distinguish at least two meanings of extensionality, as well as non-extensionality, in the substitutivity sense. The reason for doing so is that in the next section sentences turn out to be non-extensional in one of these two meanings, while turning out extensional in the other. Hence, due to the ambiguity of co-extensionality, (Df.SE) is a definition-schema: that is, a schema to construct several explicit definitions which have an analogous form.

Furthermore, the concept of *truth-value-as-extension-related* extensionality can be defined as follows:

(Df.SV) A sentence S_1 is *extensional* in the substitutivity-salva-veritate sense :⇔
S_1 is extensional in the substitutivity sense, and
truth-values are the extensions of sentences.

Finally, the notion of *state-of-affairs-as-extension-related* extensionality can be defined as follows:

(Df.SS) A sentence S_1 is *extensional* in the substitutivity-salvo-statu-rerum sense :⇔
S_1 is extensional in the substitutivity sense, and
states are the extensions of sentences.

4. Proving extensionality in the substitutivity sense of elementary sentences containing empty singular terms

My approach to proving the state-of-affairs-as-extension-related extensionality (Df.SS) – and thereby also the extensionality in the substitutivity sense (Df.SE) – of all elementary sentences containing empty singular terms, such as

(4) Δ*y*(*y* rotates)Vulcan,

proceeds from the five assumptions (A)–(E) laid out in section 2.

(a) Basic idea of proof

I first introduce a notation for predicative states that involve non-existent objects and their being-so, respectively. The construction '$s[o_1, o_2]\varepsilon$' means 'the state *s* that involves the objects o_1, o_2 (in that order) at an exemplification relation ε'. The unity of $s[o_1, o_2]\varepsilon$ comes about either by the logical possibility of the predication of the property o_2 to the individual o_1, or by the logical possibility of the predication of the combination of properties o_2 to the property o_1 (see section 2). An *exemplification relation* ε can be defined as a set of ordered pairs of its two relata o_1, o_2, where either an individual o_1 exemplifies a property o_2, or a property o_1 exemplifies a combination of properties o_2: $\varepsilon = \{\langle o_1, o_2\rangle | o_1 \in o_2\}$. Let *M* be a model for single-domain-based free logic. Then $s[o_1, o_2]\varepsilon$ obtains in a model *M* iff o_1 exemplifies o_2, that is, iff $\langle o_1, o_2\rangle \in \varepsilon$ iff $o_1 \in o_2$. The elementary sentence (4) then has one of the following states s_1, s_2, under one of the two formal-ontological interpretations, as extension (see section 2):

(i) Let *M* be a model for negative free logic. Then

s_1 = *ext*(Δ*y*(*y* rotates)Vulcan) = *s*[**v***, *ext*(Δ*y*(*y* rotates))]ε, and

v* = Vulcan, and

the hole-marker **v*** is represented by a non-element of the domain *D*, and

ext(Δ*y*(*y* rotates)) = the set of rotating individuals of the domain *D*, and

s_1 obtains in a model *M* iff **v*** exemplifies *ext*(Δ*y*(*y* rotates)), that is, iff Vulcan lies in the set of rotating individuals of the domain *D*.

Or

(ii) Let *M* be a model for single-domain-based positive free logic. Then

s_2 = *ext*(Δ*y*(*y* rotates)Vulcan) = *s*[*ext*(Δ*y*(*y* rotates)), **v***]ε, and

ext(Δ*y*(*y* rotates)) = the set of rotating individuals of the domain *D*, and

v* = the being-so of Vulcan, and

the hole-marker **v*** is represented by a set of subsets of the domain *D*, and

s_2 obtains in a model *M* iff *ext*(Δ*y*(*y* rotates)) exemplifies **v***, that is, iff the set of rotating individuals of the domain *D* lies in the being-so of Vulcan.

Substitute in (4) expressions that are *strongly* co-extensional to the empty singular term 'Vulcan', or the general term 'Δ*y*(*y* rotates)', or the elementary sentence (4). Demonstrate that both the state which is the extension of (4) under one of the two formal-ontological interpretations, and the state which is the extension of the respective result of substituting co-extensional expressions, involve the same objects. Demonstrate that both states have the same necessary and sufficient condition of obtaining. Then both states are identical according to the criterion of identity for states (E). Hence, it is always possible to substitute strongly co-extensional expressions for each other within (4) without changing the state as *extension*. Therefore, the elementary sentence (4) is extensional in the substitutivity-salvo-statu-rerum sense (Df.SS), and thus extensional in the substitutivity sense (Df.SE). Since the same applies to all other elementary sentences containing empty singular terms, such as (4), all such sentences are extensional in the substitutivity sense (Df.SE).

(b) The case of co-extensional singular terms

We must now consider empty singular terms that are co-extensional to 'Vulcan'. An empty singular term can be either irreferential or referential.

(i) In negative free logic, all empty singular terms are irreferential.

(ii) In single-domain-based positive free logic, some empty singular terms are irreferential and some are referential.

All irreferential singular terms are weakly co-extensional because they have no extension. Replacing in (4) the irreferential singular term 'Vulcan' with a *different* irreferential singular term, such as 'Selena', might result in a predication that describes (e.g., according to the second formal-ontological interpretation) something that is different from the state s_2 that (4) describes. For example,

s_3 = *ext*(Δ*y*(*y* rotates)Selena) = *s*[*ext*(Δ*y*(*y* rotates)), **s***]*ε*, and

ext(Δ*y*(*y* rotates)) = the set of rotating individuals of the domain *D*, and

s* = the being-so of Selena, and

the hole-marker **s*** is represented by a set of subsets of the domain *D*; **s*** is possibly different from the being-so of Vulcan; Selena is yet another non-existent planet of our planetary system.

Therefore, such irreferential singular terms endanger extensionality in the substitutivity-salvo-statu-rerum sense (Df.SS) as long as extensionality is based on *weak* co-extensionality. However, irreferential singular terms cannot be strongly co-extensional to the irreferential singular term 'Vulcan' because they have no extension. Hence, one cannot replace in (4) the irreferential singular term 'Vulcan' with a singular term that is strongly co-extensional to 'Vulcan', due to there being no such strongly co-extensional irreferential singular term. Therefore, such irreferential singular terms cannot endanger extensionality in the substitutivity-salvo-statu-rerum sense as long as extensionality is based on *strong* co-extensionality. For this reason, extensionality in the substitutivity-salvo-statu-rerum sense must be based on strong co-extensionality. Observe that basing extensionality on strong co-extensionality does not imply that irreferential singular terms are treated as being referential within my way of proving extensionality: irreferential singular terms remain irreferential, though the holes in the states described by elementary sentences containing such irreferential singular terms are marked by non-existent objects.

Replacing in (4) the referential singular term 'Vulcan' with singular terms that are strongly co-extensional to 'Vulcan' results in sentences which have as extensions, under the second formal-ontological interpretation, the state s_2 in (a)(ii). Hence, the sentence in (4) and the respective results of substitution describe the same state. In negative free logic, such substitutions are not possible because all empty singular terms are irreferential.

(c) The case of co-extensional elementary sentences

Recall the assumption that (4) has an extension, and then consider that there are, however, no sentences without an extension that could be strongly co-extensional to (4); the former have no extensions, whereas the latter has an extension. Therefore, such sentences without an extension cannot endanger extensionality in the substitutivity-salvo-statu-rerum sense as long as extensionality is based on strong co-extensionality.

Replacing in (4) the elementary sentence (4) with sentences that are strongly co-extensional to (4) results in sentences that have as extensions, under one of the two formal-ontological interpretations, either the state s_1 in (a)(i) or s_2 in (a)(ii). Hence, (4) and the respective results of substitution describe the same state.

(d) The case of co-extensional general terms (or predicates)

Observe that general terms (or predicates) always have an extension if there is an empty set. Hence, the co-extensionality of general terms (or predicates) is always their *strong* co-extensionality. Assume that '(Δ*yA*)' is a general term that is co-extensional to the general term 'Δ*y*(*y* rotates)'. Replacing in (4) the general term 'Δ*y*(*y* rotates)' with the co-extensional general term '(Δ*yA*)' results in the following elementary sentence:

(5) (Δ*yA*)Vulcan.

With regard to the substitution of co-extensional general terms in (4), one must distinguish two sub-cases.

(i) In negative free logic, all empty singular terms are irreferential. The extension of an elementary sentence that contains an irreferential singular term, such as 'Vulcan', is a state that involves two objects, namely, a non-existent object as hole-marker and the extension of the general term that is part of this elementary sentence. Assume that the extensions of two elementary sentences containing the same irreferential singular term involve the same non-existent object as hole-marker. Then

s_4 = *ext*(Δ*y*(*y* rotates)Vulcan) = *s*[**v***, *ext*(Δ*y*(*y* rotates))]*ε*, and

v* = Vulcan, and

the hole-marker **v*** is represented by a non-element of the domain *D*, and

ext(Δ*y*(*y* rotates)) = the set of rotating individuals of the domain *D*.

And

s_5 = *ext*((Δ*yA*)Vulcan) = *s*[**w***, *ext*(Δ*yA*)]*ε*, and

w* = Vulcan, and

the hole-markers **w*** and **v*** are represented by the same non-element of the domain *D*, and

ext(Δ*yA*) is a subset of the domain *D*.

Observe that – due to the co-extensionality of both general terms – both sets, which are their extensions, are identical. Hence, the two predications (4) and (5) have as extensions states that involve the same non-existent object (namely, Vulcan), as represented by the same non-element of the domain, and identical subsets of the domain. Since s_4 obtains iff s_5 obtains, both states have the same necessary and sufficient condition of obtaining. Therefore, the two predications (4) and (5) have the same object as extension if they have an extension. This extension is the state that the same non-existent object (Vulcan), as represented by the same non-element of the domain, lies in identical subsets of the domain. This state is, in other words, the state that the same non-existent object (Vulcan) exemplifies an identical property (to rotate).

(ii) In single-domain-based positive free logic, some empty singular terms are irreferential. The extension of an elementary sentence containing the irreferential singular term 'Vulcan' is a state that involves two objects, namely, a being-so of a non-existent object as hole-marker and the extension of the general term that is part of this elementary sentence. Assume that the extensions of two elementary sentences that contain the same irreferential singular term involve the same being-so of a non-existent object as hole-marker. Then

s_6 = *ext*(Δ*y*(*y* rotates)Vulcan) = *s*[*ext*(Δ*y*(*y* rotates)), **v***]*ε*, and

ext(Δ*y*(*y* rotates)) = the set of rotating individuals of the domain *D*, and

v* = the being-so of Vulcan, and

the hole-marker **v*** is represented by a set of subsets of the domain *D*.

And

s_7 = *ext*((Δ*yA*)Vulcan) = *s*[*ext*(Δ*yA*), **w***]*ε*, and

ext(Δ*yA*) is a subset of the domain *D*, and

w* = the being-so of Vulcan, and

the hole-markers **w*** and **v*** are represented by the same set of subsets of the domain *D*.

Observe that (due to the co-extensionality of both general terms) both subsets of the domain, which are their extensions, are identical. Hence, the two predications (4) and (5) have as extensions states that involve identical subsets of the domain and the same being-so of Vulcan, as represented by the same set of subsets of the domain. Since s_6 obtains iff s_7 obtains, both states have the same necessary and sufficient condition of obtaining. Therefore, the two predications (4) and (5) have the same object as extension if they have an extension. This extension is the state that identical subsets of the domain lie in the same being-so of Vulcan, as represented by the same set of subsets of the domain. This state is, in other words, the state that an identical property (to rotate) exemplifies the same combination of properties of the non-existent object Vulcan.

The sub-case where the empty singular term 'Vulcan' is referential was already covered in (d)(ii), though the assumption (D) is not needed here and 'Vulcan' refers to the being-so of Vulcan.

In light of the two sub-cases (i)–(ii), co-extensional general terms (or predicates) are always substitutable for each other within the predication (4) salvo statu rerum.

(e) Conclusion and consequences

As demonstrated in the three cases (b)–(d), it is always possible to substitute strongly co-extensional expressions for each other within (4) 'Δ*y*(*y* rotates)Vulcan' without changing the state as extension. Therefore, if one proceeds from the assumptions (A)–(E), then (4) turns out to be extensional in the substitutivity-salvo-statu-rerum sense (Df.SS), and thus extensional in the substitutivity sense (Df.SE).

Nevertheless, the predication (4) is trivially non-extensional in the substitutivity-salva-veritate sense (Df.SV) because truth-values are here no longer the extensions of sentences.

5. Extensionality in the substitutivity-salvo-statu-rerum sense implies Quine's truth-value-related extensionality

In what follows, the fact that two sentences have the same truth-value is not the fact that they have the same truth-value *as* extension; states already play the role of extensions of sentences, and this implies that truth-values can no longer be such extensions.

The basic idea of descripta state-of-affairs semantics for a logical system can be expressed as follows:

(DS) A sentence *S* has the truth-value *True* in a model *M* (for that logical system) iff there is a state *s* such that *S describes s* and *s obtains* in *M*.

Hence, in bivalent state-of-affairs semantics, a sentence *S* has the truth-value *False* in a model *M* if (i) there is no state *s* such that *S* describes *s*, or if (ii) there is a state *s* such that *S* describes *s*, but *s* does not obtain in *M* (however, here (i) is already excluded because – due to assumption (B) – sentences describe states).

Moreover, if one assumes this basic idea (DS), then the extensionality result of section 4 has the highly significant consequence that all elementary sentences containing empty singular terms, such as (4) 'Δy(*y* rotates)Vulcan', are extensional in Quine's sense of truth-value-related extensionality. Since the predication (4) is extensional in the substitutivity-salvo-statu-rerum sense, it is always possible to substitute strongly co-extensional expressions for each other within (4) without changing the state as extension. Hence, the predication (4) and all predications resulting from (4), by substituting strongly co-extensional expressions, describe the same state. If one assumes (DS), then the fact that these predications describe the same state implies that they have the same truth-value. These predications have the truth-value True if they describe an obtaining state. And these predications have the truth-value False if they describe a non-obtaining one. Hence, it is always possible to substitute strongly co-extensional expressions for each other within the predication (4) without changing the truth-value. Consequently, the predication (4) is extensional in the sense of Quine's conception (Df.Q), according to which a sentence is extensional iff it is always possible to substitute strongly co-extensional expressions for each other within the sentence salva veritate (that is, the notion of *truth-value-related* extensionality).[12] Since (4) was chosen arbitrarily, it follows that all elementary sentences with empty singular terms, such as the predication (4), are extensional in the truth-value-related sense (Df.Q) if they are extensional in the substitutivity-salvo-statu-rerum sense (Df.SS). Thus, there is no loss in going from salva veritate substitution of co-extensional general terms (or predicates) to salvo statu rerum substitution of co-extensional general terms (or predicates) because co-extensionality between general terms (or predicates) is always strong co-extensionality.

Finally, I would like to remark that it holds generally that a sentence's extensionality in the substitutivity-salvo-statu-rerum sense (Df.SS) implies its extensionality in the truth-value-related sense (Df.Q) (both notions of extensionality being based on strong co-extensionality). Assume that a sentence S_1 is extensional in the substitutivity-salvo-statu-rerum sense. Then, this sentence S_1 and all sentences resulting from S_1, by substituting strongly co-extensional expressions, describe the same state. Furthermore, it follows from these sentences describing the same state – by means of (DS) – that they have the same truth-value; namely, the truth-value True if the state described by these sentences obtains, and the truth-value False if this state does not obtain. Therefore, the sentence S_1 is extensional in the truth-value-related sense (Df.Q) if it is extensional in the substitutivity-salvo-statu-rerum sense (Df.SS) (both notions of extensionality being based on strong co-extensionality).

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[12] (Df.Q) A sentence S_1 is *extensional* in the truth-value-related sense :⇔ it is always possible to substitute strongly co-extensional singular and general terms (or predicates) and sentences for each other within S_1 salva veritate; that is, iff for all substitution-results S_2 and expressions A_1, A_2: if A_1 is strongly co-extensional to A_2, then S_1 has the same truth-value as S_2. Observe that the expressions capable of having extensions are confined here to singular and general terms (or predicates) and sentences because Lambert, following Quine [14], restricts them in his metasemantic argument to such expressions.

6. Summary

The five assumptions (A)–(E) imply that all elementary sentences containing empty singular terms, such as (1) 'Vulcan rotates', are extensional in the substitutivity-salvo-statu-rerum sense (Df.SS) – and thus extensional in the substitutivity sense (Df.SE). I have demonstrated that if all elementary sentences containing empty singular terms, such as (1), are extensional in the substitutivity-salvo-statu-rerum sense, then such sentences are – due to (DS) – extensional in the sense of Quine's truth-value-related conception of extensionality (Df.Q). Nevertheless, these sentences are, then, trivially non-extensional in the sense of the truth-value-as-extension-related conception of extensionality (Df.SV) because truth-values are no longer the extensions of sentences. Furthermore, I have demonstrated that substitutivity-salvo-statu-rerum extensionality (Df.SS) implies truth-value-related extensionality (Df.Q) if one presupposes the basic idea of descripta state-of-affairs semantics (DS) and assumes that extensionality is based on strong co-extensionality. Therefore, the substitutivity-salvo-statu-rerum conception of extensionality (Df.SS) serves the purposes of Quine's truth-value-related conception of extensionality (Df.Q).

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