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The Hyperintensional Variant of Kaplan's Paradox

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Abstract
David Kaplan famously argued that mainstream semantics for modal logic, which identifies propositions with sets of possible worlds, is affected by a cardinality paradox. Takashi Yagisawa showed that a variant of the same paradox arises when standard possible worlds semantics is extended with impossible worlds to deliver a hyperintensional account of propositions. After introducing the problem, we discuss two general approaches to a possible solution: giving up on sets and giving up on worlds, either in the background semantic framework or in the corresponding conception of propositions. As a result, we conclude that abandoning worlds by embracing a truthmaker-based approach offers a promising way to account for hyperintensional propositions without facing the paradoxical outcome.

Keywords Hyperintensionality · Impossible worlds semantics · Kaplan's paradox · Truthmaker semantics

1 Introduction

According to David Kaplan, a cardinality paradox lies at the core of possible worlds semantics, as long as we take propositions to be sets of possible worlds. Among the solutions proposed in the literature, the most straightforward and orthodox consists in disproving an allegedly intuitive principle about propositions. This kind of solution looks particularly desirable, since it allows to keep both the standard account of modal quantification and the corresponding analysis of propositions as sets of possible worlds.

However, standard possible worlds semantics has another widely recognized problem – it cannot distinguish between intensionally equivalent propositions. In other words, it lacks the resources to account for *hyperintensional* semantic distinctions. In order to deal with this limitation, some philosophers propose to extend the classical

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world-theoretic picture with impossible worlds. This seems a promising move, but it faces a new variant of Kaplan's paradox, as long as the background framework validates a weaker principle about propositions.

In the present work, we first introduce the original paradox and briefly discuss its main solutions (Section 2). Then, we illustrate how its new variant affects theories of propositions as sets of possible and impossible worlds (Section 3). Finally, we explore two general paths to a possible solution: giving up on sets (Section 4) and giving up on worlds (Section 5), either in the background semantic framework or in the corresponding account of propositions. As a result, we conclude that abandoning worlds by embracing a truthmaker-based approach offers a promising way to account for hyperintensional propositions without facing the paradoxical outcome.

2 The Original Paradox and its Solutions

Kaplan (1995) claims that standard possible worlds semantics for a propositional modal logic extended with quantifiers, identity and propositional variables conflicts with a somewhat intuitive principle about propositions:

(K) $\forall p \Diamond \forall s (Qs \leftrightarrow p = s)$

Since p and s are propositional variables, (K) says that for every proposition p, there is at least one possible world where p alone has Q. An informal example of Q, proposed by Davies (1981) and Lewis (1986), is the property of 'being thought by someone at time t'. If Q is assigned this meaning, then (K) amounts to the claim that every proposition could be the only proposition that is thought by someone at the given time t.

The intuitive acceptability of (K) is challenged by the following argument:

(1) There is a set W of all possible worlds.
(2) p is a proposition $:= p \in \wp(W)$, namely, each subset of W is a proposition.
(3) Modal operators are quantifiers that range over W.
(4) Propositional variables and quantifiers range over $\wp(W)$.
(5) For every p, there is at least one $w \in W$ where p alone has Q.
(6) $\therefore |W| \geq |\wp(W)|$.

Premise (5) comes from accepting (K), and together with (1)–(4) entails (6), which says that there are at least as many worlds as propositions. However, (6) violates Cantor's Theorem. To quickly see this, suppose that the cardinality of W is $|W|$. Then, the cardinality of $\wp(W)$ is $2^{|W|}$. But if we accept (1)-(5), we have to conclude that the cardinality of W is at least $2^{|W|}$, which contradicts our assumption.

According to Kaplan, the source of the problem is to be found in possible worlds semantics itself. He stresses that (K) strikes us as an intuitive truth, so we should give up on the framework that invalidates it. In assessing Q, he writes: ≪I can think of no plausible reason why logic itself should refute the existence of such a property≫ (pp. 42-43). In light of this, Kaplan proposes to drop the picture of propositions associated with possible worlds semantics, in favor of a *ramified* account. We will not dive into the technical details here, but the basic idea behind ramification is to impose *orders* on propositions, namely, some hierarchy that forbids to speak of 'all' propositions in the unrestricted sense – this amounts to denying at least premise (4) of the paradoxical argument. Such a move might have some advantages, but it requires to complicate the model quite drastically. Furthermore, as noticed by Anderson (2009), it faces the threat of being self-refuting: if we cannot speak about all propositions, then we cannot say that *all* propositions have orders.[1]

On the other hand, Lewis (1986) argues that we should not bother with properties like Q. Sticking to its informal reading, he claims that it is simply not the case that any set of possible worlds is the content of some possible thought. So, there cannot be such a property, and (K) is to be rejected:

> most sets of worlds, in fact all but an infinitesimal minority of them, are not eligible contents of thought. It is absolutely impossible that anybody should think a thought with content given by one of these ineligible sets of worlds (p. 105).

One might object that such a reply is effective only in showing that Q cannot be a property about our thoughts or some other propositional attitude, due to our limited cognitive faculties – obviously, we are not able to grasp the semantic content represented by every single set of worlds. But this fact alone does not rule out the possible existence of alternative readings of Q which do not involve propositional attitudes.

While rejecting (K) by forcing a specific reading on Q might be a questionable move, it is important to highlight that there is no real need for an informal interpretation of that property. Indeed, regardless of our *prima facie* intuitions towards the acceptability of (K), a result by Bueno et al. (2014) shows that (K) is actually a logical falsehood within the same framework adopted by Kaplan. By introducing a quantified version of the T-axiom $\forall p(\Box p \rightarrow p)$ and a simple comprehension schema for propositions $\exists r(r \leftrightarrow \varphi)$ for any formula φ of the language in which r does not occur free, the following turns out to be a theorem:

(NK) $\quad \exists p \Box \neg \forall s(Qs \leftrightarrow p = s)$

We can read (NK) as follows: for some proposition p, it is necessary that p does not uniquely have Q.[2] This is obviously equivalent to the negation of (K), as we can easily show through the interdefinability between quantifiers and modal operators: $\exists p \Box \neg \forall s(Qs \leftrightarrow p = s)$ is equivalent to $\exists p \neg \Diamond \forall s(Qs \leftrightarrow p = s)$, which is in turn equivalent to $\neg \forall p \Diamond \forall s(Qs \leftrightarrow p = s)$. By transitivity, $\exists p \Box \neg \forall s(Qs \leftrightarrow p = s)$ is then equivalent to $\neg \forall p \Diamond \forall s(Qs \leftrightarrow p = s)$. A similar argument can be found in Anderson (2009), which presents a different proof of (NK).

While the debate has evolved to encompass various perspectives on the severity of the threat posed by Kaplan's paradox, for the current purposes it is sufficient to emphasize the availability of a purely logical solution to it. Indeed, the main strategies employed to avoid the paradox without dropping the possible worlds account of propositions tend to agree with Lewis in denying (K), i.e. step (5) of the argument presented

[1] See Bacon et al. (2016) for a detailed discussion of the ramified approach.

[2] See Bueno et al. (2014), p. 25, footnote 40, for the full proof.

above. This saves the standard account of modal quantification and the identification of propositions with sets of possible worlds – maybe at the expense of some *prima facie* intuition.

3 Impossible Worlds Semantics and Kaplan's Paradox

In recent years, new formal tools have emerged to address a family of problems related to *hyperintensionality*. As it is widely recognized, possible worlds semantics collapses the distinction between necessarily equivalent propositions by identifying them with sets of possible worlds – necessary truths are identified with the set of all possible worlds, and impossibilities are identified with the empty set. Places in sentences that are sensitive to the distinction between necessary equivalents are called hyperintensional contexts. Within them, substitution *salva veritate* of intensionally equivalent expressions may fail.

Among the strategies adopted to deal with this phenomenon, we find extensions of standard possible worlds semantics which add *impossible* worlds to the recipe.[3] Impossible worlds are worlds in which some impossible state of affairs obtains. For the current purposes, we will consider an account of semantic content committed to the finest grain of distinction between necessary equivalents, in order to have a general account of propositions in terms of worlds, both possible and impossible. This amounts to including worlds that are not closed under any nontrivial logical rule, as well as worlds where metaphysically necessary equivalents such as Hesperus and Phosphorus are not one and the same object, along the lines of the accounts proposed by Yagisawa (2010) and Jago (2015). [4]

The conceptual core of such approaches is the identification of a proposition p with the set of (possible and impossible) worlds at which it is the case that p. This allows for a simple explanation of a variety of hyperintensional phenomena. For instance, consider the paradigmatic case of propositional attitude reports: it is possible that Jim believes that Hesperus is bright without believing that Phosphorus is bright. Within an impossible worlds semantics, the mismatch in truth-value between 'Jim believes that Hesperus is bright' and 'Jim believes that Phosphorus is bright' is accounted for by the existence of some impossible worlds where Hesperus is not identical to Phosphorus, so that 'Hesperus is bright' and 'Phosphorus is bright' are identified with two different sets, and thus they express two distinct propositions. Generally speaking, an account of propositions of this kind is able to capture very fine-grained distinctions between intensionally equivalent propositions. However, as first noticed by Yagisawa, it faces another serious challenge. [5]

[3] See for instance Rantala (1982); Priest (1992); Berto and Jago (2019).

[4] The main differences between the two accounts concern the ontological status of the worlds to which they are committed (in Jago's account worlds are representational *ersatz* entities, while in Yagisawa's account they are real just like the actual world); and the room for more than two truth-values (Yagisawa identifies propositions with tuples of sets of worlds instead of simply sets of worlds for this reason). The relative advantages of one approach over the other are not relevant for the present discussion.

[5] Kripke (2011) also seems to be aware of the issue. In assessing the original Kaplan's paradox, he writes: ≪if someone has a more fine-grained notion of proposition than a set of possible worlds, this only makes the problem worse≫ (p. 374).

Suppose that we extend the semantics of our modal propositional language with impossible worlds. This brings a new commitment in the truth condition for impossible statements: $\neg\Diamond\varphi$ will still be true if and only if φ is false at every *possible* world. But now, in any model fine-grained enough to meet our hyperintensional needs, there must be also some *impossible* world at which φ is true. Moreover, even if it is still false that, for every proposition p, it is possible that p uniquely has Q – as established by proving (NK) – a weaker principle seems true: [6]

(Y) $\quad \forall p(\Diamond\forall s(Qs \leftrightarrow p = s) \vee \neg\Diamond\forall s(Qs \leftrightarrow p = s))$

That is, for every proposition p, it is either possible or impossible that p uniquely has Q. In other words, for every proposition p, there is at least one world (either possible or impossible) where p alone has Q. Notice that, while this particular reading is enabled only by the presence of impossible worlds in the semantics, (Y) would be valid even in the standard framework.

Now, let W be the set of all possible worlds and I be the set of all impossible worlds. It seems that we can build a new paradoxical argument as follows:

(1_i) There is a set $W \cup I$ of all worlds.
(2_i) p is a proposition $:= p \in \wp(W \cup I)$.
(3_i) Modal operators are quantifiers that range over $W \cup I$.
(4_i) Propositional variables and quantifiers range over $\wp(W \cup I)$.
(5_i) For every p, there is at least one $w \in W \cup I$ where p alone has Q.
(6_i) $\therefore |W \cup I| \geq |\wp(W \cup I)|$.

This time, the paradox-triggering premise (5_i) comes from accepting (Y). However, as stressed by Yagisawa, (Y) is simply an instance of a truism: that a given fact about propositions is either possible or impossible. So, denying it on purely logical grounds looks far from easy. While the truth of (K) in the original setup was a contentious matter, (Y) seems relatively innocuous, so if we want our framework to properly capture modal facts, we expect it to be valid.

Furthermore, in this case we cannot appeal to the alleged impossibility of Q as a way to sidestep the paradox. In order to see this, suppose with Davies and Lewis that Q is the property of 'being thought by someone at time t'. If it is not the case that any set of worlds gives the content of a *possible* thought, as Lewis claims, then any proposition that *cannot* be thought is thought at some impossible world. But then, for every proposition p, there must be at least a possible or an impossible world at which p is thought by someone at time t. In other words, here the alleged impossibility of Q would be captured precisely by the existence of impossible worlds at which Q is instantiated by some proposition. (Y) adds a uniqueness constraint on the instantiation of Q regardless of its modal status, so (Y) cannot be rejected simply by stressing that Q is impossible.

Let us now turn to a potential reaction on behalf of friends of impossible worlds. One might stipulate that, at each impossible world where Q is uniquely instantiated, Q is both uniquely instantiated *and not* uniquely instantiated (by some proposition p). This amounts to allowing for more than one proposition to uniquely have Q at the

[6] (Y) is our original formalization, Yagisawa (2010) only states the principle informally.

same impossible world. After all, impossible worlds are not required to be closed under any logical rule, so we do not need to concede that there must be at least one world for each proposition – an individual impossible world may account for the "unique" instantiation of Q by more than one proposition (even all of them). This should suffice to block the paradox. [7]

But now recall that (NK) entails the existence of propositions that, necessarily, do not uniquely have Q. Suppose that p_1 is such a proposition and consider the following pair of sentences:

(a) p_1 uniquely has Q.
(b) p_1 uniquely has Q and p_1 does not uniquely have Q.

Both (a) and (b) are necessarily false, and they will be true at some impossible worlds. But if we adopt a model that satisfies the constraint described above (if a proposition p uniquely has Q at some impossible world w, then p does not uniquely have Q at w), (a) and (b) turn out to be true at the exact same impossible worlds. Thus, according to such a model, they would express the same content.

However, while (a) and (b) may be intensionally equivalent, they clearly express two different propositions, and impossible worlds were introduced precisely for tracking hyperintensional distinctions – including those between necessary falsehoods. In other words, if we want to keep the semantic distinction between sentences like (a) and (b), we want them to be identified with two different sets of worlds. But if we introduce the constraint to avoid the paradox, the model lacks the expressive power to account for this distinction. Since we have no independent reason to weaken the framework in a way that goes explicitly against the main point for having impossible worlds, the introduction of such constraints looks like a clumsy *ad hoc* move.

At this point, the friend of impossible worlds might either bite the bullet and claim that sentences like (a) and (b) do in fact express the same proposition, or go against the received view and attempt to develop an even more sophisticated framework which invalidates (NK), in order to allow sentences like (a) to be true at some possible worlds. We will not explore this latter option here.

Summing up, no matter what model-theoretic preferences one might have, (1_i)–(6_i) seems to constitute a serious challenge to the analysis of propositions as sets of possible and impossible worlds associated with impossible worlds semantics. In what follows, we will explore some alternative strategies to avoid this new version of Kaplan's Paradox (HKP from now on), without giving up on a hyperintensional picture of propositions.

4 Giving up on Sets

The only explicit attempt to avoid (HKP) is presented by Yagisawa (2010). His solution consists in denying that it is possible to speak of a set of all propositions: [8]

[7] We are indebted to Matteo Plebani for suggesting us this possible reply.

[8] In both his account of propositions and his phrasing of the paradox Yagisawa speaks of 'collections', which he takes to be a more neutral notion than 'set'.

> I reject the absolutely unrestricted sense of a quantifying expression. [...] It is improper to speak of the collection of all propositions in the absolutely unrestricted sense of 'all'. We may not suppose the cardinality of such a collection to be anything at all (p. 238).

In other words, Yagisawa's solution amounts to denying at least (4_i): if there is no set of all propositions, then propositional quantifiers do not range over $\wp(W \cup I)$ and the paradoxical inference is blocked (notice that, if Yagisawa rejects *any* use of 'all' in the unrestricted sense, then it is also improper to speak of a set of all worlds, which would amount to denying (1_i) as well).

The prohibition to speak of a set of all propositions seems to be motivated primarily by worries concerning the problem of absolute generality.[9] One might think that the domains at stake here – that of worlds and that of propositions – are *restricted* generalizations, namely, universal generalizations about absolutely every member of a comparatively limited kind. *Prima facie*, such generalizations do not seem to quantify over everything in the absolutely unrestricted sense, which is how the debate is typically framed. [10] But as we are about to see, this is not a real concern for the current purposes.

Instead of speaking of the set of all propositions, Yagisawa proposes to restrict the range of propositional quantification to specific collections of propositions, defined by the *type* of content that they express (e.g. a collection T_1 of propositions with non-modal intentional content, a collection T_2 of propositions with modal intentional content, and so on). Then, he claims that such collections of different types of propositions shall not be mixed together to form a single collection. Yagisawa does not provide independent reasons to justify the hierarchy, and it is easy to notice how his move resembles ramification. Indeed, an analogous objection might be moved against it: if we cannot speak of all propositions, then we cannot say that all of them have types.

Yagisawa might then reply that we can speak of the properties of each member of a collection without speaking of the whole collection as a single entity (a set or a class). In order to achieve this, an obvious option is to employ plural quantification. Yagisawa does not explicitly take this path, but it may be interesting for the current purposes to explore how to use pluralities to solve (HKP). So, let us try to sketch a plural approach to the problem and check whether it fares better than its set-theoretic counterpart.

Plural quantifiers do not bear a commitment to sets. Instead, they range directly over *pluralities* of entities, without taking them as individual objects. Assuming a plurality ww of all worlds, both possible and impossible, each sub-plurality of ww will represent a proposition. We can now replace our standard propositional quantifiers and modal operators with plural propositional quantifiers and plural modal operators. The latter will range over ww, while propositional quantifiers and variables will range

[9] See Yagisawa (2010), p. 55.

[10] Nevertheless, Williamson (2003) argues that even restricted generalizations ultimately need to quantify over absolutely everything.

over the sub-pluralities of ww, which taken together form the super-plurality www. [11]

Even with our new plural domains in play, we can stick to (Y) as an adequate formalization of the principle that, for every proposition p, there is at least one world where p alone has Q. [12] This new setup enables a rephrasing of the paradoxical argument:

(1_p) There is a plurality ww of all worlds.
(2_p) pp is a proposition $:= pp \preceq www$, namely, pp is a sub-plurality of ww.
(3_p) Modal operators range over ww.
(4_p) Propositional variables and quantifiers range over www.
(5_p) For every proposition pp, there is at least one world where pp alone has Q.
(6_p) $\therefore ww \geq www$.[13]

At first glance, (6_p) is just as paradoxical as (6) and (6_i). But now one might argue that, since there are no sets involved in (1_p)–(5_p), cardinality concerns are not in play anymore, therefore we shall simply conclude that there are in fact at least as many worlds as propositions.

That conclusion would be too hasty. Florio and Linnebo (2021) present two proofs of a plural equivalent of Cantor's Theorem, according to which for any plurality with two or more members, its sub-pluralities are strictly more numerous than its members. More precisely, given our plurality of worlds ww and our corresponding super-plurality www of propositions, there is no surjection from the former to the latter and no injection from the latter to the former. But this means that the only scenarios where (1_p)–(5_p) does not lead to a contradictory outcome are those where there are fewer than two worlds in the model. Obviously, such models would not bear sufficient expressive power for any semantic need, so any reasonable model would eventually run into paradox. It seems that, even if we gave up on sets in favor of plural domains, a plural variant of (HKP) would nevertheless survive.

An alternative option in the spirit of abandoning sets could be assuming that, instead of forming a set, the collection of all propositions forms a proper class. If that is the case, it cannot be identified with $\wp(W \cup I)$, as long as $W \cup I$ is itself considered a set. Hence, the collection of all worlds should not be a set as well.[14] There might be even some independent reasons to believe this, as Kripke (2011) notices:

> it seems to me to be reasonable to suppose [...] that for every cardinality κ it is possible that there are exactly κ individuals. But then it would immediately follow that the possible worlds cannot form a set (p. 378).

[11] A super-plurality is a higher-order plurality, or a *plurality of pluralities*. Despite its being a controversial notion (see Florio and Linnebo (2021) for an in-depth discussion), for the current purposes it is fit to capture the analogy with the notion of powerset in order to sketch a plural approach to the problem.

[12] For the sake of argument, we assume that such a smooth transition from the standard framework to the plural one is available, but this might not be so obvious. A possible example in this direction is Hewitt (2012), which adds modal operators to PFO+, the same framework in which Florio and Linnebo (2021) prove a plural version of Cantor's Theorem.

[13] This is meant to be read as follows: the plurality of worlds is equally or more numerous than the plurality of propositions. In other words, there must be at least as many worlds as propositions.

[14] This particular view is defended in Pruss (2001).

The point obviously extends to the collection of all worlds, since it includes the collection of all possible worlds. We might then conclude that $W \cup I$ should be a proper class as well. This would require to adopt an unorthodox framework where proper classes are employed as domains of quantification, as long as we want to formally capture the full range of hyperintensional distinctions among propositions. One might then argue that this would simply shift the problem to a higher level of mathematical abstraction, but we shall not explore this issue here.

One way or another, giving up on sets requires drastic and complex revisions to the framework, which do not seem to guarantee a straightforward solution to the paradox. In light of this, a safer way of approaching the issue may consist in revising the ontology of worlds instead, as we shall see in the next section.

5 Giving up on Worlds

An easier way to avoid (HKP) may not lie in the disposal of sets, but in the disposal of *worlds*, either from the background semantic framework or from the theory of propositions (assuming that an account of propositions can be given independently from a background semantics). The former approach amounts to denying (1_i) and, consequently, (2_i), (3_i) and (4_i). The latter amounts to denying at least (2_i) and (4_i).

5.1 Modal Quantification without Worlds

The first path is explicitly taken by Dunaway (2013), who develops an account of modal operators that is not committed to an ontology of worlds. According to Dunaway, by introducing primitive quantification into predicate position, paired with a primitive hyperintensional connective, we can have the benefits of modal quantification without committing to worlds: a lighter ontology at the cost of a more complex ideology.

Within this framework, modal expressions like $\Diamond p$ are analyzed as second-order equivalents of 'there is some possible way for things to be W, and things being W entails p'. *Ways* are maximal predicates that might have been instantiated, and the entailment relation is a hyperintensional relation $\Rightarrow$ employed to avoid the notion of truth-at-a-world. It roughly says that, for a maximal predicate W to be instantiated, p has to obtain. So, the definitions of modal operators will be the following:

$\Diamond p := \exists W$ things are $W \Rightarrow p$.
$\Box p := \forall W$ things are $W \Rightarrow p$.

Quantifiers are treated as primitive in order to avoid any kind of ontological commitment – according to this analysis, there is no domain over which they range. This means that the maximal predicates adopted here do not specify a corresponding set of properties to ground the truth of the predications. This may sound counterintuitive, but Dunaway claims that there might be independent reasons to accept primitive second-order quantification, in addition to the advantage of discharging our modal talk from the commitment to problematic entities.[15] As far as (HKP) is concerned, this proposal amounts to successfully denying (1_i) and thus avoiding the paradox.

Nonetheless, the need for a primitive hyperintensional connective '⇒' is problematic. As clarified by Dunaway:

> by 'hyperintensional' I mean simply substituting cointensional arguments of the connective does not necessarily preserve truth value (p. 168).

The intension of a piece of language, at least in the context of formal semantics, is usually defined as a function from *worlds* to objects. If Dunaway's understanding of hyperintensionality involves an explicit reference to intensions, how are we supposed to understand them in a world-free setting like the one he is proposing? They cannot be functions from maximal predicates, since such predicates do not define a domain of objects, as stressed by Dunaway himself. But appealing to a primitive notion of intensionality, which is contextually employed as a technical term, does not seem to be a viable option. Since hyperintensional resources are required in his own account of modal quantification, Dunaway should not adopt a world-theoretic notion of hyperintensionality in the first place, on pain of contradiction. Moreover, his definition of hyperintensionality appeals also to necessity, which is the very phenomenon that is supposed to be analyzed in terms of modal quantification. In other words, it seems that he may also have a problem of circularity.

A further limitation of this proposal is that it cannot give an account of propositions in terms of sets of worlds, as explicitly recognized by Dunaway:

> if we give up on quantification over worlds, we cannot say that worlds are constituents of propositions (p. 166).

This may not be a problem in its own right, but it means that the account is not suited for the task of delivering a hyperintensional picture of propositions: it does not provide the resources for distinguishing between cointensional propositions, since such distinction is already assumed in the analysis of modal operators.

5.2 Propositions as Truthmaker Conditions

If, in the context of modal quantification, kicking the worlds out by the door only invites them back through the window, we can still try to deal with the paradox by focusing on the theory of propositions first. An alternative way to avoid (HKP) while keeping a hyperintensional picture of propositions may then consist in switching directly to an account that does not identify them with sets of worlds. However, since we may still want worlds to account for modal quantification, a background semantic framework which allows to uniformly recapture the notion of world is arguably to be preferred.

We may then try to modify the framework in order to allow partial entities to represent the truth conditions of a sentence, following the tradition of relevance logics and situation semantics. In what follows, we will focus on a specific approach of

[15] In particular, Dunaway appeals to the existence of Ostrich Nominalism, the Quinean idea according to which simple predication does not need to be analyzed in terms of properties. If that is the case, then second-order generalizations likewise can be free of such an analysis.

this kind, which has been recently endorsed within the context of exact truthmaker semantics: the theory of propositions as *truthmaker conditions*. [16]

In the previous sections we identified a proposition with the set containing all and only the worlds at which it obtains. This is defined by a characteristic function taking worlds as input and giving 'true' or 'false' as output. A truthmaker condition is a characteristic function that takes also possible partial entities as input (rather than only complete worlds), and gives 'yes' or 'no' as output. A 'yes' answer means that the input entity is a possible truthmaker for the proposition in question, and this amounts to identifying a proposition p with the set $|p|^+$ of all its possible (i.e. actual and non-actual) truthmakers. [17]

How do propositions as truthmaker conditions avoid (HKP)? Answering by merely pointing out that, according to this picture, (2_i) and (4_i) are false (since propositions are not sets of worlds anymore) would not be enough. As observed above, we may adopt a principle of ontological parsimony to the effect that, in order to account for modal quantification, we build a space of worlds using the entities already employed for analyzing propositions – namely, possible truthmakers. As a number of authors have suggested, we can identify worlds with *maximal* truthmakers. We can characterize a maximal truthmaker as a state s such that, for every proposition p, either s verifies p or s verifies its negation. [18]

The set of worlds as maximal truthmakers W will include impossible worlds as well, since the maximality constraint by itself does not rule out maximal *inconsistent* truthmakers.[19] Furthermore, a general consistency constraint on truthmakers seems to go against our hyperintensional needs, since inconsistent truthmakers are arguably required in order to provide a sufficiently fine-grained picture of semantic content.

But then, if *any* set of truthmakers qualifies as a proposition, propositional variables and quantifiers will range over the powerset $\wp(T)$ of the set of all truthmakers T. As a result, we would face a new, truthmaker-based variant of (HKP):

(1_t) There is a set T of all truthmakers.
(2_t) There is a set $W \subset T$ of all worlds (the set of all maximal truthmakers).

[16] While Jago (2017) has the only explicit account of propositions of this kind currently in the literature, it is a picture that naturally follows from the adoption of a full-fledged truthmaker semantics, like the one presented in Fine (2017b). However, here we do not assume that a theory of propositions must always depend on a background semantics.

[17] This kind of propositions are not 'fully' hyperintensional: even if they distinguish between necessary truths, they might not be able to distinguish between all necessary falsehoods. For instance, $a \wedge \neg a$ and $b \wedge \neg b$ might not differ with respect to their truthmakers but might differ with respect to their falsitymakers. In order to track such distinctions as well, we might choose to pair the sets of possible truthmakers with the sets of possible falsitymakers for each proposition, resulting in *double* propositions. A double proposition p is then a pair $\{|p|^+; |p|^-\}$ such that $|p|^+$ is the set of all its truthmakers and $|p|^-$ is the set of all its falsitymakers. For the sake of simplicity, in what follows we will focus on a picture with single propositions only.

[18] See for instance Plantinga (1978) and Restall (1996), as well as the idea of a *modalized* state space in Fine (2017a).

[19] A maximal inconsistent truthmaker can be characterized as a maximal truthmaker that includes at least two incompatible truthmakers, where two truthmakers s and t are said to be incompatible if and only if their fusion $s \sqcup t$ is an impossible truthmaker. Of course, this kind of characterization requires to employ a primitive notion of possibility in the metaphysics of truthmakers. See Fine (2017a) for a discussion.

(3_t) p is a proposition $:= p \in \wp(T)$, namely, p is a subset of T.
(4_t) Modal operators are quantifiers that range over W.
(5_t) Propositional variables and quantifiers range over $\wp(T)$.
(6_t) For every p, there is at least one $w \in W$ where p alone has Q.
(7_t) $\therefore |W| \geq |\wp(T)|$.

The problem here would be that, since W is a proper subset of T, its cardinality must be strictly smaller than the cardinality of $\wp(T)$.

One might think that this is not a serious worry: we should simply keep the task of finding a suitable semantics for modal logic separate from the task of providing a theory of propositions in terms of truthmakers – even though the adoption of a semantic framework often motivates the adoption of a corresponding theory of propositions. But even if we take a less liberal stance towards the interdependence of those two tasks, the just sketched account may still have access to the resources for avoiding the paradox.

A fundamental feature of a truthmaker space is that it must be endowed with a mereological structure, which determines a partial order over T. In other words, truthmakers can be fused together and be parts of each other, according to a reflexive, transitive and anti-symmetric relation on T. Given this formal structure, if we want to extract an account of propositions from subsets of T, there are some reasonable conditions that we may want to place on them.

Where s, t and u are arbitrary truthmakers in T, in order for a subset of T to count as a proposition, we may require it to be (i) upwards closed with respect to fusion: if $s, t \in |p|^+$, then their fusion $s \sqcup t \in |p|^+$; and (ii) *convex*: if $s, u \in |p|^+$ and some part t of u has s as a part, then $t \in |p|^+$. Since it is not the case that every subset of T satisfies such conditions, we can deny (3_t) and (5_t): even if we take worlds to be maximal truthmakers, (HKP) is apparently rejected.

Now, there seems to be independent justification for placing such conditions for propositionhood. [20] However, as remarked also by Fine (2017a), employing them weakens the expressive power of the account. If our goal is to provide a picture of semantic content that captures the full range of hyperintensional distinctions, restricting the domain of propositions in this setting might backfire similarly to how placing constraints on impossible worlds backfired in Section 3.

Moreover, regardless of what criteria for propositionhood we might place, we still need to be careful in employing the set of worlds *qua* maximal truthmakers as the domain for modal quantification. With some constraints in play, propositions will represent only a proper subset of $\wp(T)$ – for instance, those which conform to the criteria discussed above. Let us call the resulting set of all propositions R.

If the singleton of each world counts as a proposition (i.e. world-singletons are closed under fusion and convex), then the set of world-singletons S must be a proper subset of R. Obviously, S and W have the same cardinality. Now, either S has the same cardinality of R, or the cardinality of R is strictly greater than the cardinality of S. The former case does not lead to problems with Cantor's Theorem, but it may still appear undesirable since it forces to admit that there are exactly as many worlds as

[20] See Fine (2017a) and Jago (2017).

propositions. On the other hand, if we want to resist that conclusion and assume that $|R|$ is strictly greater than $|S|$, we run again into troubles:

(1_{ti}) There is a set $W \subset T$ of all worlds (the set of all maximal truthmakers).
(2_{ti}) There is a set $R \subset \wp(T)$ of all propositions.
(3_{ti}) There is a set $S \subset R$ of all world-singletons.
(4_{ti}) $|R| > |S|$.
(5_{ti}) $|W| = |S|$.
(6_{ti}) Modal operators are quantifiers that range over W.
(7_{ti}) Propositional variables and quantifiers range over R.
(8_{ti}) For every p, there is at least one $w \in W$ where p alone has Q.
(9_{ti}) $|W| \geq |R|$.
(10_{ti}) $\therefore |R| \leq |S|$.

The conclusion contradicts our assumption (4_{ti}). It seems that, as long as we build worlds out of truthmakers and (8_{ti}) is a correct reading of (Y), we run into some variant of the paradox, unless we concede that there must be exactly as many worlds as propositions. At this point, one might suggest to place some further conditions for propositionhood in order to exclude world-singletons from the domain of propositions – a blatant *ad hoc* move that would further reduce the expressive power of the framework.

Luckily, there is an easier way out: all we need to do is to exclude impossible worlds from the domain of modal quantification. We can do so by placing a restricted consistency constraint on maximal truthmakers: in order for a truthmaker to count as a (possible) world, it must be maximal and it must not contain incompatible truthmakers. If our domain of modal quantification contains only *possible* worlds, then (6_t) and (8_{ti}) will not be acceptable readings of (Y) anymore. It would still be true that, for every proposition p, it is either possible or impossible that p uniquely instantiates Q. But this would not amount to claiming that, for every p, there is at least one world at which p uniquely has Q. The reason is pretty obvious: there will not be impossible worlds in the account, so modal quantifiers will behave just like they do in the classical setting. More precisely, $\neg\Diamond\varphi$ will be true if and if only if there is no possible world at which φ is true. As a result, in the new framework the validity of (Y) is guaranteed by the fact that for every p, at no possible world (i.e. maximally consistent truthmaker) p uniquely has Q.

Notice that, unlike the restriction on impossible worlds discussed in Section 3, the just sketched solution would not constitute an *ad hoc* move – we simply do not need impossible worlds anymore. Recall that, in Section 3, impossible worlds were introduced precisely to provide a hyperintensional account of propositions. But here we already did that in terms of truthmaker conditions. Worlds have been subsequently defined in order to account for modal quantification, but they were not required in the first place. Within this setting, we can effectively provide the truth conditions for modal claims in terms of standard quantification over possible worlds, identified with maximally consistent truthmakers, without losing expressive power.

We close this section by highlighting two further advantages of the solution just sketched: first, it does not force to concede that there are exactly as many worlds as propositions. And, most importantly, it is not committed to any particular constraint for propositionhood. This allows for attaining the full expressive power guaranteed by the identification of semantic contents with sets of truthmakers, at no additional cost.

Conclusion

We presented a version of Kaplan's paradox that affects one of the most popular hyperintensional accounts of propositions – propositions as sets of possible and impossible worlds. After discussing why the main solutions proposed for the original Kaplan's paradox fall short when applied to its hyperintensional variant, we explored two general paths to a possible solution: giving up on sets and giving up on worlds.

Our diagnosis is that abandoning sets altogether in favor of plural quantification is not a viable option, and the same applies to abandoning worlds in the analysis of modal operators. On the other hand, giving up on worlds in the account of propositions in favor of a partial notion of truth-supporting circumstances allows for a simple solution to the paradox. Most importantly, this is true even if we opt to keep worlds in the account of modal quantification.

The present work aimed to explore the interaction between modal semantics and related conceptions of propositions, and it was not meant to defend any particular approach. In order to fulfill the need for both a hyperintensional account of propositions and a worldly picture of modal quantification (which represent the two key ingredients for the paradox), here we focused on the theory of propositions associated with exact truthmaker semantics, which allows for a straightforward recapturing of the classical notion of (im)possible world. However, our case study does not rule out that approaches rooted in alternative semantic backgrounds could also effectively avoid the paradox.

Therefore, it may be worthy to explore how different hyperintensional accounts (such as structured propositions, two-dimensional semantics and related Fregean approaches) interact with the paradox. We leave questions concerning how it might affect them and their possible solutions open for future work.

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Declarations

Competing Interests The author has no competing interests to declare that are relevant to the content of this article.

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